Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®

ELSEVIER

Journal of Combinatorial Theory, Series B 90 (2004) 139–145

Journal of Combinatorial Theory Series B

http://www.elsevier.com/locate/jctb

Reduction of the 5-Flow Conjecture to cyclically 6-edge-connected snarks

Martin Kochol

MÚ SAV, Štefánikova 49, 814 73 Bratislava 1, Slovakia

Received 3 August 2001

Abstract

We show that a smallest counterexample to the 5-Flow Conjecture of Tutte (every bridgeless graph has a nowhere-zero 5-flow) must be a cyclically 6-edge-connected cubic graph.

Keywords: Nowhere-zero 5-flow; Cyclic edge-connectivity; Girth; Snark

1. Introduction

A graph admits a *nowhere-zero k-flow* (k is an integer $\geqslant 2$) if its edges can be oriented and assigned numbers $\pm 1, \ldots, \pm(k-1)$ so that for every vertex, the sum of the incoming values equals the sum of the outgoing ones. It is well-known that a graph with a bridge (1-edge-cut) does not have a nowhere-zero k-flow for any $k \geqslant 2$ (see, e.g., [3,11]). The famous 5-*Flow Conjecture* of Tutte [13] is the statement that every bridgeless graph has a nowhere-zero 5-flow.

An edge cut of a graph is called *cyclic* if deleting its edges results in a graph having at least two cyclic components. A graph is called *cyclically k-edge-connected* if it has no cyclic cut of cardinality smaller than k.

It is well-known (see cf. [3]) that a smallest counterexample to the 5-Flow Conjecture must be a *snark* which is a cyclically 4-edge-connected cubic graph without an edge-3-coloring and with girth (the length of the shortest cycle) at least 5. Furthermore by Celmins [1], this counterexample must be cyclically 5-edge-connected and have girth at least 7 (see also [3,9]). This result was very interesting because until recently, no snarks with girth at least 7 were known. Indeed, Jaeger and

E-mail address: kochol@savba.sk.

0095-8956/$ - see front matter
doi:10.1016/S0095-8956(03)00080-7

Swart [4] conjectured that such snarks do not exist. In [6] we disproved this conjecture constructing cyclically 5-edge-connected snarks with arbitrary large girth (see also [11]).

In this paper we prove that a smallest counterexample to the 5-Flow Conjecture must be cyclically 6-edge-connected. Furthermore, in an accompanied paper [10] we have proved that it must have girth at least 9. Since all cyclically 6-edge-connected snarks known until now have girth 6 (see [2,5,7,11,12], such a counterexample belongs to a class of graphs for which we do not know whether it is empty.

2. Preliminaries

The graphs considered in this paper are all finite and undirected. Multiple edges and loops are allowed. If G is a graph, then $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. By a *multi-terminal network*, briefly a *network*, we mean a pair (G, U) where G is a graph and $U = (u_1, \dots, u_n)$ is an n-tuple of pairwise distinct vertices of G. If no confusion can occur, we denote by U also the set $\{u_1, \dots, u_n\}$. The vertices from U and $V(G)\backslash U$ are called the *outer* and *inner* vertices of the network (G, U), respectively. We allow $n = 0$, i.e., $U = \emptyset$.

We associate with each edge of G two distinct arcs, distinct for distinct edges (see also [11]). If one of the arcs corresponding to an edge is denoted by x, the other is denoted by x^{-1}. If the ends of an edge e are the vertices u and v, one of the arcs corresponding to e is said to be *directed from u to v* (and the other from v to u). In particular, a loop corresponds to two distinct arcs both directed from a vertex to itself. Let $D(G)$ denote the set of arcs on G. Then $|D(G)| = 2|E(G)|$. If $v \in V(G)$, then $\omega_G(v)$ denotes the set of arcs of G directed from v to $V(G)\backslash\{v\}$.

If G is a graph and A is an additive abelian group, then an *A-chain* in G is a mapping $\varphi : D(G) \to A$ such that $\varphi(x^{-1}) = -\varphi(x)$ for every $x \in D(G)$. Furthermore, the mapping $\partial\varphi : V(G) \to A$ such that

$$\partial\varphi(v) = \sum_{x \in \omega_G(v)} \varphi(x) \quad (v \in V(G))$$

is called the *boundary* of φ. An A-chain φ in G is called *nowhere-zero* if $\varphi(x) \neq 0$ for every $x \in D(G)$. If (G, U) is a network, then an A-chain φ in G is called an *A-flow* in (G, U) if $\partial\varphi(v) = 0$ for every inner vertex v of (G, U). The following statement is proved in [8,11].

