Functional Differential Equations of Retarded and Neutral Type: Analytic Solutions and Piecewise Continuous Controls

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Received March 20, 1981; revised December 21, 1981

1. Introduction

In this paper we investigate analyticity properties of solutions of differential equations of retarded and of neutral type. The analyticity results are applied in establishing existence of piecewise continuous controls for hereditary control systems. The observations made here improve results of Banks and Jacobs in [3].

We establish piecewise analyticity of solutions of the equations

\[ \dot{x}(t) = \int_{t-\tau}^{t} A(t, s) x(s) \, ds + \sum_{j=1}^{l} [A_{ij}(t) x(t-\tau_{j}) + A_{ij}(t) \dot{x}(t-\tau_{j})] \]

(1)

\( t \in [0, T] \) (rather than quasi-piecewise analyticity, as obtained in Banks and Jacobs [3] for the non-neutral case, i.e., when \( A_{ij} = 0, j = 1, \ldots, l \)). This is done under the assumption that both the coefficients, the initial function and the initial derivative are analytic. (The choice of the interval \([0, T]\) rather than any finite interval \([t_0, t_1]\) is done merely for convenience and for simplicity in notation.)

Analyticity of the solution might fail, in general, but we find a condition on the initial data, which we call compatibility, and which guarantees an unbroken analyticity of the solution.

A key step in our consideration is the determination by the structure of the equation of a finite subset \( S \) of \([0, T]\) which contains all points which might ruin the analyticity of the solutions. This observation proves useful throughout. In particular it enables us to drop the assumption made by Banks and Jacobs that the lags \( \tau, \tau_1, \ldots, \tau_l \) are commensurate. It also implies that solutions of the equation

\[ \dot{x}(t) = \int_{t-\tau}^{t} A(t, s) x(s) \, ds + B(t) x(t) \]

(2)
\( t \in [0, T] \), are analytic when \( A \) and \( B \) are analytic functions, regardless of analyticity properties of the initial function.

Our techniques apply to more general equations. We establish analyticity properties of solutions for equations of the type

\[
\dot{x}(t) = \int_{t-\tau}^{t} A(t, s) x(s) ds + \sum_{j=1}^{\infty} \left[ A_{0j}(t) x(t - \tau_j) + A_{1j}(t) \dot{x}(t - \tau_j) \right]
\]

\( t \in [0, T] \), when the "lags" \( \tau_j \) belong to a class of analytic functions. Indeed, the results concerning Eq. (1) will be presented as corollaries of the more general statement.

(Note that by adding a term of the form \( \int_{1-T}^{t} A_1(t, s) \dot{x}(s) ds \) on the right-hand side of Eq. (3), one does not obtain any higher degree of generality. Indeed, assuming \( A_1 \) is analytic, integration by parts would yield

\[
\int_{1-T}^{t} A_1(t, s) \dot{x}(s) ds = A_1(t, t) x(t) - A_1(t, t - \tau) x(t - \tau)
\]

and the original form of Eq. (3) is now regained.)

The analytic results are applied in the study of hereditary control systems. We improve results of Banks and Jacobs, and find piecewise continuous (rather than almost piecewise continuous) controls for the systems

\[
\dot{x}(t) = \int_{1-T}^{t} A(t, s) x(s) ds + \sum_{j=1}^{l} \left[ A_{0j}(t) x(t - \tau_j) + A_{1j}(t) \dot{x}(t - \tau_j) \right] + \int_{1-T}^{t} B(t, s) h(s, u(s)) ds + \sum_{j=1}^{k} B_j(t) u(t - \sigma_j)
\]

\( t \in [0, T] \), and

\[
\dot{x}(t) = \int_{1-T}^{t} A(t, s) x(s) ds + \sum_{j=1}^{l} \left[ A_{0j}(t) x(t - \tau_j) + A_{1j}(t) \dot{x}(t - \tau_j) \right] + \int_{1-T}^{t} B(t, s) h(s, u(s)) d\mu(s)
\]

\( t \in [0, T] \), when the coefficients \( A, B, A_{0j}, A_{1j}, j = 1, \ldots, l, B_j, j = 1, \ldots, k \), and the function \( h \) are analytic, where \( I \) is a real interval and where \( \mu \) is a finite atomless positive measure on \( I \).

Our techniques, and in particular the observation that the set of pathological points can be determined by properties of the system, are applicable also in treating the more general cases of infinitely many
functional delays. For the sake of clarity, we prefer, however, to restrict the discussion to Eqs. (4) and (5).

Note that in Eq. (4) we consider, as in Banks and Jacobs [3], a system in which the domain of the integral in the controlled part is \([-\tau, t]\). Similar considerations to those we present here would readily yield sufficient conditions for the existence of piecewise continuous controls in other cases, e.g., in case that the interval of integration is of a fixed length, \([t - \tau, t]\).

