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Perfect Modules over Cohen–Macaulay Local Rings

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1. INTRODUCTION

Let A be a Cohen–Macaulay local ring, and let $C^r(A)$ be the class of all finitely generated A-modules which are of finite projective dimension and of codimension at least r. The corresponding Grothendieck group $K^r(A)$ has a generator $[M]$ for each isomorphic class of modules M in $C^r(A)$, with a relation $[M]=[M']+[M'']$ for each exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of modules in $C^r(A)$. In Section 4, we will show that $K^r(A)$ is generated by the elements $[A/\mathfrak{a}]$ defined by the cyclic A-modules which are perfect of codimension r.

If $x_1,..., x_r$ is an A-regular sequence then the cyclic module $A/(x_1,..., x_r)$ is perfect of codimension r. For $r=0$, 1, and 2, the group $K^r(A)$ is generated by the elements $[A/(x_1,..., x_r)]$, but this is not so for $r=3$. In [2], Dutta, Hochster, and McLaughlin exhibit a module M which is of finite length and finite projective dimension over a Cohen–Macaulay local ring A of dimension three, such that M has negative Serre intersection multiplicity with an A-module of the form $A/\mathfrak{p}$, where $\mathfrak{p}$ is prime of height one in A. Since $A/\mathfrak{p}$ has zero intersection multiplicity with every module of the form $A/(x_1, x_2, x_3)$, it follows that $[M]$ does not belong to the subgroup of $K^3(A)$ generated by the elements $[A/(x_1, x_2, x_3)]$. As we will show, there is a cyclic module $A/\mathfrak{a}$ of finite length and finite projective dimension over A, such that $[M]=[A/\mathfrak{a}]$ in $K^3(A)$, modulo elements $[A/(x_1, x_2, x_3)]$. It follows that $A/\mathfrak{a}$ has negative intersection multiplicity with $A/\mathfrak{p}$. This fact is of interest in geometry.

We would like to express our thanks to the referee and to Melvin Hochster, for their suggestions improving this paper. When this paper was first submitted, [2] had not appeared, and we were not aware of the

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results. The referee did know of these results, and pointed out that the method of Section 4 could be used to replace the module M of [2] with a cyclic module.

2. Perfect Modules

This section will provide some background. For more detail, the reader may consult Serre [4].

Let A be a Cohen–Macaulay local ring. The codimension of a nonzero finitely generated A-module M is the minimum of the heights of the prime ideals of A in the support of M, and it is also the length of every maximal A-regular sequence contained in the annihilator of M. The codimension and the dimension of M are related by $\operatorname{codim}(M) + \dim(M) = \dim(A)$. If M is of finite projective dimension over A then the projective dimension and the depth of M are related by $\operatorname{pd}(M) + \operatorname{depth}(M) = \dim(A)$. We always have $\operatorname{depth}(M) \leqslant \dim(M)$, equality meaning that M is Cohen–Macaulay. It follows that $\operatorname{codim}(M) \leqslant \operatorname{pd}(M)$, with equality if and only if M is Cohen–Macaulay and of finite projective dimension. An A-module with these last two properties is *perfect*. If $x_1,..., x_r$ is an A-regular sequence then the module $A/(x_1,..., x_r)$ is perfect of codimension r over A. An A-module of finite length and finite projective dimension is perfect of codimension equal to $\dim(A)$.

Let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of finitely generated A-modules. Localizing at the primes of A of heights less than r, we see that $\operatorname{codim}(M) \geqslant r$ if and only if $\operatorname{codim}(M') \geqslant r$ and $\operatorname{codim}(M'') \geqslant r$. If $\operatorname{pd}(M') \leqslant r$ and $\operatorname{pd}(M'') \leqslant r$, then $\operatorname{pd}(M) \leqslant r$. Thus if M' and M'' are perfect of codimension r, so is M. Similarly, if M and M'' are perfect of codimension r, so is M'.