Lemma 1. *If φ is an A-flow in a network (G, U), then $\sum_{u \in U} \partial\varphi(u) = 0$.*

By a (*nowhere-zero*) *A-flow* in a graph G we mean a (nowhere-zero) A-flow in the network $(G, \emptyset)$. Our concept of nowhere-zero flows in graphs coincides with the usual definition of nowhere-zero flows as presented in [3]. By Tutte [13,14], a graph has a nowhere-zero k-flow if and only if it has a nowhere-zero A-flow for some Abelian group A of order k. Thus the study of nowhere-zero 5-flows is in certain sense equivalent with the study of nowhere-zero $\mathbb{Z}_5$-flows. We shall use this fact and deal only with $\mathbb{Z}_5$-flows because they are easier to handle than integral flows.

3. Simple networks

A network (G,U), $U=(u_1,\dots,u_n)$, is called *simple* if the vertices $u_1,\dots,u_n$ have valency 1. If φ is a nowhere-zero $\mathbb{Z}_5$-flow in (G,U), then denote by $\partial\varphi(U)$ the n-tuple $(\partial\varphi(u_1),\dots,\partial\varphi(u_n))$. By Lemma 1, $\partial\varphi(U)$ belongs to the set

$$S_n=\{(s_1,\dots,s_n);\ s_1,\dots,s_n\in\mathbb{Z}_5-0,\ s_1+\cdots+s_n=0\}.$$

For every $s\in S_n$, denote by $\Phi_{G,U}(s)$ the set of nowhere-zero $\mathbb{Z}_5$-flows φ in (G,U) satisfying $\partial\varphi(U)=s$ and define $F_{G,U}(s)=|\Phi_{G,U}(s)|$.

Let $P=\{Q_1,\dots,Q_r\}$ be a partition of the set $\{1,\dots,n\}$. If one of $Q_1,\dots,Q_r$ is a singleton, P is called *trivial*, otherwise it is called *nontrivial*. Let $\mathscr{P}_n$ denote the set of nontrivial partitions of $\{1,\dots,n\}$.

If $s=(s_1,\dots,s_n)\in S_n$, $P=\{Q_1,\dots,Q_r\}\in\mathscr{P}_n$, and $\sum_{i\in Q_j}s_i=0$ for $j=1,\dots,r$, then we say that P and s are *compatible*. (For example, $\{\{1,2\},\{3,4,5\}\}\in\mathscr{P}_5$ is compatible with $(1,4,1,2,2)\in S_5$.)

If $|\mathscr{P}_n|=m$, then $\mathscr{P}_n$ can be considered as an m-tuple $(P_{n,1},\dots,P_{n,m})$. For any $s\in S_n$, denote by $\boldsymbol{\chi}_n(s)$ the integral vector $(\chi_{s,1},\dots,\chi_{s,m})$ such that $\chi_{s,i}=1$ $(\chi_{s,i}=0)$ if $P_{n,i}$ is (is not) compatible with s, $i=1,\dots,m$.

Lemma 2. *Let (G,U), $U=(u_1,\dots,u_n)$, be a simple network and $|\mathscr{P}_n|=m$. Then there exists an integral vector $\mathbf{x}_{G,U}=(x_1,\dots,x_m)$ such that for every $s\in S_n$, $F_{G,U}(s)=\sum_{i=1}^{m}\chi_{s,i}\cdot x_i$ where $(\chi_{s,1},\dots,\chi_{s,m})=\boldsymbol{\chi}_n(s)$.*

Proof. We can assume that no two outer vertices of (G,U) are joined by an edge (otherwise subdivide each such edge by a new vertex of valency 2) and G has no isolated vertex. For such networks we apply induction on $|E(G)|\geqslant n$.

