A Maximum Principle for Nonsmooth Optimal-Control Problems with State Constraints

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Optimality conditions are derived in the form of a maximum principle governing solutions to an optimal control problem which involves state constraints. The conditions, which apply in the absence of differentiability assumptions on the data, are stated in terms of Clarke's generalized Jacobians. Although not the most general available, the conditions are derived by a novel method: this involves removal of the state constraints by introduction of a penalty term and application of Ekeland's variational principle.

1. INTRODUCTION

In [3] Clarke derives a maximum principle associated with the control problem

$$\min \left\{ \int_0^1 l(t, x(t), u(t)) \, dt : \dot{x}(t) = f(t, x(t), u(t)), x(0) \in C_0, x(1) \in C_1 \right\}$$

when the functions $l(t, x, u), f(t, x, u)$ are merely Lipschitz continuous in their $x$-dependence. The approach followed in [3] was to obtain optimality conditions by application of Ekeland's variational principle to a related more tractable problem and by use of a limiting process.

Now the results in [3], as they stand, do not apply when state constraints of the form

$$g(t, x(t)) \leq 0, \quad t \in [0, 1]$$

(1.1)

are present. It is the object of this paper to show that Clarke's methodology can be adapted to permit such constraints. Again, a maximum principle is derived via Ekeland's variational principle and a limiting process.

The underlying idea is a simple, indeed, rather obvious one. We replace constraint (1.1) by a penalty term added to the cost
for some $k > 0$. We thereby obtain a problem to which the results of [3] are applicable. The costate equation for this new problem is

$$-\dot{p}(t) \in p(t) \partial_x f + c \partial_x l + ck \partial_x \max\{0, g\}$$

(1.3)

in which the generalized Jacobians $\partial_x f(t, x, u), \partial_x l(t, x, u), \partial_x \max\{0, g(t, x)\}$ are evaluated along the solution. Equation (1.3) may be written

$$-dp(t) \in p(t) \partial_x f + c \partial_x l + ck(\partial_x g) a(t)$$

for some function $a(t) \geq 0$ or, symbolically,

$$-dp(t) \in p(t) \partial_x f dt + c \partial_x l dt + \partial_x g dt dv(t)$$

(1.4)

with

$$dv(t) = cka(t) dt.$$

Equation (1.4) resembles the costate equation for the state-constraint problem of interest. We cannot expect the state-constraint problem to be equivalent to minimization of (1.2). Under an appropriate well-posedness hypothesis (H4), however, the problems are equivalent to within $\epsilon$, for $k$ sufficiently large. We may now use Ekeland's result to obtain the costate equation for the original problem as a limit of Eq. (1.4).

Notice that our approach involves use of optimality conditions for problems involving nondifferentiable functions in an essential way; even if the original problem involves only smooth functions, the cost function with penalty term (1.2) is not differentiable. A selector of the multifunction $ck \partial_x \max\{0, g\}$ supplies the state-constraint multiplier in the limit.

A number of papers have recently appeared ([8, 9, 11]) which provide maximum principles for nonsmooth problems with state constraints. None of these invoke our well-posedness hypothesis (H4). Halkin and Warga introduce conditions which are, in other respects, more restrictive than ours. Ioffe, however, gives a result in [9] which holds under conditions that are more general. Ioffe's result, too, is in a sense more precise than the maximum principle given here.

We shall lay stress then in this paper, not on the conditions under which we shall derive the maximum principle which are not the most general available, but on the novel methods employed whereby optimality conditions for problems with state constraints can be determined from those for state constraint-free problems. The machinery developed here will possibly be
relevant in other contexts, e.g., in deriving a maximum principle for a control problem associated with a differential inclusion (as treated by Clarke [4]), when state constraints are introduced.

2. NOTATION, ETC.

The Borel subsets of \( \mathbb{R}^k \) will be written \( \mathcal{B}^k \). The Lebesgue subsets of \([0, 1]\) are denoted by \( \mathcal{L} \). The terms *measurable* and *almost every* (a.e.) are understood with respect to Lebesgue measure.

We shall refer to signed, Radon measures on \([0, 1]\) briefly, as *measures*. The class of \( k \)-tuples of measures is denoted by \( C^*(\mathbb{R}^k) \) (or simply \( C^* \)) and the subclass of positive measures by \( C^\oplus(\mathbb{R}^k) \) (or simply \( C^\oplus \)).

All norms are written \( | \cdot | \). The norm is the Euclidean norm for points \( s \in \mathbb{R}^k \), \( (\sum_j s_{ij}^2)^{1/2} \) for points \( s = \{s_{ij}\} \) in the space of \( l \times k \) matrices and the sum of the total variations of the components of \( s \) for \( s \in C^*(\mathbb{R}^k) \).

Take \( \mu \in C^\oplus \). A Borel set \( B \subset [0, 1] \) is a \( \mu \) continuity set \([2]\) if \( \mu(B) = 0 \). Here \( \partial B \) is the boundary of \( B \).

A useful property of the continuity sets is the following: Let \( \{\mu_1, \mu_2, \ldots\} \) be a countable family in \( C^\oplus \). Let \( \mathcal{G}_i \) be the collection of \( \mu_i \) continuity sets, \( i = 1, 2, \ldots \). Then \( \bigcap_i \mathcal{G}_i \) generates the Borel sets. This property is easily derived from the fact that \( \mathcal{G}_i \) contains the sets

\[
[0, 1], \quad [0, b), \quad (a, b), \quad \text{and} \quad (a, 1],
\]

where \( a \) and \( b \) belong to the complement of the countable set of points \( e \) such that \( \{e\} \) is a \( \mu_i \) atom.

Let \( g: \mathbb{R}^k \rightarrow \mathbb{R}^l \) be locally Lipschitz continuous at the point \( s \). The generalized Jacobian \([5]\) \( \partial g(s) \) of \( g \) at \( s \) is defined as the convex hull of the set of accumulation points of sequences \( \{Dg(s_i)\} \), where we consider all sequences \( \{s_i\} \) converging to \( s \) such that the usual Jacobian matrices \( Dg(s_i) \), \( i = 1, 2, \ldots \), exist.

We note that if \( g \) is Lipschitz continuous in some neighbourhood of the point \( s \), with Lipschitz constant \( L \), then \( |g| \leq L \) for all \( y \in \partial g(s) \).