THEOREM 2.1. *Every module in $C^r(A)$ has a finite resolution by perfect modules of codimension r.*

Proof. Let M be a module in $C^r(A)$. If $\operatorname{pd}(M) \leqslant r$ then M is perfect of codimension r, and we are done. If $\operatorname{pd}(M) > r$, choose an A-regular sequence $x_1,..., x_r$ in the annihilator of M and let $B = A/(x_1,..., x_r)$. Then M is a finitely generated B-module, and for some n there is an exact sequence

$$0 \to M' \to B^n \to M \to 0.$$

As an A-module, B'' is perfect of codimension r, and M' is a module in $C^r(A)$ with $\mathrm{pd}(M') < \mathrm{pd}(M)$. ∎

It follows that $K^r(A)$ is generated by the elements $[M]$, with M perfect of codimension r. The group $K^0(A)$ is generated by the element $[A]$, because the perfect modules of codimension zero over A are the free modules of finite rank. Localizing at a minimal prime of A shows that $K^0(A)$ is free on $[A]$.

3. Codimension One

In this section we will see that $K^1(A)$ is generated by the elements $[A/(x)]$, with x regular on A. The basic result, Theorem 3.1, may be deduced from general results of Bass [1], but we will give an elementary argument.

If M is a perfect module of codimension one over a Cohen–Macaulay local ring A, then M has a resolution

$$0 \to A^n \xrightarrow{\alpha} A^n \to M \to 0.$$

Considering α as an $n \times n$ matrix, we essentially want to reduce it to upper triangular form, using row and column operations. To apply a row operation to α is to replace α by $\beta\alpha$, where β is the result of applying the given row operation to the $n \times n$ identity matric. The effect of the row operation on $M = \mathrm{Coker}(\alpha)$ is expressed in the exact sequence

$$\begin{aligned} 0 \to \mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta\alpha) &\xrightarrow{\alpha} \mathrm{Ker}(\beta) \\ \to \mathrm{Coker}(\alpha) \xrightarrow{\beta} \mathrm{Coker}(\beta\alpha) &\to \mathrm{Coker}(\beta) \to 0. \end{aligned} \tag{*}$$

Adding a multiple of one row of α to another is an invertible operation, and does not change $\mathrm{Coker}(\alpha)$. Multiplication of a row of α by an element x of A replaces α by $\beta\alpha$, where

$$\beta = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & & x & & \\ & & & & 1 & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}.$$

The cokernel of β is isomorphic to $A/(x)$. If x is regular on A then $\operatorname{Ker}(\beta)=0$, and

$$0 \to \operatorname{Coker}(\alpha) \to \operatorname{Coker}(\beta\alpha) \to A/(x) \to 0$$

is exact. Column operations behave in a similar fashion.

THEOREM 3.1. *Let M be perfect of codimension one over A, and let*

$$0 \to A^n \xrightarrow{\alpha} A^n \to M \to 0$$

be exact. Then $\det(\alpha)$ *is regular on A and* $[M]=[A/(\det(\alpha))]$ *in* $K^1(A)$.

Proof. This is clear if $n=1$, so assume $n>1$. Choose x in the annihilator of M regular on A. Multiplication by x maps A^n into the image of α, so there exists $\gamma\colon A^n \to A^n$ such that $\alpha\gamma = x1_n$. Thus $\det(\alpha)\det(\gamma)=x^n$, and it follows that $\det(\alpha)$ is regular on A. The relation $\alpha\gamma=x1_n$ implies that x belongs to the ideal generated by the elements $a_{11},\ldots, a_{1n}$ of the first row of α. Using invertible column operations, we reduce to the case that a_{11} is regular on A. Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_m$ be the minimal primes of A. If a_{11} is contained in none of these primes, we are done. Otherwise, let $\mathfrak{p}_i$ be the first of the primes which contains a_{11}. Since x is not in $\mathfrak{p}_i$, some element a_{1j} of the first row of α is not in $\mathfrak{p}_i$. Choose c in $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{i-1}$, c not in $\mathfrak{p}_i$, and add to the first column of α the jth column multiplied by c. The new element $a_{11}+ca_{1j}$ is not in any of the primes $\mathfrak{p}_1,\ldots,\mathfrak{p}_i$. Continuing, we eventually get a_{11} regular on A. We have not changed $\operatorname{Coker}(\alpha)$ or $\det(\alpha)$.