If $|E(G)|=n$, then G is bipartite with partition sets $\{u_1,\dots,u_n\}$ and $\{v_1,\dots,v_r\}$. Thus, there exists a partition $P=\{Q_1,\dots,Q_r\}$ of $\{1,\dots,n\}$ such that $i\in Q_j$ iff u_i is adjacent to v_j ($i\in\{1,\dots,n\}$, $j\in\{1,\dots,r\}$). If P is trivial (i.e., (G,U) has an inner vertex of valency 1), then $F_{G,U}(s)=0$ for every $s\in S_n$, and we can choose $\mathbf{x}_{G,U}$ to be the zero vector of $\mathbb{Q}^m$. If $P\in\mathscr{P}_n$, then there exists $j\in\{1,\dots,m\}$ such that $P=P_{n,j}$. Now $F_{G,U}(s)=\chi_{s,j}$ for every $s\in S_n$. Thus $\mathbf{x}_{G,U}$ equal the jth standard basis vector of $\mathbb{Q}^m$ satisfies the assumptions of lemma.

If $|E(G)|>n$, then there exists an edge e of G with ends $v_1,v_2\notin U$.

Assume that $v_1\neq v_2$. Let $G-e$ and G/e be the graphs arising from G after deleting and contracting e, respectively. We claim that for every $s\in S_n$,

$$F_{G,U}(s)=F_{G/e,U}(s)-F_{G-e,U}(s). \tag{1}$$

Clearly, any $\varphi \in \Phi_{G-e,U}(s)$ can be considered as a flow from $\Phi_{G/e,U}(s)$ and any $\varphi \in \Phi_{G,U}(s)$ can be transformed to exactly one flow from $\Phi_{G/e,U}(s)$. On the other hand, any $\varphi \in \Phi_{G/e,U}(s)$ can be considered as a $\mathbb{Z}_5$-chain in $G-e$, which is a nowhere-zero $\mathbb{Z}_5$-flow in the network $(G-e,(v_1,v_2,u_1,\dots,u_n))$, whence by Lemma 1, $\partial\varphi(v_1)+\partial\varphi(v_2)=0$. If $\partial\varphi(v_1)=\partial\varphi(v_2)=0$, then φ is from $\Phi_{G-e,U}(s)$, otherwise φ can be extended to exactly one flow from $\Phi_{G,U}(s)$. In this way, we get a bijective mapping from $\Phi_{G/e,U}(s)$ to $\Phi_{G,U}(s)\cup\Phi_{G-e,U}(s)$. This implies (1) because $\Phi_{G,U}(s)\cap\Phi_{G-e,U}(s)=\emptyset$.

We have $|E(G)|>|E(G/e)|,|E(G-e)|$. Thus, by the induction hypothesis, there are integral vectors $\mathbf{x}_{G/e,U}=(x'_1,\dots,x'_m)$ and $\mathbf{x}_{G-e,U}=(x''_1,\dots,x''_m)$ such that for every $s\in S_n$, $F_{G/e,U}(s)=\sum_{i=1}^m \chi_{s,i}x'_i$ and $F_{G-e,U}(s)=\sum_{i=1}^m \chi_{s,i}x''_i$, whence by (1), $F_{G,U}(s)=F_{G/e,U}(s)-F_{G-e,U}(s)=\sum_{i=1}^m \chi_{s,i}(x'_i-x''_i)$. Thus the vector $\mathbf{x}_{G,U}=\mathbf{x}_{G/e,U}-\mathbf{x}_{G-e,U}$ satisfies the assumptions of lemma.

Assume that $v_1=v_2$, i.e., e is a loop of G. Then every nowhere-zero $\mathbb{Z}_5$-flow in $G-e$ can be extended to exactly four nowhere-zero $\mathbb{Z}_5$-flows in G. Thus for every $s\in S_n$, $F_{G,U}(s)=4\cdot F_{G-e,U}(s)$, whence the vector $\mathbf{x}_{G,U}=4\mathbf{x}_{G-e,U}$ satisfies the assumptions of lemma. □

More details about this topic can be found in [10].

Let $\mathscr{A}$ denote the automorphism group of $\mathbb{Z}_5$. The elements of $\mathscr{A}$ are $\alpha_0=\mathrm{id}$, $\alpha_1=(1,2,4,3)$, $\alpha_2=(1,4)(2,3)$ and $\alpha_3=(1,3,4,2)$. If $s=(s_1,\dots,s_n)\in S_n$ and $\alpha\in\mathscr{A}$, then denote $\alpha(s)=(\alpha(s_1),\dots,\alpha(s_n))\in S_n$. Clearly, $\boldsymbol{\chi}_n(s)=\boldsymbol{\chi}_n(\alpha(s))$, whence by Lemma 2, $F_{G,U}(s)=F_{G,U}(\alpha(s))$ for every simple network (G,U) with n outer vertices.