Now let \( E \) be a closed subset of \( \mathbb{R}^k \). We introduce the Lipschitz continuous function

\[
d_e(x) = \min \{|x - y| : y \in E\}.
\]

Let \( e \) be a point in \( E \). We define the cone of normals \( N(e) \) and the extended cone of normals \( N^+_e(e) \) at \( e \) as

\[
N_e(e) = \{ p : sp \in \partial d_e(e), s > 0 \}
\]

\[
N^+_e(e) = \{ x : (x, e) \in \text{closure graph} \{ a \rightarrow N_e(a) \} \}
\]
3. A Maximum Principle

We are given functions $f: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $l: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $g: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, closed sets $C_0, C_1, X \subseteq \mathbb{R}^n$, a closed set $I \subseteq [0, 1]$, and a multifunction $U: [0, 1] \to \mathbb{R}^m$.

Consider the differential equation with control term
\[ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [0, 1]. \] (3.1)

A control is a measurable function $u$ such that
\[ u(t) \in U(t) \quad \text{a.e. } t \in [0, 1]. \]

A trajectory (corresponding to control $u$) is an absolutely continuous function $x$ which satisfies (3.1). A pair $(u, x)$ of a control $u$ and a corresponding trajectory $x$ is an admissible pair when
\[ x(0) \in C_0, \quad x(1) \in C_1, \quad \text{and } x(t) \in X \quad \text{for all } t \in [0, 1]. \]

We say that $(u, x)$ is interior if, additionally, $x(t)$ is contained in the interior of $X$, for all $t \in [0, 1]$. We study the following control problem:

\[ (P) \quad \min \left\{ \int_0^1 l(t, x(t), u(t)) \, dt : g(t, x(t)) \leq 0, \, t \in I \right\}. \]

It is understood that minimization is conducted over admissible pairs $(u, x)$ for which $t \to l(t, x(t), u(t))$ is integrable. A minimizing pair is termed a solution of $(P)$.

We shall approach derivation of optimality conditions through study of the family of problems $P_k, k \geq 0$,

\[ (P_k) \quad \min \left\{ \int_0^1 l(t, x(t), u(t)) \, dt + k \int_t g^+(t, x(t)) \, dt \right\} \]

(the function $g^+$ is defined as $g^+(x, t) = \max\{0, g(x, t)\}$). Again the minimization is conducted over admissible pairs. We denote the value of problem $(P_k)$ by $\inf \{P_k\}$, etc.

A maximum principle which governs solutions to problem $(P)$ will be obtained under the following hypotheses:

(H1) There exists an integrable function $\alpha$ and a number $K$ such that, for all $s_1, s_2$ in a neighbourhood of $X$, $t \in [0, 1]$, $u \in U(t)$
\[ |f(t, s_1, u) - f(t, s_2, u)| + |l(t, s_1, u) - l(t, s_2, u)| \leq \alpha(t) |s_1 - s_2| \]
and
\[ |g(t, s_1) - g(t, s_2)| \leq K |s_1 - s_2|. \]

(H2) For each \( s \) in a neighbourhood of \( X \), \( l(t, s, u) \) are measurable with respect to the product \( \sigma \)-algebra \( \mathcal{L} \times \mathcal{B}^m \), and \( t \to g(t, s) \) is upper semicontinuous.

(H3) The graph of \( U \) is \( \mathcal{L} \times \mathcal{B}^m \)-measurable.

(H4) \( \lim_{k \to \infty} \inf |P_k| = \inf |P| \).

**Theorem 3.1.** Let \((v, z)\) be an admissible, interior pair which solves problem \((P)\), and suppose that (H1)–(H4) are true. Then there exist a left continuous function of bounded variation \( p, v \in C^b(\mathbb{R}) \) which has support in \( \{t: g(t, z(t)) = 0\} \cap I \) and a nonpositive number \( c \) (\( p, v, \) and \( c \) not all zero) such that
\[
-p(t) \in p(t),
\]
\[
\partial_x f(t, z(t), v(t)) dt + c \partial_x l(t, z(t), v(t)) dt - \partial_x g(t, z(t)) dv(t), \tag{3.2}
\]
\[
p(0) \in N_{c_0}(z(0)), -p(1) - \partial_x g(t, z(t)) v(\{1\}) \cap N_{c_1}(z(1)) \neq \emptyset, \tag{3.3}
\]
\[
p(t) f(t, z(t), v(t)) + c l(t, z(t), v(t)) = 0. \tag{3.4}
\]

In the theorem \( \partial_x f \) is the generalized Jacobian of \( s \to f(t, s, u) \), etc. The extended cone of normals \( N_{c_0} \) was introduced in Section 2. The multifunction \( \partial_x g \) is defined as
\[
\partial_x g(t, s) = \{ \xi: (\xi, (t, s)) \in \text{closure graph}(t', s') \to \partial_x g(t', s') \}.
\]

The inclusions (3.2) and (3.3) are interpreted as follows: There exist measurable functions \( \Phi, \lambda, \) and a Borel measurable function \( \gamma \) such that
\[
\Phi(t) \in \partial_x f(t, z(t), v(t)), \quad \lambda(t) \in \partial_x l(t, z(t), v(t)) \quad \text{a.e.} \quad t \in [0, 1],
\]
\[
\gamma(t) \in \partial_x g(t, z(t)), \quad v \text{ a.e.}, \quad t \in [0, 1].
\]
and
\[
-p(t) + p(0) = \int_{[0, t]} (p(\tau) \Phi(\tau) + c\lambda(\tau)) d\tau
\]
\[
-\int_{[0, t]} \gamma(\tau) dv(\tau), \quad t \in [0, 1],
\]
\[
p(0) \in N_{c_0}(z(0)), -p(1) - \int_{[1]} \gamma(t) dv(t) \in N_{c_1}(z(1)).
\]
We shall comment on the hypotheses. (H1)–(H3) are the hypotheses under which Clarke derives a maximum principle in the absence of state constraints. Hypothesis (H4) is in essence a condition on the well posedness of the value of $P$ with respect to perturbations of the state constraints. Indeed let us define the function $\theta$ by

$$\theta(\alpha) = \inf \left\{ \int_0^1 l(t, x(t), u(t)) \, dt : (u, x) \text{ admissible}, \int_0^1 g^+(t, x(t)) \, dt \leq \alpha \right\}.$$ 

Then it is easy to show that the conditions

$$\inf \left\{ \int_0^1 l(t, x(t), u(t)) \, dt : (u, x) \text{ admissible} \right\} > -\infty$$

and

$$\lim_{\alpha \to 0} \theta(\alpha) = \theta(0)$$

imply (H4). We may replace (H4) by the hypothesis that the *extended velocity set* is convex, provided that (H2) and (H3) are strengthened somewhat (see section 6).