With a_{11} regular on A, we reduce to the case that the remaining elements of the first column of α are zero. Multiply the second row of α by a_{11}, replacing α by $\beta\alpha$. Since

$$0 \to \operatorname{Coker}(\alpha) \to \operatorname{Coker}(\beta\alpha) \to A/(a_{11}) \to 0$$

is exact, we have $[\operatorname{Coker}(\alpha)]=[\operatorname{Coker}(\beta\alpha)]-[A/(a_{11})]$ in $K^1(A)$. Also, since $\det(\beta\alpha)=a_{11}\det(\alpha)$, the sequence

$$0 \to A/(\det(\alpha)) \to A/(\det(\beta\alpha)) \to A/(a_{11}) \to 0$$

is exact, and $[A/(\det(\alpha))]=[A/(\det(\beta\alpha))]-[A/(a_{11})]$ in $K^1(A)$. Hence $[\operatorname{Coker}(\alpha)]=[A/(\det(\alpha))]$ will follow from $[\operatorname{Coker}(\beta\alpha)]=[A/(\det(\beta\alpha))]$. Subtracting a multiple of the first row of $\beta\alpha$ from the second, we make the first element of the second row of $\beta\alpha$ equal to zero. Continuing, we eventually get α in the form

$$\alpha = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ 0 & \\ \vdots & \alpha' \\ 0 & \end{pmatrix}.$$

The diagram

$$\begin{array}{ccccccccc} 0 \to & A & \to & A^n & \to & A^{n-1} & \to 0 \\ & \downarrow{\scriptstyle a_{11}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} & \\ 0 \to & A & \to & A^n & \to & A^{n-1} & \to 0 \end{array}$$

commutes, and

$$0 \to A/(a_{11}) \to \mathrm{Coker}(\alpha) \to \mathrm{Coker}(\alpha') \to 0$$

is exact. Hence $[\mathrm{Coker}(\alpha)] = [A/(a_{11})] + [\mathrm{Coker}(\alpha')]$. Since $\det(\alpha) = a_{11}\det(\alpha')$, we have also $[A/(\det(\alpha))] = [A/(a_{11})] + [A/(\det(\alpha'))]$. Inductively, $[\mathrm{Coker}(\alpha')] = [A/(\det(\alpha'))]$, so $[\mathrm{Coker}(\alpha)] = [A/(\det(\alpha))]$. ∎

It follows that $K^1(A)$ is generated by the elements $[A/(x)]$ with x regular on A.

4. Cyclic Modules

In codimension $r > 1$, the exact sequence (*) of the last section still has a role to play. It may be used to reduce the number of generators of a perfect module.

Lemma 4.1. *Let M be the cokernel of a map α: $A^n \to A^m$, and let $x_1, \ldots, x_r$ be an A-regular sequence in the annihilator of M. Assuming $r \leqslant m$, there is an exact sequence*

$$0 \to M \to N \to A/(x_1, \ldots, x_r) \to 0,$$

where N is the cokernel of a map $\beta\alpha$: $A^n \to A^{m-r+1}$.

Proof. Let $\beta: A^m \to A^{m-r+1}$ be the map with matrix

$$\begin{pmatrix} x_1 & \cdots & x_r & 0 & \cdots & 0 \\ 0 & & 0 & 1 & & \\ \vdots & & \vdots & & \ddots & 0 \\ & & & 0 & & \\ 0 & \cdots & 0 & & & 1 \end{pmatrix},$$