4. The main result

$\mathscr{P}_5$ contains the following partitions (considering the sums $i+1$, $i+2$, $i+3$, $i+4 \bmod 5$):

$$P_{5,i}=\{\{i,i+1\},\{i+2,i+3,i+4\}\},\quad (i=1,\dots,5),$$
$$P_{5,5+i}=\{\{i,i+2\},\{i+1,i+3,i+4\}\},\quad (i=1,\dots,5),$$
$$P_{5,11}=\{1,2,3,4,5\}.$$

If $s=(s_1,\dots,s_5)\in S_5$, then define $\pi(s)=(s_5,s_1,s_2,s_3,s_4)$. Let $\mathbf{e}_i$ denote the ith standard basis vector of $\mathbb{Q}^{11}$.

Theorem 3. *Suppose that G is a counterexample to the* 5*-Flow Conjecture of minimal order. Then G is a cyclically* 6*-edge-connected snark.*

Proof. By Celmins [1], G is a cyclically 5-edge-connected snark. Suppose that G has a cyclic 5-edge cut $\{f_1,\dots,f_5\}$. Then $G-\{f_1,\dots,f_5\}$ has exactly two components G'

and G'', which are cyclic and bridgeless. For $i = 1, \dots, 5$, let v_i' and v_i'' be the end of f_i contained in G' and G'', respectively. Add to G' (G'') five vertices $u_1, \dots, u_5$ ($w_1, \dots, w_5$) and, for $i = 1, \dots, 5$, join u_i with v_i' (w_i with v_i''). We get from G' (G'') a new graph X (Y). Consider the simple networks (X, U), $U = (u_1, \dots, u_5)$, and (Y, W), $W = (w_1, \dots, w_5)$.

$F_{X,U}(s)$, $F_{Y,W}(s) \geqslant 0$ for every $s \in S_5$. If there exists $s \in S_5$ such that $F_{X,U}(s)$, $F_{Y,W}(s) > 0$, then (X, U) and (Y, W) have nowhere-zero $\mathbb{Z}_5$-flows φ_1 and φ_2, respectively, such that $\partial\varphi_1(U) = s$ and $\partial\varphi_2(W) = \alpha_2(s)$ (because $F_{Y,W}(\alpha_2(s)) = F_{Y,W}(s) > 0$) which can be "pieced together" into a nowhere-zero $\mathbb{Z}_5$-flow in G, a contradiction. Thus $F_{X,U}(s) \cdot F_{Y,W}(s) = 0$ for every $s \in S_5$.

By Lemma 2, there exist integers x_i and y_i, $i = 1, \dots, 11$, such that for every $s \in S_5$, $F_{X,U}(s) = \sum_{i=1}^{11} \chi_{s,i} x_i$ and $F_{Y,W}(s) = \sum_{i=1}^{11} \chi_{s,i} y_i$ where $(\chi_{s,1}, \dots, \chi_{s,11}) = \boldsymbol{\chi}_5(s)$.

Now $F_{X,U}(1,1,1,1,1) = x_{11}$ and $F_{Y,W}(1,1,1,1,1) = y_{11}$. Thus $x_{11}, y_{11} \geqslant 0$ and $x_{11} \cdot y_{11} = 0$. Without loss of generality we can assume $x_{11} = 0$.

Suppose that $x_i \geqslant 0$ for every $i = 1, \dots, 10$. If $x_1 = \cdots = x_{11} = 0$, then $F_{X,U}(s) = 0$ for every $s \in S_5$. Identify u_1 with u_2 and u_3 with u_4, u_5 in X and suppress the vertex of valency 2. The resulting cubic graph is bridgeless, has order smaller than G, and does not have a nowhere-zero 5-flow (otherwise $F_{X,U}(s) > 0$ for some $s \in S_5$), which contradicts the minimality of G. Hence at least one x_i must be positive. We can choose the ordering of edges $f_1, \dots, f_5$ so that $x_1 > 0$. Thus $F_{X,U}(s) > 0$ and $F_{Y,W}(s) = 0$ if $s \in S_5$ and $\boldsymbol{\chi}_5(s)$ has first coordinate 1. Then identifying w_1 with w_2 and w_3 with w_4, w_5 in Y and suppressing the vertex of valency 2 we get a bridgeless cubic graph without a nowhere-zero 5-flow and of order smaller than G, a contradiction.