The work is constructed as follows: Part I, which consists of Sections 3–8, is dedicated to the study of analyticity properties of solutions. In Section 3 we state our standing hypotheses concerning Eq. (3), and define the set \(F\). We give a sufficient condition for our hypotheses to hold and illustrate the discussion with a few examples. In Section 4 we demonstrate that solutions of Eq. (3) are piecewise analytic. In Section 5 we present the notion of compatible initial data and prove that the solution is analytic over the whole interval if the initial data is compatible. The structure of Eq. (2) is such that the solution is analytic regardless of the analyticity of the initial data. This is discussed in Section 6. Examples of systems in which we allow analyticity breaking of the coefficients and of the initial functions are given in Section 7. In deriving our results of Sections 4–6, we employ successive approximations of the solutions over domains in the complex plane. The proof of convergence of the successive approximations is deferred to Section 8.

In part II we briefly demonstrate the application of our results to control theory. In Section 9 we prove our main statements concerning the existence of piecewise continuous controls. In establishing these results we apply a representation formula for solutions, which we derive in Section 10.

Prior to both parts, we give, in Section 2, some function theoretic preliminaries which we use throughout.

2. Function Theoretic Preliminaries

A real function \(f\) on a real interval \(I\) is said to be analytic if there exist a complex open neighborhood \(D\) of \(I\) and an analytic continuation of \(f\) on \(D\). (In the sequel we shall not distinguish between a function and its analytic continuation.) This property is equivalent to \(f\) agreeing with its power series expansion in a neighborhood of each point of \(I\) (Ahlfors [1, pp. 38, 179]). The function \(f\) is called piecewise analytic if there exists a partition of \(I\) into subintervals such that:

(i) The ends of these subintervals do not accumulate (hence if \(I\) is compact the partition is finite), and

(ii) The restriction of \(f\) to the closure of each subinterval is analytic.

Following Banks and Jacobs [3], we also introduce the notion of quasi-
piecewise analyticity: $f$ is \textit{quasi-piecewise analytic} if a partition of $I$ into subintervals exists such that condition (i) holds, and

(iii) The restriction of $f$ to the interior of each of the subintervals is analytic.

If for some point $\theta$ in $I$ and some positive $\varepsilon$, the function $f$ is analytic either on $[\theta - \varepsilon, \theta]$ or on $[\theta, \theta + \varepsilon]$ or on both these intervals, and yet $f$ is not analytic on any open neighborhood of $\theta$, then we say that the analyticity of $f$ breaks at $\theta$.

A real function of several variables $f$ is \textit{analytic} if there exists a complex open neighborhood of its domain on which $f$ admits an analytic continuation (that is, if the domain of $f$ is in $\mathbb{R}^l$ then $f$ admits an analytic continuation on a $\mathbb{C}^l$ open neighborhood). Again, this property is equivalent to $f$ agreeing with its power series expansion in a neighborhood of each point in its domain (Hörmander [9, pp. 26–27]). Hartog's theorem [9, p. 28] states that $f$ is analytic if and only if it is analytic in each variable separately.

We use the letters $r, s$ and $t$ as both real and complex variables. Similarly, we use the symbol $j(t)$ to denote the derivative with respect to a real and a complex variable (although $d/dt$ is used sometimes too). In each case, it is transparent from the context which is the correct meaning.

The following known results are used throughout.

\textbf{Lemma 2.1.} Let $f$ be an analytic function on an open simply connected complex domain $D$. Then, for each point $\theta$ in $D$, the function $F(t) = \int_\theta^t f(s) \, ds$ does not depend on the path of integration and is analytic on $D$.

\textbf{Proof.} It follows from Cauchy's integral theorem (Ahlfors [1, p. 141]) that $F$ is well defined. Indeed, any two curves from $\theta$ to $t$ in $D$ form a closed curve on which the integral of $f$ vanishes. Since obviously $F$ is the primitive function of $f$, it is analytic. Q.E.D.

\textbf{Lemma 2.2.} Let $f(t, s)$ be a bounded measurable function on some convex and open $C^2$-domain $D$, and assume it is analytic in the variable $t$. Let $I$ be a compact interval included in the projection of $D$ on the $s$-plane, and let $D_I$ be the convex, open complex neighborhood

$$D_I = \{t: (t, s) \in D \text{ for each } s \text{ in } I\}.$$

Then the function $F(t) = \int_I f(t, s) \, ds$ is analytic on $D_I$.

\textbf{Proof.} Let $\Gamma$ be any closed curve in $D_I$. By Cauchy's theorem, for each $s$ in $I$,

$$\int_\Gamma f(t, s) \, dt = 0.$$
Fubini's theorem implies
\[
\int_I \int_J f(t, s) \, ds \, dt = \int_J \int_I f(t, s) \, dt \, ds = 0.
\]
Therefore, it follows from Morera's theorem (Ahlfors [1, p. 122]) that \( F \) is analytic. Q.E.D.

