Multiplicative Designs
II. Uniform Normal and Related Structures

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We continue the study of multiplicative designs concentrating primarily on uniform, normal designs with two replications. A structure result and consequent finiteness theorem is obtained and certain classes are related to symmetric block designs, supplementary difference sets, and three eigenvalue graphs.

1. INTRODUCTION

A multiplicative design [4, 5, 10, 12] is a family of \( n \) subsets of an \( n \) set whose \((0, 1)\) incidence matrix \( A \) satisfies

\[
A^t A = D + \alpha \alpha^t,
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_n)^t \) is a real vector with positive entries and \( D = \text{diag}(k_1 - \alpha_1^2, \ldots, k_n - \alpha_n^2) \), \( k_j \) being the \( j \)th column sum of \( A \). These designs introduced by Ryser [12] are direct generalizations of \( \lambda \)-designs [10, 11, 17] where \( \lambda_j = \lambda^{1/2} \) for all \( i \) and \( (v, k, \lambda) \) configurations [13] where also \( D \) is scalar. Such a design is called normal if some incidence matrix is normal, reducible if some incidence matrix is reducible, and uniform if \( D \) is a scalar matrix. We have previously determined all reducible multiplicative designs and found all nonuniform normal designs modulo certain symmetric designs [4]. We are concerned here mainly with uniform normal designs so that (1.1) becomes

\[
A^t A = A A^t = d I + \alpha \alpha^t.
\]

Here the number of block sizes, the number of replications and the number of distinct components in \( \alpha \) are, of course, the same as \( k_i = r_i = d + \alpha_i^2 \). After some preliminaries we concentrate in Section 2 on the case of two block sizes relating this to the dimension of a certain \( A \)-invariant subspace which in turn provides a rather strong structure statement—Theorem 2.2—about

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these 2-class uniform normal designs. From this we are able to see that for a
given pair of distinct numbers $\alpha_1$ and $\alpha_2$ there are only finitely many 2-class
designs with $\alpha_1$ and $\alpha_2$ as the distinct components of $\alpha$. (This result for
$\alpha_1 - \alpha_2 > 1$ is an intriguing conjecture for symmetric block designs.)
We exhibit all feasible parameter sets with $\alpha_1 < \alpha_2 \leq 5$ and in Section 3
construct several of these designs. Curiously for these designs with $\alpha_1 \neq \alpha_2$
the order $d$ (1.2) must be a square so that rational congruence provides no
exclusions. Some of our constructions utilize 2-supplementary difference sets
[15, 16] and were found by computer search. Most of the remaining designs
we have been able to construct are related to symmetric designs with appro-
priate substructure. In fact, an interesting by-product in this study is the
suggestion of the existence of certain $(v, k, \lambda)$-designs with appropriate
tactical decompositions.

In Section 4 multiplicative graphs are discussed and shown to be equivalent
to a certain family of three eigenvalue graphs.
Throughout the paper $I$ will denote the identity matrix, $J$ the matrix of
ones, and 0 the zero matrix with subscripts denoting sizes where necessary.
If $G$ is a graph by the cone over $G$ we mean the graph obtained from $G$ by
adjoining a vertex and joining it to all the vertices of $G$. We use $A(G)$ for the
adjacency matrix of $G$ and say generally that a matrix $B$ carries a graph or
design if it is the adjacency or incidence matrix of same.