Therefore at least one x_i is negative. We can choose the ordering of edges $f_1, \dots, f_5$ so that $x_1 < 0$. Consider

$$
\begin{aligned}
p_1 &= (1,4,1,2,2), \quad \boldsymbol{\chi}_5(p_1) = \mathbf{e}_{11} + \mathbf{e}_1 + \mathbf{e}_2,\\
p_2 &= (1,4,2,1,2), \quad \boldsymbol{\chi}_5(p_2) = \mathbf{e}_{11} + \mathbf{e}_1 + \mathbf{e}_7,\\
p_3 &= (1,4,2,2,1), \quad \boldsymbol{\chi}_5(p_3) = \mathbf{e}_{11} + \mathbf{e}_1 + \mathbf{e}_{10},\\
p_4 &= (4,1,1,2,2), \quad \boldsymbol{\chi}_5(p_4) = \mathbf{e}_{11} + \mathbf{e}_1 + \mathbf{e}_6,\\
p_5 &= (4,1,2,1,2), \quad \boldsymbol{\chi}_5(p_5) = \mathbf{e}_{11} + \mathbf{e}_1 + \mathbf{e}_9,\\
p_6 &= (4,1,2,2,1), \quad \boldsymbol{\chi}_5(p_6) = \mathbf{e}_{11} + \mathbf{e}_1 + \mathbf{e}_5.
\end{aligned} \tag{2}
$$

Since $F_{X,U}(p_i) \geqslant 0$ for $i = 1, \dots, 6$, we have $x_2, x_5, x_6, x_7, x_9, x_{10} \geqslant -x_1 - x_{11} = -x_1 > 0$. If one of x_3, x_4, x_8 is negative, we can choose the ordering of edges f_3, f_4, f_5 so that $x_3 < 0$. For $i = 1, \dots, 6$, replacing p_i with $\pi^2(p_i)$ in (2) and using the fact that $F_{G,U}(\pi^2(p_i)) \geqslant 0$, we get $x_2, x_4, x_6, x_7, x_8, x_9 \geqslant -x_3 - x_{11} = -x_3 > 0$. Thus, without abuse of generality, we can assume that exactly one of the following cases occurs:

(i) $x_1 < 0$, $x_2, x_5, x_6, x_7, x_9, x_{10} \geqslant -x_1$, and $x_3, x_4, x_8 \geqslant 0$;
(ii) $x_1, x_3 < 0$, $x_2, x_5, x_6, x_7, x_9, x_{10} \geqslant -x_1$, and $x_2, x_4, x_6, x_7, x_8, x_9 \geqslant -x_3$.

Let S be the set of permutations $(s_1, s_2, s_3, s_4, s_5)$ of $1, 1, 1, 3, 4$. S is a proper subset of S_5 (for instance $(1, 1, 1, 1, 1)$ and $(1, 1, 2, 2, 4)$ belong to S_5 but not to S). We claim that $F_{X,U}(s) > 0$ for every $s \in S$.

Let $s = (s_1, \ldots, s_5) \in S$. Then $\boldsymbol{\chi}_5(s) = \mathbf{e}_a + \mathbf{e}_b + \mathbf{e}_c + \mathbf{e}_{11}$ and $F_{X,U}(s) = x_a + x_b + x_c$ where a, b, c are pairwise distinct elements from $\{1, \ldots, 10\}$ such that partitions $P_{5,a}, P_{5,b}, P_{5,c}$ are of the form $\{\{i_1, i_2\}, \{i_3, i_4, i_5\}\}$ where $s_{i_1} + s_{i_2} = 0$.

If $a = 1$, i.e., $s_1 + s_2 = 0$, then $b, c \notin \{3, 4, 8\}$ (otherwise at least one of the sums $s_3 + s_4$, $s_4 + s_5$, $s_3 + s_5$ equals 0, whence at least one of s_5, s_3, s_4 is 0, a contradiction). Since $x_2, x_5, x_6, x_7, x_9, x_{10} \geqslant -x_1$, we have $x_a + x_b + x_c \geqslant -x_1 > 0$.