**Corollary 2.3.** Let \( f(t, s) \) be an analytic function on the open, convex \( C^2 \) domain \( D \), and let \( \theta \) be a point in the projection of \( D \) on the \( s \)-plane. Then the function \( F(t) = \int_\theta^t f(t, s) \, ds \) is well defined and analytic on the open neighborhood

\[ D_\theta = \{ t : \text{both} (t, \theta) \text{ and} (t, t) \text{ are points of} \ D \}. \]

**Proof.** Let us define a function of two variables \( G(t, r) = \int_\theta^r f(r, s) \, ds \). It follows from Lemmas 2.1 and 2.2 and from Hartog's theorem that \( G \) is well defined and analytic in both variables on the open domain \( \{ (t, r) \in C^2 : (r, t) \text{ and} (r, \theta) \text{ are both points of} \ D \} \). Thus \( F(t) = G(t, t) \) is analytic on \( D_\theta \). Q.E.D.

Finally, given an open complex domain \( D \), we denote by \( C'(D) \) the space of analytic functions on \( D \) which are, together with their first derivatives, uniformly bounded. We endow \( C'(D) \) with the norm

\[
|f| = \sup_{t \in D} |f(t)| + \sup_{t \in D} \left| \frac{df}{dt} (t) \right|.
\]

(Throughout this work the symbol \( | \cdot | \) stands for both the absolute value of a complex variable, the norm of a vector, a matrix or of an element of any specified normed space. The exact meaning will be transparent from the context.)

3. **Standing Hypotheses**

The kernel \( A \) and the coefficients \( A_{aj} \) and \( A_{ij} \) in Eqs. (1)–(3) are real \( n \times n \) matrix-valued functions, while the "lags" \( a_j \) are scalar mappings from \( [0, T] \) into \( [-\tau, T] \) with the property that \( a_j(t) \) lies in \( [t - \tau, t] \) for each \( t \) in \( [0, T] \) and each index \( j \). Given an initial function \( \varphi \), an initial derivative \( \psi \) (both are functions on \( [-\tau, 0] \)) and an initial value \( x_0 \in \mathbb{R}^n \), a solution is a
pair of functions \((\dot{x}(t), x(t))\) which satisfies the equation on \([0, T]\) and such that
\[
\begin{align*}
\dot{x}(t) &= \psi(t), & t &\in [-\tau, 0] \\
x(t) &= \begin{cases} 
\varphi(t), & t \in [-\tau, 0) \\
\int_0^t \dot{x}(s) \, ds, & t \in [0, T].
\end{cases}
\end{align*}
\]

The context within which we consider our equation is a priori that of real analysis. However, seeking analytic solutions (i.e., solutions which admit analytic continuations on complex neighborhoods of the time interval) we prefer to work on complex systems.

The following Assumptions 1–4 refer to properties of the functions \(A, \varphi, \psi, A_{0j}, A_{1j}\) and \(a_j, j = 1, 2, \ldots\), on their real domains, and to properties of their analytic continuations on neighborhoods in the complex plane. In Section 4 we shall demonstrate that these assumptions guarantee the piecewise analyticity of the solution. Simple and computationally applicable hypotheses will be mentioned in Proposition 3.2. We, however, prefer to list and work with the more complicated assumptions, since they clarify the structure of the proofs of Theorems 4.1 and 5.1.

**Assumption 1.** (Existence of common domains of analyticity). There exists a convex, open neighborhood \(D\) of \([-\tau, T]\) in the complex plane, and a positive constant \(\delta\) such that:

(i) The coefficients \(A_{0j}, A_{1j}\) and the functions \(a_j\) admit bounded analytic continuations on the domain

\[
D_\tau = \{t \in D: \Re t > -\delta\}.
\]

Moreover on this neighborhood, the series \(\sum |A_{0j}|\) and \(\sum |A_{1j}|\) converge uniformly and \(\sum |A_{1j}| \leq N\), for some positive \(N\), strictly smaller than one. (For convenience, we consider a norm of a matrix, denoted by \(|\cdot|\), to be the operator norm subordinated to a given vector norm.)

(ii) The initial function \(\varphi\) and the initial derivative \(\psi\) admit bounded analytic continuations on

\[
D_0 = \{t \in D: \Re t < \delta\}.
\]

(iii) The kernel \(A\) admits a bounded analytic continuation on the \(C^2\) region

\[
D_{\tau S} = \{(t, s): t \in D_\tau, s \in D \text{ and } \Re s \in (\Re t - \tau - \delta, \Re t + \delta)\}.
\]

(iv) Each of the functions \(a_j\) maps \(D_\tau\) into \(D\).
This next remark refers to the meaningfulness of the various demands in Assumption 1.

Remark 3.1. The demand that the series $\sum |A_{ij}|$ should be bounded by a constant $N < 1$ is made in order to avoid degeneracy with respect to $x(t)$, as in the trivial equation $x(t) = x(t)$. (This problem is demonstrated by another simple instance in Example 3.6 below.) It is, however, a superfluous condition in many cases. In particular, when the delays $t - a_j(t)$ are all uniformly bounded away from zero, as in Eq. (1), then uniform convergence of $\sum |A_{ij}(t)|$ will do. Since for the presentation of a weaker condition we need more complicated structures and terminology which will be presented only in the following Section 4, we shall be satisfied, for the moment, with the present form of Assumption 1(i).

