NORTH-HOLLAND

# Stabilizing Solutions of the $H_\infty$ Algebraic Riccati Equation

Anton A. Stoorvogel*

*Department of Mathematics and Computing Science*
*Eindhoven University of Technology*
*P.O. Box 513*
*5600 MB Eindhoven, The Netherlands*

Submitted by Hans Schneider

ABSTRACT

The algebraic Riccati equation studied in this paper is related to the suboptimal state feedback $H_\infty$ control problem. It is parametrized by the $H_\infty$-norm bound $\gamma$ we want to achieve. The objective of this paper is to study the behavior of the solution to the Riccati equation as a function of $\gamma$. It turns out that a stabilizing solution exists for all but finitely many values of $\gamma$ larger than some *a priori* determined bound $\gamma_-$. On the other hand, for values smaller than $\gamma_-$ there does not exist a stabilizing solution. The finite number of exception points can be characterized as switching points where eigenvalues of the stabilizing (symmetric) solution can switch from negative to positive with increasing $\gamma$. After the final switching point the solution will be positive semidefinite. We obtain the following interpretation: The Riccati equation has a stabilizing solution with $k$ negative eigenvalues if and only if there exists a static feedback such that the closed-loop transfer matrix has $k$ unstable poles and an $L_\infty$ norm strictly less than $\gamma$.

## 1. INTRODUCTION

The algebraic Riccati equation has a long history. The algebraic Riccati equation with a sign-definite quadratic term has played an important role in control theory. It was used in linear quadratic control, Kalman filtering, and the combination of the two: the linear quadratic Gaussian or $H_2$, control problem (see e.g. [1, 2, 7, 15, 25]). The specific properties of this Riccati equation have also been studied extensively (see e.g. [21]).

But also a more general form of the algebraic Riccati equation has appeared in the literature. In this case, the quadratic term is not necessarily

*E-mail: wscoas@win.tue.nl.



0024-3795/96/$15.00
655 Avenue of the Americas, New York, NY 10010 SSDI 0024-3795(94)00195-J

sign-definite. This more general Riccati equation first appeared in the game-theory literature (see e.g. [4, 16, 17]). More recently, it turned out to play an important role in $H_\infty$ control theory (see e.g. [9, 20, 24]). In the latter case the Riccati equation is parametrized by a parameter $\gamma$. It turns out that there exists a state feedback which makes the closed-loop $H_\infty$ norm strictly less than $\gamma$ if and only if there exists a positive semidefinite, stabilizing solution to the algebraic Riccati equation. An iterative search then determines the minimal achievable $H_\infty$ norm, say $\gamma_*$. In the process of determining $\gamma_*$ one also checks for existence of stabilizing solutions for values of $\gamma$ smaller than $\gamma_*$. It turned out that the solution either does not exist or is indefinite. The objective of this paper is to study the behavior and existence of stabilizing solutions of the algebraic Riccati equation also for values of $\gamma$ smaller than $\gamma_*$.

It turns out that a stabilizing solution exists for all but finitely many values of $\gamma$ larger than some *a priori* determined bound $\gamma_-$. On the other hand, for values smaller than $\gamma_-$ there does not exist a stabilizing solution. The finite number of exception points can be characterized as switching points where eigenvalues of the stabilizing (symmetric) solution can switch from negative to positive with increasing $\gamma$. After the final switching point the solution will be positive semidefinite. We obtain the following interpretation: the Riccati equation has a stabilizing solution with $k$ negative eigenvalues if and only if there exists a static feedback such that the closed-loop transfer matrix has $k$ unstable poles and a closed-loop $L_\infty$ norm strictly less than $\gamma$. Note that the $L_\infty$ norm is equal to the $H_\infty$ norm for stable systems; in other words, the $L_\infty$ norm is an extension of the $H_\infty$ norm to the larger class of possibly unstable transfer matrices.

This is not a purely theoretical exercise. This study might help to find more efficient ways to perform the abovementioned $\gamma$-iteration. In general, this Riccati equation plays such an important role in present-day controller design that it is important to study its properties. Some related results have been obtained in [11, 14].

We denote by $H_\infty^i$ the set of transfer matrices with $i$ unstable poles, i.e. such that the McMillan degree of the unstable part is equal to $i$. Moreover $\mathcal{G}H_\infty$ denotes those transfer matrices in $H_\infty$ that are invertible and whose inverses are again in $H_\infty$.

## 2. PROBLEM FORMULATION

This paper studies the following Riccati equation:

$$\begin{aligned} 0 = &\, A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C \\ &-(PB + C^{\mathrm{T}}D)(D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C) + \gamma^{-2}PEE^{\mathrm{T}}P. \end{aligned} \tag{2.1}$$

In this paper we only study stabilizing solutions of this equation, i.e., solutions for which the following matrix is asymptotically stable:

$$A - B(D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C) + \gamma^{-2}EE^{\mathrm{T}}P. \tag{2.2}$$

One of the main reasons for studying this Riccati equation is related to the $H_\infty$ control problem. Consider the following system:

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew, \\ z &= Cx + Du, \end{aligned} \tag{2.3}$$

where $x \in \mathbb{R}^n$. The following theorem can be found e.g. in [9, 20, 24]:

THEOREM 2.1. *Consider the system* (2.3), *and let* $\gamma > 0$. *Assume that the system* $(A, B, C, D)$ *has no invariant zeros on the imaginary axis and* $D$ *is injective. Then the following statements are equivalent:*

(i) *There exists a static feedback law* $u = Fx$ *such that after applying this compensator to the system* (2.3) *the resulting closed-loop system is internally stable and the closed-loop transfer matrix* $G_F$ *has* $H_\infty$ *norm less than* $\gamma$, *i.e.,* $\|G_F\|_\infty < \gamma$.