2. The Normal, Uniform Case

We consider now those multiplicative designs which are both normal and
uniform. Here the incidence matrix, $A$, satisfies

$$AA^t = A^tA = dI + \alpha \alpha^t.$$  \hspace{1cm} (2.1)

For an arbitrary multiplicative design it is not difficult to see that the com-
ponents of $\alpha$ must lie in a quadratic extension of the rational field, in fact,
$\alpha_i = s_m^{1/2}$, where $m$ is a square free integer ($1 \leq i \leq n$). In the uniform
case we may conclude that each $\alpha_i^2$ is an integer whence so is $d$. In the uniform
normal case we have, in addition to (2.1), with

$$\mu = (d + \alpha \cdot \alpha)^{1/2}$$ \hspace{1cm} (2.2)

that

$$A\alpha = A^t\alpha = \mu \alpha.$$ \hspace{1cm} (2.3)

Further putting

$$A1 = A^t1 = R,$$ \hspace{1cm} (2.4)
we have

\[ AR = A^tR = d\mathbf{1} + (\alpha \cdot \mathbf{1})\alpha \]  

(2.5)

from (2.1) by multiplying by \( \mathbf{1} \). Now the form of the \( \alpha_i \) and (2.3) will force \( \mu \) to be rational and so an integer. (This need not be the case for nonnormal uniforms.) Summing coordinates in (2.3) gives the relation

\[ R \cdot \alpha = \mu(\alpha \cdot \mathbf{1}). \]  

(2.6)

We have, of course,

\[ R = d\mathbf{1} + (\alpha_1^2, \ldots, \alpha_n^2)^t \]  

(2.7)

and taking inner products here with the vector \( \alpha \) and using (2.6) we find

\[ \mu = \left( d + \sum_{i=1}^{n} \alpha_i^2 \right)^{1/2} = d + \left( \frac{1}{\mu} \right) \mu. \]  

(2.8)

This last relation (2.8) is the fundamental parameter relation for multiplicative designs found by Ryser [12] specialized to the normal, uniform case.

Now (2.3), (2.4), and (2.5) show that

\[ W = \langle 1, R, \alpha \rangle, \]  

(2.9)

the span of \( \{ 1, R, \alpha \} \) is \( A \)-invariant. In fact if \( \theta^a = d \) one easily checks that

\[ X_\theta = R + \frac{\alpha \cdot \mathbf{1}}{\theta - \mu} \]  

(2.10)

satisfies

\[ AX_\theta = \theta X_\theta. \]  

(2.11)

Finally multiply (2.7) by \( A \) and use (2.5) for the term \( AR \) to obtain

\[ R = 1 + \frac{\alpha \cdot \mathbf{1}}{d} \alpha - \frac{1}{d} A(\alpha_1^2, \ldots, \alpha_n^2)^t. \]  

(2.12)

Now easily the space \( W \) of (2.9) is one dimensional if and only if \( A \) carries a symmetric block design. The two-dimensional case has the following characterization.

**Lemma 2.1.** Let \( A \) carry a uniform normal multiplicative design. Then with \( W \) as in (2.9) the following are equivalent:

(i) The \( \alpha_i \) take two values,

(ii) \( A \) has two row sums,

(iii) \( \dim W = 2. \)
Proof. The equivalence of (i) and (ii) is clear from (2.7) and \( \alpha_i > 0 \). Also (i) implies the dependence of the vectors \( 1, \alpha, (\alpha_1^2, ..., \alpha_n^2)' \) so that (2.7) gives a dependency on \( R, \alpha, \) and \( 1 \) and \( \dim W = 2 \). Conversely if \( \dim W = 2 \), \( R \in \langle 1, \alpha \rangle \) and (2.7) gives a quadratic equation which each \( \alpha_i \) must satisfy.

We confine our attention to the case \( \dim W = 2 \) referring to the design as a 2-class normal uniform design of order \( d \). In this case we have from (2.10) that \( \alpha, X_{d/2} \) and \( X_{-d/2} \) all lie in \( W \). But as these vectors lie in different eigenspaces of \( A \) and so are independent we must conclude that one of them is zero. Let us define \( \sigma \) by \( \sigma^2 = d \) and \( X_{-\sigma} = 0 \) so that \( \sigma \) is an eigenvalue of \( A \) and from (2.7), (2.10):

\[
R = \sigma 1 + \frac{\alpha \cdot 1}{\sigma + \mu} \alpha = d 1 + (\alpha_1^2, ..., \alpha_n^2)'.
\]  