Our proof of Theorem 3.1 readily adapts to provide a maximum principle for problems which involve a collection of $r$ state constraints [10],

$$g_k(t, x(t)) \leq 0, \quad t \in I_k, \quad k = 1, \ldots, r.$$ 

4. **Preliminary Lemmas**

Our starting point will be a maximum principle for solutions to problem $(P_k)$.

**Lemma 4.1.** Let $(\bar{v}, \bar{z})$ solve problem $(P_k)$. Suppose (H1)–(H3) are true. Then there exist an absolutely continuous function $p$, measurable functions $\Phi, \lambda, \gamma,$ and $\beta,$ and a nonpositive number $c$ such that $|p(0)| + |c| \neq 0$,

$$\Phi(t) \in \partial_x f, \quad \lambda(t) \in \partial_x l, \quad \gamma(t) \in \partial_x g, \quad \text{a.e. } t,$$

(the generalized Jacobians are evaluated at $$(t, \bar{z}(t), \bar{v}(t))$$)

$$\beta(t) \geq 0 \quad \text{a.e. } t \in [0, 1],$$

$$\beta(t) = 0 \quad \text{a.e. } t \notin \{e; \ g(e, \bar{z}(e)) = 0\} \cap I,$$

$$-\dot{p}(t) = p(t) \Phi(t) + c\lambda(t) - \gamma(t) \beta(t),$$

$$p(0) \in N_{C_0}(\bar{z}(0)), \quad -p(1) \in N_{C_1}(\bar{z}(1)), \quad \quad (4.1)$$
and

\[
p(t)f(t, \bar{z}(t), \bar{v}(t)) + \text{cl}(t, \bar{z}(t), \bar{v}(t)) = \max_{u \in U(t)} \{ p(t)f(t, \bar{z}(t), u) + \text{cl}(t, \bar{z}(t), u) \} \quad \text{a.e. } t \in [0, 1]. \tag{4.2}
\]

**Proof.** By [3, Corollary 2; 5, Proposition 5], there exists an absolutely continuous function \( p \) and a nonpositive number \( c \) \((p \text{ and } c \text{ not both zero})\) such that (4.1) and (4.2) are satisfied, and such that

\[
-\dot{\rho}(t) \in p(t)\partial_x f(t, \bar{z}(t), \bar{v}(t)) + c \partial_x I(t, \bar{z}(t), \bar{v}(t)) + k\chi(t) \partial_x g^+(t, \bar{z}(t)),
\]

(4.3)

where \( \chi(t) = 1 \) for \( t \in I \) and 0, otherwise.

We know [6] that

\[
\partial_x g^+(t, s) = \{ y \in \partial_x g(t, s), e \in [0, 1] \}, \quad \text{for } g(t, s) = 0, \quad t \in I,
\]

\[
\subset \{ 0 \}, \quad \text{for } g(t, s) < 0, \quad t \in I.
\]

Let us now define the multifunction \( \Sigma \),

\[
\Sigma(t) = [0, 1], \quad \text{when } g(t, \bar{z}(t)) = 0, \quad t \in I,
\]

\[
= \{ 0 \}, \quad \text{otherwise}.
\]

Define also

\[
G(t, \sigma) = p(t) \Phi + c\lambda + ck\gamma \beta \quad (\sigma = (\Phi, \lambda, \gamma, \beta))
\]

and

\[
\Omega(t) = \partial_x f \times \partial_x I \times \partial_x g \times \Sigma(t)
\]

(the generalized Jacobians are evaluated at \((t, \bar{z}(t), \bar{v}(t))\)). Then (4.3) may be expressed as \( \dot{\rho}(t) \in \{ G(t, \sigma); \sigma \in \Omega(t) \} \).

The function \( G(t, \sigma) \) is measurable in \( t \) and continuous in \( \sigma \). The multifunction \( \Omega \) takes values compact subsets and is measurable, as may be shown.

It follows now from a well-known selection theorem (see, e.g. [2, Theorem 1.7.6]) that there exists a measurable function \( \sigma = (\Phi, \lambda, \gamma, \beta) \) such that \( \sigma(t) \in \Omega(t), \dot{\rho}(t) \in G(t, \sigma(t)) \) a.e. We now define \( \beta(t) = -ck\beta(t) \). The number \( c \) and the functions \( p, \Phi, \lambda, \gamma, \beta \) have the required properties. \( \blacksquare \)

The costate function \( p \) of Theorem 3.1 will emerge as the limit of a sequence of costate functions for the \((P_k)\) problems. We derive now some results which will be useful in justifying the limiting process.
LEMMA 4.2. Let $h : [0, 1] \to \mathbb{R}$ be a continuous function and let $\{\mu_i\}$ be a sequence in $C^\infty(\mathbb{R}^n)$ such that

$$\mu_i \to \mu_0 \quad \text{weakly}^*$$

for some $\mu_0 \in C^\infty$. Then

$$\int_A h(t) \, d\mu_i(t) \to \int_A h(t) \, d\mu_0(t)$$

for all $\mu_0$ continuity sets $A$.

($\mu_0$ continuity set was defined in Section 2.)

Proof. Suppose that $h \geq 0$. (The general case is treated in an obvious way by adding a constant function.) Define $v_i = i = 1, 2, \ldots$, by $d v_i = h \, d \mu_i$. Since $h$ is continuous, $v_i \to v_0$ weakly* in $C^\infty$, with $v_0 = h \, d \mu_0$. Now let $A$ be any $\mu_0$ continuity set. Then $A$ is also a $v_0$ continuity set. By [2, Theorem 2.1] then

$$\int_A d v_i \to \int_A v_0$$

which is just another way of stating the result.

LEMMA 4.3. Let $\{\mu_i\}$ be a sequence in $C^*(\mathbb{R}^n)$ such that

$$\mu_i \to \mu_0 \quad \text{weakly}^*$$

for some $\mu_0 \in C^*(\mathbb{R}^n)$. Then for some subsequence and some countable set $\mathcal{S}$.

$$\int_{[0, t]} d\mu_i \to \int_{[0, t]} d\mu_0, \quad t \in [0, 1] \setminus \mathcal{S}. \quad (4.4)$$

Proof. Let $\mu_i^+$ and $-\mu_i^-$ be the positive and negative measures associated with $\mu_i$ through the Jordan decomposition of $\mu_i$. Since $|\mu_i^+|, |\mu_i^-| \leq |\mu_i|$, the sequences are bounded in total variation and, by limiting attention to a subsequence of $\{\mu_i\}$, we may arrange that $\mu_i^+ \to \mu_0^+, \mu_i^- \to \mu_0^-$ (weakly*) for $\mu_0^+, \mu_0^- \in C^\infty(\mathbb{R}^n)$. We have $\mu_0^- = -\mu_0^-$. Take $\mathcal{S}$ to be the countable set

$$\mathcal{S} = \{t : [0, t) \text{ is not a } \mu_0^+, \mu_0^- \text{ continuity set}\}.$$ 

