Recent results on blow-up and quenching for nonlinear Volterra equations
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Received 2 October 2005

Abstract

In this survey paper, the author examines nonlinear Volterra integral equations of the second kind with solutions that blow-up or quench. The focus is on analytical results, although a few words about numerical solutions for such equations are provided. The integral equations arise in the mathematical modeling of thermal processes within a reactive–diffusive medium. The scope of this review is on the published literature between 1997 and 2005, serving as an update to a previous review by the same author.

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MSC: 45D05; 45G05; 35K55; 34A12
Keywords: Nonlinear integral equations; Volterra; Blow-up; Explosion; Quenching; Reaction-diffusion; Difference kernels; Volterra integral equations of the second kind; Literature review

1. Introduction

The equations under consideration are nonlinear Volterra equations of the second kind with singular difference kernels. The overriding objective is to investigate the phenomena of blow-up or quenching within a reactive–diffusive medium. The results draw on the theory of functional and asymptotic analysis, as well as numerical analysis.

Research on such problems has been active for some 20 years. A review paper by the author surveyed the literature on what are known as “blow-up problems” through 1997 [58]. In 2003, Kirk and Roberts [35] reviewed quenching problems that lend themselves to analysis in the context of nonlinear Volterra equations. Here, the author will discuss results from the intervening years. Researchers who have been active recently include Baker, Brunner, Kirk, Mydlarczyk, Okrasiński, Olmstead, Souplet, Roberts, among others.

Typically, the solution of the Volterra equation represents the temperature of the evolving thermal properties of the underlying physical problem. For blow-up problems, we seek to determine conditions under which we are guaranteed the existence of a unique, nonnegative solution that becomes unbounded in finite time. In the case of quenching the solution remains bounded, whereas the first time derivative becomes unbounded in finite time. When such specialized solutions exist, an added challenge lies in establishing the value of the critical time when the blow-up or quenching occurs. Functional analysis can provide an interval with upper and lower bounds for this critical time, but numerical
methods are best for identifying this critical time more precisely. Additionally, asymptotic and numerical analysis can provide insight into how the solutions behave as this critical time is approached.

2. Preliminaries

This research is closely related to that of parabolic partial differential equations with blow-up or quenching solutions. Under certain conditions, it is advantageous to recast these models in terms of a Volterra integral equation. Although such a reformulation removes the spatial dependence, this limitation is overshadowed by the benefit of being able to analyze nonlinearities that are not necessarily smooth or well behaved. For instance, this is the case when a nonlinear source term has a strongly concentrated and localized spatial dependence. This would be found, for example, in a model for heating a combustible material via a thin wire. This connection is described more fully in [56]. In general, the class of nonlinear Volterra equations under consideration will be of the form

\[ u(t) = Tu(t) \equiv \int_0^t k(t - s)r(s)g[u(s) + h(s)] \, ds, \quad t \geq 0. \tag{1} \]

Typical forms for \( g(u) \) arising in applications to thermal problems include a power-law function or an Arrhenius-type exponential function. The given nontrivial functions \( r(t) \) and \( h(t) \) are allowed to enhance the explosive behavior by being nondecreasing. The function \( h(t) \), generally related to the initial data in applications, is usually nonnegative, continuous, and bounded. The kernel, which is determined by a Green’s function for the related parabolic problem, is generally continuously differentiable, except possibly for endpoint discontinuities. These assumptions are consistent with applications. It is worth noting that if one writes \( v(t) = u(t) + h(t) \), then Eq. (1) becomes \( v(t) = h(t) + \int_0^t k(t - s)r(s)g[v(s)] \, ds \), which is another familiar form for the nonlinear Volterra integral equations of the second kind that are under consideration here.

These integral equations arise in models of explosive behavior in diffusive media. In combustion applications where ignition occurs within a very confined spatial region, integral equations within the class given by (1) are appropriate models. When a material such as steel is placed under high strain, localized shear bands can form with a temperature ignition occurs within a very confined spatial region, integral equations within the class given by (1) are appropriate models. When a material such as steel is placed under high strain, localized shear bands can form with a temperature

For these types of problems, the goal is to determine sufficient conditions to ensure existence of a blow-up or quenching solution. Analytical and numerical estimates for the critical time to the onset of the solution singularity, as well as the asymptotic behavior of the solution in the key limit, are also of interest.