(ii) *There exists a positive semidefinite solution* $P$ *of the Riccati equation* (2.1) *such that the matrix in* (2.2) *is asymptotically stable.*

*If* $P$ *satisfies the conditions in part* (ii), *then a controller satisfying the conditions in part* (i) *is given by*

$$F := -(D^{\mathrm{T}}D)^{-1}(D^{\mathrm{T}}C + B^{\mathrm{T}}P). \tag{2.4}$$

For later use, we define the infimal achievable $H_\infty$ norm via a stabilizing state feedback by $\gamma_*$. It is our objective to extend the above result. We will show that for some $\gamma < \gamma^*$ there still exists a stabilizing solution of the algebraic Riccati equation, but the solution is not positive semidefinite and the number of negative eigenvalues determines the number of unstable poles we have to admit to guarantee an $L_\infty$ performance bound of $\gamma$. More precisely stated, the main result of this paper is the following theorem:

THEOREM 2.2. *Consider the system* (2.3), *and let* $\gamma > 0$. *Assume that the system* $(A, B, C, D)$ *has no invariant zeros on the imaginary axis and* $D$ *is injective. Then for all but finitely many* $\gamma$ *the following statements are equivalent:*

(i) *There exists a static feedback law* $u = Fx$ *such that after applying this compensator to the system* (2.3), *the resulting closed-loop system has at most* $i$ *unstable eigenvalues and the closed-loop transfer matrix* $G_F$ *has* $L_\infty$ *norm less than* $\gamma$, *i.e.,* $\|G_F\|_\infty < \gamma$. *Moreover, there does not exist a static feedback law* $u = Fx$ *such that after applying this compensator to the system* (2.3) *the resulting closed-loop system has less than* $i$ *unstable eigenvalues while the closed-loop transfer matrix* $G_F$ *has* $L_\infty$ *norm less than* $\gamma$.

(ii) *There exists a solution* $P$ *of the Riccati equation* (2.1), *such that the matrix in* (2.2), *is asymptotically stable. Moreover,* $P$ *has* $i$ *negative eigenvalues.*

*If* $P$ *satisfies the conditions in part* (ii), *then a controller satisfying the conditions in part* (i) *is given by* (2.4).

REMARK. Note that we do not suggest that people should start designing controllers which do not stabilize the system. The importance of the above theorem lies in the fact that it tells us a great deal about the algebraic Riccati equation and the behavior of its stabilizing solution as a function of $\gamma$. For large $\gamma$ the equation has a positive semidefinite stabilizing solution. Then after a switching point, $\gamma_*$ as defined above, the Riccati equation may still have a solution, but it will have at least one negative eigenvalue. These switching points are the only values of $\gamma$ where the number of positive eigenvalues of the stabilizing solution can change. The finitely many values of $\gamma$ for which the above theorem might not hold are precisely these switching points. Hence we also know *a priori* that there are no more than $n$ values of $\gamma$ for which the theorem might not be true.

We will show via an example that these switching points do indeed occur and that the equivalence between (i) and (ii) in the above theorem might not hold true for such a switching point.

EXAMPLE 2.3. Consider the following system:

$$\dot{x} = x + u + w,$$
$$z = u.$$

It is easy to check that the Riccati equation (2.1) has two solutions for all $\gamma \neq 1$: $P_1 = 0$ and $P_2 = 2\gamma^2/(\gamma^2 - 1)$. $P_1$ is not stabilizing, while $P_2$ yields a stable matrix in (2.2) for $P = P_2$. For $\gamma = 1$ there is only one solution: $P = 0$, which is not stabilizing.

For all $\gamma > 1$, there exists a positive definite stabilizing solution to the algebraic Riccati equation, and the feedback (2.4) with $P = P_2$ satisfies the $H_\infty$-norm bound of part (i). On the other hand, for $\gamma < 1$ there exists a negative definite stabilizing solution, and (2.4) with $P = P_2$ yields a closed-loop system with one unstable eigenvalue and an $L_\infty$ norm less than $\gamma$; in other words, part (i) is satisfied for this feedback with $i = 1$.

But for $\gamma = 1$ the implication (i) $\Rightarrow$ (ii) does not hold. It is easy to see that $u = 0$ satisfies part (i) for $\gamma = 1$ and $i = 1$ but part (ii) is not satisfied: there does not exist a stabilizing solution.

Note that these switching points are not always present. For instance, the system

$$\begin{aligned} \dot{x} &= -x + u + w, \\ z &= u \end{aligned}$$

yields a stabilizing solution 0 for all $\gamma$, and no switching occurs.

## 3. THE BOUNDED REAL LEMMA

In our derivation of Theorem 2.2, the bounded real lemma will play a role. But this is a different version than the classical result (see e.g. [3, 25]). Instead of a test to check whether a stable transfer matrix has $H_\infty$ norm strictly less than $\gamma$, we derive a test whether an arbitrary (not necessarily stable) rational matrix has $L_\infty$ norm strictly less than $\gamma$. Reference [25] already basically contains this result, although our assumptions are slightly weaker.

THEOREM 3.1. *Let a stabilizable and detectable realization* $[A, B, C, D]$ *of the transfer matrix* $G$ *be given. Then the following statements are equivalent:*

(i) *We have* $\|G\|_\infty < \gamma$.

(ii) *We have* $D^{\mathrm{T}}D < \gamma^2 I$. *Moreover, there exists a solution* $P$ *of the algebraic Riccati equation*

$$\begin{aligned} 0 = {} & A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C \\ & + (PB + C^{\mathrm{T}}D)(\gamma^2 I - D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C) \end{aligned} \tag{3.1}$$

*such that the following matrix is asymptotically stable:*

$$A + B(\gamma^2 I - D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C).$$

(iii) *We have* $DD^{\mathrm{T}} < \gamma^2 I$. *Moreover, there exists a solution* $Q$ *of the algebraic Riccati equation*

$$\begin{aligned} 0 = {} & AQ + QA^{\mathrm{T}} + BB^{\mathrm{T}} \\ & + (QC^{\mathrm{T}} + BD^{\mathrm{T}})(\gamma^2 I - DD^{\mathrm{T}})^{-1}(CQ + DB^{\mathrm{T}}) \end{aligned} \tag{3.2}$$

*such that the following matrix is asymptotically stable.*

$$A + (QC^{\mathrm{T}} + BD^{\mathrm{T}})(\gamma^2 I - DD^{\mathrm{T}})^{-1}C.$$

*If $P$ satisfies condition* (ii) *or $Q$ satisfies condition* (iii), *then it has the same number of negative eigenvalues as the number of unstable eigenvalues of $A$.*

*Proof.* Conditions (ii) and (iii) are clearly dual to each other. Hence it suffices to prove equality between conditions (i) and (iii).