the direct sum of the row matrix $(x_1 \cdots x_r)$ with the $(m-r)\times(m-r)$ identity matrix. The claim is that the kernel of β is contained in the image of α, so that the connecting homomorphism in the sequence (*) of Section 3 is zero. Since $x_1,\dots, x_r$ annihilate M, there are maps $\gamma_j: A^m \to A^n$ such that $\alpha\gamma_j = x_j 1_m$, for $j = 1,\dots, r$. If $e_1,\dots, e_r$ are the first r elements of the natural basis of A^m, then for $1 \leqslant i < j \leqslant r$, the elements $\alpha(\gamma_j(e_i) - \gamma_i(e_j)) = x_j e_i - x_i e_j$ of the image of α generate the kernel of β. It follows that

$$0 \to \operatorname{Coker}(\alpha) \to \operatorname{Coker}(\beta\alpha) \to \operatorname{Coker}(\beta) \to 0$$

is exact, and this is the sequence of the lemma. ∎

The restriction $r \leqslant m$ is not serious, since we can always increase m. If M is perfect of codimension r, so is N, and N has $r-1$ fewer generators than M. In codimension $r > 1$, we can reduce to a cyclic module N by assuming at the outset that $m = k(r-1)+1$ for some k. Thus M can be embedded in a cyclic module N in such a way that N/M has a filtration with quotients of the form $A/(x_1,\dots, x_r)$, where $x_1,\dots, x_r$ is an A-regular sequence in the annihilator of M. Therefore,

THEOREM 4.2. *For every $r \geqslant 0$, the group $K^r(A)$ is generated by the elements $[A/\mathfrak{a}]$, where $A/\mathfrak{a}$ is perfect of codimension r.*

5. Codimension Two

In this section, we will see that $K^2(A)$ is generated by the elements $[A/(x_1, x_2)]$, where x_1 and x_2 form an A-regular sequence. This fact was known at least to Hochster, who gave a partial proof in [3]. But the method of the previous section leads to a proof which is quite simple.

LEMMA 5.1. *Let $\mathfrak{a} = (y_1,\dots, y_n)$ be an ideal of a Cohen–Macaulay local ring A, and suppose that $A/\mathfrak{a}$ is of codimension r over A. Then $r \leqslant n$, and $\mathfrak{a} = (x_1,\dots, x_n)$, where $x_1,\dots, x_r$ form an A-regular sequence.*

Proof. If $r=0$, there is nothing to do, so assume $r>0$. Then $\mathfrak{a}$ contains a regular element. As in the proof of 3.1, we can use invertible operations on the row $(y_1 \cdots y_n)$ and replace y_1 by a regular element x_1. Let $B=A/(x_1)$, and let $\mathfrak{b}=\mathfrak{a}/(x_1)$. Then $\mathfrak{b}=(\bar{y}_2,\dots,\bar{y}_n)$, and $B/\mathfrak{b}\approx A/\mathfrak{a}$ is of codimension $r-1$ over B. Inductively, $r-1\leqslant n-1$, and we can suppose that $\mathfrak{b}=(\bar{x}_2,\dots,\bar{x}_n)$, where $\bar{x}_2,\dots,\bar{x}_r$ is a B-regular sequence. Lifting $\bar{x}_2,\dots,\bar{x}_n$ to elements $x_2,\dots,x_n$ of A, we have $\mathfrak{a}=(x_1,\dots,x_n)$, and the sequence $x_1,\dots,x_r$ is A-regular. ∎

THEOREM 5.2. *The group $K^2(A)$ is generated by the elements $[A/(x_1,x_2)]$, where x_1 and x_2 form a regular sequence on A.*

Proof. We begin with any module M which is perfect of codimension two over A, and choose a resolution

$$0\to A^{n-m}\to A^n\to A^m\to M\to 0.$$

Using Lemma 4.1, we reduce to

$$0\to A^{n-1}\to A^n\xrightarrow{\alpha} A\to A/\mathfrak{a}\to 0.$$

In $K^2(A)$, $[M]=[A/\mathfrak{a}]$, modulo elements $[A/(x_1,x_2)]$. By 5.1, we can assume that $\alpha=(x_1\ x_2\cdots x_n)$, where x_1 and x_2 form an A-regular sequence. If $n=2$, we are done. If $n>2$, inject the Koszul complex defined by x_1 and x_2 into the complex resolving $A/\mathfrak{a}$, and form the exact sequence of complexes