3. Connection to parabolic equations

The literature reflects an active history in the study of nonlinear parabolic partial differential equations with blow-up solutions. An excellent introduction to combustion problems and their early analysis can be found in a 1989 monograph by Bebernes and Eberly [8]. Research on blow-up traces back to 1966, when Fujita considered the initial value problem in \( \mathbb{R}^N \) for \( u_t = \Delta u + u^p \) with \( p > 1 \). He showed that if \( p < 1 + 2/N \), then the initial value problem does not have any nontrivial, nonnegative solutions but that if \( p > 1 + 2/N \), then there exist positive solutions that are nonglobal [22]. The literature reflects the subsequent efforts of those who have worked on new theorems associated with Fujita’s first problem and the role of this critical exponent \( p \). In [37], Levine surveyed the analytical results on blow-up that had appeared in the literature prior to 1990. Deng and Levine updated their review for work published through 2000 in [18]. In 1998, a special issue of this journal titled Nonlinear Problems with Blow-up Solutions: Applications and Numerical Analysis was edited in [11]. That issue contained several articles, from both the perspective of partial
differential equations and integral equations, that are relevant to this discussion. It is recommended that the interested reader consult this special issue.

In 1996, Wang and Wang examined the existence and nonexistence of global positive solutions for three nonlocal reaction–diffusion problems in [64]. Soon after, Martel sought to characterize the blow-up set for a certain parabolic problem in [40], where he showed that a blow-up solution cannot be extended in any sense beyond the critical blow-up time. Also in 1997, Galaktionov and Vázquez examined the possible continuation of solutions for the nonlinear heat equation with a power-law nonlinearity after the blow-up time has been reached [23]. They showed that blow-up is complete for a wide class of initial data. They also established conditions that lead to a scenario of incomplete blow-up, where the blow-up region is bounded and propagates with finite speed.

In [12], Chadam et al. examined the blow-up property of solutions to a parabolic partial differential equation with a localized reaction term. A path from a partial differential equation to an integral equation formulation comes from using a Green’s function. In a physical model where the heat energy is input into the system at a point, the reaction term of a parabolic equation could be modeled by a function localized around that point. In [12], this reaction term was necessarily smooth. In the more extreme case where this reaction term is highly spatially localized (modeled by a Dirac delta function, for example), the resulting Green’s function solution at the point of blow-up would be expressed as a nonlinear Volterra integral equation in time belonging to the class described in (1).

In [7], Bandle and Brunner surveyed the literature on blow-up solutions for quasilinear reaction–diffusion equations. Their paper presents some numerical algorithms and results. An English translation of a 1987 Russian monograph by Samarskii et al. [60] was published in 1995. This work discusses results for blow-up in quasilinear parabolic equations, emphasizing applications in porous media with a variety of source terms. The primary methods used in this analysis include differential inequalities, as well as comparison and maximum principles. Souplet [63], who has been active in this area for several years, published a paper in 2004 that examined semilinear parabolic equations with nonlinear memory. In other work, Souplet [61,62] uses some integral equation techniques in his examination of gradient blow-up for nonlinear parabolic equations with general boundary conditions or with a nonlocal nonlinearity.

The literature of research that examines the phenomenon of quenching continues to grow. Models are constructed with nonlinear parabolic partial differential equations. This work began in 1975 with Karawada [28], who considered \( u_t = u_{xx} + 1/(1 - u) \) on the interval \( 0 < x < l \) for \( t \geq 0 \). He determined conditions under which the solution would remain bounded, while the time derivative became unbounded. Two reviews by Levine can be found in [36,38]. In 2001, Chan [13] edited a special volume titled Advances in Quenching that brings together several new results for nonlinear parabolic equations. Chan discusses both blow-up and quenching for problems that have a concentrated nonlinear source term in [14,15], as well as with coauthor Jiang in [16]. In [20,21], Deng and Zhao derive criteria for finite time quenching and blow-up for a parabolic equation. It is precisely these types of problems that connect to Volterra equations given by (1).

4. Analytical results for blow-up

Consideration of the existence of blow-up solutions in the context of nonlinear Volterra integral equations first appeared in the literature in a monograph by Miller [41] in 1971. Miller posed open problems regarding the maximal interval of existence for the solution to a class of nonlinear Volterra integral equations. These problems were later investigated in [4,25]. In [25], sufficient conditions for the existence of a blow-up solution were presented. A systematic mathematical analysis of Volterra equations arising out of explosion phenomena was introduced in 1983 by Olmstead in [49]. In this paper, the leading order perturbation of temperature for the heat equation was governed by a nonlinear integral equation of the form

\[
u(t) = \int_{-\infty}^{t} \frac{e^{u(s)+s}}{\sqrt{\pi(t-s)}} \, ds, \quad t \geq -\infty.
\]

Olmstead proved that the existence of a unique, continuous and bounded solution could only be guaranteed for a finite length of time. The results established criteria involving the kernel and the nonlinearity for the solution to experience blow-up. For a fuller description of the early work on blow-up for these types of problems, see [58, Section 5].

