A discrete scheme of Laplace–Beltrami operator and its convergence over quadrilateral meshes

Dan Liu, Guoliang Xu*, Qin Zhang

State Key Laboratory of Scientific and Engineering Computing, Institute of Computational Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing, 100080, China

Accepted 25 April 2007

Abstract

Laplace–Beltrami operator and its discretization play a central role in the fields of image processing, computer graphics, computer aided geometric design and so on. In this paper, a discrete scheme for Laplace–Beltrami operator over quadrilateral meshes is constructed based on a bilinear interpolation of the quadrilateral. Convergence results for the proposed discrete scheme are established under some conditions. Numerical results which justify the theoretical analysis are also given.

Keywords: Laplace–Beltrami operator; Mean curvature; Quadrilateral meshes; Discretization; Convergence; Bilinear interpolation

1. Introduction

Let $\mathcal{M} \subset \mathbb{R}^3$ be a given sufficiently smooth surface. Laplace–Beltrami operator (LBO) over $\mathcal{M}$, denoted by $\Delta_{\mathcal{M}}$ in this paper, is a generalization of the classical Laplacian $\Delta$ from flat space to $\mathcal{M}$. Laplace–Beltrami operator, which relates closely to the mean curvature normal $H$ of surface $\mathcal{M}$ by the relation $\Delta_{\mathcal{M}} p = 2H(p)$ ($p \in \mathcal{M}$), plays a central role in many areas, such as image processing (see [1–3]), surface processing (see [4,5] for references) and the study of geometric partial differential equations (see [6]). In these application areas, the objective surfaces to be processed are usually represented as discrete meshes. Hence, there are comprehensive needs in practice to discretize the LBO and the mean curvature normal $H$.

Previous work. It is well-known that the most often used and studied meshes in surface processing are triangular and quadrilateral. For the triangular meshes, which are even more popular than quadrilateral meshes, several discrete schemes of LBO have been proposed and used (see [7–13]). These schemes can be expressed by weighted averages over the neighborhood of mesh vertices. Specifically, let $M$ be a triangulation of surface $\mathcal{M}$ with vertices $\{p_i\}$, $f$ be...
a smooth function defined on $\mathcal{M}$. Then the approximate LBO acting on $f$ is expressed as

$$\Delta_{\mathcal{M}} f(p_i) \approx \sum_{j \in N(i)} w_{ij} (f(p_j) - f(p_i)), \quad (1.1)$$

where $N(i)$ is the index set of one-ring neighbor vertices of $p_i$, and the weights $w_{ij}$ can be chosen in many ways. For instance, $w_{ij}$ can be taken as $1/\parallel p_j - p_i \parallel$ (see [8]), $(\cot \alpha_{ij} + \cot \beta_{ij})/\sum_j (\cot \alpha_{ij} + \cot \beta_{ij})$ (see [7]), $3(\cot \alpha_{ij} + \cot \beta_{ij})/(2A(p_i))$ or $(\cot \alpha_{ij} + \cot \beta_{ij})/(2A_M(p_i))$ (see [9]), and so on. Here $\alpha_{ij}$ and $\beta_{ij}$ are the two angles opposite to the edge in two triangles sharing the edge $[p_ip_j]$ (see Fig. 1.1(a)), $A(p_i)$ is the summation of areas of triangles surrounding vertex $p_i$ and $A_M(p_i)$ is the area of the Voronoi region of $p_i$ (see Fig. 1.1(b)).

The convergence problem of discrete LBO over triangular meshes has been studied recently in [13,14]. None of the discrete schemes aforementioned have been proved to be convergent over any triangulated surfaces. But some of them converge to the exact LBO under particular conditions. The most important and popular one is the following Desbrun et al.'s discretization:

$$\Delta_{\mathcal{M}} f(p_i) = \frac{3}{2A(p_i)} \sum_j (\cot \alpha_{ij} + \cot \beta_{ij}) [f(p_j) - f(p_i)].$$

It converges to the LBO under the conditions that the valence of the vertex $p_i$ is 6 and $p_i = F(q_i), p_j = F(q_j)$ for a smooth parametric surface $F$ and the relations $q_{j+3} + q_j = 2q_i$ ($j = 1, 2, 3$) hold, where $q_j$ ($j = 1, \ldots, 6$) are one-ring neighbors of vertex $q_i$ on the 2D domain (see [14] for details).

It is obvious that quadrilateral meshes can be processed as triangular meshes by subdividing each quadrilateral into two triangles. Hence, the discrete schemes of LBO over triangular meshes could be easily applied to quadrilateral ones. However, two ways of subdividing each quadrilateral into triangles often lead to different computational results even though the same discrete scheme is applied to the same quadrilateral. Therefore, it is necessary to construct a discrete scheme which can be used to compute the LBO and the mean curvature normal directly over quadrilateral meshes.

**Our work.** Discrete schemes of LBO and mean curvature operator are usually derived by minimizing surface mesh area. However, since the vertices of a quadrilateral may not locate on a plane, there is an uncertainty in determining the surface area. The basic idea in the construction of our discrete scheme of LBO in this paper is to use a bilinear interpolation surface of a quadrilateral to represent the polygon. The discrete LBO in the form (1.1) as well as the mean curvature normal is then derived by locally minimizing the area of the interpolation surface. Thus the discrete scheme is uniquely determined. Furthermore, we show that the discrete operators converge to the exact ones in a quadratic rate under some conditions. A preliminary version of this work was reported in a conference [15].

The rest of the paper is organized as follows. In Section 2, we first propose the discretization scheme and then present the convergence results. Simplified discrete schemes are also provided in this section. Since the proofs of these convergence results involve lengthy derivations, we separate them into a single section (Section 3). In Section 4, some numerical experiments are given to show the convergence properties of our discrete scheme. Section 5 includes the conclusion of this paper and future work.
2. LBO and its discretization

This section includes the main results of this paper. A discrete scheme of mean curvature normal and LBO based on a bilinear interpolation over quadrilateral meshes is first proposed and the detailed derivation is described. We then present some convergence results and simplified discrete scheme.