(2.13)

We relabel so that

\[
\alpha = (\alpha_1, ..., \alpha_1, \alpha_2, ..., \alpha_2)',
\]

where there are \( e \) occurrences of \( \alpha_1 \) and \( f = n - e \) occurrences of \( \alpha_2 \). Then of course, \( R \) takes the similar form

\[
R = (r_1, ..., r_1, r_2, ..., r_2)'
\]  

(2.14)

and \( W \) becomes

\[
W = \langle (1, ..., 1, 0, ..., 0)', (0, ..., 0, 1, ..., 1)' \rangle,
\]  

(2.15)

where these vectors have \( e \) and \( f \) ones, respectively. We collect our conclusions in:

**Theorem 2.2.** Let \( D \) be a 2-class normal uniform design of order \( d \) carried by the matrix \( A \) with parameters \( (\alpha_1, \alpha_2, \sigma, \mu, r_1, r_2, e, f) \) as above. Then

1. \( \mu \) is an integer and an eigenvalue of \( A \),
2. The order \( d \), must be an integral square.
3. The following relations must hold among the parameters:

   (i) \( (\mu - \sigma - e) \alpha_1 = (f - \mu - \sigma) \alpha_2 \),
   (ii) \( \alpha_1 \alpha_2 = d - \sigma \),
   (iii) \( e\alpha_1(r_1 - \mu) = f\alpha_2(\mu - r_2) \),
   (iv) \( \mu = d + (e\alpha_1^3 + f\alpha_2^3)/(e\alpha_1 + f\alpha_2) \),
   (v) \( \mu^2 = d + e\alpha_1^2 + f\alpha_2^2 \).
The carrying matrix $A$ has the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

(2.17)

where $A_{11}$ is $e \times e$, $A_{22}$ is $f \times f$, and each $A_{ij}$ is a 1-design with row sums $r_{ij}$ and column sums $k_{ij}$ determined by

(i) $r_{11} + r_{12} = r_1$, $r_{21} + r_{22} = r_2$,

(ii) $r_{11}a_1 + r_{12}a_2 = \mu a_1$,

(iii) $r_{11} + r_{22} = \mu + \sigma$,

(iv) $k_{12} = r_{21}$, $k_{21} = r_{12}$.

(2.18)

If $\alpha_1 < \alpha_2$ then $d + \alpha_1^2 < \mu < d + \alpha_2^2$.

Proof. Condition (1) was established in the Introduction and the remarks following (2.5). Condition (2) will follow from (2.16ii) as this makes $\sigma = \pm d^{1/3}$ an integer. We obtain (3i) and (3ii) from the quadratic implicit in (2.13). Condition (3iii) is (2.6) and (3iv) is (2.8). Condition (3v) is just the definition of $\mu$. Condition (4) is the invariance of the space $W$ and the relations (2.18) are mostly counting. Note that (2.18ii) comes from $A\alpha = \mu \alpha$ and (2.18iii) is most easily seen as the trace equation for $A$ restricted to $W$ where the eigenvalues are $\mu$ and $\sigma$. Finally (5) is the standard bound for the spectral radius where equality must be strict if $R$ is not a constant vector. This may also be seen as a direct consequence of (3iii).

COROLLARY 2.3. Given $\alpha_2 > \alpha_1 > 0$ there are only finitely many 2-class uniform normal designs with $\alpha_1$ and $\alpha_2$ as the components of $\alpha$.

Proof. (2.16ii) since $\sigma^2 = d$ gives at most two $\sigma$ values from a fixed pair $\alpha_1$, $\alpha_2$. Then (5) gives for each corresponding $d$ a finite number of possible values of $\mu$. Now (2.16i) and (2.16v) provide a rank 2 linear system for $e$ and $f$ if $\alpha_1 \neq \alpha_2$. Since $e + f = n$ the result is clear.