Due to detectability, there exists a solution $Y$ of

$$AY + YA^{\mathrm{T}} - YC^{\mathrm{T}}CY = 0$$

such that $A - YC^{\mathrm{T}}C$ is asymptotically stable. Define a transfer matrix $H$ with realization

$$[A - YC^{\mathrm{T}}C, B - YC^{\mathrm{T}}D, C, D].$$

It is then easy to check that $G^{\sim}G = H^{\sim}H$, where $G^{\sim}(s) = G^{\mathrm{T}}(-s)$, and it is immediate that $G$ and $H$ have the same $L_\infty$ norm. On the other hand, since $H$ is stable, we know from the classical small-gain theorem that $H$ has $H_\infty$ norm less than $\gamma$ if and only if $DD^{\mathrm{T}} < \gamma$ and if there exists a solution $Z$ of

$$\begin{aligned} 0 = {} & [A - YC^{\mathrm{T}}C]Z + Z[A - YC^{\mathrm{T}}C]^{\mathrm{T}} + [B - YC^{\mathrm{T}}D][B - YC^{\mathrm{T}}D]^{\mathrm{T}} \\ & + (ZC^{\mathrm{T}} + [B - YC^{\mathrm{T}}D]D^{\mathrm{T}})(\gamma^2 I - DD^{\mathrm{T}})^{-1}(CZ + D[B - YC^{\mathrm{T}}D]^{\mathrm{T}}) \end{aligned}$$

such that the following matrix is stable:

$$[A - YC^{\mathrm{T}}C] + (ZC^{\mathrm{T}} + [B - YC^{\mathrm{T}}D]D^{\mathrm{T}})(\gamma^2 I - DD^{\mathrm{T}})^{-1}C.$$

The proof is completed by noting via some algebraic manipulations that $Z$ satisfies the above equations if and only if $P := Z - \gamma^2 Y$ satisfies the conditions of Theorem 3.1.

To prove the part regarding the number of unstable eigenvalues of $P$ and $Q$, we view (3.1) and (3.2) as Lyapunov equations, and since $(A, B)$ is stabilizable and $(C, A)$ is detectable, we obtain the result directly from [6]. ∎

## 4. RELATION OF $H_\infty$ CONTROL PROBLEMS TO $J$-SPECTRAL FACTORIZATION

In this section we will show the relation between the existence of suitable $H_\infty$ control problems and $J$-spectral factorization. This section is strongly based on the paper [13]. We basically study how the results change if we allow unstable closed-loop poles. We study the classical one- and two-block problems and relate the existence of a controller with at most $i$ unstable poles to the existence of a $J$-spectral factorization with a specific additional feature. In the next section we relate $J$-spectral factorization to Riccati equations according to a theorem of [5, 13]. Finally, in the section after, we use our results for the two-block problem to prove our Theorem 2.2.

### *4.1. The Nehari Problem*

Let $R \in L_\infty$ be given with Hankel singular values $\sigma_1^H \geq \cdots \geq \sigma_i^H \geq \cdots \geq \sigma_n^H$. Then we have the following theorem:

THEOREM 4.1. *The following statements are equivalent:*

(i) $\sigma_{i+1}^H > \gamma > \sigma_i^H$.
(ii) *There exists* $L \in H_\infty^i$ *such that* $\|R + L\|_\infty < \gamma$. *Moreover, there does not exist* $L \in H_\infty^{i-1}$ *such that* $\|R + L\|_\infty < \gamma$.
(iii) *There exists* $W \in \mathcal{G}H_\infty$ *where* $W_{11}$ *is invertible with* $W_{11}^{-1} \in H_\infty^i$ *satisfying*

$$G^\sim JG = W^\sim JW, \tag{4.1}$$

*where*

$$G = \begin{pmatrix} I & R \\ 0 & I \end{pmatrix}, \qquad J = \begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I \end{pmatrix}, \qquad W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix},$$

*and all block decompositions are of compatible sizes w.r.t.* (4.1).

*Proof.* The equivalence of (i) and (ii) has been shown in [12].

(i) $\Rightarrow$ (iii): We split $R = R_+ + R_-$ where $R_+$ is stable while $R_-$ is strictly proper and antistable and $R_-$ has minimal realization $[A_-, B_-, C_-, 0]$. Let $P$ and $Q$ be the controllability and observability gramians of $R_-$:

$$\begin{aligned} A^{\mathrm{T}}Q + QA + C^{\mathrm{T}}C &= 0, \\ AP + PA^{\mathrm{T}} + BB^{\mathrm{T}} &= 0. \end{aligned}$$

Since $\gamma \neq \sigma_j^H$ $(j = 1, \ldots, n)$ we have that $N := (I - \gamma^{-2}PQ)^{-1}$ is well defined. Define $X$ by

$$X := \left[ -A_-^{\mathrm{T}}, (C_-^{\mathrm{T}} \quad QB_-), \gamma^{-2} \begin{pmatrix} -C_-PN \\ B_-^{\mathrm{T}}N \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \right].$$

Then it can be easily checked that $G_-^{\sim} J G_- = X^{\sim} J X$ and $X \in \mathcal{G}H_\infty$, where

$$G_- := \begin{pmatrix} I & R_- \\ 0 & I \end{pmatrix}.$$

Moreover the $(1,1)$ block of $X$, denoted by $X_{11}$, is invertible, and $X_{11}^{-1}$ has realization

$$\left[ -A_-^{\mathrm{T}} + \gamma^{-2} C_-^{\mathrm{T}} C_- PN, C_-^{\mathrm{T}}, \gamma^{-2} C_- PN, I \right].$$

We have

$$\widehat{A}(P^{-1} - \gamma^{-2} Q) + (P^{-1} - \gamma^{-2} Q)\widehat{A}^{\mathrm{T}} + (P^{-1} B_- B_-^{\mathrm{T}} P^{-1} + \gamma^{-2} C_-^{\mathrm{T}} C_-) = 0,$$

where $\widehat{A} = -A_-^{\mathrm{T}} + \gamma^{-2} C_-^{\mathrm{T}} C_- PN$. Moreover, $P^{-1} - \gamma^{-2} Q$ is invertible and has precisely $i$ negative eigenvalues, and finally, $(C_-, A_-)$ is observable. Then it follows from [6] that $\widehat{A}$ has precisely $i$ eigenvalues in the closed right half plane and hence $X_{11}^{-1} \in H_\infty^i$.