Laplace–Beltrami operator. Let \( \mathcal{M} \subset \mathbb{R}^3 \) be a 2D manifold, which is locally parameterized by \( \{ p(\xi_1, \xi_2) : (\xi_1, \xi_2) \in \Omega \subset \mathbb{R}^2 \} \). Then the Laplace–Beltrami operator \( \Delta_\mathcal{M} \) applying to \( f \in C^2(\mathcal{M}) \) is given by

\[
\Delta_\mathcal{M} f = \frac{1}{\sqrt{\det(G)}} \sum_{ij} \frac{\partial}{\partial \xi_i} \left( g^{ij} \sqrt{\det(G)} \frac{\partial f}{\partial \xi_j} \right),
\]

where \( G = (g_{ij})_{i,j=1}^2 \) is locally positively definite, \( G^{-1} = (g^{ij})_{i,j=1}^2 \) with \( g_{ij} = \langle t_i, t_j \rangle \) and \( t_i = \frac{\partial p}{\partial \xi_i} \). Let \( p \) be a surface point of \( \mathcal{M} \), then it is well-known that (see [16, p. 151])

\[
\Delta_\mathcal{M} p = 2H(p) \in \mathbb{R}^3,
\]

where \( H(p) \) is the mean curvature normal at \( p \).

2.1. Discretization

The derivation of the discrete mean curvature normal in the following is based on a formula in differential geometry:

\[
\lim_{\text{diam}(R) \to 0} \frac{2\nabla A}{A} = -H(p),
\]

where \( A \) is the area of a region \( R \) with diameter \( \text{diam}(R) \) of the surface around point \( p \), and \( \nabla A \) is the gradient of \( A \) with respect to the \((x, y, z)\) coordinates of \( p \).

Let \( M \) be a quadrilateral mesh in \( \mathbb{R}^3 \) and \([p_i, p_j, p_{j+1}, p_{j'}]\) be a neighbor quadrilateral of the vertex \( p_i \) (see Fig. 2.1(a)). Then a bilinear parametric surface \( S \) that interpolates four vertices of the quadrilateral can be defined as:

\[
S(u, v) = (1 - u)(1 - v)p_i + v(1 - u)p_j + u(1 - v)p_{j+1} + uvp_{j'}, \quad (u, v) \in [0, 1]^2.
\]

Let \( A_j \) denote the area of surface \( S(u, v) \) for \((u, v) \in [0, 1]^2\). Then it can be expressed as:

\[
A_j = \int_0^1 \int_0^1 \sqrt{\|S_u\|^2\|S_v\|^2 - (S_u, S_v)^2} \, du \, dv,
\]

where the tangents of the surface are

\[
S_u(u, v) = (1 - v)(p_{j+1} - p_i) + v(p_{j'} - p_j),
\]

\[
S_v(u, v) = (1 - u)(p_{j'} - p_i) + u(p_{j} - p_{j+1}).
\]

Then we have

\[
\nabla A_j = \int_0^1 \int_0^1 \nabla \sqrt{\|S_u\|^2\|S_v\|^2 - (S_u, S_v)^2} \, du \, dv
= \frac{1}{2} \int_0^1 \int_0^1 \frac{\nabla \left[ \|S_u\|^2\|S_v\|^2 - (S_u, S_v)^2 \right]}{\sqrt{\|S_u\|^2\|S_v\|^2 - (S_u, S_v)^2}} \, du \, dv
\]
\[ \begin{aligned}
&= \int_0^1 \int_0^1 S_u(S_v, (v - 1)S_v - (u - 1)S_u) + S_v(S_u, (u - 1)S_u - (v - 1)S_v) \\
&\quad \times \frac{\sqrt{\|S_u\|^2\|S_v\|^2 - (S_u, S_v)^2}}{\sqrt{\|S_u\|^2\|S_v\|^2 - (S_u, S_v)^2}} \, du \, dv \\
&= \alpha_{j+1i}(p_{j+1} - p_i) + \alpha_{j}(p_{j'} - p_j) + \alpha_{ji}(p_j - p_i) + \alpha_{j'j+1}(p_{j'} - p_{j+1}),
\end{aligned} \]

where

\[ \begin{aligned}
\alpha_{j+1i} &= \int_0^1 \int_0^1 \frac{(1 - v)(S_v, (1 - u)S_u - (1 - v)S_v)}{\sqrt{\|S_u\|^2\|S_v\|^2 - (S_u, S_v)^2}} \, du \, dv, \\
\alpha_{j} &= \int_0^1 \int_0^1 \frac{v(S_v, (1 - u)S_u - (1 - v)S_v)}{\sqrt{\|S_u\|^2\|S_v\|^2 - (S_u, S_v)^2}} \, du \, dv, \\
\alpha_{ji} &= \int_0^1 \int_0^1 \frac{(1 - u)(S_u, (1 - v)S_v - (1 - u)S_u)}{\sqrt{\|S_u\|^2\|S_v\|^2 - (S_u, S_v)^2}} \, du \, dv, \\
\alpha_{j'j+1} &= \int_0^1 \int_0^1 \frac{u(S_u, (1 - v)S_v - (1 - u)S_u)}{\sqrt{\|S_u\|^2\|S_v\|^2 - (S_u, S_v)^2}} \, du \, dv.
\end{aligned} \]

Let

\[ \begin{aligned}
\alpha_i &= \alpha_{j+1i} + \alpha_{ji}, \\
\alpha_j &= -\alpha_{ji} + \alpha_{j'}, \\
\beta_{j+1} &= -\alpha_{j+1i} + \alpha_{j'j+1}, \\
\gamma_{j'} &= -\alpha_{j'j} - \alpha_{j'j+1}.
\end{aligned} \]

Noticing that \(\alpha_i + \alpha_j + \beta_{j+1} + \gamma_{j'} = 0\), from (2.2), we can rewrite the gradient of the area as

\[ \nabla A_j = \alpha_j(p_i - p_j) + \beta_{j+1}(p_j - p_{j+1}) + \gamma_{j'}(p_{j'} - p_j). \]