The existence of common domains of analyticity is evident when the number of analytic functions which participate in Eq. (3), i.e., $\varphi, \psi, A, A_{0j}, A_{ij}$ and $a_j, j = 1, 2, \ldots$, is finite. When infinitely many functions are involved, in the general case, we have to assume explicitly the existence of these complex regions, as we do in hypotheses (i)-(iii). This is demonstrated in Example 3.4 at the end of this section.

Since we allow more general forms than $t - \tau_j$, of the delay functions $a_j(t)$, the invariance requirement (iv) is also needed.

Summing up, in the case of Eq. (1) we can simplify Assumption 1, requiring that the coefficients $A, A_{0j}$ and $A_{ij}, j = 1, \ldots, l$, and the initial function and derivative $\varphi$ and $\psi$ would be analytic.

Prior to stating our next assumption, we construct the subset $\mathcal{F}$ of $[0, T]$ which we shall later prove to contain all those points in which analyticity of solutions might break. We set

$$
\mathcal{F}_0 = \{t \in [0, T] : t = lt, l = 0, 1, 2, \ldots\},
$$

$$
\mathcal{F}_i = \{t \in [0, T] : t - \tau \in \mathcal{F}_{i-1}, \text{ or } a_j(t) \in \mathcal{F}_{i-1}, \text{ for some } j\}
$$

and

$$
\mathcal{F} = \bigcup \mathcal{F}_i.
$$

Assumption 2. The set $\mathcal{F}$ is finite.

In the following two hypotheses we refer to properties of the analytic continuations of the functions $a_j$.

For convenience we order the points of the finite set $\mathcal{F}$ and the points $-\tau$ and $T$ as follows:

$$
-\tau = \theta_{-1} < 0 = \theta_0 < \theta_1 < \cdots < \theta_q = T.
$$

Assumption 3. There exist open and convex complex neighborhoods $D_i$, $i = 1, \ldots, q$, with $[\theta_{i-1}, \theta_i] \subseteq D_i \subseteq \{t \in D : \text{Re } t \in (\theta_{i-1} - \delta, \theta_i + \delta)\}$, and such
that if for some \(i\) between 1 and \(q\) and for some \(l\) between 0 and \(i\), the function \(t - \tau\) or one of the "lags" \(a_j(t)\) maps the interval \([\theta_{i-1}, \theta_i]\) into \([\theta_{i-1}, \theta_i]\), then the analytic continuation of this function maps \(D_i\) into \(D_l\). (The neighborhood \(D_0\) has already been fixed in Assumption 1.)

**Assumption 4.** For each integer \(j\) and for each \(i\) between 1 and \(q\), if the function \(a_j\) maps the interval \([\theta_{i-1}, \theta_i]\) into itself, then for each \(t\) in \(D_i\) (as defined in Assumption 3), the following inequality holds:

\[
|a_j(t) - \theta_{i-1}| \leq |t - \theta_{i-1}|
\]

In the next proposition we give sufficient conditions for some of our hypotheses to hold.

**Proposition 3.2.** Assume that the requirements (i)–(iii) in Assumption 1 hold and that, moreover, the first complex derivatives \(\frac{d a_j}{d t}\) are all absolutely bounded by 1 on \(D_T\). Then there exists a neighborhood \(D^* \subset D\) of \([-r, T]\) with respect to which:

(i) Hypothesis (iv) of Assumption 1, Assumption 3 and Assumption 4 are satisfied.

(ii) If only finitely many distinct lags appear and if the functions \(a_j\) are increasing on \([0, T]\), then Assumption 2 is met.

**Proof:** (i) Since the functions \(a_j\) are all analytic, we have for any two points \(t_1\) and \(t_2\) in \(D_T\) and each \(j = 1, 2, \ldots\), the equality

\[
a_j(t_1) - a_j(t_2) = \int_{t_1}^{t_2} \frac{d a_j}{d t}(s) d s.
\]

The integral on the right-hand side is well defined, regardless of a particular choice of the curve of integration in \(D_T\). In particular, we can choose a segment and obtain

\[
|a_j(t_1) - a_j(t_2)| \leq |t_1 - t_2| \max_t \left| \frac{d a_j}{d t}(t) \right| \leq |t_1 - t_2|.
\]  

(6)

Now each of the functions \(a_j\) maps the interval \([0, T]\) into \([-r, T]\). Hence, for some small enough positive constant \(\delta\), Eq. (6) implies that \(a_j\) maps also \([-\delta, T + \delta]\) into \([-r - \delta, T + \delta]\) and

\[
\{t \in \mathbb{C}: d(t, [-\delta, T + \delta]) < \delta\} \text{ into } D^* = \{t \in \mathbb{C}: d(t, [-r - \delta, T + \delta]) < \delta\},
\]

where \(d\) denotes the euclidean distance on \(\mathbb{C}\). Thus, in particular, the functions \(a_j\) map the neighborhood \(D^*_T = \{t \in D^* : \text{Re } t > -\delta\}\) into \(D^*\), as required in Assumption 1(iv).
Assume that some of the functions $a_j(t)$ (or $t - \tau$) map the interval $[\theta_{i-1}, \theta_i]$ into $[\theta_{i-1}, \theta_i]$. We set

$$D_i^* = \{ t \in \mathbb{C} : d(t, [\theta_{i-1}, \theta_i]) < \delta \} \quad \text{and} \quad D_i^* = \{ t \in \mathbb{C} : d(t, [\theta_{i-1}, \theta_i]) < \delta \}.$$ 