The relations given in Theorem 2.2 are not independent. The vector $\alpha$, i.e., $\alpha_1$, $\alpha_2$, $e$, and $f$ will, for example, determine all remaining parameters. Likewise $\alpha_1$, $\alpha_2$, $\mu$, and $\sigma$ are sufficient. We record for reference the formulas

$$e = \frac{(\sigma + \mu)(\sigma^2 + \alpha_2^2 - \mu)}{\alpha_1(\alpha_2 - \alpha_1)},$$

$$f = \frac{(\sigma + \mu)(\mu - \sigma^2 - \alpha_1^2)}{\alpha_2(\alpha_2 - \alpha_1)},$$

$$r_{11} = \frac{\sigma \alpha_2 + \alpha_1 \alpha_2^2 + \alpha_1^2 \alpha_2 - \mu \alpha_1}{\alpha_2 - \alpha_1}.$$
TABLE I
Basically feasible parameter sets for 2-class normal uniform designs with $\alpha_1 < \alpha_2 < 5$

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* These designs are constructed in this paper.
Table I lists the 34 feasible parameter sets for 2-class normal uniform designs with $\alpha_1 < \alpha_2 \leq 5$. In addition to the previously discussed relations we have imposed the basic feasibility requirements $0 \leq r_{ij}, r_{ii} \leq e$, $r_{ii} \leq f$.

### 3. Constructions and Related Structures

In this section we give some general constructions for 2-class designs, produce several of the possibilities appearing in Table I and indicate some connections among uniform multiplicative designs, three eigenvalue graphs, and generalized difference sets.

We begin with the observation that whenever $\alpha_1$, $\alpha_2$, and $\sigma$ are compatible, i.e., $\alpha_1 \alpha_2 = \sigma^2 - \sigma$ (2.16ii) there is always at least one $\mu$ which provides a basically feasible parameter set:

$$
\begin{align*}
\mu &= \alpha_1^2 | \alpha_2^2 | \sigma, \\
e &= f = \mu + \sigma, \\
r_{ii} &= \sigma + \alpha_i^2 \quad (i = 1, 2), \\
r_{12} &= r_{21} = \alpha_1 \alpha_2.
\end{align*}
$$

We denote the family of designs with parameters (3.1) and $\sigma > 0$ by $\Gamma_+(\alpha_1, \alpha_2)$ and the corresponding family with $\sigma < 0$ by $\Gamma_-(\alpha_1, \alpha_2)$. The parameters (3.1) are characterized by the condition $e = f$ (equicardinal classes). There is a basic relation between the $\Gamma_+$ and $\Gamma_-$ families.

**Theorem 3.1.** If $m \geq 2$ is a natural number the families $\Gamma_+(m - 1, m)$ and $\Gamma_-(m, m + 1)$ coexist.

**Proof.**

If $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \Gamma_-(m, m + 1)$

if and only if

$$
B = \begin{pmatrix} A_{12} & A_{11} \\ J - A_{22} & J - A_{21} \end{pmatrix} \in \Gamma_+(m - 1, m)
$$

as is easily checked by direct computation.

Theorem 3.1 pairs, for example design numbers 2 and 7, 6 and 17, and 15 and 34 in Table I.

Suppose now that $\alpha_1$, $\alpha_2$, and $\sigma$ give a feasible parameter set and for the corresponding $e$ we have commuting normal $e \times e (0, 1)$-matrices $A$ and $B$ satisfying $AA^t + BB^t = \sigma^2 I + \alpha_1^2 J$ and $BJ = \alpha_1 \alpha_2 J$. Then

$$
\begin{pmatrix} A & B \\ B^t & J - A^t \end{pmatrix}
$$

(3.2)
belongs to $\Gamma_d(\alpha_1, \alpha_2)$. This observation provides some further families of examples.