Then the discrete mean curvature normal is given by

\[ H(p_i) \approx \frac{2}{A(p_i)} \sum_j \left[ \alpha_j(p_j - p_i) + \beta_{j+1}(p_{j+1} - p_i) + \gamma_{j'}(p_{j'} - p_i) \right], \]

where \(A(p_i) = \sum_j A_j\) is the total area of the quadrilaterals around \(p_i\) and the summation is carried out for all quadrilaterals of \(M\) around \(p_i\). Using the relation (2.1), the discretization of LBO applying to a function \(f\) is obtained as

\[ \Delta_M f(p_i) \approx \frac{4}{A(p_i)} \sum_j \left[ \gamma_{j'}(f(p_{j'}) - f(p_i)) + \alpha_j(f(p_j) - f(p_i)) + \beta_{j+1}(f(p_{j+1}) - f(p_i)) \right]. \]

In practice, a mesh considered may be in a mixed form, that means it contains both triangles and quadrilaterals. In such a case, discrete schemes (2.4) and (2.5) can be alternated by replacing the summation with two summations. One is for quadrilaterals and the other is for triangles. For a triangle \([p_i, p_j, p_{j+1}]\) (see Fig. 2.1(b)), it is easy to derive that (see [9])

\[ \nabla A_j^T = \frac{1}{2} [\cot a_j(p_{j+1} - p_i) + \cot b_j(p_j - p_i)], \]

where \(A_j^T\) is the area of triangle \([p_i, p_j, p_{j+1}]\). Hence, the summation for triangles is

\[ \sum_j \frac{\cot a_j(p_{j+1} - p_i) + \cot b_j(p_j - p_i)}{2}. \]

It should be pointed out that the areas used in the summation for triangles are the summation of all areas of triangles \(A_j^T(p_i)\).

### 2.2. Convergence

For the discretization (2.4) and (2.5), we have the following convergence results:
Theorem 2.1. Let \( p_i \) be a vertex of a quadrilateral mesh \( M \) with valence 4, \( p_1, \ldots, p_4 \) be its neighbor vertices and \( p_{1'}, \ldots, p_{4'} \) be its opposite vertices in the quadrilateral \([p_i p_j p'_j p_{j+1}]\) (see Fig. 2.1(a)). Suppose \( p_i, p_j (j = 1, \ldots, 4) \) and \( p_{j'} (j' = 1', \ldots, 4') \) are on a sufficiently smooth parametric surface \( F(\xi_1, \xi_2) \in \mathbb{R}^3 \), and there exist \( q_i, q_1, \ldots, q_4, q_{1'}, \ldots, q_{4'} \in \mathbb{R}^2 \) (see Fig. 2.2) such that

\[
\begin{align*}
p_i &= F(q_i), \quad p_j = F(q_j), \quad p_{j'} = F(q_{j'}), \\
q_j &= q_j + q_{j+1} - q_i, \quad j = 1, \ldots, 4, \\
q_{j+2} - q_i &= -(q_j - q_i), \quad j = 1, 2.
\end{align*}
\]  

(2.6)

Then
\[
G(p_i, h) = \frac{2}{A(p_i, h)} \sum_j \left[ \gamma_{j'}(h)(p_{j'}(h) - p_i) + \alpha_j(h)(p_j(h) - p_i) + \beta_{j+1}(h)(p_{j+1}(h) - p_i) \right]
\]
\[
= H(p_i) + O(h^2), \quad \text{as } h \to 0,
\]  

(2.7)

where
\[
p_j(h) = F(q_j(h)), \quad q_j(h) = q_i + h(q_j - q_i), \quad j = 1, \ldots, 4, 1', \ldots, 4'.
\]

Theorem 2.2. Let \( f \) be a sufficiently smooth function over surface \( F(\xi_1, \xi_2) \). Then under the conditions in Theorem 2.1, we have

\[
\frac{4}{A(p_i, h)} \sum_j \left[ \gamma_{j'}(h)(f(p_{j'}(h)) - f(p_i)) + \alpha_j(h)(f(p_j(h)) - f(p_i)) + \beta_{j+1}(h)(f(p_{j+1}(h)) - f(p_i)) \right]
\]
\[
= \Delta_M f(p_i) + O(h^2), \quad \text{as } h \to 0.
\]  

(2.8)

Since the proofs of the theorems require a lot of fine derivations, we put them into a separate section. These theorems say that the discrete LBO and mean curvature normal converge in a quadratic rate under conditions in Theorem 2.1.

The computation of the coefficients \( \alpha_j(h), \beta_{j+1}(h) \) and \( \gamma_{j'}(h) \) involves integrations over the unit square domain \([0, 1]^2\). These integrations can be computed by numerical integration formulas. The following corollary indicates how accurate these integrals need to be computed.

Corollary 2.1. Under the conditions in Theorems 2.1 and 2.2, if the coefficients \( \alpha_j(h), \beta_{j+1}(h), \gamma_{j'}(h) \) and the area \( A_j(h) \) are approximately evaluated by numerical integration formulas with an algebraic precision at least one, then the conclusions of Theorems 2.1 and 2.2 still hold.