With this choice of $\mathcal{S}$, (4.4) follows from Lemma 4.2.
Lemma 4.4. Let \( \{\pi_i\} \) and \( \{\eta_i\} \) be sequences in \( C^*(\mathbb{R}^n) \), \( \{q_i\} \) a sequence of measurable vector-valued functions and \( \{\Phi_i\} \) a sequence of measurable matrix-valued functions. It is assumed that there exist \( a_i \in L^1 \) and a number \( K_i \), such that
\[
|q_i(t)| \leq K_i, \quad |\Phi_i(t)| \leq \alpha_i(t) \quad \text{a.e. } t \in [0,1)
\]
\( i = 1, 2, \ldots \). Suppose that
\[
\int_B d\pi_i(t) = \int_B q_i(t) \Phi_i(t) \, dt + \int_B d\eta_i(t)
\]
for every Borel set \( B \), \( i = 1, 2, \ldots \), and
\[
\pi_i \to \pi_0, \quad \eta_i \to \eta_0, \quad \text{weakly}^*
\]
\[
q_i \to q_0, \quad \text{a.e.}
\]
\[
\Phi_i \to \Phi_0, \quad \text{weakly in } L^1
\]
for some \( \pi_0, \eta_0, q_0, \Psi_0 \). Then
\[
\int_B d\pi_0(t) = \int_B q(t) \Phi_0(t) \, dt + \int_B d\eta_0(t)
\]
for every Borel set \( B \).

Proof. Decompose \( \pi_i \) and \( \eta_i \) into their positive and negative components:
\( \pi_i = \pi_i^+ - \pi_i^- \), \( \eta_i = \eta_i^+ - \eta_i^- \). Limiting attention to subsequences we have
\( \pi_i^+ \to \pi_0^+ \), etc., for \( \pi_0^+, \pi_0^-, \eta_0^+ \), \( \eta_0^- \in C^0(\mathbb{R}^n) \), and \( \pi_0 = \pi_0^+ - \pi_0^- \), \( \eta_0 = \eta_0^+ - \eta_0^- \).

Now let \( \mathcal{F} \) be the family of sets which are simultaneously \( \pi_0^+, \pi_0^-, \eta_0^+, \eta_0^- \) continuity sets. Then by Lemma 4.2
\[
\int_B d\pi_i(t) \to \int_B d\pi_0(t), \quad \int_B d\eta_i(s) \to \int_B d\eta_0(s)
\]
for all \( B \in \mathcal{F} \).

For any Borel set \( B \), however,
\[
\int_B q_i(t) \Phi_i(t) \, dt = \int_B q_0(t) \Phi_i(t) \, dt + \int_B (q_i(t) - q_0(t)) \Phi(t) \, dt.
\]
The right-hand side is readily seen to converge to \( \int_B q_0(t) \Phi_0(t) \, dt \) in view of
the hypotheses and by application of the dominated convergence theorem to
the second term. It follows that

$$\int_B d\pi_0(t) = \int_B q_0(t) \Phi_0(t) \, dt + \int_B \eta_0(t)$$  \tag{4.5}

for all $B \subseteq \mathcal{F}$. But $\mathcal{F}$ generates the $\sigma$-algebra of Borel sets of $[0, 1]$ (see comments in Section 2). It follows then that (4.5) holds for all Borel sets $B$.

**Lemma 4.5.** Let $G$ be a function on $[0, 1] \times \mathbb{R}^n$ which takes values compact, convex subsets of $\mathbb{R}^n$. Let $\{v_i\}$ be a sequence in $C^\infty(\mathbb{R})$, $\{x_i\}$ a sequence of continuous $\mathbb{R}^n$-valued functions and $\{\gamma_i\}$ a sequence of $\mathbb{R}^n$-valued Borel measurable functions.

We assume that $G$ has closed graph and that

$$\gamma_i(t) \in G(t, x_i(t)), \quad v_i \text{ a.e., } i = 1, 2, \ldots.$$  \tag{4.6}

Now suppose that the sequence $\{\eta_i\}$ in $C^*(\mathbb{R}^n)$ is defined by

$$d\eta_i = \gamma_i \, dv_i.$$  \tag{4.6}

Suppose further that $v_i \rightharpoonup v_0$ weakly*, that $x_i \to x_0$ uniformly and that for some neighbourhood $S$ of range $\{x_0\}$ there exists a number $K_1$ such that

$$|G(t, s)| \leqslant K_1, \quad \text{all } (t, s) \in [0, 1] \times S.$$  \tag{4.7}

Then there exists a Borel measurable function $\gamma_0$, which is $v_0$ integrable, such that

$$\gamma_0(t) \in G(t, x_0(t)), \quad v_0 \text{ a.e.}$$  \tag{4.8}

and (for some subsequence)

$$\eta_i \rightharpoonup \eta_0 \quad \text{weakly*},$$

where $\eta_0$ is defined by $d\eta_0 = \gamma_0 \, dv_0$.

**Proof.** Since the weak* convergent sequence $\{v_i\}$ is bounded in total variation and since, by (4.7), $|\gamma_i(t)| \leqslant K_1$, $v_i$-a.e. for $i$ sufficiently large, it follows that $\{\eta_i\}$ defined by (4.6) is bounded in total variation.

Decompose $\eta_i$ into its positive and negative components, $\eta_i = \eta_i^+ - \eta_i^-$. Limiting attention to subsequences we have

$$\eta_i \rightharpoonup \eta_0, \quad \eta_i^+ \rightharpoonup \eta_0^+, \quad \eta_i^- \rightharpoonup \eta_0^-$$

for $\eta_0 \in C^*(\mathbb{R}^n)$, $\eta_0^+ \in C^\infty(\mathbb{R}^n)$, and $\eta_0 = \eta_0^+ - \eta_0^-$. We show that $\eta_0$ is absolutely continuous with respect to $v_0$. 

Let $\mathcal{C}$ be the family of Borel sets which are $\eta_0^+$, $\eta_0^-$, and $\nu_0$-continuity sets. Then, by Lemma 4.2, for $E \in \mathcal{C}$,

$$\left| \int_E d\eta_0 \right| = \lim_{i \to \infty} \left| \int_E d\eta_i \right| \leq K_1 \lim_{i \to \infty} \int_E d\nu_i = K_1 \int_E d\nu_0.$$ 

But $\mathcal{C}$ generates the Borel sets. It follows that $\left| \int_E d\eta_0 \right| \leq K_1 \int_E d\nu_0$ for all Borel sets $E$. In other words, $\eta_0$ is $\nu_0$-absolutely continuous as claimed.