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\binom{x_2}{-x_1}"] \arrow[d] & A^2 \arrow[r, "(x_1\ x_2)"] \arrow[d, "\iota"] & A \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & A^{n-1} \arrow[r] \arrow[d] & A^n \arrow[r, "\alpha"] \arrow[d, "\pi"] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q \arrow[r] & A^{n-2} \arrow[r] & 0. &
\end{tikzcd}$$

The map ι is the injection into the first two coordinate positions of A^n, and π is the projection from the last $n-2$. The homology of the quotient complex is $\mathfrak{a}/(x_1,x_2)$ in degree one, and zero in degree two, because the homology exact sequence is

$$0\to\mathfrak{a}/(x_1,x_2)\to A/(x_1,x_2)\to A/\mathfrak{a}\to 0.$$

The module $\mathfrak{a}/(x_1, x_2)$ is perfect of codimension two, and $[A/\mathfrak{a}] = [A/(x_1, x_2)] - [\mathfrak{a}/(x_1, x_2)]$ in $K^2(A)$. From the diagram we see that $\mathfrak{a}/(x_1, x_2)$ has a resolution

$$0 \to A \to A^{n-1} \to A^{n-2} \to \mathfrak{a}/(x_1, x_2) \to 0.$$

We are back at the point of beginning, but with n replaced by $n-1$. ∎

The theorem has a consequence worth mentioning. The inclusion of $C^{r+1}(A)$ in $C^r(A)$ defines a group homomorphism $K^{r+1}(A) \to K^r(A)$. If A is regular, then $C^r(A)$ is the class of all finitely generated A-modules of codimension at least r, and the theory of associated primes shows that

$$K^{r+1}(A) \to K^r(A) \to Z^r(A) \to 0$$

is exact, where $Z^r(A)$ is the free abelian group on the primes on height r of A. By 5.2, the map $K^2(A) \to K^1(A)$ is the zero map, because

$$0 \to A/(x_1) \xrightarrow{x_2} A/(x_1) \to A/(x_1, x_2) \to 0$$

is an exact sequence of modules in $C^1(A)$, and $[A/(x_1, x_2)] = 0$ in $K^1(A)$. Thus $K^1(A)$ is free on the primes of height one of A.

6. Codimension Three

In [2], Dutta, Hochster, and McLaughlin exhibit a module M which is of finite length and finite projective dimension over a Cohen–Macaulay local ring A of dimension three, such that the Serre intersection multiplicity

$$\chi(M, A/\mathfrak{p}) = \sum_i (-1)^i \operatorname{length}(\operatorname{Tor}_i^A(M, A/\mathfrak{p}))$$

satisfies $\chi(M, A/\mathfrak{p}) = -1$. Here, $\mathfrak{p}$ is a prime of height one of A. As the authors point out, it follows immediately that $[M]$ cannot belong to the subgroup of $K^3(A)$ generated by the elements $[A/(x_1, x_2, x_3)]$, where x_1, x_2, and x_3 form an A-regular sequence. For $\chi(A/(x_1, x_2, x_3), A/\mathfrak{p}) = 0$. Hence $K^3(A)$ is not generated by the elements $[A/(x_1, x_2, x_3)]$.

It follows from Lemma 4.1 that M may be embedded in a cyclic module $A/\mathfrak{a}$, also of finite length and finite projective dimension over A, such that $[M] = [A/\mathfrak{a}]$ in $K^3(A)$, modulo elements $[A/(x_1, x_2, x_3)]$. Hence $\chi(A/\mathfrak{a}, A/\mathfrak{p}) = -1$. The cyclic module $A/\mathfrak{a}$ is far from unique. Beginning with the module M^* of [2], and using the construction of Lemma 4.1 twice, we get a cyclic module $A/\mathfrak{a}$ such that the ideal $\mathfrak{a}$ has 16 generators. The length of a cyclic module obtained in this way seems to be unreasonably large. A direct construction, as in [2], may give a better result.

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