Remark 3.2. There exists a symmetric block design with parameters $(\lambda^2(\lambda + 2), \lambda(\lambda + 1), \lambda)$ if and only if $\Gamma_\frac{\lambda + 1}{2}(\lambda^2, \lambda^1/2(\lambda + 1)) \neq \emptyset$. Here taken $A = 0$ and $B$ the incidence matrix of the SBD in the above discussion. We obtain numbers 4 and 21 of Table I from the $(3, 2, 1)$- and $(16, 6, 2)$-designs, for example.

A natural choice in using (3.2) would be $e \times e$ circulants for $A$ and $B$. This amounts to finding an appropriate 2-supplementary difference set [15, 16]. Specifically we need two sets of residues modulo $e$ so that every nonzero residue appears $\alpha_1^2$ times among the differences within $S$ and $T$ together. While supplementary difference sets have been constructed and used in connection with Hadamard matrices most do not have the parameters required to give a 2-class multiplicative design. We do have some examples of suitable sets however.

Remark 3.3. The following 2-supplementary difference sets give the indicated designs from Table I:

(a) $\{2, 7\}, \{0, 1, 2, 3, 4, 6\}$ mod 9, design number 7,
(b) $\emptyset$, $\{1, 2\}$ mod 3, design number 4,
(c) $\{0\}, \{0, 1, 3, 4, 5, 9\}$ mod 11, design number 13,
(d) $\{0, 1, 2, 5, 8, 15\}, \{0, 2, 8, 12, 17\}$ mod 26, design number 19.

Note that (a) above provides a solution of number 2 as well via Theorem 3.1. That design number 6 does not have a difference set solution has been determined by computer search.

Designs numbered 1, 5, 11, and 25 have solutions as given in [4] constructed from the symmetric designs $(21, 5, 1), (40, 13, 4)$ and their complements by fixing a point block pair $(p_0, B_0)$ and replacing all the blocks, $B$, containing $p_0$ by $B + B_0$ (symmetric difference) and deleting $p_0$ and $B_0$.

Numbers 8, 12, and 22 may be obtained from the following construction. Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

be a tactical decomposition of a $(v, k, \lambda)$-design where $A_{ij} = r_{ij}J$ and $A_{11}$ is $e \times e$, $A_{11}A_{21}^t = aJ$ with

$$\lambda(\lambda + e - 2r_{11}) = (r_{21} + \lambda - 2a)^2.$$  (3.3)

Then

$$\tilde{A} = \begin{pmatrix} J - A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$
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is a 2-class uniform normal multiplicative design. Choosing $A$ a $(21, 16, 12)$-design where $A_{22}$ is a $6 \times 6$ $J$ matrix arising from an oval in the complementary plane of order 4 gives $r_{11} = 12$, $r_{31} = 10$, $e = 15$, $a = 8$ and (3.3) is met. The resulting $\hat{A}$ provides a solution to number 12.

From [1] there exists a $(45, 12, 3)$-design in the form (3.2) where $A_{11}$ is a $9 \times 9$ zero matrix. The above construction yields design number 8.

Design number 22 will exist if and only if a $(70, 24, 8)$ SBD can be found with a decomposition (3.2) with $A_{22} = 0$ of size $10 \times 10$. This possibility is unknown to us.

Of course the table entries with $e$ or $f$ equal to 1 indicate simple borderings. Numbers 3, 10, and 20 are of this type and easily constructed.

Finally we exhibit a solution to number 14. Let $P$ be an incidence matrix for the projective plane of order 2 and put

$$\hat{A} = \begin{pmatrix}
0_8 & 1 & 0 \\
1^t & J - P \\
0^t & J - P^t
\end{pmatrix},$$

where $X = J - I$ of order 7. Note that $A$ represents a graph with eigenvalues $\mu = 14$, 2 with multiplicity 7 and $-2$ with multiplicity 14. This is easily recognizable as one of the exceptional graphs with least eigenvalue $-2$ associated with the root system $E_6$ [6]. We now discuss those multiplicative designs which are also graphs.