The proof of the implication is completed by noting that

$$W := X \begin{pmatrix} I & R_+ \\ 0 & I \end{pmatrix}$$

satisfies all the requirements of the above theorem.

(iii) $\Rightarrow$ (ii): Suppose a $W$ exists satisfying the conditions of part (iii). Define $V = W^{-1}$, and partition $V$ conformably with $W$. Define $L = V_{12} V_{22}^{-1}$. It is easy to check that $L \in H_\infty^i$. Moreover,

$$(R+L)^{\sim}(R+L) - \gamma^2 I = -\gamma^2 (V_{22} V_{22}^{\sim})^{-1} < 0.$$

This implies part (ii). ∎

*4.2. The Two-Block Problem*

THEOREM 4.2. *Let $S, T \in H_\infty$ be given, where $T$ has full row rank on the imaginary axis. For all but finitely many $\gamma$, the following statements are equivalent:*

(i) *There exists $L \in H_\infty^i$ such that*

$$\|T + SL\|_\infty < \gamma. \tag{4.2}$$

*Moreover, there does not exist $L \in H_\infty^{i-1}$ satisfying* (4.2).

(ii) *There exists* $W \in \mathcal{G}H_\infty$ *where* $W_{11}$ *is invertible with* $W_{11}^{-1} \in H_\infty^i$ *satisfying*

$$G^\sim JG = W^\sim JW, \tag{4.3}$$

*where*

$$G = \begin{pmatrix} S & T \\ 0 & I \end{pmatrix}, \qquad J = \begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I \end{pmatrix}, \qquad W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix},$$

*and all block decompositions are of compatible sizes w.r.t.* (4.3).

*Proof.* Factorize $T = T_i T_o$ where $T_o \in \mathcal{G}H_\infty$ and $T_i$ is inner. Choose $T_\perp \in H_\infty$ such that $[T_i \ T_\perp]$ is square and inner. Then we have (4.2) if and only if

$$(R_1 + T_o L)^\sim (R_1 + T_o L) < \gamma^2 I - R_2^\sim R_2,$$

where $R_1 = T_i^\sim S$ and $R_2 = T_\perp^\sim S$. Therefore there exists an $L \in H_\infty^i$ such that (4.2) is satisfied if and only if there exists $N \in \mathcal{G}H_\infty$ such that

$$N^\sim N = \gamma^2 I - R_2^\sim R_2 \tag{4.4}$$

and

$$\|R_1 N^{-1} + \widehat{L}\|_\infty < 1 \tag{4.5}$$

(where $\widehat{L} = T_o L N^{-1}$). If $1 \neq \sigma_j^H(R_1 N^{-1})$ $(j = 1, \ldots, n)$, we can apply Theorem 4.1. We get that (4.5) is satisfied for some $\widehat{L} \in H_\infty^i$ if and only if there exist $X$ such that

$$\begin{pmatrix} I & R_1 N^{-1} \\ 0 & I \end{pmatrix}^\sim \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} I & R_1 N^{-1} \\ 0 & I \end{pmatrix} = X^\sim \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} X \tag{4.6}$$

with $X \in \mathcal{G}H_\infty$ with $X_{11}$ invertible and $X_{11}^{-1} \in H_\infty^i$. Finally, $X$ satisfies the above properties if and only if

$$W := X \begin{pmatrix} T_o & 0 \\ 0 & \gamma^{-1} N \end{pmatrix}$$

satisfies (4.3) with $W \in \mathcal{G}H_\infty$ with $W_{11}$ invertible and $W_{11}^{-1} \in H_\infty^i$.

The proof is complete if we show that the existence of $W$ satisfying the conditions of part (ii) implies the existence of $N$ satisfying (4.4). The latter follows because

$$G^{\sim}JG = \begin{pmatrix} T_o & 0 \\ S^{\sim}T_i & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R_2^{\sim}R_2 - \gamma^2 I \end{pmatrix} \begin{pmatrix} T_o & T_i^{\sim}S \\ 0 & I \end{pmatrix}. \tag{4.7}$$

Because $W$ has full rank on the imaginary axis, (4.3) implies that $G^{\sim}JG$ evaluated on the imaginary axis has the same inertia as $J$. According to (4.7) this requires that

$$R_2^{\sim}R_2 - \gamma^2 I < 0,$$

which in turn implies the existence of the required $N$.

We see that $\gamma$ should not be such that $R_1N^{-1}$ has a Hankel singular value equal to 1. It is easy to check that the Hankel singular values of $R_1N^{-1}$ are decreasing functions of $\gamma$. Hence the number of exception points is no more than the McMillan degree of $R_1N^{-1}$. ■

## 5. $J$-SPECTRAL FACTORIZATION

In this section we would like to show the relation between the existence of a $J$-spectral factorization and the existence of a solution to the algebraic Riccati equation. Also, since the factorization is not unique, we show that the number of unstable poles of the inverse of the (1,1) block of the $J$-spectral factor is independent of the specific choice for the $J$-spectral factor. This is needed because that number played an important role in the previous section.