The inequality (6) implies that these functions map also the region $D_i^*$ into $D_i^*$. Likewise, if the interval $[\theta_{i-1}, \theta_i]$ is mapped into itself by some of the functions $a_j(t)$, then, in particular, they map the point $\theta_{i-1}$ to itself, and by (6)

$$|a_j(t) - \theta_{i-1}| = |a_j(t) - a_j(\theta_{i-1})| \leq |t - \theta_{i-1}|$$

for all $t$ in $D_i^*$. This proves that Assumptions 3 and 4 are met.

(ii) Assume that only finitely many distinct delays are involved and that the corresponding functions $a_j$ are increasing on $[0, T]$. The claim is that $\mathcal{E}$ is a finite set.

Prior to proving our claim, we shall make the following observation: For any point $\theta$ in $(0, T]$, the functions $t - a_j(t)$, $j = 1, 2, \ldots$, are uniformly bounded below by some positive constant, when $t$ varies in $[\theta, T]$. Indeed, each of these analytic functions may vanish at most finitely many times on $[0, T]$, being strictly positive elsewhere on that interval. Since the derivative of each of the functions is non-negative, once the function is positive, it will remain so.

We now go back to our assertion. The set $\mathcal{E}$ is given by

$$\mathcal{E} = \bigcup \mathcal{E}_i,$$

where each of the sets $\mathcal{E}_i$ is finite. We thus wish to demonstrate that there is no increment in the sets $\mathcal{E}_i$ for $i$ large enough. Since the identity map, $a(t) \equiv t$, adds no such increment (recall the definition of $\mathcal{E}_j$), we assume without loss of generality that none of the functions $a_j$ is the identity map. For the same reason we also assume that the delay constant $\tau$ is positive.

We now suppose that our claim is false. Then for each $i = 1, 2, \ldots$, the set $\mathcal{E}_{i+1} \setminus \mathcal{E}_i$ is non-empty. We denote by $s_i$ the minimal element in this set. Therefore the bounded sequence $\{s_i\}$ contains a convergent subsequence. We shall obtain a contradiction by verifying that the differences $s_{i+1} - s_i$ are uniformly bounded below by a positive constant. Indeed, given $i = 1, 2, \ldots$, the set $\mathcal{E}_{i+1} \setminus \mathcal{E}_i$ is contained within $\{ t \in [0, T] : t - \tau \in \mathcal{E}_i \setminus \mathcal{E}_{i-1} \text{ or } a_j(t) \in \mathcal{E}_j \setminus \mathcal{E}_{j-1} \}$ for some $j$. Therefore, either $s_{i+1} - \tau$, or, for some $j$, $a_j(s_{i+1})$ is a point in $\mathcal{E}_j \setminus \mathcal{E}_{j-1}$. In the first case, $s_{i+1} - s_i$ is not smaller than $\tau$. In the second case, $s_{i+1} - s_i$ is bounded below by $s_{i+1} - a_j(s_{i+1})$ and by the observation we made above, $\min \{ t - a_k(t) : t \in [s_k, T], k = 1, 2, \ldots \}$ should be a positive number.

Q.E.D.
COROLLARY 3.3. If the functions $A$, $\phi$, $\psi$, $A_{0j}$ and $A_{ij}$, $j = 1, \ldots, l$, are analytic, then Eq. (1) satisfies Assumptions 1–4, except perhaps the superfluous demand $\sum_j |A_{ij}| \leq N < 1$ (recall Remark 3.1).

We conclude this section with a few examples.

EXAMPLE 3.4. We show that, not as in the case of Eq. (1), once infinitely many constant lags are involved, then a statement in the spirit of Corollary 3.3 fails to hold.

We consider equations of the form

$$x(t) = \int_{t-\tau}^{t} A(t, s) x(s) ds + \sum_{j=1}^\infty A_j(t) [x(t-\tau_j) + \dot{x}(t-\tau_j)]$$

$t \in [0, T]$, and assume that all the functions $A$ and $A_j$, $j = 1, 2, \ldots$, are analytic. One problem that may arise is that of a lack of a common domain of analyticity: For instance, let $A_j(t)$ be the function $[2e^{j!}(t'+1/j)]^{-1}$, $j = 1, 2, \ldots$. Each of these functions is analytic, and the series converge uniformly on $[0, T]$ and is bounded by $\frac{1}{4}$. However, there is no open neighborhood of the origin in the complex plane, on which all the functions admit analytic continuations.

Another problem, originated by the structure of the lags, is described by the following assertion:

Claim. If the terminal time $T$ is larger than $\lim_{j \to \infty} \tau_j$, then Eq. (7) does not satisfy Assumption 2.