By the Radon–Nikodym theorem, there exists a $\mathbb{R}^n$-valued, Borel measurable, $\nu_0$-integrable function $\gamma_0$ on $[0, 1]$ such that

$$\int_E d\eta_0(t) = \int_E \gamma_0(t) \, d\nu_0(t)$$

for all Borel sets $E$. It remains to show that $\gamma_0$ satisfies (4.8).

Fix $q \in \mathbb{R}^n$. The function $s_q(t, x) = \max\limits_{d \in G(t, x)} \{qd: d \in G(t, x)\}$ is upper semicontinuous and bounded above on $[0, 1] \times S$ under the assumptions. It follows [1, p. 222] that there exists a sequence of continuous functions $\{\zeta_j\}$ with the properties

$$s_q(t, x) \leq \zeta_j(t, x) \quad \text{all} \quad (t, x) \in [0, 1] \times S, \quad j = 1, 2, \ldots \quad (4.9)$$

and

$$\lim_{j \to \infty} \zeta_j(t, x) = s_q(t, x), \quad (t, x) \in [0, 1] \times S. \quad (4.10)$$

Choose $E \subset \mathcal{C}$. We have, by (4.9),

$$q \int_E d\eta_0(t) \leq \int_E s_q(t, x_i(t)) \, d\nu_i(t) \leq \int_E \zeta_j(t, x_i(t)) \, d\nu_i(t).$$

It follows that

$$q \int_E d\eta_0(t) \leq \int_E \zeta_j(t, x_0(t)) \, d\nu_i(t) + \int_E (\zeta_j(t, x_i(t)) - \zeta_j(t, x_0(t))) \, d\nu_i(t).$$

The integrand in the last term converges uniformly to zero as $i \to \infty$ (for fixed $j$) by continuity of $\zeta_j$ and the uniform convergence of the $x_i$'s. Since $\{\nu_j\}$ is bounded in total variation, the last term, therefore, has limit zero. The remaining terms converge by Lemma 4.2, since $E \in \mathcal{C}$, to give

$$q \int_E d\eta_0(t) = q \int_E \gamma_0(t) \, d\nu_0(t) \leq \int_E \zeta_j(t, x_0(t)) \, d\nu_0(t). \quad (4.11)$$
We readily deduce from the fact that $\mathcal{B}$ generates the Borel sets that, in fact, (4.11) holds for all Borel sets. It follows that

$$q\gamma_0(t) \leq \zeta_j(t, x_0(t)), \quad \nu_0 \text{ a.e.}$$

Taking the limit $j \to \infty$, we have (see (4.10))

$$q\gamma_0(t) \leq \max\{qd: d \in G(t, x_0(t))\}, \quad \nu_0 \text{ a.e.} \quad (4.12)$$

But then (4.12) holds for all $q$ belonging to some dense subset, $\nu_0$ a.e. The term $G$, however, has bounded values on $[0, 1] \times S$. The function $q \to s_q(t, x)$ is, therefore, continuous, and we can conclude that (4.12) holds for all $q$, $\nu_0$ a.e. Since $G$ assumes closed convex values, it follows that

$$\gamma_0(t) \in G(t, x_0(t)), \quad \nu_0 \text{ a.e.}$$

### 5. Proof of Theorem 3.1

Throughout this section we shall take $(v, z)$ to be a solution to problem $P$. Theorem 3.1 will be proved initially when the following hypotheses are in force, in addition to (H1)–(H4):

- **(H5)** $C_0 = \{c_0\}$ some $c_0 \in \mathbb{R}^n$
- **(H6)** There exists some integrable function $\alpha$ (which for convenience we shall take to be the same as that introduced in (H1)) such that

$$|f(t, z(t), u)|, \ |l(t, z(t), u)| \leq \alpha(t)$$

for $t \in [0, 1], u \in U(t)$.

**Proof Under Supplementary Hypotheses (H5) and (H6).**

Let $\{k_i\}$ be a sequence of positive real numbers tending to infinity. Define $V$ to be the set of controls $u$ which yield an admissible trajectory $x_u$. (In consequence of the hypotheses, $u$ uniquely defines $x_u$.) Following Ekeland [7], we provide $V$ with the metric $\delta(\cdot, \cdot)$

$$\delta(u, w) = \text{Lebesgue measure}\{t: u(t) \neq w(t)\}.$$ 

Define the functions $J_i$ on $V$

$$J_i(u) = \int_0^1 l(t, x_u(t), u(t)) \, dt + \int_1^t k_i g^+(t, x_u(t)) \, dt, \quad i = 1, 2, \ldots$$
and set
\[ \varepsilon_i = J_i(v) - \inf\{J_i(u) : u \in V\}. \]

Here, \((v, z)\) is our solution to problem \((P)\). Since \((v, z)\) satisfies \(g(t, z(t)) \leq 0\) on \(I\), \(J_i(v) = \inf\{P\}\) and we may express \(\varepsilon_i\) as
\[ \varepsilon_i = \inf\{P\} - \inf\{P_{k_i}\}. \]

By hypothesis \((H4)\), then
\[ \varepsilon_i \to 0. \tag{5.1} \]

We readily check that \(V\) is a complete metric space and that the functions \(J_i : V \to \mathbb{R}\) are continuous (c.f., proofs of \([3, \text{Lemmas } 8 \text{ and } 9]\)).

The conditions are satisfied then under which Ekeland's variational principle \([7]\) applies. This yields a sequence \(\{v_i\}\) in \(V\) such that
\[ \delta(v_i, v) \leq \varepsilon_i^{1/2} \tag{5.2} \]
and
\[ \tilde{J}_i(v_i) \leq \tilde{J}_i(u) \quad \text{for all } u \in V, \tag{5.3} \]
with \(\tilde{J}_i(u) = J_i(u) + \varepsilon_i^{1/2} \delta(u, v_i).\) Let \(z_i = x_{v_i}.