Remark. The convergence results are established under particular conditions. However, this particular case is very useful and important, because many numerical simulations of geometric partial differential equations are conducted over a domain grid formed by a uniform two-directional partition. This kind of domain grid satisfies the condition (2.6) in Theorem 2.1.
2.3. Simplified scheme

Corollary 2.1 implies that we can use the following one-point integration formula

\[
\int_0^1 \int_0^1 f(u, v) du dv \approx f \left( \frac{1}{2}, \frac{1}{2} \right)
\]  

(2.9)

to compute \( \alpha_j(h), \beta_{j+1}(h), \gamma_{j'}(h) \) and \( A(p_i, h) \). The algebraic precision of formula (2.9) is one. Using this formula, we can derive that

\[
A_j = \sqrt{\|S_u\|^2 \|S_v\|^2 - \langle S_u, S_v \rangle^2}
\]

\[
\alpha_{j+1} = \frac{\langle S_v, S_u - S_v \rangle}{4A_j}, \quad \alpha'_{j+1} = \frac{\langle S_u, S_v - S_u \rangle}{4A_j}, \quad \alpha_j = \frac{\langle S_u, S_u - S_v \rangle}{4A_j}, \quad \alpha'_j = \frac{\langle S_v, S_v - S_u \rangle}{4A_j},
\]

where

\[
S_u = \frac{1}{2}(p_{j+1} - p_i) + \frac{1}{2}(p_j - p_i), \quad S_v = \frac{1}{2}(p_j - p_i) + \frac{1}{2}(p_j' - p_{j+1}).
\]

Therefore, from (2.3) we have

\[
\alpha_j = \frac{\|S_u\|^2 - \|S_v\|^2}{4A_j}, \quad \beta_{j+1} = \frac{\|S_v\|^2 - \|S_u\|^2}{4A_j}, \quad \gamma'_{j'} = \frac{\|S_u - S_v\|^2}{4A_j}.
\]  

(2.10)

These formulas are simple, neat and easy to implement. The numerical examples provided in Section 4 show that the discretized schemes derived from one-point integration formula have similar accuracy to the schemes derived from higher order integration rules.

3. Proofs of the convergence results

In this section, we give the proofs of the convergence results in detail. Readers, who are not interested in the derivations, may skip this section.

**Proof of Theorem 2.1.** Without loss of generality, we may assume that \( q_1 - q_i = (1, 0) \). Then there exist a constant \( a > 0 \) and an angle \( \theta > 0 \) (see Fig. 2.2) such that

\[
q_2 - q_i = a(\cos \theta, \sin \theta), \quad q_{1'} - q_i = (1 + a \cos \theta, a \sin \theta), \quad q_{2'} - q_i = (a \cos \theta - 1, a \sin \theta),
\]

and \( q_{j+2} - q_i = -(q_j - q_i), j = 1, 2, 1', 2' \). Let

\[
q_j - q_i = s_j d_j \quad \text{with} \quad s_j = ||q_j - q_i||, \quad d_j = (q_j - q_i)/||q_j - q_i||, \quad j = 1, \ldots, 4, 1', \ldots, 4'.
\]

Let \( F_{d_j}^k = F_{d_j}^k(q_i) \) denote the directional derivative of \( F \) at \( q_i \) of order \( k \) and in the direction \( d_j \), then we can expand \( p_j(h), p_{j'}(h) \) into the following form

\[
p_j(h) = p_i + h s_j F_{d_j} + \frac{1}{2} h^2 s_j^2 F_{d_j}^2 + O(h^3),
\]

\[
p_{j'}(h) = p_i + h s_j F_{d_j} + \frac{1}{2} h^2 s_j^2 F_{d_j}^2 + O(h^3)
\]

\[
= p_i + h s_j F_{d_j} + \frac{1}{2} h^2 s_j^2 F_{d_j}^2 + h s_j F_{d_{j+1}} + \frac{1}{2} h^2 s_j^2 F_{d_{j+1}}^2 + h^2 s_j s_{j+1} F_{d_{j+1}}^2 + O(h^3).
\]

Hence

\[
S_u = (1 - v)(p_{j+1}(h) - p_i) + v(p_j(h) - p_j(h))
\]

\[
= (1 - v) \left( h s_{j+1} F_{d_{j+1}} + \frac{1}{2} h^2 s_{j+1}^2 F_{d_{j+1}}^2 \right) + v \left( h s_{j+1} F_{d_{j+1}} + \frac{1}{2} h^2 s_{j+1}^2 F_{d_{j+1}}^2 \right)
\]

\[
+ h^2 s_j s_{j+1} F_{d_{j+1}}^2 + O(h^3)
\]

\[
= h s_{j+1} F_{d_{j+1}} + \frac{1}{2} h^2 s_{j+1}^2 F_{d_{j+1}}^2 + v h^2 s_j s_{j+1} F_{d_{j+1}}^2 + O(h^3).
\]
\[ S_v = (1 - u)(p_j(h) - p_i) + u(p_{j'}(h) - p_{j+1}(h)) \]
\[ = (1 - u)\left( h s_j F_{d_j} + \frac{1}{2} h^2 s_j^2 F_{d_j}^2 \right) + u \left( h s_j F_{d_{j'}} + \frac{1}{2} h^2 s_j^2 F_{d_{j'}}^2 + h^2 s_j s_{j+1} F_{d_{d_{j+1}}}^2 \right) + O(h^3) \]
\[ = h s_j F_{d_j} + \frac{1}{2} h^2 s_j^2 F_{d_j}^2 + u h^2 s_j s_{j+1} F_{d_{d_{j+1}}}^2 + O(h^3), \]

then we have
\[ \langle S_u, S_u \rangle = h^2 s_j^2 s_{j+1}^2 \| F_{d_{j+1}} \|^2 + h^3 s_j^3 s_{j+1} \langle F_{d_{j+1}}, F_{d_{d_{j+1}}}^2 \rangle + 2 u h^3 s_j s_{j+1} \langle F_{d_{j+1}}, F_{d_{d_{j+1}}}^2 \rangle + O(h^4) \]
\[ = h^2 \Delta^2_j + h^3 \Delta^3_j + 2 u h^3 \Delta^{2(u)}_j + O(h^4), \]
\[ \langle S_v, S_v \rangle = h^2 s_j^2 \| F_{d_j} \|^2 + h^2 s_j^2 \langle F_{d_j}, F_{d_{d_{j+1}}}^2 \rangle + 2 u h^3 s_j^2 s_{j+1} \langle F_{d_j}, F_{d_{d_{j+1}}}^2 \rangle + O(h^4) \]
\[ = h^2 \Delta^2_j + h^3 \Delta^2_j + 2 u h^3 \Delta^{2(u)}_j + O(h^4). \]