Proof. Without loss of generality, we prove for the case $\lim_{j \to \infty} \tau_j = 0$. Then already the set $\mathcal{I}$ is infinite, as it contains an infinite subsequence of $\{\tau_j\}$ which converges to zero. One can easily check that in fact $\mathcal{E}$ is a dense subset of $[0, T]$.

Q.E.D.

EXAMPLE 3.5. By the following simple example we wish to explain the motivation for the assumption $|da_j/dt| \leq 1$, in Proposition 3.2, with respect to Assumption 2: Suppose that the terminal time is $T = 1$, $\tau = 1$ and that only one delay $a(t) = 2t - 1$ is involved. Then the set $\mathcal{E}$ contains the infinite sequence $\{(2^k - 1)/2^k : k = 1, 2, \ldots\}$ and Assumption 2 is not met.

In general, our claim is not that Assumptions 1–4 are necessary conditions for piecewise analyticity of the solution. In this case, however, one can easily check that solutions of the equation

$$\dot{x}(t) = x(2t - 1)$$

$t \in [0, 1]$, which correspond to non-zero constant initial functions, are not analytic over the whole interval $[0, 1]$. Indeed, the analyticity will break at the points $0, \frac{1}{2}, \frac{3}{4}, \ldots$.
Example 3.6. This example demonstrates again (as promised in Remark 3.1 above) the meaningfulness of the hypothesis $\sum |A_{ij}| \leq N < 1$ in Assumption 1(i). Consider the following equation (with $\tau = \frac{1}{2}$)

$$\dot{x}(t) = \dot{x}(\frac{1}{2}t)$$

$t \in [0, 1]$. Given any integrable function $f$ on $[0, 1]$ which satisfies the equality $f(\frac{1}{2}t) = f(t)$, a solution $x(t) = x_0 + \int_0^t f(s) \, ds$ is obtained. The solution is not unique and, in general, certainly not analytic.

We complete the list of examples presenting an equation which satisfies Assumptions 1–4.

Example 3.7. Let $\{\varepsilon_j\}$ be a sequence of constants in $[0, 1]$. By Proposition 3.2, a sufficient condition for the equation (with $\tau = 2\pi$)

$$\dot{x}(t) = \int_{t-2\pi}^t x(s) \, ds + \sum_{j=1}^{\infty} 2^{-\lfloor j/2 \rfloor} [x(\frac{1}{2}(t - \varepsilon_j \sin t)) + \dot{x}(\frac{1}{2}(t - \varepsilon_j \sin t))]$$

(9)

to satisfy Assumptions 1–4 on any interval $[0, T]$ is that all the constants $\varepsilon_j$ should be uniformly bounded away from 1. Indeed then the derivatives of the delay functions $a_j(t) = \frac{1}{2}(t - \varepsilon_j \sin t)$, $j = 1, 2, \ldots$, are all absolutely bounded by 1 on some open strip $\{t \in \mathbb{C}: |\text{Im } t| < \delta\}$. The set $\mathcal{F}$ consists in this case of the points $0, 2\pi, 4\pi, \ldots$.

By direct computation one can show, however, that Eq. (9) satisfies Assumptions 1–4 under less restrictive conditions, e.g., when the $\varepsilon_j - s$ belong to the closed interval $[0, 1]$.

4. PIECEWISE ANALYTIC SOLUTIONS

Theorem 4.1. Suppose that Assumptions 1–4 are satisfied. Then the unique solution of Eq. (3) is piecewise analytic and its analyticity may break only at points of $\mathcal{F}$.

The proof uses ideas similar to those used in the proof of Banks and Jacobs in [3]. It includes, however, one more observation (see Claims a and b below) which enables us to deduce piecewise, rather than quasi-piecewise, analyticity.

Proof. In the previous section we denoted the points of $\mathcal{F}$ as follows:

$$-\tau = \theta_{-1} < 0 = \theta_0 < \theta_1 < \cdots < \theta_q = T.$$  

The proof is by induction on the intervals $[\theta_i, \theta_{i+1}]$. Assume that for some integer $e$ between 0 and $q - 1$, a solution $x$ of Eq. (3) exists uniquely on
Assume furthermore that the restriction of this solution to each of the subintervals \([\theta_{l-1}, \theta_l]\), \(l = 0, 1, \ldots, e\), is analytic with a \(C^1(D_l)\)-analytic continuation on the open neighborhood \(D_l\), mentioned in Assumptions 1, 3 and 4. (For \(l = 0\) this refers to the initial function and derivative \(\varphi\) and \(\psi\).) We shall demonstrate the existence of a unique solution also on \([\theta_e, \theta_{e+1}]\) with a \(C^1(D_{e+1})\)-analytic continuation.