Now define \(D_i = \{t : v_i(t) = v(t)\}\). Notice that \(5.2\) may be expressed as Lebesgue measure
\[ |D_i| > 1 - \varepsilon_i^{1/2}. \tag{5.4} \]

We deduce from \([3, \text{Lemma } 9]\) that
\[ z_i \to z \quad \text{uniformly.} \tag{5.5} \]

Following Clarke \([3]\) we interpret \(5.3\) as meaning that \(v_i\) solves a control problem, labelled \((\tilde{P}_i)\). \((\tilde{P}_i)\) is the same as problem \((P_{k_i})\) of Section 3 except that the cost integrand \(l\) is replaced by \(\tilde{l}\):
\[ \tilde{l}(x, t, u) = l(x, t, u) + \varepsilon_i^{1/2} \chi_i(t, u). \]

Here
\[ \chi_i(t, u) = 1, \quad \text{if } u = v_i(t), \]
\[ = 0, \quad \text{otherwise.} \]

Lemma 4.1 applies to \((\tilde{P}_i)\) (see \([3]\) for verification that the hypotheses still hold following modification of \(l\)). It is convenient to express the conclusions
of Lemma 4.1 as follows: For each $i$ there exist $\pi_i \in C^*(\mathbb{R}^n)$, $v_i \in C^0(\mathbb{R})$ having support in $\{t : g(t, z_i(t)) = 0\} \cap Q$, measurable functions $\Phi_i$ and $\lambda_i$, a Borel measurable function $\gamma_i$, a left-continuous function of bounded variation $p_i$, a nonpositive number $c_i$, and $b_i \in \mathbb{R}^n$ such that

\begin{align*}
\Phi_i(t) &\in \partial_x f(t, z_i(t), v_i(t)) \quad \text{a.e.,} \\
\lambda_i(t) &\in \partial_x l(t, z_i(t), v_i(t)) \quad \text{a.e.,} \\
\gamma_i(t) &\in \partial_x g(t, z_i(t)) \quad v_i \text{ a.e.,}
\end{align*}

\begin{equation}
- \int_B d\pi_i(t) = \int_B \left( p_i(t) \Phi_i(t) + c_i \lambda_i(t) \right) dt - \int_B \gamma_i(t) dv_i(t) \tag{5.9}
\end{equation}

for all Borel sets $B$,

\begin{equation}
p_i(t) = b_i + \int_{[0, t]} d\pi_i(r), \quad t \in (0, 1], \tag{5.10}
\end{equation}

\begin{equation}
b_i \in \overline{N_{c_i}(z_i(0))}, \quad -b_i - \int_{[0, 1]} d\pi_i(t) \in \overline{N_{c_i}(z_i(1))}, \tag{5.11}
\end{equation}

\begin{equation}
|b_i| + |c_i| + |v_i| = 1, \tag{5.12}
\end{equation}

\begin{equation}
p_i(t) f(t, z_i(t), v(t)) + c_i l(t, z_i(t), v(t)) = \max_{u \in U(t)} \{ p_i(t) f(t, z_i(t), u) + c_i l^{-}(t, z_i(t), u) \} \quad \text{a.e. } t \in D \tag{5.13}
\end{equation}

In the above, $p_i$ is actually absolutely continuous and is the function $p$ of Lemma 4.1; $\pi_i$ and $v_i$ are the measures defined by

\begin{equation}
d\pi_i(t) = \dot{p}(t) \, dt, \quad dv_i(t) = \beta(t) \, dt,
\end{equation}

$b_i = p(0), \Phi_i = \Phi$, etc. (with, again, $p$ and also $\Phi, \lambda, \gamma, c$ as in Lemma 4.1).

Notice that we can take $\gamma_i$ (i.e., $\gamma$ in Lemma 4.1) Borel measurable by adjustment on a (Lebesgue) null set. Inclusion (5.8) applies $v_i$ a.e. because $v_i$ is absolutely continuous with respect to Lebesgue measure. The measure $v_i$ is in $C^0(\mathbb{R})$ because $\beta(t) \geq 0$, and has support in $\{t : g(t, z_i(t)) = 0\} \cap Q$ because $\beta$ is zero on the complement of this set. Finally, we observe that (5.12) may be achieved by scaling since $b_i$ and $c_i$ are not both zero.

We shall show that there exist $\pi$, $p$, $v$, $\Phi$, $\lambda$, $\gamma$, $c$, and $b$ which satisfy (5.6)–(5.13) (with $D_i$ and $r$'s deleted). It will follow in particular that $p$, $v$, and $c$ have the properties asserted in Theorem 3.1.

By (H1) and by (5.6) and (5.7)

\begin{equation}
|\Phi_i(t)|, |\lambda_i(t)| \leq \alpha(t) \tag{5.14}
\end{equation}
for \( i \) sufficiently large (\( \alpha \) as in (H1)). By replacing \( \{ k_i \} \) by a subsequence if necessary (we shall refer to such a process as extracting subsequences) we may arrange then that

\[
\Phi_i \to \Phi \quad \text{and} \quad \lambda_i \to \lambda \quad \text{weakly in} \quad L^1 \quad (5.15)
\]

for \( \Phi, \lambda \in L^1 \).

Because of (5.12) we may also arrange that

\[
v_i \to v \quad \text{weakly*}, \quad b_i \to b \quad \text{and} \quad c_i \to c \quad (5.16)
\]

for some \( v, b, c \). Since \( v_i \) is a sequence in \( C^\infty, \ |v_i| \to |v| \). It follows from (5.12) that

\[
|b| + |c| + |v| - 1.
\]

A standard argument which involves the use of Gronwall's inequality supplies a number \( K_1 \) (which depends only on \( \alpha \) and \( K \) of (H1)) such that

\[
|\pi_i| \leq K_1.
\]

From (5.10) and (5.12), then

\[
|p_i(t)| \leq 1 + K_1, \quad t \in [0, 1], \quad i = 1, 2, \ldots.
\]

Following extraction of subsequences we have

\[
\pi_i \to \pi \quad \text{weakly*}
\]

for some \( \pi \in C^*(\mathbb{R}^n) \). By weak*-lower semicontinuity of the dual norm

\[
|\pi| \leq K_1.
\]

Since \( b_i \to b \), we conclude from Lemma 4.3 that

\[
p_i(t) \to p(t) \quad \text{a.e.,}
\]

where the function of bounded variation \( p \) is given by

\[
p(t) = b + \int_{[0,t]} d\pi,
\]

and

\[
b_i + \int_{[0,1]} d\pi_i \to b + \int_{[0,1]} d\pi. \quad (5.17)
\]

The scene is now set for application of Lemmas 4.4 and 4.5. These lemmas (applied with \( q_i \Psi_i = p_i \Phi_i + c_i \lambda_i \) and \( d\eta_i = \gamma_i dv_i \)) give a Borel measurable, \( \nu \)-integrable function \( \gamma \) such that the adjoint equation ((5.9) and
(5.10)) is satisfied (with $i$'s deleted). The inclusions (5.6)–(5.8) hold (with $i$'s deleted); this we readily deduce from (5.4), (5.5), (5.14), and (5.15) and from [3, Lemma 5]. The boundary conditions (5.11) are satisfied (with $i$'s deleted) because of (5.16) and (5.17) and because the multifunctions $s \rightarrow N_{c_i}(s), s \rightarrow N_{c_i}(s)$ have closed graphs.