We have:

THEOREM 5.1. *Let $S$ and $T$ have realizations $[A, B_1, C, D]$ and $[A, B_2, C, 0]$ respectively with $A$ stable. Then there exists $W \in \mathcal{G}H_\infty$ such that (4.3) is satisfied if and only if there exists a solution $P$ of the algebraic Riccati equation*

$$\begin{aligned} 0 = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C - (PB_1 + C^{\mathrm{T}}D)(D^{\mathrm{T}}D)^{-1} \\ \times \left(B_1^{\mathrm{T}}P + D^{\mathrm{T}}C\right) + \gamma^{-2}PB_2B_2^{\mathrm{T}}P \end{aligned} \tag{5.1}$$

*such that the following matrix is asymptotically stable:*

$$A - B_1(D^{\mathrm{T}}D)^{-1}(B_1^{\mathrm{T}}P + D^{\mathrm{T}}C) + \gamma^{-2}B_2B_2^{\mathrm{T}}P. \tag{5.2}$$

*Proof.* This is a direct result of [5, 13]. ■

Next, we focus on the question whether the existence of one $J$-spectral vector $W$ of $G^{\sim}JG$ for which $W_{11}$ is invertible with $W_{11}^{-1} \in H_\infty^i$ implies that every spectral factor of $G^\sim JG$ has this property. We first need a preliminary lemma:

LEMMA 5.2. *Let $H \in L_\infty$ be a given rational matrix with $\|H\|_\infty < 1$. Then $(I+H)^{-1}$ exists and has the same number of unstable poles as $H$.*

*Proof.* Let $[A, B, C, D]$ be a minimal realization of $H$. Then $(I+H)^{-1}$ has a realization

$$[A - B(I+D)^{-1}C, B(I+D)^{-1}, -C(I+D)^{-1}, (I+D)^{-1}].$$

Since $H$ has norm less than 1, we can apply Theorem 3.1. In other words, there exists a matrix $Q$ of the algebraic Riccati equation (3.2). Then, after some algebraic manipulations, we get

$$[A - B(I+D)^{-1}C]Q + Q[A - B(I+D)^{-1}C]^{\mathrm{T}} + S = 0, \tag{5.3}$$

where

$$\begin{aligned} S = {} & [QC^{\mathrm{T}} + B(I+D)^{-1}(I+D^{\mathrm{T}})](I - DD^{\mathrm{T}})^{-1} \\ & \times [CQ + (I+D)(I+D^{\mathrm{T}})^{-1}B^{\mathrm{T}}] \geq 0. \end{aligned}$$

We know that $(A, B)$ is controllable and that $A$ has no imaginary-axis eigenvalues. Hence, if we view (3.2) as a Lyapunov equation, we get that the number of unstable eigenvalues of $A$ is equal to the number of negative eigenvalues of $Q$. Moreover, $Q$ is not singular.

It is immediate that $A - B(I+D)^{-1}C$ has no eigenvalues on the imaginary axis. Hence, using some classical results for the Lyapunov equation (see e.g. [19]), we find that the Lyapunov equation (5.3) implies that $A - B(I+D)^{-1}C$ has as many unstable poles as $A$. ■

The above is for SISO systems a direct consequence of the classical theorem by Rouché (see [22]). This result allows us to derive the following theorem establishing that the number of unstable zeros of the (1, 1) block of a $J$-spectral factorization is independent of the specific factorization chosen.

THEOREM 5.3. *Let $G$ be given as in Theorem 4.2. Let $V, W \in \mathcal{G}H_\infty$ be two spectral factors of $G^\sim JG$, i.e.*

$$V^\sim JV = G^\sim JG = W^\sim JW.$$

*Then $V_{11}^{-1}$ and $W_{11}^{-1}$ both exist, and they have the same number of unstable poles.*

*Proof.* Note that (4.3) together with $S$ full row rank implies that

$$W_{11}^{\sim} W_{11} - \gamma^2 W_{21}^{\sim} W_{21} = S^{\sim} S > 0.$$

Hence $W_{11}$ is invertible and $\|W_{21} W_{11}^{-1}\|_\infty < \gamma^{-1}$. Also note that the number of unstable zeros of $W_{21} W_{11}^{-1}$ is equal to the number of unstable poles of $W_{11}^{-1}$, i.e., no pole-zero cancellations can occur.

It is easy to show (see [13]) that $J$-spectral factors are unique up to a constant $J$-unitary matrix. In other words, there exists a constant matrix $A$ such that $V = AW$ where

$$A^{\sim} J A = J.$$

This condition for $A$ implies

$$A_{11}^{\mathrm{T}} A_{11} - \gamma^2 A_{21}^{\mathrm{T}} A_{21} = I,$$

and therefore $A_{11}$ is invertible and $\|A_{11}^{-1} A_{12}\| < \gamma$. We find $\|A_{12} W_{21} W_{11}^{-1} A_{11}^{-1}\|_\infty < 1$ and

$$\begin{aligned} V_{11}^{-1} &= (A_{11} W_{11} + A_{12} W_{21})^{-1} \\ &= \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} A_{11} W_{11} + A_{12} W_{21} & 0 \\ \alpha I & I \end{pmatrix}^{-1} \end{aligned}$$

for any value for $\alpha \neq 0$. Therefore the number of unstable poles of $V_{11}^{-1}$ is equal to the number of unstable poles of

$$\begin{pmatrix} A_{12} W_{21} & 0 \\ \alpha I & 0 \end{pmatrix} \begin{pmatrix} A_{11} W_{11} + A_{12} W_{21} & 0 \\ \alpha I & I \end{pmatrix}^{-1}, \tag{5.4}$$

since $A_{12} W_{21}$ is stable and $\alpha \neq 0$. Define $H$ by

$$H := \begin{pmatrix} A_{12} W_{21} W_{11}^{-1} A_{11}^{-1} & 0 \\ \alpha I & 0 \end{pmatrix},$$

and choose $\alpha$ small enough that $\|H\|_\infty < 1$. Then the matrix in (5.4) is equal to

$$H(I+H)^{-1} = I - (I+H)^{-1}$$

Therefore Lemma 5.2 guarantees that $V_{11}^{-1}$ has as many unstable eigenvalues as $H$. Moreover the number of unstable eigenvalues of $H$ is clearly equal to the number of eigenvalues of $W_{11}^{-1}$. ∎

Using the above, we can extend Theorem 5.1 to include the number of unstable poles of the inverse of the $(1,1)$ block.