Let $a, b, c \neq 1$ and case (i) occur. Then $\{3, 4, 8\} \neq \{a, b, c\}$ (otherwise $s_3 + s_4 = s_4 + s_5 = s_3 + s_5 = 0$, whence $s_3 = s_4 = s_5 = 0$, a contradiction). Thus x_a, x_b, x_c are nonnegative integers and at least one of them is positive, whence $x_a + x_b + x_c > 0$.

Let $a, b, c \neq 1$ and case (ii) occur. If $3 \in \{a, b, c\}$, then we can choose the ordering of edges $f_1, \ldots, f_5$ so that we get the case $a = 1$. If $3 \notin \{a, b, c\}$, then x_a, x_b, x_c are positive, and so is $x_a + x_b + x_c$.

Therefore for every $s \in S$, $F_{X,U}(s) > 0$ and $F_{Y,W}(s) = 0$. Consider

$$\begin{aligned} p_7 &= (4, 3, 1, 1, 1), \quad \boldsymbol{\chi}_5(p_7) = \mathbf{e}_{11} + \mathbf{e}_5 + \mathbf{e}_6 + \mathbf{e}_9, \\ p_8 &= (4, 1, 3, 1, 1), \quad \boldsymbol{\chi}_5(p_8) = \mathbf{e}_{11} + \mathbf{e}_5 + \mathbf{e}_1 + \mathbf{e}_9, \\ p_9 &= (4, 1, 1, 3, 1), \quad \boldsymbol{\chi}_5(p_9) = \mathbf{e}_{11} + \mathbf{e}_5 + \mathbf{e}_1 + \mathbf{e}_6. \end{aligned} \tag{3}$$

Since $p_7, p_8, p_9 \in S$, we have

$$\begin{aligned} 0 &= F_{Y,W}(p_7) - F_{Y,W}(p_8) = y_6 - y_1, \\ 0 &= F_{Y,W}(p_7) - F_{Y,W}(p_9) = y_9 - y_1, \\ 0 &= F_{Y,W}(p_8) - F_{Y,W}(p_9) = y_9 - y_6. \end{aligned} \tag{4}$$

Therefore $y_1 = y_6 = y_9$. For $i = 1, 2, 3, 4$, replacing p_7, p_8, p_9 with $\pi^i(p_7)$, $\pi^i(p_8)$, $\pi^i(p_9)$, respectively, in (3) and (4) we get $y_2 = y_7 = y_{10}$, $y_3 = y_8 = y_6$, $y_4 = y_9 = y_7$, $y_5 = y_{10} = y_8$, whence $y_1 = \cdots = y_{10}$. Furthermore, $F_{X,U}(1, 1, 2, 2, 4) = x_5 + x_{10} + x_{11} > 0$ in both cases (i) and (ii). Thus $0 = F_{Y,W}(1, 1, 2, 2, 4) = y_5 + y_{10} + y_{11}$ and $0 = F_{Y,W}(1, 1, 1, 3, 4) - F_{Y,W}(1, 1, 2, 2, 4) = y_8$. Therefore $y_1 = \cdots = y_{11} = 0$ and we get a smaller counterexample in a similar way as in the case $x_1 = \cdots = x_{11} = 0$. This proves the statement. □

5. Concluding remarks

If $n \in \{2, 3\}$, then $\mathscr{P}_n$ contains exactly one partition and $\boldsymbol{\chi}_n(s) = (1)$ for every $s \in S_n$. Thus, by Lemma 2, $F_{G,U}(s) = F_{G,U}(s')$ for every $s, s' \in S_n$ and every simple network (G, U) with n outer vertices. This implies that a smallest counterexample to the 5-flow conjecture must be cyclically 4-edge-connected (see also [3]).