Consider next the support of $v$. Suppose that, contrary to our assertions, $v(\{t: g(t, z(t)) < 0\}) > 0$. By $\sigma$-additivity of the measure, there exists $e > 0$ such that $v(E) > 0$, where $E = \{t: g(t, z(t)) < -e\}$. By (5.5) and (H1)

$$\{t: g(t, z(t)) < -e\} \subseteq \{t: g(t, z_i(t)) \leq -\frac{1}{2}e\}$$

for sufficiently large $i$. But this last set is contained in the complement of the support of $v_i$, whence $v_i(E) = 0$. Since $g$ is upper semicontinuous it follows that $E$ is an open set. But $v_i \rightarrow v$ weakly*. By [2], $v(E) = 0$, a contradiction. We have shown that $v(\{t: g(t, z(t)) < 0\}) = 0$. We show, likewise, that $v([0, 1] \setminus I) = 0$. Thus $v$ has support in $\{t: g(t, z(t)) = 0\} \cap I$.

Consider finally (5.13) in the limit. Let us arrange by extracting subsequences that

$$\sum_{i=1}^{\infty} e_i^{1/2} < \infty.$$  

(5.18)

Define $F \subset [0, 1]$ to be the set of $i$'s in $[0, 1]$ such that $v_i(t) = v(t)$ for all $i$ sufficiently large. Then $F = \bigcup_i \bigcap_{i' > i} D_i$ (recall that $D_i = \{t: v(t) = v_i(t)\}$). The subset $F$ has full measure by (5.4) and (5.18). The set of $t \in F$ such that $p_i(t) \rightarrow p(t)$ and such that (5.13) holds for $i$ sufficiently large also has full measure; we select such a $t$. Choose $u \in U(t)$. For sufficiently large $i$, $v_i(t) = v(t)$ and

$$p_i(f(t, z_i(t)), v(t)) + c_i I(t, z_i(t), v(t))$$

$$\geq p_i(f(t, z(t)), u) + c_i I(t, z(t), u) - e_i^{1/2}. \quad (5.19)$$

But $p_i(f(t, z_i(t)), u) \rightarrow p(t)$ $f(t, z(t), u)$ etc., and we may take the limit in (5.19) obtaining

$$p(f(t, z(t)), v(t)) + c I(t, z(t), v(t))$$

$$\geq p(f(t, z(t)), u) + c I(t, z(t), u).$$

This is the inequality required.

We draw attention to the fact that we have shown, in the course of the proof, that the measure $\pi$ satisfies

$$|\pi| \leq K_1,$$
where the real number $K_1$ is determined by $a$ and $K$ of (H1). This will be needed shortly.

Removal of Supplementary Hypotheses (H5) and (H6).

We dispose first of (H6) by using an argument due to Clarke [3]. Suppose that $(v, z)$ is a solution to problem (P) when (H1)-(H5) hold. Then $(v, z)$ also solves each of a family of problems in which $U(t)$ is replaced by

$$U_i(t) = \{u \in U(t): |f(t, z(t), v(t)) - f(t, z(t), u)| \leq i, \text{ etc.} \}$$

$i = 1, 2, \ldots$. For $i = 1, 2, \ldots$, the modified problem additionally satisfies (H6), and the preceding results apply. It follows that there exist $p_i, \pi_i, v_i$, etc., such that (5.6)-(5.13) are satisfied (with $v_i = v$, $z_i = z$). By (5.20),

$$|\pi_i| \leq K_1, \quad i = 1, 2, \ldots \quad (5.21)$$

We now deduce the conclusions of the theorem, when (H6) is removed, using essentially the same arguments as before; indeed existence of $p, \pi, v$, etc., such that (5.6)-(5.12) hold (with $i$'s deleted) follows, as was shown, from (5.6)-(5.12) and the bound (5.21). The modifications to previous arguments needed to establish (5.13) (with $i$'s deleted) are obvious.

It remains to remove (H5). This is accomplished in two stages. Let us suppose that $(v, z)$ solves problem (P) when (H1)-(H4) hold and when

(H5) the closed set $C_0$ is convex.

Following an idea of Warga [13], we consider a reformulated problem in which controls, written $(u, w)$, take values in $\mathbb{R}^{m+n}$ and the underlying time interval is $[-1, +1]$.

$$\min \int_{[0,1]} l(t, x(t), u(t)) \, dt$$

$$\dot{x}(t) = w(t) - z(0), \quad t \in [-1, 0),$$

$$= f(t, x(t), u(t)), \quad t \in [0, 1],$$

$$g(t, x(t)) \leq 0, \quad t \in I; \quad x(-1) = z(0), \quad x(1) \in C_1,$$

$$(u(t), w(t)) \in \{0\} \times C_0, \quad t \in [-1, 0),$$

$$\in U(t) \times \{0\}, \quad t \in [0, 1].$$

It is clear that the pair $(\bar{z}, \bar{v})$ solves this problem, where

$$\bar{z}(t) = z(0), \quad t \in [-1, 0), \quad \bar{v}(t) = (0, z(0)), \quad t \in [-1, 0),$$

$$= z(t), \quad t \in [0, 1], \quad = (v(t), 0), \quad t \in [0, 1].$$
The new problem satisfies (H1)–(H4), under which the theorem has been shown to apply (that the underlying time interval is now \([-1, +1]\), not \([0, 1]\), is clearly immaterial). Applying the theorem to the new problem we easily deduce the conclusions of the theorem for the original problem. We merely remark here that, for \(t \in [0, 1]\), the left-continuous function \(p(t) = p(0)\), and it follows from the maximization of the Hamiltonian a.e. on \([-1, 0]\) that
\[
p(0) z(0) \geq p(0) s \quad \text{for all } s \in C_0.
\]

The inequality implies that \(p(0) \in N_{C_0}(z(0))\) \((N_{C_0}(z(0))\) coincides with the normal cone of \(C_0\) at \(z(0)\), in the sense of convex analysis, when \(C_0\) is convex). We have obtained the boundary condition at \(t = 0\) on \(p\) restricted to \([0, 1]\).