THEOREM 5.4. *Let $T$ and $S$ have realizations $[A, B_1, C, D]$ and $[A, B_2, C, 0]$, respectively, with $A$ stable. Then there exists $W \in \mathcal{G}H_\infty$ where $W_{11}$ is invertible with $W_{11}^{-1} \in H_\infty^i$ such that* (4.3) *is satisfied if and only if there exists a solution $P$ of the algebraic Riccati equation* (5.1) *such that the matrix in* (5.2) *is asymptotically stable and $P$ has $i$ negative eigenvalues.*

*Proof.* It is easily checked that if a matrix $P$ satisfying the conditions of Theorem 5.1 exists, then one particular $J$-spectral factor $W \in \mathcal{G}H_\infty$ is given by

$$W = [A, (B_1 \;\; B_2), C_W, D_W],$$

where

$$C_W := \begin{pmatrix} (D^{\mathrm{T}}D)^{-1/2}(D^{\mathrm{T}}C + B_1^{\mathrm{T}}P) \\ -\gamma^{-2}B_2^{\mathrm{T}}P \end{pmatrix},$$

$$D_W := \begin{pmatrix} (D^{\mathrm{T}}D)^{1/2} & 0 \\ 0 & I \end{pmatrix}.$$

Therefore we find the following realization for $W_{11}^{-1}$:

$$W_{11}^{-1} := \left[A_W, -B_1(D^{\mathrm{T}}D)^{-1/2}, (D^{\mathrm{T}}D)^{-1}(D^{\mathrm{T}}C + B_1^{\mathrm{T}}P), (D^{\mathrm{T}}D)^{-1/2}\right],$$

where $A_W := A - B_1(D^{\mathrm{T}}D)^{-1}(D^{\mathrm{T}}C + B_1^{\mathrm{T}}P)$. The algebraic Riccati equation for $P$ can be rewritten as

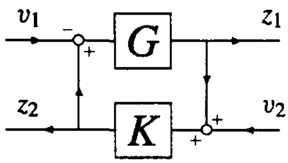
$$0 = A_W^{\mathrm{T}}P + PA_W + C_{W,1}^{\mathrm{T}}C_{W,1} + \gamma^{-2}PB_2B_2^{\mathrm{T}}P, \tag{5.5}$$

where $C_{W,1} := C - D(D^{\mathrm{T}}D)^{-1}(D^{\mathrm{T}}C + B_1^{\mathrm{T}}P)$. Treating this equation as a Lyapunov equation and noting that $(A_W, PB_2)$ is stabilizable, [6] tells us that the number of negative eigenvalues of $P$ is equal to the number of unstable eigenvalues of $A_W$. In other words, the number of unstable poles of $W_{11}^{-1}$ is equal to the number of negative eigenvalues of $P$. Because of Theorem 5.3, it is sufficient to prove the result for one particular $J$-spectral factorization, and hence the proof is complete. ∎

## 6. YOULA PARAMETRIZATION

The Youla parametrization is an often used tool in modern control theory (see e.g. [8, 10, 26]). However, since we allow for a fixed number of unstable poles, we need to extend this theory.

First of all, we need to define the unstable closed-loop poles of the closed-loop system. Suppose we have the following interconnection:




(6.1)

The closed-loop transfer matrix from $v$ to $z$ is equal to

$$T(G,K) := \begin{pmatrix} -G(I-KG)^{-1} & G(I-KG)^{-1}K \\ -K(I-GK)^{-1}G & (I-KG)^{-1}K \end{pmatrix}.$$

Our standing assumption in this paper is that $G$ is stabilizable and detectable. Then we define the unstable closed-loop poles as the unstable poles of $T(G,K)$, and the number of unstable poles as the McMillan degree of the unstable part of $T(G,K)$.

We obtain left and right coprime factorizations over $H_\infty$ of $K$ and $G$:

$$\begin{aligned} K &= \widetilde{V}^{-1}\widetilde{U} = UV^{-1}, \\ G &= \widetilde{M}^{-1}\widetilde{N} = NM^{-1}. \end{aligned} \tag{6.2}$$

Then it is easy to show that a right coprime factorization of $T(G,K)$ is given by

$$T(G,K) = \begin{pmatrix} -N & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix}^{-1}.$$

Therefore the number of unstable poles of the closed-loop system is equal to the number of unstable zeros of

$$\begin{pmatrix} M & U \\ N & V \end{pmatrix}.$$

We can now derive the following theorem:

THEOREM 6.1. *The set of all proper controllers $K$ which, when applied to the system $G$, yield a closed-loop system with $i$ unstable poles is parametrized by*

$$K = (Y - ML)(X - NL)^{-1} = (\widetilde{X} - L\widetilde{N})^{-1}(\widetilde{Y} - L\widetilde{M}), \qquad L \in H_\infty^i,$$

*where* $N, M, \widetilde{M}, \widetilde{N}, X, Y, \widetilde{X}, \widetilde{Y}$ *form a doubly coprime factorization of* $G$, *i.e.,* (6.2) *is satisfied and*

$$\begin{pmatrix} \widetilde{X} & -\widetilde{Y} \\ -\widetilde{N} & \widetilde{M} \end{pmatrix} \begin{pmatrix} M & Y \\ N & X \end{pmatrix} = I.$$

## 7. PROOF OF THEOREM 2.2

Using the classical technique from [10], we transform the state feedback $H_\infty$ control problem into a model-matching problem.

The following result is a direct consequence of our extended Youla parametrization as given in Theorem 6.1 and an explicit expression for the doubly coprime factorization which can be found in e.g. [10, 18]:

THEOREM 7.1. *A (possibly dynamic) feedback* $u = Kx$ *yields, when applied to* (2.3), *a closed-loop system with* $i$ *unstable poles if and only if there exists a* $L \in H_\infty^i$ *such that*

$$K = (Y - ML)(X - NL)^{-1},$$

*where* $M, N, Y, X$ *are defined by:*

$$\begin{aligned} M &:= [A + BF, B, F, I], \\ N &:= [A + BF, B, I, 0], \\ Y &:= [A + BF, -H, F, 0], \\ X &:= [A + BF, -H, I, I], \end{aligned}$$

*and* $F$ *and* $H$ *are such that* $A + BF$ *and* $A + H$ *are asymptotically stable. Moreover, the resulting closed-loop transfer matrix is equal to*

$$G_{cl} := T_1 - T_2 L T_3,$$

*where*

$$\begin{aligned} T_1 &= \left[ \begin{pmatrix} A + BF & -BF \\ 0 & A + H \end{pmatrix}, \begin{pmatrix} E \\ E \end{pmatrix}, \begin{pmatrix} C + DF & -DF \end{pmatrix}, 0 \right], \\ T_2 &= [A + BF, B, C + DF, D], \\ T_3 &= [A + H, E, I, 0]. \end{aligned}$$