4. REMARKS ON MULTIPLICATIVE GRAPHS

The incidence matrix $A$ of a multiplicative design may be the adjacency matrix of an ordinary graph ($A = A^t$, trace $A = 0$). From [4, Theorem 4.3] the design will be uniform or a cone over a $(v, k, \lambda)$-graph (i.e., obtained by adjoining a vertex connected to all the vertices of the $(v, k, \lambda)$-graph). Hence, we assume that $A$ is uniform and we have

$$A^2 = dI + \alpha A^t$$  \hspace{1cm} (4.1)

and refer to $A$ as a multiplicative graph. This is, of course, a direct generalization of the $(v, k, \lambda)$-graph notion [2, 3] and some of the following remarks will be seen to generalize observations about such graphs. From (4.1) we have that the underlying graph is connected with minimum polynomial $(X - \mu)(X^2 - d)$ (neglecting the complete graph). Easily $d$ must be a square by the usual trace consideration. In our previous notation this says $\sigma$ is an integer, $d = \sigma^2$. The eigenvalues of $A$ are then $\mu > \sigma > -\sigma$ taking $\sigma > 0$. 
Conversely if \( G \) is a connected graph with three distinct eigenvalues \( y > x > -x \) we may conclude as, e.g., in [8] with \( A = A(G) \) that \((A - xl)(A + xl)\) is a rank 1 symmetric matrix with column space generated by the Perron–Frobenius positive eigenvector corresponding to \( y \). Thus \( A^2 = x^2I + \alpha x^4 \) and \( A \) carries a uniform multiplicative design. We have then

**Theorem 4.1.** Let \( A \) be the adjacency matrix of an ordinary connected graph, \( G \). \( G \) is a multiplicative graph if and only if \( A \) has three distinct eigenvalues \( \mu, \sigma, -\sigma \) with \( \mu > \sigma > -\sigma \).

Note that if \( A \) has three eigenvalues \( y, x, -x \) with \( x > y \) easily \( y = 0 \) and \( G \) is complete bipartite [7].

**Remark 4.2.** There is another obvious connection between three eigenvalue graphs and multiplicative designs. Suppose \( G \) is a graph with three eigenvalues \( y, x, -x - 2 \). With \( A = A(G) + I \) we see \( A \) has eigenvalues \( y + 1, x + 1, -(x + 1) \) and the above reasoning shows \( A \) will carry a uniform multiplicative design.

Let \( G \) be a multiplicative graph on \( v \) vertices with spectrum \( \mu^{(1)}, \sigma^{(m)}, (-\sigma)^{(v-1-m)} \) indicating multiplicities in the superscripts. From the trace equation we have

\[
\mu = \sigma(v - 2m - 1). \tag{4.2}
\]

Since \( A^g = dA + m \alpha x^4 \) has even diagonal entries we must have \( m \alpha_i^g \) even for each \( i \). Further as trace \( A^3 = 0 \) (mod 3), we must have \( \mu(\sum_{i=1}^v \alpha_i^7) = 0 \) (mod 6).

In addition to design number 14 in the previous section from [4, Theorem 4.3] it follows that the cone over a \((\lambda^3 + 2\lambda^2, \lambda^2 + \lambda, \lambda)\)-graph is multiplicative. The parameters will be

\[
\begin{align*}
v &= \lambda^3 + 2\lambda^2 + 1, \\
\sigma &= \lambda, \\
\mu &= \lambda(\lambda + 2), \\
\alpha_i &= \lambda(\lambda + 1)^{1/2}, \\
\alpha_i &= (\lambda^3 + 1)^{1/2} \quad (i > 1), \\
m &= \frac{1}{3}(\lambda^3 - 1)(\lambda + 2). \tag{4.3}
\end{align*}
\]

These graphs exist for \( \lambda \) a prime power [1]. There are, however, multiplicative graphs which are cones not of this type as we shall see.