Finally, we remove (H5). Suppose \((v, z)\) is a solution to problem (P) when (H1)–(H4) hold. We consider a control problem in which the trajectories, written \((x, y)\), are \(\mathbb{R}^{n+1}\)-valued
\[
\min \int_0^1 l(t, x(t), u(t)) \, dt,
\]
\[
\dot{x}(t) = f(t, x(t), u(t)),
\]
\[
\dot{y}(t) = 0,
\]
\[
(x(0), y(0)) \in \{(s, s) : s \in \mathbb{R}^n\}, \quad (x(1), y(1)) \in C_0 \times C_1,
\]
\[
u(t) \in U(t) \quad \text{a.e.},
\]
\[
g(t, x(t)) \leq 0, \quad t \in I.
\]

\((z, v)\) solves the new problem with \(\dot{z}(t) = (z(t), z(0))\), \(t \in [0, 1]\). The new problem satisfies (H1)–(H5) and (H5). We may therefore apply the theorem. It is easy to see that we recover the conclusions of the theorem for the original problem.

6. **Directly Verifiable Hypotheses**

Finally, we shall show that Theorem 3.1 leads to a maximum principle which applies subject to directly verifiable hypotheses; here the well-posedness hypothesis (H4) is replaced by the requirements that \(l\) and \(f\) be continuous in their \(u\) dependence, that \(U\) have values compact sets and, most notably, that the extended velocity set be convex.

**Corollary 6.1.** Theorem 3.1 remains true when (H1)–(H4) are replaced by the verifiable hypotheses (V1)–(V4):
(V1) This is the same as condition (H1).

(V2) For each \( s \) in a neighbourhood of \( X, t \in \mathbb{R} \) and \( u \in \mathbb{R}^m \), \( f(s, \cdot, u) \) and \( l(s, \cdot, u) \) are measurable, \( f(s, t, \cdot) \) and \( l(s, t, \cdot) \) are continuous and \( g(\cdot, s) \) is upper semicontinuous.

(V3) \( U \) has values compact subsets of \( \mathbb{R}^m \) and is measurable (in the sense that \( \{ t \in [0, 1] : U(t) \cap B \neq \emptyset \} \) is measurable for every closed ball \( B \)).

(V4) For all \( s \in X, t \in [0, 1] \)

\[
V(s, x) = \{(l(t, s, u), f(t, s, u)) : u \in U(t)\}
\]

is a convex subset of \( \mathbb{R}^{n+1} \).

Proof: Suppose that, in addition to (V1)–(V4), (H5) and (H6) of Section 5 are true. As in Section 3 we define the function \( \theta \) on \( [0, \infty) \)

\[
\theta(\alpha) = \inf \left\{ \int_0^1 l(t, x(t), u(t)) \, dt : (u, t) \text{ admissible}, \int_0^1 g^+(t, x(t)) \, dt \leq \alpha \right\}.
\]

From (H5) and (H6)

\[
\inf \left\{ \int_0^1 l(t, x(t), u(t)) \, dt : (u, x) \text{ admissible} \right\} > -\infty \quad (6.1)
\]

We shall show that

\[
\lim_{\alpha \to 0} \theta(\alpha) = \theta(0). \quad (6.2)
\]

Hypotheses (V1)–(V3) imply (H1)–(H3). As commented on in Section 3, (6.1) and (6.2) imply (H4). It will follow then from Theorem 3.1 that the maximum principle applies (subject to (V1)–(V4), (H5), and (H6)). We may now remove (H5) and (H6), in the manner shown at the end of Section 5.

It remains then to prove (6.2) (when we assume (VI)–(V4), (H5), and (H6)). Let \( \{\alpha_i\} \) be an arbitrary sequence of real numbers decreasing to zero. By definition of \( \theta \), we may choose a corresponding sequence of admissible pairs \( \{(u_i, x_i)\} \) such that

\[
\int_0^1 g^+(t, x_i(t)) \, dt \leq \alpha_i \quad (6.3)
\]

and

\[
\int_0^1 l(t, x_i(t), u_i(t)) \, dt \leq \theta(\alpha_i) + \alpha_i. \quad (6.4)
\]

Define the \( \mathbb{R}^{n+1} \) valued function \( \tilde{f} : \tilde{f}(t, s, u) = (l(t, s, u), f(t, s, u)) \).
By (H5) and (H6), and by application of the Dunford-Pettis criterion, we deduce that \{x_i\} is a bounded, equicontinuous family of continuous functions and \{\tilde{f}(\cdot, x_i(\cdot), u_i(\cdot))\} is weakly precompact in \(L^1\). It is easy to show now that there exists \(\zeta(=\zeta^0, \xi^1)\) \(\in L^1\) such that, after extraction of subsequences,

\[
\tilde{f}(\cdot, x_i(\cdot), u_i(\cdot)) \rightarrow \zeta \quad \text{weakly in } L^1 \quad (6.5)
\]

and

\[
x_i \rightarrow x \quad \text{uniformly.} \quad (6.6)
\]

where \(x\) satisfies

\[
\dot{x}(t) = \zeta^1(t), \quad \text{a.e. } t \in [0, 1], \quad x(0) = c_0. \quad (6.7)
\]

Properties (6.5) and (6.6), and hypothesis (V1) imply that

\[
\int_E \tilde{f}(t, x(t), u(t)) \, dt \rightarrow \int_E \zeta(t) \, dt
\]

for any measurable set \(E\). By arguments similar to those used at the end of the proof of Lemma 4.5 (which apply in view of (V4)), we deduce that

\[
\zeta(t) \in \tilde{f}(t, x(t), u(t)), \quad \text{a.e. } t \in [0, 1].
\]

The conditions under which the measurable selection theorem [2, Theorem 1.7.6] apply are fulfilled; there exists then a control \(u\) such that

\[
\zeta(t) = \tilde{f}(t, x(t), u(t)), \quad \text{a.e. } t \in [0, 1]. \quad (6.8)
\]

It is clear now from (6.6)-(6.8), and from the closedness of \(X\), that \((u, x)\) is an admissible pair.

Equations (6.3) and (6.6), and hypothesis (V1) imply that

\[
\int_0^1 g^+(t, x(t)) \, dt = 0.
\]

It follows that \(\int_0^1 l(t, x(t), u(t)) \, dt \geq \theta(0)\). Taking note of (6.4), (6.5), and (6.8) we obtain

\[
\theta(0) \geq \lim_{i \rightarrow \infty} \theta(a_i) \geq \lim_{i \rightarrow \infty} \int_0^1 l(x_i(t), t, u_i(t)) \, dt
\]

\[
= \int_0^1 l(x(t), t, u(t)) \, dt \geq \theta(0).
\]
But \( \{\alpha_i\} \) was an arbitrary sequence. Hence, \( \lim_{\alpha \to 0} \theta(\alpha) = \theta(0) \), as required.

More general results, in the spirit of Corollary 6.1 are to be found in [10].

**REFERENCES**