In [58, Section 3], the reader will find a description of the analysis used by Olmstead, Kirk and Roberts to establish the existence of an unbounded solution to nonlinear Volterra equations given by (1). In order to establish that solutions become unbounded in finite time, it is demonstrated that any continuous and differentiable solution must be positive
and increasing. Contraction arguments are then used to prove the existence of a unique and bounded solution for a restricted range of the independent variable, $t$. The upper limit on this range serves as a lower limit for the critical time to blow-up. If blow-up is to occur, it must happen after this point. Next, a contradiction argument is used to show that a continuous solution fails to exist beyond a certain value of $t$. This value serves as the upper limit on the blow-up time. Altogether, these results suggest criteria under which the nonlinear Volterra integral equation has a blow-up solution. Newcomers to this type of analysis are encouraged to begin with [58] in order that the subsequent results in this paper make the most sense to the reader.

Since 1997, new contributions to the literature have been made by Kirk and Olmstead, who have explored the influence of a moving heat source in a reactive–diffusive medium. Additionally, Mydlarczyk, Okrasiński and Roberts have made several contributions in recent years. This section will provide some summary comments on their work.

Recently, Kirk and Olmstead have considered the influence of a moving heat source on blow-up. By allowing the source to move, it becomes possible to avoid a blow-up that would occur if the source were stationary. When the source location is fixed, the heat it is providing to the system increases the temperature in the immediate neighborhood of the source. When the source is moving through the reactive–diffusive medium, however, it is constantly being exposed to new surroundings that are relatively cool. This increases the ability to diffuse the heat and possibly prevent a blow-up.

In [29], Kirk and Olmstead consider the case of two distinct heat sources that move with a constant velocity. These sources are located physically within a one-dimensional reaction–diffusion medium, moving in the same direction. In part due to the fact that the leading source preheats the medium, it is shown that blow-up occurs at the site of the trailing source. In [29], the analysis proceeds after converting the initial boundary value problem to a pair of nonlinear Volterra equations for the temperatures at the two distinct source points. Conditions for blow-up involve the speed and the separation distance between the two heat sources. In addition, the asymptotic behavior of the solution near the critical time is examined.

In [30,31], the same authors consider one concentrated heat source that moves more generally through a reactive–diffusive medium. In this case, the source can change its position and velocity over time. Physically, there is a competition between the diffusive properties of the medium and the heat that is being input by the source. If the source moves quickly, then there is insufficient time for the heat energy to build up. Sufficient conditions are found for the existence of a blow-up solution, which would occur when the source is moving slowly enough that the heat energy can overwhelm the diffusion in the medium. These conditions depend on parameters such as the velocity and position of the moving heat source. Two special cases, that of a constant velocity and that of periodic motion, are examined in detail. In the case of periodic motion, it is shown that the upper bound on the blow-up time increases with increasing amplitude as well as with increasing frequency.

In [32,33], Kirk and Olmstead generalize the problem to consider the heat equation in two dimensions with a single moving source. As a physical motivation, consider a laser beam that is moving across the flat surface of a combustible material during a cutting, welding or heating treatment. The nonlinear source term is no longer a point source, but it is generalized to a small two-dimensional region centered around a reference point. It can change position and velocity. The problem is cast as a nonlinear Volterra equation, capitalizing on the localized nature of the nonlinear source term. Analysis of the integral equation leads to conditions on the size, strength and motion of the localized heat source. Two special cases, that of a constant velocity and that of periodic motion, are examined in detail. In the case of periodic motion, it is shown that the upper bound on the blow-up time increases with increasing amplitude as well as with increasing frequency.

In [32,33], Mydlarczyk and coauthors have also made several recent contributions to the literature on blow-up of Volterra equations. In [42], Mydlarczyk considers Eq. (1) with $k(t - s) = (t - s)^{2-1}$ for $a > 0$. Using a comparison theorem, he finds conditions for the existence of a maximal positive solution. He also finds conditions under which this solution will blow up in finite time.