Let \( A_j(h) \) denote the area of the quadrilateral \( \{ p_i p_j p_{j+1}(h) p_{j'}(h) \} \). Using the area formula, we obtain
\[ A_j(h) = \int_0^1 \int_0^1 \sqrt{\| S_u \|^2 \| S_v \|^2 - \langle S_u, S_v \rangle^2} \, du \, dv \]
\[ = \int_0^1 \int_0^1 \left[ h^2 \delta_j^{(0)} + h^5 \delta_j^{(1)} + 2 u h^4 \delta_j^{(1)(u)} + O(h^6) \right] \, du \, dv \]
\[ = \int_0^1 \int_0^1 \left[ h^2 \delta_j^{(0)} + \frac{1}{2} h^3 \delta_j^{(1)} + 2 u h^4 \delta_j^{(1)(u)} + O(h^6) \right] \, du \, dv, \]

where
\[ \delta_j^{(0)} = s_j^2 s_{j+1}^2 \left( \| F_{d_j} \|^2 \| F_{d_{j+1}} \|^2 - \langle F_{d_j}, F_{d_{j+1}} \rangle^2 \right), \]
\[ \delta_j^{(1)} = s_j^3 s_{j+1}^2 \left[ \langle F_{d_j}, F_{d_{d_{j+1}}}^2 \rangle \| F_{d_{j+1}} \|^2 - \langle F_{d_j}, F_{d_{j+1}} \rangle \langle F_{d_{d_{j+1}}}, F_{d_{d_{j+1}}} \rangle \right] + s_j^2 s_{j+1}^2 \left[ \langle F_{d_{d_{j+1}}}, F_{d_{d_{j+1}}}^2 \rangle \| F_{d_j} \|^2 - \langle F_{d_{d_{j+1}}}, F_{d_{d_{j+1}}} \rangle \langle F_{d_j}, F_{d_{d_{j+1}}} \rangle \right], \]
\[ \delta_j^{(1)(u)} = s_j^2 s_{j+1} \left[ \langle F_{d_j}, F_{d_{d_{j+1}}} \rangle \| F_{d_{j+1}} \|^2 - \langle F_{d_j}, F_{d_{j+1}} \rangle \langle F_{d_{d_{j+1}}}, F_{d_{d_{j+1}}} \rangle \right], \]
\[ \delta_j^{(1)(v)} = s_j^2 s_{j+1} \left[ \langle F_{d_{d_{j+1}}}, F_{d_{d_{j+1}}} \rangle \| F_{d_j} \|^2 - \langle F_{d_{d_{j+1}}}, F_{d_{d_{j+1}}} \rangle \langle F_{d_j}, F_{d_{d_{j+1}}} \rangle \right]. \]

Therefore
\[ A_j(h) = h^2 \left( \delta_j^{(0)} + \frac{1}{2} h^3 \delta_j^{(1)} + \delta_j^{(1)(u)} + \delta_j^{(1)(v)} \right) \left( \delta_j^{(0)} + \frac{1}{2} h^3 \delta_j^{(1)} + \delta_j^{(1)(u)} + \delta_j^{(1)(v)} \right)^{-\frac{1}{2}} + O(h^4). \]

Let \( t_i = \frac{\partial F}{\partial d_i}, \ t_{ij} = \frac{\partial^2 F}{\partial d_i \partial d_j}, \ g_{ij} = \langle t_i, t_j \rangle, \ g_{ijk} = \langle t_i, t_j, t_k \rangle \), then we have
\[ F_{d_i} = t_i, \quad F_{d_i}^2 = t_{11}, \quad F_{d_2} = t_1 \cos \theta + t_2 \sin \theta, \quad F_{d_2}^2 = t_{11} \cos^2 \theta + 2 t_{12} \cos \theta \sin \theta + t_{22} \sin^2 \theta, \]
\[ F_{d_{d_{j+1}}}^2 = t_{11} \cos \theta + t_{12} \sin \theta, \quad F_{d_{d_{j+1}}}^2 = - t_{11} \cos \theta - t_{12} \sin \theta. \]

It follows from (3.1) that
\[ \delta_j^{(0)} = a^2 \sin^2 \theta \det(G) = \delta. \]

Using the fact that
\[ s_{j+2} = s_j, \quad d_{j+2} = - d_j, \quad j = 1, 2, \]
we have

\[ F_{d_j+2}^k = (-1)^k F_{d_j}^k, \quad j = 1, 2, \quad k = 1, 2. \]  

(3.3)

\[ \delta_j^{(1)(u)} = -\delta_j^{(1)(u)}, \quad \delta_j^{(1)(v)} = -\delta_j^{(1)(v)}, \quad \delta_j^{(m)} = (-1)^m \delta_j^{(m)}, \quad j = 1, 2, \quad m = 0, 1. \]  

(3.4)

Therefore

\[ A(p_j, h) = \sum_{j=1}^{4} A_j(h) = 4h^2 \sqrt{\delta} + O(h^4). \]

Now let us compute the coefficients \( \alpha_j(h) \), \( \beta_j+1(h) \) and \( \gamma_j(h) \). It is easy to obtain

\[
\frac{\langle S_u, S_u \rangle}{\sqrt{\|S_u\|^2 \|S_v\|^2 - \langle S_u, S_v \rangle^2}} = \Delta_j^{1+} + \frac{h}{\sqrt{\delta}} \left[ \delta \left( \Delta_j^2 + 2v \Delta_j^{2(u)} \right) - \frac{1}{2} \Delta_j^1 \left( \delta_j^{(1)} + 2u \delta_j^{(1)(u)} + 2v \delta_j^{(1)(v)} \right) \right] + O(h^2),
\]

\[
\frac{\langle S_v, S_v \rangle}{\sqrt{\|S_v\|^2 \|S_v\|^2 - \langle S_u, S_v \rangle^2}} = \Delta_j^{1+} + \frac{h}{\sqrt{\delta}} \left[ \delta \left( \Delta_j^2 + 2u \Delta_j^{2(u)} \right) - \frac{1}{2} \Delta_j^1 \left( \delta_j^{(1)} + 2u \delta_j^{(1)(u)} + 2v \delta_j^{(1)(v)} \right) \right] + O(h^2),
\]