Since the results of Celmins [1] are not published, we also sketch a proof of the statement that a smallest counterexample to the 5-flow conjecture has no cyclic

4-edge cut. Otherwise in a similar way as in Theorem 3 construct simple networks (X, U), $U = (u_1, \dots, u_4)$, and (Y, W), $W = (w_1, \dots, w_4)$, which satisfy $F_{X,U}(s)$, $F_{Y,W}(s) \geqslant 0$ and $F_{X,U}(s) \cdot F_{Y,W}(s) = 0$ for every $s \in S_4$. $\mathscr{P}_4$ contains partitions $P_{4,1} = \{\{1,2\},\{3,4\}\}$, $P_{4,2} = \{\{2,3\},\{4,1\}\}$, $P_{4,3} = \{\{1,3\},\{2,4\}\}$, and $P_{4,4} = \{\{1,2,3,4\}\}$. By Lemma 2, there exist integers $x_1, \dots, x_4$ and $y_1, \dots, y_4$ such that for every $s \in S_4$, $F_{X,U}(s) = \sum_{i=1}^4 \chi_{s,i} x_i$ and $F_{Y,W}(s) = \sum_{i=1}^4 \chi_{s,i} y_i$ where $(\chi_{s,1}, \dots, \chi_{s,4}) = \boldsymbol{\chi}_4(s)$. Since $F_{X,U}(1,1,1,2) = x_4$ and $F_{Y,W}(1,1,1,2) = y_4$, we have $x_4, y_4 \geqslant 0$ and $x_4 \cdot y_4 = 0$. Suppose that $x_4 = 0$. Then $F_{X,U}(1,4,2,3) = x_1 \geqslant 0$, $F_{X,U}(1,2,3,4) = x_2 \geqslant 0$, $F_{X,U}(1,2,4,3) = x_3 \geqslant 0$. If $x_1 = \cdots = x_4 = 0$, then identifying u_1 (u_3) with u_2 (u_4) in X and suppressing the vertices of valency 2, we get a smaller counterexample. Therefore, at least one x_i must be positive and without loss of generality we can assume that $x_1 > 0$. Then $F_{X,U}(s) > 0$ and $F_{Y,W}(s) = 0$ if $\boldsymbol{\chi}_4(s)$ has first coordinate 1 $(s \in S_4)$. Thus identifying w_1 (w_3) with w_2 (w_4) in Y and suppressing the vertices of valency 2 we get a smaller counterexample, concluding the proof.

References

[1] U.A. Celmins, On cubic graphs that do not have an edge-3-colouring, Ph.D. Thesis, Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada, 1984.

[2] R. Isaacs, Infinite families of nontrivial trivalent graphs which are not Tait colorable, Amer. Math. Monthly 82 (1975) 221–239.

[3] F. Jaeger, Nowhere-zero flow problems, in: L.W. Beineke, R.J. Wilson (Eds.), Selected Topics in Graph Theory 3, Academic Press, New York, 1988, pp. 71–95.

[4] F. Jaeger, T. Swart, Conjecture 1, in: M. Deza, I.G. Rosenberg (Eds.), Combinatorics 79, Annals of Discrete Mathematics, Vol. 9, North-Holland, Amsterdam, 1980, p. 305.

[5] M. Kochol, Constructions of cyclically 6-edge-connected snarks, Technical Report TR-II-SAS-07/93-05, Institute for Informatics, Slovak Academy of Sciences, Bratislava, Slovakia, 1993.

[6] M. Kochol, Snarks without small cycles, J. Combin. Theory Ser. B 67 (1996) 34–47.

[7] M. Kochol, A cyclically 6-edge-connected snark of order 118, Discrete Math. 161 (1996) 297–300.

[8] M. Kochol, Hypothetical complexity of the nowhere-zero 5-flow problem, J. Graph Theory 28 (1998) 1–11.

[9] M. Kochol, Cubic graphs without a Petersen minor have nowhere-zero 5-flows, Acta Math. Univ. Comenian. (N.S.) 68 (1999) 249–252.

[10] M. Kochol, Forbidden subgraphs for the smallest counterexample to the 5-flow conjecture, preprint 9/2001, Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia (http://www.mat.savba.sk/preprints).

[11] M. Kochol, Superposition and constructions of graphs without nowhere-zero k-flows, European J. Combin. 23 (2002) 281–306.

[12] M. Kochol, Constructions of graphs without nowhere-zero flows from Boolean formulas, Ars Combin., to appear.

[13] W.T. Tutte, A contribution to the theory of chromatic polynomials, Canad. J. Math. 6 (1954) 80–91.

[14] W.T. Tutte, A class of Abelian groups, Canad. J. Math. 8 (1956) 13–28.