It turns out that the parametrization as obtained from [10] can be simplified by replacing $L$ with $L + F$. Clearly $L + F \in H_\infty^i$ if and only if $L \in H_\infty^i$. In this way we obtain the following corollary:

COROLLARY 7.2. *A (possibly dynamic) feedback $u = Kx$ yields, when applied to* (2.3)*, a closed-loop system with $i$ unstable poles if and only if there exists a $\widehat{L} \in H_\infty^i$ such that*

$$K = (\widehat{Y} - M\widehat{L})(\widehat{X} - N\widehat{L})^{-1},$$

*where $M, N, Y, X$ are defined by*

$$\begin{aligned} M &:= [A + BF, B, F, I], \\ N &:= [A + BF, B, I, 0], \\ \widehat{Y} &:= [A + BF, -H + BF, F, F], \\ \widehat{X} &:= [A + BF, -H + BF, I, I]. \end{aligned}$$

*Moreover, the resulting closed-loop transfer matrix is equal to*

$$G_{cl} := \widehat{T}_1 - T_2 \widehat{L} T_3,$$

*where*

$$\begin{aligned} \widehat{T}_1 &= [A + BF, E, C + DF, 0], \\ T_2 &= [A + BF, B, C + DF, D], \\ T_3 &= [A + H, E, I, 0]. \end{aligned}$$

The implication (i) $\Rightarrow$ (ii) in Theorem 2.2 is now a direct consequence of the above corollary, Theorems 5.4 and 4.2. After all, the existence of a suitable feedback implies according to the above corollary the existence of a matrix $\widehat{L} \in H_\infty^i$ such that $\|\widehat{T}_1 - T_2 \widehat{L} T_3\|_\infty < \gamma$. Hence $\widetilde{L} := \widehat{L} T_3$ satisfies $\|\widehat{T}_1 - T_2 \widetilde{L}\|_\infty < \gamma$, and according to Theorem 4.2 this implies the existence of a certain $J$-spectral factorization for all but finitely many $\gamma$. By Theorem 5.4, this $J$-spectral factorization exists if and only if there exists a solution to an algebraic Riccati equation. Finally, it is easily checked that the solution of this Riccati equation satisfies all the requirements of part (ii) of Theorem 2.2.

The implication (ii) $\Rightarrow$ (i) in Theorem 2.2 is almost immediate. The feedback given by (2.4) results in a closed-loop system $[A_W, E, C_W, 0]$ where

$$\begin{aligned} A_W &:= A - B(D^{\mathrm{T}} D)^{-1}(D^{\mathrm{T}} C + B^{\mathrm{T}} P), \\ C_W &:= C - D(D^{\mathrm{T}} D)^{-1}(D^{\mathrm{T}} C + B^{\mathrm{T}} P). \end{aligned}$$

It is easy to check that the algebraic Riccati equation for $P$ can be rewritten as

$$0 = A_W^{\mathrm{T}} P + P A_W + C_W^{\mathrm{T}} C_W + \gamma^{-2} P E E^{\mathrm{T}} P. \tag{7.1}$$

Moreover,

$$A_W + \gamma^{-2} E E^{\mathrm{T}} P$$

is asymptotically stable. Given (7.1), classical inertia theory then tells us that the number of unstable eigenvalues of $A_W$ equals the number of negative eigenvalues of $P$. It is then a direct consequence of Theorem 3.1 that this feedback satisfies the conditions of part (i) of Theorem 2.2.

## 8. EXISTENCE OF A STABILIZING SOLUTION TO THE RICCATI EQUATION

We define $\gamma_-$ as the unique value for $\gamma$ such that for all but finitely many $\gamma$ larger than $\gamma_-$ there exists a stabilizing solution to the algebraic Riccati equation. Moreover, the stabilizing solution does not exist for $\gamma$ smaller than $\gamma_-$. According to Theorem 2.2, $\gamma_-$ is the minimal achievable $L_\infty$ norm of the closed-loop system without any stability requirements. According to Corollary 7.2, we have

$$\gamma_- = \inf_{\widehat{L} \in L_\infty} \|\widehat{T}_1 - T_2 \widehat{L} T_3\|_\infty.$$

Since $T_3$ is minimum-phase and $\widehat{T}_1$ is strictly proper, it is easy to see that

$$\gamma_- = \inf_{\widehat{L} \in L_\infty} \|\widehat{T}_1 - \widehat{T}_2 \widehat{L}\|_\infty,$$

where $\widehat{T}_2 := T_2 (D^{\mathrm{T}} D)^{-1/2}$. We still have the freedom to pick $F$. We choose $F$ such that $\widehat{T}_2$ becomes co-inner. In other words, $F = -(D^{\mathrm{T}} D)^{-1} (B^{\mathrm{T}} R + D^{\mathrm{T}} C)$ where $R$ is a stabilizing solution of

$$0 = A^{\mathrm{T}} R + R A + C^{\mathrm{T}} C - (R B + C^{\mathrm{T}} D)(D^{\mathrm{T}} D)^{-1} (B^{\mathrm{T}} R + D^{\mathrm{T}} C).$$

Then we can obtain the following result:

$$\gamma_- = \|\widehat{T}_{2,\perp}^{\sim} \widehat{T}_1\|_\infty, \tag{8.1}$$

where

$$\widehat{T}_{2,\perp} := [A + BF, -R^+ C^{\mathrm{T}} D_\perp, C + DF, D_\perp]$$

with $D_\perp$ such that $[D \; D_\perp]$ is square and unitary and $R^+$ denotes the Moore-Penrose inverse of $R$. The transfer matrix $\widehat{T}_{2,\perp}$ is constructed such that $[\widehat{T}_2 \ \widehat{T}_{2,\perp}]$ is square and inner. Therefore

$$\begin{aligned}\|\widehat{T}_1 - \widehat{T}_2\widehat{L}\|_\infty &= \|[\widehat{T}_2 \ \widehat{T}_{2,\perp}]^\sim(\widehat{T}_1 - \widehat{T}_2\widehat{L})\|_\infty \\ &= \left\| \begin{bmatrix} \widehat{T}_2^\sim \widehat{T}_1 - \widehat{L} \\ \widehat{T}_{2,\perp}^\sim \widehat{T}_1 \end{bmatrix} \right\|_\infty ,\end{aligned}$$

and (8.1) follows immediately. We obtain the following realization for $\widehat{T}_{2,\perp}^\sim \widehat{T}_1$:

$$[-(A+BF)^{\mathrm{T}}, RE, D_\perp^{\mathrm{T}} CR^+, 0]. \tag{8.2}$$

In conclusion, the minimal achievable $L_\infty$ norm is equal to the $L_\infty$ norm of $\widehat{T}_{2,\perp}^\sim \widehat{T}_1$ whose realization is given by (8.2). Note that the Moore-Penrose inverse is only needed if the realization for $\widehat{T}_1$ and $\widehat{T}_2$ is nonminimal (otherwise $R$ is invertible). A numerically more reliable way to determine $\gamma_-$ is therefore based on applying model reduction to $\widehat{T}_1$ and $\widehat{T}_2$.