\[
\frac{\langle S_u, S_v \rangle}{\sqrt{\|S_u\|^2 \|S_v\|^2 - \langle S_u, S_v \rangle^2}} = \Delta_j^{1+} + \frac{h}{\sqrt{\delta}} \left[ \delta \left( \Delta_j^2 + 2u \Delta_j^{2(u)} + v \Delta_j^{2(v)} \right) - \frac{1}{2} \Delta_j^1 \left( \delta_j^{(1)} + 2u \delta_j^{(1)(u)} + 2v \delta_j^{(1)(v)} \right) \right] + O(h^2),
\]

then we have

\[
\alpha_{j+1}(h) = \int_{0}^{1} \int_{0}^{1} \frac{1}{v(1-u)-(1-v)\langle S_u, S_v \rangle - (1-v)^2 \langle S_v, S_v \rangle} \, dv \, du
\]

\[ = \Delta_j^{1+} + \frac{h}{\sqrt{\delta}} \left[ \delta \left( \Delta_j^2 + 2u \Delta_j^{2(u)} + v \Delta_j^{2(v)} \right) - \frac{1}{2} \Delta_j^1 \left( \delta_j^{(1)} + 2u \delta_j^{(1)(u)} + 2v \delta_j^{(1)(v)} \right) \right] + O(h^2), \]

(3.5)

\[
\alpha_{j'}(h) = \int_{0}^{1} \int_{0}^{1} \frac{v(1-u)\langle S_u, S_v \rangle - v(1-v)\langle S_v, S_v \rangle}{\sqrt{\|S_u\|^2 \|S_v\|^2 - \langle S_u, S_v \rangle^2}} \, dv \, du
\]

\[ = \Delta_j^{1+} + \frac{h}{\sqrt{\delta}} \left[ \delta \left( \Delta_j^2 + 2u \Delta_j^{2(u)} + v \Delta_j^{2(v)} \right) - \frac{1}{2} \Delta_j^1 \left( \delta_j^{(1)} + 2u \delta_j^{(1)(u)} + 2v \delta_j^{(1)(v)} \right) \right] + O(h^2), \]

(3.6)

\[
\alpha_{ji}(h) = \int_{0}^{1} \int_{0}^{1} \frac{(1-u)(1-v)\langle S_u, S_v \rangle - (1-u)^2 \langle S_u, S_u \rangle}{\sqrt{\|S_u\|^2 \|S_v\|^2 - \langle S_u, S_v \rangle^2}} \, dv \, du
\]

\[ = \Delta_j^{1+} + \frac{h}{\sqrt{\delta}} \left[ \delta \left( \Delta_j^2 + 2u \Delta_j^{2(u)} + v \Delta_j^{2(v)} \right) - \frac{1}{2} \Delta_j^1 \left( \delta_j^{(1)} + 2u \delta_j^{(1)(u)} + 2v \delta_j^{(1)(v)} \right) \right] + O(h^2), \]

(3.7)
\[ \alpha_{j'}(h) = -\frac{\Delta_1'}{4\sqrt{\delta}} + \frac{h}{\delta \sqrt{\delta}} \left[ \delta \left( \frac{1}{4} \Delta_j' + \frac{1}{6} \Delta_j^{(a)} + \frac{1}{12} \Delta_j^{(v)} \right) - \frac{1}{2} \Delta_j' \left( \frac{1}{4} \delta_j^{(1)} + \frac{1}{3} \delta_j^{(a)} + \frac{1}{6} \delta_j^{(v)} \right) \right] + O(h^2). \]

It follows from (3.5)–(3.8) that the coefficients

\[ \alpha_j(h) = -\alpha_{j+1}(h) + \alpha_{j'}(h) \]

\[ = \frac{h}{\delta \sqrt{\delta}} \left[ \delta \left( \Delta_j^{(2a)} - \frac{1}{12} \Delta_j^{(a)} \delta_j^{(1)} + \delta_j^{(v)} + \frac{1}{3} \delta_j^{(a)} \delta_j^{(v)} + \frac{1}{6} \delta_j^{(v)} \right) \right] + \frac{\Delta_1}{3\sqrt{\delta}} \]

\[ + \frac{h}{3\delta \sqrt{\delta}} \left[ \delta \left( \Delta_j^{(2a)} + \Delta_j^{(a)} - \frac{1}{12} \Delta_j^{(a)} \delta_j^{(1)} + \delta_j^{(v)} + \frac{1}{3} \delta_j^{(a)} \delta_j^{(v)} + \frac{1}{6} \delta_j^{(v)} \right) \right] \]

\[ - \frac{\Delta_1}{6\sqrt{\delta}} - \frac{h}{6\delta \sqrt{\delta}} \left[ \delta \left( \Delta_j^{(2a)} + \Delta_j^{(a)} - \frac{1}{12} \Delta_j^{(a)} \delta_j^{(1)} + \delta_j^{(v)} + \frac{1}{3} \delta_j^{(a)} \delta_j^{(v)} + \frac{1}{6} \delta_j^{(v)} \right) \right] + O(h^2), \]

\[ \beta_j(h) = -\alpha_{j+1}(h) + \alpha_{j'}(h) \]

\[ = -\frac{\Delta_1}{2\sqrt{\delta}} - \frac{h}{\delta \sqrt{\delta}} \left[ \delta \left( \frac{1}{4} \Delta_j^{(2a)} + \frac{1}{6} \Delta_j^{(a)} + \frac{1}{12} \Delta_j^{(v)} - \frac{1}{4} \Delta_j' \left( \frac{1}{4} \delta_j^{(1)} + \frac{1}{3} \delta_j^{(a)} + \frac{1}{6} \delta_j^{(v)} \right) \right] \]

\[ + \frac{\Delta_1}{6\sqrt{\delta}} + \frac{h}{6\delta \sqrt{\delta}} \left[ \delta \left( \frac{1}{4} \Delta_j^{(2a)} + \frac{1}{6} \Delta_j^{(a)} + \frac{1}{12} \Delta_j^{(v)} - \frac{1}{4} \Delta_j' \left( \frac{1}{4} \delta_j^{(1)} + \frac{1}{3} \delta_j^{(a)} + \frac{1}{6} \delta_j^{(v)} \right) \right] \]

\[ + \frac{\Delta_1}{6\sqrt{\delta}} - \frac{h}{6\delta \sqrt{\delta}} \left[ \delta \left( \frac{1}{4} \Delta_j^{(2a)} + \frac{1}{6} \Delta_j^{(a)} + \frac{1}{12} \Delta_j^{(v)} - \frac{1}{4} \Delta_j' \left( \frac{1}{4} \delta_j^{(1)} + \frac{1}{3} \delta_j^{(a)} + \frac{1}{6} \delta_j^{(v)} \right) \right] \right] + O(h^2). \]