The principal results in [45,43], by Mydlarczyk and Okrasiński, address the existence and blow-up behavior of nontrivial solutions of the nonlinear integral equation

$$u(t) = \int_{-\infty}^{t} (t - s)^{2-1} g(u(s)) \, ds, \quad t \geq -\infty.$$  

For $a \geq 1$, the authors prove the existence of nonnegative blow-up solutions. In [43], Mydlarczyk shows this equation always has nontrivial, nonnegative solutions for $0 < a < 1$. 

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Two recent papers examine a system of nonlinear Volterra equations with convolution kernels. In [47], Mydlarczyk and Okrasiński and, in [48] joined by Roberts, consider

\[ u(t) = \int_0^t (t - s)^{n-1}[v(s)]^\gamma \, ds, \]
\[ v(t) = \int_0^t (t - s)^{m-1} g(u(s)) \, ds, \]

where \( m, n \) are natural numbers and \( \gamma > 0 \). The function \( g \) is a nondecreasing continuous function, positive for \( x > 0 \) such that \( g(0) = 0 \). Estimates are given for the blow-up time when conditions are such that the solution is known to become unbounded in finite time. For two examples that arise in combustion problems, numerical estimates of blow-up time are presented. Additionally, the asymptotic behavior of the blow-up solution in the key limit is derived for the power-law and exponential nonlinearity cases. Several lemmas are established in order to derive the conditions for blow-up solutions.

When a nonlinear Volterra equation is known to have a blow-up solution, it is natural to be curious about the nature of that growth under varying conditions. From a physical standpoint, this information can provide insight into how to control or delay the onset of ignition. In [49], Olmstead investigated the leading order asymptotic behavior of the blow-up solution. It was established that the exponential nonlinearity resulted in a blow-up solution with a logarithmic growth rate. More recently in [57], Roberts examined the asymptotic behavior of the growing solution for equations that are known to blow up. Specifically, the author assumes that the nonlinearity in (1) has the asymptotic behavior

\[ g(u) \sim u^m (\log u)^{-n} \exp(u^p), \quad u \to \infty, \]

where \( m, n, p \geq 0 \) and the kernel has the asymptotic behavior

\[ k(t - s) \sim k_0(t - s) \equiv \frac{1}{\Gamma(\mu)} (t - s)^{\mu-1}, \]

where \( \Gamma(\mu) \) is the Gamma function. For this class of kernels and nonlinearities, the leading self-consistent asymptotic approximation of the blow-up solution as \( t \) approaches the blow-up time was obtained. The analysis relied on converting the Volterra integral to one in the complex plane by use of the Parseval formula for Mellin transforms. The dominant contributions to the integral could then be determined, with this information contributing to our understanding of how various nonlinearities and kernels affect the growth rate for the solution.

Two other recent papers that address the existence of a blow-up solution for a class of Volterra equations include [65] by Yang and [50] by Olmstead. In [44,46], Mydlarczyk and Okrasiński do not directly address blow-up, but do consider an equation in the class, namely \( g(u) = u^p \). They establish conditions to guarantee that this equation has nontrivial solutions and that the nontrivial solution is unique. Li in [39] examines the existence of a blow-up solution and the nature of the growth rate for a nonlinear parabolic equation with nonlinear memory using the integral equation formulation for his analysis.

5. Analytical results for quenching

The phenomenon of quenching has certain intrinsic mathematical similarities to that of blow-up. In quenching, the first order time derivative becomes unbounded and blows up in finite time. The analytical techniques developed for blow-up have been modified for use in some quenching problems, first in [19,53]. In [53], Olmstead and Roberts considered a scenario in which the nonlinear source term had a singular part governed by a nonlocal effect. In [19], Deng and Roberts considered a nonlinearity that was spatially concentrated at a location within a finite domain. In each paper, criteria for global existence and finite time quenching of the solution were established. The growth rate and an estimate on the quenching time were also provided for a certain class of nonlinearities. In [35], Kirk and Roberts reviewed recent progress in the analysis of quenching phenomena for problems that lend themselves to analysis in the context of nonlinear Volterra equations. It considers research through 2002 and contains an extensive bibliography to place this work into the broader context of existing research on nonlinear partial differential equations. Although discussed in [35], the following three papers represent the most recent research on quenching from the Volterra perspective. In [59], Roberts and Olmstead examine a quenching problem for a one-dimensional heat equation on a strip is of
finite length. The boundary condition at one end can be of either local or nonlocal type. The other end has either a homogeneous Neumann or Dirichlet boundary condition. This leads to four cases. The analysis is conducted in the context of a nonlinear Volterra equation. The existence of a unique solution is established using contraction mapping arguments, which also provide a lower bound on the quenching time. Differential inequalities are used to establish sufficient conditions for the nonexistence of a solution, which provides for an upper bound on the critical time. The fact that quenching is shown to always occur in the Neumann-type problems implies that all solutions are nonglobal. For the cases with the Dirichlet condition, it is found that either quenching or nonquenching is possible. Sufficient conditions for the existence of a stationary solution are provided.