Let \( G \) be a multiplicative graph which is the cone over a graph \( H \). Let \( B = A(H) \) and note it is easy to see that if \( H \) is regular it is a \((\lambda^3 + 2\lambda^2,
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**THEOREM 4.2.** Let $A$ carry a multiplicative graph $G$ which is the cone over a graph $H$. Then either:

(i) $H$ is a $(\lambda^3 + 2\lambda^2, \lambda^2 + \lambda^3)$-graph or,

(ii) for a suitable ordering of the vertices of $G$ we have

(a) $x^t = (a_1, a_2, \ldots, a_2, a_3, \ldots, a_3)$,

(b) $a_1 = a_2 + a_3$,

(c) $A = \begin{pmatrix} 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & \vdots & A_1 & A_2 \\ 1 & A_3^t & A_3 \\ 1 \\ \vdots & \vdots & \vdots \\ 1 & P & Q & T \end{pmatrix}$, blocked according to $\alpha$ where each $A_i$ has constant row and column sums. Here the final claim regarding the tactical decomposition is a consequence of $A\alpha = \mu\alpha$ and the form of $\alpha$.

We have two examples of possibility (ii) in Theorem 4.2. The $(45, 12, 3)$-graph constructed in [1] can be put in tactical form

\[
\begin{bmatrix} 0 & X & P \\ X & 0 & Q \\ P^t & Q^t & I \end{bmatrix}, \tag{4.4}
\]

where the diagonal 0 blocks are $9 \times 9$, $T$ is $27 \times 27$, and $X = X^t$ has full trace, 9. Further one may verify $XP = XQ = J$. We put

\[
A = \begin{pmatrix} 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & \vdots & J - X & J & P \\ 1 & \vdots & J & J - X & Q \\ 1 & \vdots & P^t & Q^t & T \end{pmatrix}, \tag{4.5}
\]
and obtain, as a consequence of (4.4) a multiplicative graph on 46 vertices with valences 45, 25, and 13, \( \alpha_1 = 6, \alpha_2 = 4, \alpha_3 = 2 \). Its eigenvalues are \( \mu = 21 \) and \( \pm 3 \).

A similar though slightly more subtle construction applied to the \((96, 20, 4)\)-design in [1] produces a multiplicative graph on 97 vertices with \( \alpha_1 = 4(5)^{1/2}, \alpha_2 = 5^{1/2}, \alpha_3 = 3(5)^{1/2} \), and spectrum \( 56(1), 4(11), -4(56) \).

5. PROBLEMS AND CONCLUDING REMARKS

While we have been able to construct several of the possibilities in Table I there remain many curious unsettled possibilities. Design number 18 for example on 56 points with replications 11 and 27 and \( \alpha_1^2 = 2 \) bears a striking resemblance to a \((56, 11, 2)\)-design. Indeed if such a multiplicative design exists and in (2.17) we replace \( A_{xx} \) by its complement we get a design with all block sizes 11 but which a little calculation shows cannot be a \((56, 11, 2)\) SBD. As noted in Section 3 design number 22 begs the question of the existence of a \((70, 24, 8)\)-design with a nice tactical decomposition. Various other possible multiplicative designs seem to have associated symmetric designs as well.

We have concentrated here on 2-class designs. In fact we know of no multiplicative design with more than three distinct components in the vector \( \alpha \). We know of no multiplicative graphs with three \( \alpha_i \) values which are not cones.

In general, the determination of multiplicative graphs seems a natural area of investigation closely related to strongly regular graphs and admiring a nice spectral characterization.

Finally, all the examples known to us of uniform multiplicative designs which are not normal are, in fact, reducible designs. Such designs have uniform duals and an irreducible example would be interesting.

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