From (3.1)–(3.4), we obtain

\[ G(p_i, h) = \frac{2}{A(p_i, h)} \sum_{j=1}^{4} \left[ \gamma_j(h)(p_j(h) - p_i) + \alpha_j(h)(p_j(h) - p_i) + \beta_j+1(h)(p_{j+1}(h) - p_i) \right] \]

\[ = \frac{1}{2} \text{det}(G)^2 (c_{11} t_1 + c_{22} t_2 + c_{12} t_1 t_2 + c_{22} t_1 t_2 + c_{22} t_2) + O(h^2), \]

where

\[ c_1 = -2g_{12}^2 g_{21} - g_{22} (g_{11} g_{22} + g_{22} g_{11}) + g_{12} (2g_{22} g_{11} + g_{22} g_{21} + g_{11} g_{22}), \]

\[ c_2 = -2g_{12}^2 g_{12} + g_{12} (g_{12} g_{11} + g_{11} g_{22} + g_{22} g_{12} + g_{22} g_{21} + g_{11} g_{22}), \]

\[ c_{11} = g_{22} (g_{11} g_{22} - g_{12}^2), \]

\[ c_{12} = -2g_{12} (g_{11} g_{22} - g_{12}^2), \]

\[ c_{22} = g_{11} (g_{11} g_{22} - g_{12}^2). \]
Therefore

\[
G(p_1, h) = \frac{1}{2 \det(G)} (g_{22}t_{11} + g_{11}t_{22} - 2g_{12}t_{12}) + O(h^2)
- \frac{1}{2 \det(G)^2} [g_{22}(g_{22}g_{11} - g_{12}g_{21}) + g_{11}(g_{22}g_{12} - g_{12}g_{22}) - 2g_{12}(g_{22}g_{12} - g_{12}g_{22})] t_1
- \frac{1}{2 \det(G)^2} [g_{22}(g_{11}g_{21} - g_{12}g_{21}) + g_{11}(g_{11}g_{22} - g_{12}g_{22}) - 2g_{12}(g_{11}g_{22} - g_{12}g_{22})] t_2
= \frac{1}{2 \det(G)} \left\{ (g_{22}t_{11} + g_{11}t_{22} - 2g_{12}t_{12}) - [t_1, t_2] G^{-1} \left[ \begin{array}{c} g_{11} \\ g_{22} \end{array} \right] \right\} + O(h^2)
+ g_{11} \left[ \begin{array}{c} g_{12} \\ g_{22} \end{array} \right] - 2g_{12} \left[ \begin{array}{c} g_{12} \\ g_{22} \end{array} \right] \right\} + O(h^2) = \frac{1}{2 \det(G)} (I - [t_1, t_2] G^{-1} [t_1, t_2]^T) [g_{22}t_{11} + g_{11}t_{22} - 2g_{12}t_{12}] + O(h^2).
\]

(3.9)

The first term on the right hand side of (3.9) is the mean curvature normal (see [17]). Hence the proof is completed.

**Proof of Theorem 2.2.** Using the relations

\[
\left[ \frac{\partial f(p)}{\partial \xi_1}, \frac{\partial f(p)}{\partial \xi_2} \right]^T = [t_1, t_2]^T \nabla f(p), \quad \frac{\partial \nabla f(p)}{\partial \xi_1} = \nabla^2 f(p)t_1, \quad \frac{\partial \nabla f(p)}{\partial \xi_2} = \nabla^2 f(p)t_2,
\]

\[\Delta_M f\] can be rewritten as the following form

\[
\Delta_M f(p) = 2H(p)^T \nabla f(p) + \frac{1}{\det(G)} \left[ (g_{22}t_{11} - g_{12}t_{22})^T \nabla^2 f(p)t_1 + (g_{11}t_{12} - g_{12}t_{12})^T \nabla^2 f(p)t_2 \right].
\]

Furthermore

\[
f(p_j(h)) = f(p_i) + \nabla f(p_i) (p_j(h) - p_i)^T + \frac{1}{2}(p_j(h) - p_i)^T \nabla^2 f(p_i)(p_j(h) - p_i) + O(h^3),
\]

then the left hand side of (2.8) can be written as

\[
\frac{4\nabla f(p_i)}{A(p_i, h)} \sum_{j=1}^4 [\gamma_j(h) (p_j(h) - p_i)^T + \alpha_j(h)(p_j(h) - p_i)^T + \beta_{j+1}(h)(p_{j+1}(h) - p_i)^T]

+ \frac{4h^2}{A(p_i, h)} \sum_{j=1}^2 \left[ \gamma_j(h)s_{j}^2 F_d^T \nabla^2 f(p_i) F_d + \alpha_j(h)s_{j}^2 F_d^T \nabla^2 f(p_i) F_d + \beta_{j+1}(h)s_{j+1}^2 F_{d+1}^T \nabla^2 f(p_i) F_{d+1} \right]

= 2H(p)^T \nabla f(p_i) + \frac{1}{\det(G)} \left[ (g_{22}t_{11} - g_{12}t_{22})^T \nabla^2 f(p_i)t_1 + (g_{11}t_{12} - g_{12}t_{12})^T \nabla^2 f(p_i)t_2 \right] + O(h^2)

= \Delta_M f(p_i) + O(h^2).
\]

We thus complete the proof.