Finally, we would like to note that in [23] a similar problem has been studied where the smallest value of $\gamma$ for which a stabilizing solution exists is characterized.

## 9. CONCLUSION

In this paper we have established a very general result regarding the existence of stabilizing solutions to the algebraic Riccati equation. The stabilizing solution exists for all but finitely many $\gamma$ larger than $\gamma_-$. The stabilizing solution does not exist for $\gamma$ smaller than $\gamma_-$. Moreover, we related the number of negative eigenvalues of the stabilizing solution to the number of unstable poles needed to achieve the required $L_\infty$ performance.

Using the techniques of this paper, one can also derive conditions for the $L_\infty$ control problem with measurement feedback, where we look for dynamic controllers which yield no more than $i$ unstable closed-loop poles and achieve an *a priori* given bound on the $L_\infty$ norm of the closed-loop. However, this seems to be mainly of theoretical interest.

*The author would like to thank Professor Hautus for numerous discussions related to this paper. The research of Dr. Stoorvogel has been made possible by a fellowship of the Royal Netherlands Academy of Sciences and Arts.*

## REFERENCES

1 B. D. O. Anderson and J. B. Moore, Optimal filtering, in *Information and System Sciences*, Prentice-Hall, Englewood Cliffs, N.J., 1979.

2 ———, *Optimal Control: Linear Quadratic Methods*, Prentice-Hall, Englewood Cliffs, N.J., 1989.

3 B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis: A Modern Systems Theory Approach*, Prentice-Hall, Englewood Cliffs, N.J., 1973.

4 T. Başar and G. J. Olsder, *Dynamic Noncooperative Game Theory*, Academic Press, New York, 1982.

5 H. Bart, I. Gohberg, and M. A. Kaashoek, *Minimal Factorization of Matrix and Operator Functions*, Oper. Theory Adv. Appl. 1, Birkhäuser, Basel, 1979.

6 S. Bittanti and P. Colaneri, Lyapunov and Riccati equations: Periodic inertia theorems, *IEEE Trans. Automat. Control* 31:659–661 (1986).

7 R. W. Brockett, *Finite Dimensional Linear Systems*, Wiley, 1969.

8 C. A. Desoer, R. W. Liu, J. Murray, and R. Saeks, Feedback system design: The Fractional Representation Approach, *IEEE Trans. Automat. Control* 25:399–412 (1980).

9 J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, State space solutions to standard $H_2$ and $H_\infty$ control problems, *IEEE Trans. Automat. Control* 34:831–847 (1989).

10 B. A. Francis, *A Course in $H_\infty$ Control Theory*, Lecture Notes in Control and Inform. Sci. 88, Springer-Verlag, 1987.

11 P. Gahinet, On the game Riccati equations arising in $H_\infty$ control problems, *SIAM J. Control Optim.* 32:635–647 (1994).

12 K. Glover, All optimal Hankel-norm approximations of linear multivariable systems and their $L^\infty$-error bounds, *Internat. J. Control* 39:1115–1193 (1984).

13 M. Green, K. Glover, D. J. N. Limebeer, and J. C. Doyle, A $J$-spectral factorization approach to $H_\infty$ control, *SIAM J. Control Optim.* 28:1350–1371 (1990).

14 E. A. Jonckheere, J. C. Juang, and L. M. Silverman, Spectral theory of the linear quadratic and $H^\infty$ problems, *Linear Algebra Appl.* 122/123/124:273–300 (1989).

15 H. Kwakernaak and R. Sivan, *Linear Optimal Control Theory*, Wiley, 1972.

16 E. F. Mageirou, Values and strategies for infinite time linear quadratic games, *IEEE Trans. Automat. Control* 21:547–550 (1976).

17 J. Medanic, Bounds on the performance index and the Riccati equation in differential games, *IEEE Trans. Automat. Control* 34:613–614 (1967).

18 C. N. Nett, C. A. Jacobson, and M. J. Balas, A connection between state-space and doubly coprime fractional representation, *IEEE Trans. Automat. Control* 29:831–832 (1984).

19 A. Ostrowski and H. Schneider, Some theorems on the inertia of general matrices, *J. Math. Anal. Appl.* 4:72–84 (1962).

20 I. R. Petersen and C. V. Hollot, A Riccati equation approach to the stabilization of uncertain linear systems, *Automatica* 22:397–411 (1986).

21 A. C. M. Ran and R. Vreugdenhill, Existence and comparison theorems for algebraic Riccati equations for continuous- and discrete-time systems, *Linear Algebra Appl.* 99:63–83 (1988).

22 W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, 1987.
23 C. Scherer, $H_\infty$ control by state feedback and fast algorithms for the computation of optimal $H_\infty$ norms, *IEEE Trans. Automat. Control* 35:1090–1099 (1990).
24 A. A. Stoorvogel, *The $H_\infty$ Control Problem: A State Space Approach*, Prentice-Hall, Englewood Cliffs, N.J., 1992.
25 J. C. Willems, Least squares stationary optimal control and the algebraic Riccati equation, *IEEE Trans. Automat. Control* 16:621–634 (1971).
26 D. C. Youla, H. A. Jabr, and J. J. Bongiorno, Jr., Modern Wiener-Hopf design of optimal controllers: Part II, *IEEE Trans. Automat. Control* 21:319–338 (1976).

*Received 8 December 1993; final manuscript accepted 29 August 1994*