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A counterexample for the global separation principle for discrete-time nonlinear systems

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Abstract

In the control systems literature, it is well known that a separation principle holds *locally* for nonlinear control systems, when exponential feedback stabilizers and exponential observers are used. In this paper, we present a counterexample to show that the global separation principle need not hold for nonlinear control systems. Our example demonstrates that global stability might be lost when an exponential observer is introduced into the nonlinear feedback loop associated with an exponentially stabilizing feedback control law.

Keywords: Separation principle; Nonlinear control systems; Discrete-time systems; Exponential observers

1. Introduction

In the control systems literature, it is well known that in the case of a linear system, the *separation principle* assures that an estimate of the state may be used in lieu of the state provided that the error between the estimate and the actual state decays exponentially [1]. As far as feedback stabilization of nonlinear control systems is concerned, a similar *separation principle* holds locally around a state equilibrium [2,3]. In this paper, we basically establish that the global separation principle need not be true for nonlinear control systems. Explicitly, we present a counterexample to show that global stability might be lost when an exponential observer is introduced into the feedback loop of the nonlinear control

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system. Our discrete-time example is similar to the results of Glad [4] for continuous-time nonlinear control systems.

2. Main result

In this section, we present our new counterexample for the global separation principle for discrete-time nonlinear control systems.

Consider the scalar discrete-time nonlinear control system described by

$$x_{k+1} = x_k + x_k^3 + u_k, \tag{1}$$

where $x \in \mathbb{R}$ is the *state*, and $u \in \mathbb{R}$ is the *input* of the nonlinear control system. It is easy to see that the system (1) is globally exponentially stabilizable. Indeed, the feedback control law

$$u_k = -x_k - x_k^3 \tag{2}$$

globally exponentially stabilizes the nonlinear plant (1) with the closed-loop dynamics

$$x_{k+1} = 0. \tag{3}$$

Note that the closed-loop dynamics (3) is globally exponentially stable.

Now, we assume that the state x is replaced by an estimate z from a nonlinear observer. Let the estimation error e be defined by

$$e \triangleq x - z.$$

Then the observer-based feedback control law is given by

$$u_k = -z_k - z_k^3 = -(x_k - e_k) - (x_k - e_k)^3. \tag{4}$$

Assume that the observer error e decays exponentially according to the dynamics

$$e_{k+1} = \alpha e_k, \qquad (0 < \alpha < 1). \tag{5}$$

Note that the observer-based control law (4) leads to

$$x_{k+1} = x_k + x_k^3 - (x_k - e_k) - (x_k - e_k)^3$$

or

$$x_{k+1} = 3x_k e_k (x_k - e_k) + e_k + e_k^3.$$

Consider the composite dynamics

$$\begin{aligned} x_{k+1} &= 3x_k e_k (x_k - e_k) + e_k + e_k^3, \\ e_{k+1} &= \alpha e_k. \end{aligned} \tag{6}$$

Next, consider the quantity

$$\mu = xe.$$

Then we have

$$\mu_{k+1} = 3\alpha \mu_k (\mu_k - e_k^2) + \alpha e_k^2 + \alpha e_k^4. \tag{7}$$

Define the set

$$M_a \triangleq \left\{(x, e) \in R^2 : x > 0, 0 < e \leq a, \mu \geq a^2 + \frac{1}{3\alpha}\right\}.$$

We claim that the set M_a is invariant under the flow of the composite system (6). This can be seen easily from an induction argument.

Let $(x_0, e_0) \in M_a$. Assume that $(x_k, e_k) \in M_a$ for some non-negative integral value of k. We shall establish that $(x_{k+1}, e_{k+1}) \in M_a$ as well.

By the induction hypothesis, it follows that

$$x_k > 0, \qquad 0 < e_k \leq a \qquad \text{and} \qquad \mu_k \geq a^2 + \frac{1}{3\alpha}. \tag{8}$$

To show that $(x_{k+1}, e_{k+1}) \in M_a$, we must show that

$$x_{k+1} > 0, \qquad 0 < e_{k+1} \leq a \qquad \text{and} \qquad \mu_{k+1} \geq a^2 + \frac{1}{3\alpha}. \tag{9}$$

Now, by (8), it follows that $x_k > 0$ and $\mu_k = x_k e_k \geq a^2 + \frac{1}{3\alpha}$. Hence, we have

$$x_k \geq \frac{a^2 + \frac{1}{3\alpha}}{e_k}.$$

Therefore,

$$x_k - e_k \geq \frac{a^2 + \frac{1}{3\alpha}}{e_k} - e_k = \frac{a^2 + \frac{1}{3\alpha} - e_k^2}{e_k} > 0$$

as $0 < e_k \leq a$.

Since $x_k - e_k > 0$, $x_k > 0$ and $e_k > 0$, it is immediate from the dynamics (6) that

$$x_{k+1} = 3x_k e_k(x_k - e_k) + e_k + e_k^3 > 0.$$

Next, as $e_{k+1} = \alpha e_k$ with $0 < \alpha < 1$ and $0 < e_k \leq a$, it is immediate that

$$0 < e_{k+1} \leq a.$$

Finally, as $\mu_k > 0$, it follows that

$$\begin{aligned}
\frac{\mu_{k+1}}{\mu_k} &= 3\alpha(\mu_k - e_k^2) + \frac{\alpha e_k^2 + \alpha e_k^4}{\mu_k} \\
&\geq 3\alpha(\mu_k - e_k^2) \\
&\geq 3\alpha(\mu_k - a^2) \\
&\geq 1
\end{aligned}$$

where, in the last inequality, we have used the induction hypothesis (8), which states that $\mu_k \geq a^2 + \frac{1}{3\alpha}$. Hence, it follows that

$$\mu_{k+1} \geq \mu_k \geq a^2 + \frac{1}{3\alpha}.$$

Thus, (9) is proved. By induction, it follows that $(x_k, e_k) \in M_a$ for all positive integral values of k, if $(x_0, e_0) \in M_a$. Hence, M_a is an invariant set under the flow of the composite system (6).

Thus, all the solutions (x_k, e_k) starting in M_a will remain in M_a for all values of time. Since the error dynamics $e_{k+1} = \alpha e_k$ is globally exponentially stable, we know that

$$e_k \to 0 \text{ exponentially as } k \to \infty \text{ for all } e_0 \in \mathbb{R}.$$

Note also that

$$\mu_k = x_k e_k \geq a^2 + \frac{1}{3\alpha} > a^2.$$

Hence, if $(x_0, e_0) \in M_a$, then it is immediate that $x_k \to \infty$ as $k \to \infty$. Since this holds for any $a > 0$, we conclude that the system (6) fails to be globally stable even if the initial observation error e_0 is arbitrarily small. We note, however, that the closed-loop control system is locally exponentially stable as the linearization matrix of the composite system (6) at $(x, e) = (0, 0)$ is given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & \alpha \end{bmatrix}$$

which has the eigenvalues 0 and α both of which are inside the open unit disc $|\lambda| < 1$ of the complex plane.

Our example essentially illustrates that global stability might be lost even when we use an exponential observer with arbitrarily small exponential decay. Hence, the global separation principle need not hold for nonlinear control systems.

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