**Proof of Corollary 2.1.** If the used integration rule for computing \(a_j(h), \beta_{j+1}(h), \gamma_j(h)\) and \(A_j(h)\) has algebraic precision one, then the computing errors are bounded by \(O(h^2)\) for \(a_j(h), \beta_{j+1}(h)\) and \(\gamma_j(h)\), and bounded by \(O(h^3)\) for \(A_j(h)\), since \(A_j(h) = O(h^2)\). It follows from the proofs of the theorems that these errors do not affect the lower order terms of these coefficients and \(A_j(h)\). Hence, the convergence results are still valid.

### 4. Numerical experiments

In this section, we demonstrate the numerical behaviors of the discrete mean curvature defined by (2.7) on quadrilateral meshes and compare this scheme with Desbrun et al.’s discretization (see [7]) over triangular meshes.
by subdividing quadrilaterals into triangles (see Fig. 4.2). We select several two variable functions:

\[ F_1 = \sqrt{4 - (x - 0.5)^2 - (y - 0.5)^2}, \]
\[ F_2 = \exp \left( -\frac{81}{4} [(x - 0.5)^2 + (y - 0.5)^2] \right), \]
\[ F_3 = \sin(5x - 5y), \]
\[ F_4 = e^{x+y} \]

over xy-plane as 3D surfaces. Both the exact and discrete mean curvatures are computed at the selected points \( q_{ij} = (x_i, y_j) \) defined as \((x_i, y_j) = (\frac{i}{20}, \frac{j}{20})\) for \( i = 1, \ldots, 19, \) \( j = 1, \ldots, 19. \) First we divide the domain around \( q_{ij} \) into quadrilateral meshes, and then map the planar quadrilateral meshes onto the surface by these bivariate functions, therefore we get the quadrilateral meshes over surfaces. The quadrilateral mesh around \( q_{ij} \) is defined as shown in Fig. 2.2, where

\[ \theta = \frac{\pi}{3}, \quad q_1 = q_{ij} + (r, 0), \quad q_2 = q_{ij} + r(\cos \theta, \sin \theta), \quad q_3 = q_{ij} + (r + r \cos \theta, r \sin \theta), \]
\[ q_4 = q_{ij} + (r \cos \theta - r, r \sin \theta), \quad q_{j+1} - q_i = -(q_j - q_i), \quad j = 1, 2, 1', 2' \]

and the edge length \( r \) is selected as \( \frac{1}{6} \), where \( n = 8, 32, 128, \ldots \).

Fig. 4.1 shows the negative natural logarithm of the maximal errors of the discrete mean curvature computed by (2.7) and the exact mean curvature computed from the continuous surfaces. We use the following four-point rule to compute the integrations

\[
\int_0^1 \int_0^1 f(u, v) du dv \approx \frac{f(s_1) + f(s_2) + f(s_3) + f(s_4)}{4}, \quad (4.1)
\]

where

\[ s_1 = \left( \frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} - \frac{\sqrt{3}}{6} \right), \quad s_2 = \left( \frac{1}{2} + \frac{\sqrt{3}}{6}, \frac{1}{2} - \frac{\sqrt{3}}{6} \right), \]
\[ s_3 = \left( \frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} + \frac{\sqrt{3}}{6} \right), \quad s_4 = \left( \frac{1}{2} + \frac{\sqrt{3}}{6}, \frac{1}{2} + \frac{\sqrt{3}}{6} \right). \]

The integration rule (4.1) has accuracy \( O(h^4) \) and the maximal errors are plotted in red line with crosses. The maximal errors using one-point rule to compute the integration (see (2.10)) are plotted in blue line with diamonds. The results in this figure show that the discrete scheme using one-point rule has almost the same accuracy as those using four-point rule in general. Surprisingly, approximation results of one-point rule are better than those of four-point rule for \( F_1 \) and \( F_4 \).

Fig. 4.1. The maximal errors of (2.7) using four-point and one-point formulas. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
Fig. 4.2. Two kinds of subdivision of quadrilaterals over \(xy\)-plane. Domain (a) and (d): triangular mesh with valence 6. Domain (b): triangular mesh with valence 8. Domain (c): triangular mesh with valence 4.

To compare the numerical behavior of our scheme with a similar scheme over triangular mesh, we subdivide quadrilaterals into triangles. Since each quadrilateral can be subdivided in two ways, there are \(2^n\) cases for subdividing a quad mesh with \(n\) quadrilaterals. Fig. 4.2 shows four cases for four quadrilaterals. In the following, we compute the discrete LBO using Desbrun et al.’s discretization (see [7]) for the triangulation domain (a) and (b) in Fig. 4.2.

\[
\Delta_M f(p_i) \approx \frac{3}{A(p_i, h)} \sum_j \left( \cot \alpha_{ij} + \frac{\cot \beta_{ij}}{2} \right) [f(p_j) - f(p_i)].
\] (4.2)

The numerical results are shown in Fig. 4.3.

The red line with crosses in Fig. 4.3 shows that the discrete scheme (4.2) is convergent for triangulation domain (a). The convergence property of Desbrun et al.’s discretization over triangular mesh with valence 6 has been proved (see [14]). As indicated by the blue line with diamonds, no convergence result is observed for discrete scheme (4.2) for triangulation domain (b). Similarly, for triangulation domain (d), the discrete scheme (4.2) is convergent, but for triangulation domain (c) the discrete scheme is not convergent. From these numerical results, we can see that the convergence properties are very different for the same discrete scheme applying to the same quadrilateral mesh with different subdivision strategies.

5. Conclusions and future work

Based on a bilinear interpolation approach, discretized schemes for both Laplace–Beltrami operator and mean curvature normal have been derived for quadrilateral meshes. The theoretical analysis and numerical experiments
have shown that the proposed discretized operators converge to the exact ones as the mesh size goes to zero, under some special conditions (see Theorem 2.1).

The future work includes applying these schemes to processing quadrilateral surface meshes, such as denoising or fairing, surface recovering and free-form surface design, based on solving geometric partial differential equations.

References