The restriction of Eq. (3) to the interval \([\theta_e, \theta_{e+1}]\) is equivalent to the equation

\[
x(t) = x(\theta_e) + \int_{\theta_e}^{t} A(r, s) x(s) \, ds \\
+ \sum_{j=1}^{\infty} \left[ A_{0j}(r) x(a_j(r)) + A_{1j}(r) \dot{x}(a_j(r)) \right] \, dr \tag{10}
\]

\(t \in [\theta_e, \theta_{e+1}]\). In this setting, we wish to distinguish between those parts of the right-hand side of (10) which are determined by the already known values of \(x\) on \([-\tau, \theta_e]\) and the yet unknown values on \([\theta_e, \theta_{e+1}]\). To this effect we define the set \(J\) of all indices \(j\) such that the function \(a_j\) maps the interval \([\theta_e, \theta_{e+1}]\) into itself. It then follows from the definition of the set \(J\) that for each \(j\) not in \(J\), the range of the function \(a_j\) on \([\theta_e, \theta_{e+1}]\) is in one and only one of the intervals \([\theta_{l-1}, \theta_l]\), \(l = 0, 1, \ldots, e\). In accordance with the definition of \(J\) we define the function \(z\) on \([\theta_e, \theta_{e+1}]\) by

\[
z(t) = x(\theta_e) + \int_{\theta_e}^{t} A(r, s) x(s) \, ds \\
+ \sum_{j \in J} \left[ A_{0j}(r) x(a_j(r)) + A_{1j}(r) \dot{x}(a_j(r)) \right] \, dr \tag{11}
\]

and the operator \(\mathcal{F}\) by

\[
\mathcal{F}y(t) = \int_{\theta_e}^{t} A(r, s) y(s) \, ds \\
+ \sum_{j \neq J} \left[ A_{0j}(r) y(a_j(r)) + A_{1j}(r) \dot{y}(a_j(r)) \right] \, dr. \tag{12}
\]

Using these conventions we can rewrite Eq. (10) in the form

\[
(\mathcal{J} - \mathcal{F}) x(t) = z(t) \tag{13}
\]

\(t \in [\theta_e, \theta_{e+1}]\), where \(\mathcal{J}\) is the identity operator. Here all the information on values of \(x\) over \([-\tau, \theta_e]\) is given in the right-hand side.

It is our aim now to extend the meaning of Eq. (13) also for values of \(t\) in a complex domain. This is done in the following two assertions.
Claim a. The function $z$, as defined in Eq. (11), has an analytic continuation in $C^1(D_{e+1})$.

Claim b. The operator $\mathcal{F}$, given in Eq. (12), can be extended to map the space $C^1(D_{e+1})$ into itself. The definition of $\mathcal{F}y(t)$ for complex $t$ does not depend on the choice of particular curves of integration in $D_{e+1}$.

Given that these two statements are true, we complete the proof of Theorem 4.1: By Proposition 8.1 (see Section 8 below), the operator $\mathcal{F} - \mathcal{F}$ is invertible on $C^1(D_{e+1})$. Hence the unique solution of Eq. (13), $x = (\mathcal{F} - \mathcal{F})^{-1} z$, admits an analytic continuation in $C^1(D_{e+1})$, while continuing the solution of Eq. (3) on $[\theta_{e}, \theta_{e+1}]$.

Proof of Claim a. By Assumption 3, each of the functions $a_j(t)$ and $t - \tau$ maps the neighborhood $D_{e+1}$ into one and only one of the regions $D_l$, $l = 0, 1, \ldots, e + 1$. Thus we can make the following definition: For each $j \in J$, if the function $a_j$ maps $D_{e+1}$ into $D_l$, for some $l$ between 0 and $e$, then the function $x(a_j(t))$ (similarly, $\dot{x}(a_j(t))$) on $D_{e+1}$ agrees with the analytic continuation of $x(t)$ ($\dot{x}(t)$) from $[\theta_{l-1}, \theta_l]$ to $D_l$. Let $k$ be the integer for which $|t - \tau; t \in D_{e+1}| \subset D_k$. Accordingly we set

$$
z(t) = x(\theta_e) + \int_{\theta_e}^{t} \int_{r-\tau}^{\theta_e} A(r, s) x(s) \, ds + \int_{\theta_e}^{t} A(r, s) \dot{x}(s) \, ds
$$

$$
+ \sum_{j \in J} \left[ A_{0j}(r) x(a_j(r)) + A_{1j}(r) \dot{x}(a_j(r)) \right] \, dr \tag{14}
$$

$t \in D_{e+1}$, where the $r$-curve from $\theta_e$ to $t$ can be arbitrarily specified within $D_{e+1}$, the $s$-curve from $r - \tau$ to $\theta_e$ is arbitrarily chosen in $D_k \cap \{ \text{Re } s > \text{Re } r - \tau - \delta \}$, and the $s$-curve from $\theta_e$ to $\theta_e$ is simply the real interval which joins these two points.

Due to Lemma 2.1, it suffices to show that each of the four summands in the integrand on the right-hand side of Eq. (14) is a bounded analytic function of the variable $r$ on $D_{e+1}$. We shall check them one by one.