In [54], Olmstead and Roberts examine a one-dimensional heat equation with a nonlinear, concentrated quenching source that moves with constant speed through the diffusive medium, once again in the context of a nonlinear Volterra equation. By allowing the energy source to move, it is perpetually exposed to new surroundings in the medium that are relatively cool. Thus, if the source moves at a sufficiently high rate, it may not be able to supply enough energy at any fixed site to achieve quenching. Bounds are established for a critical speed above which quenching will not occur. When quenching does occur, the analysis provides bounds for the quenching time. For the special case of a power-law nonlinearity, the growth rate near quenching is derived. In [34], Kirk and Roberts consider the heat equation in a one-dimensional strip of finite width with special nonlinear boundary conditions. Specifically, the boundary condition at one end represents nonlinear absorption of heat and the boundary condition at the other end represents nonlinear heat loss. The interactions between the diffusion, the heat behavior at the boundaries, and the length of the domain are studied to determine conditions under which quenching will or will not occur. The usual conversion (as described in [58]) from a partial differential equation to a Volterra equation does not yield a system that is amenable to the established techniques. A Green’s function is used to express the solution of the problem, leading to a new system of nonlinear Volterra equations. It is shown that because heat loss is allowed at one end, the occurrence of quenching or nonquenching can be controlled.

6. Numerical results on blow-up

Numerical investigations for this class of Volterra equations have made progress in the past decade. A book, authored by Brunner [10] and published in 2004, presents the state of the art for collocation methods for Volterra equations. It presents several open problems for future research and includes an extensive bibliography. This book demonstrates the broad role of Volterra equations in many areas of analysis and applications.

In [6], Baker gives an introduction and short overview of the classical and recent results in the numerical analysis of Volterra equations. This paper discusses the stability for Volterra equations and how to address discretization. It is a welcome update to his 1977 monograph [5], which included a chapter on Volterra equations. Although Baker’s numerical work with Volterra equations has focussed on integro-differential equations or those with periodic solutions, the references provided in [6] provide an excellent introduction to the issues surrounding the numerical evaluation of integral equations. The numerical investigation of nonlinear Volterra equations such as (1) are particularly challenging. In the case of blow-up, the solutions become unbounded, whereas in the case of quenching, the derivative is unbounded. Moreover, the convolution kernels in (1) arise from the Green’s function for the heat operator and are unbounded at $t = 0$. As Brunner describes in [10], this singular behavior impacts on the convergence results for usual collocation schemes that are conducted in piecewise polynomial spaces on uniform meshes. Alternative approaches are necessary and include the use of adaptive methods, moving-mesh methods, graded meshes or nonpolynomial collocation spaces on uniform meshes.

To direct one’s attention to the particular challenges of discretization of Volterra equations with regular and weakly singular kernels, a good place to start is with Brunner’s [9] 1996 paper. He presents open problems that focus on issues of convergence, accuracy and stability.

7. Future directions

Because of their strong connections to Volterra equations, it is worth noting that partial differential equation models for blow-up and quenching remain of continuing interest. Open problems for parabolic equations can be found in [18]. There has been some progress recently in the examination of hyperbolic partial differential equations with solutions that exhibit blow-up. This work began with [3,24,27,37], with several new papers being published since 1999 by Ackleh
and Deng [1,2,17]. Integro-differential equations with explosion solutions is an area that has been receiving attention. In 1980, Pao examined the instability of an integro-differential system arising in nuclear reactor dynamics in [55]. The conditions that resulted in a blow-up solution held for a large class on nonlinear functions, including one that was considered the most physically compelling. In [26], Hirata examines a semilinear integro-differential equation of parabolic type that arises in the theory of nuclear reactor kinetics.

Much of the work reported here was on Volterra equations derived from one-dimensional models; clearly higher dimensional problems are of tremendous interest. In the case of spatially localized nonlinearities, this work would involve consideration of the multidimensional distribution function. This has been addressed in [32] for quenching, but much work is yet to be done in this area. This author anticipates that new papers on multidimensional blow-up and quenching will soon appear in the literature.

References