The open neighborhood $\{(r, s): r \in D_{e+1}, s \in D_k \text{ and } \text{Re } s \in (\text{Re } r - \tau - \delta, \text{Re } r + \delta)\}$ is a subset of the domain of analyticity of $A$, $D_{TS} = \{(t, s): t \in D_T, s \in D \text{ and } \text{Re } s \in (\text{Re } t - \tau - \delta, \text{Re } t + \delta)\}$, and by Assumption 3, for each $r \in D_{e+1}$, both $(r, r - \tau)$ and $(r, \theta_k)$ are members of this set. It thereby follows from Corollary 2.3 that the first summand, $\int_{\theta_k}^{\theta_e} A(r, s) x(s) \, ds$, is bounded and analytic on $D_{e+1}$.

The kernel $A(r, s)$ is an analytic function of the variable $r$ and is bounded on $D_{e+1} \times [\theta_k, \theta_e]$. Therefore Lemma 2.2 implies that the second term in the integrand, $\int_{\theta_e}^{t} A(r, s) \dot{x}(s) \, ds$, is bounded and analytic on $D_{e+1}$.

Finally, since the analytic continuation of the solution $x$ on each of the regions $D_l$, $l = 0, 1, \ldots, e$, is in $C^1(D)$, the uniform convergence of $\sum |A_{0j}|$ and
of $\sum |A_{1j}|$ imply that the series $\sum_{j \in J} |A_{0j}(r) x(a_j(r)) + A_{1j}(r) \dot{x}(a_j(r))|$ of analytic functions converges uniformly to an analytic and bounded function. This completes the proof of Claim a.

The proof of Claim b follows similar arguments and is therefore omitted. Q.E.D.

The next remark refers to the hypothesis $\sum |A_{1j}| \leq N < 1$ in Assumption 1(i).

**Remark 4.2.** Proving Claim a, we used the uniform convergence of the series $\sum_{j \in J} |A_{1j}(t)|$ on $D_{e+1}$. We did not take into account any particular bound (e.g., $N < 1$) on this subseries. The boundedness by a constant $N < 1$ is needed in the proof of invertibility of the operator $\mathcal{F}$ (in Section 8). There, however, we are concerned only with the complementing subseries $\sum_{j \in J} |A_{1j}(t)|$. Thus, we can rephrase the uniform boundedness demand as follows: For each index $e = 1, \ldots, q$, for the respective set of integers $J = J_e$ (as defined in the proof of Theorem 4.1) and for some constant $N$, $0 < N < 1$, we have $\sum_{j \in J} |A_{1j}(t)| \leq N$, uniformly on $D_e$.

**Remark 4.3.** The reader may note that the place of Claims a and b in our proof is similar to that of the Induction Lemma in the proof of Banks and Jacobs. Indeed, the main contribution here is in observing that the functions $z$ and $\mathcal{F}y$ admit analytic continuations on a neighborhood of $\{ t \in D_T : \Re t \in [\theta_e, \theta_{e+1}] \}$ instead of $\{ t \in D_T : \Re t \in (\theta_e, \theta_{e+1}) \}$ as a statement in the spirit of Banks and Jacob's result would yield.

### 5. Compatible Initial Data and Analytic Solutions

The function

\[
F(\varphi, \psi, t) = \int_{t-\tau}^{t} A(t, s) \varphi(s) \, ds + \sum_{j=1}^{\infty} \left| A_{0j}(t) \varphi(a_j(t)) + A_{1j}(t) \psi(a_j(t)) \right|
\]

is analytic on some neighborhood of the origin. This is in view of the analyticity of each of the functions $A$, $\varphi$, $\psi$, $A_{0j}$, $A_{1j}$ and $a_j$, $j = 1, 2, \ldots$, the boundedness of the analytic continuation of $\varphi$ and $\psi$ and the uniform convergence of the series $\sum |A_{0j}|$ and $\sum |A_{1j}|$. In particular $F(\varphi, \psi, t)$ is well defined on short real intervals $[-\varepsilon, 0]$. We introduce the following definition: We say that the analytic initial data $\varphi$, $\psi$ and $x_0$ are **compatible with the**
equation if \( \varphi(0) = x_0 \) and if there exists a positive constant \( \varepsilon \) such that the equalities

\[
\psi(t) = \phi(t) = \int_{t-\tau}^{t} A(t, s) \varphi(s) \, ds \\
+ \sum_{j=1}^{\infty} \left[ A_{0j}(t) \varphi(a_j(t)) + A_{1j}(t) \psi(a_j(t)) \right]
\]

hold on \([-\varepsilon, 0]\). In other words, \( \varphi, \psi \) and \( x_0 \) are compatible if \( \varphi(0) = x_0 \), \( \psi - \phi \) and if the restriction of \( \varphi \) to \([-\varepsilon, 0]\) is the solution of Eq. (3) which corresponds to the initial function and derivative \( \varphi \) and \( \psi \) on \([-\tau - \varepsilon, -\varepsilon]\).

Theorem 5.1. Suppose that Assumptions 1–4 hold and that the initial data \( \varphi, \psi \) and \( x_0 \) are compatible, then the solution of Eq. (3) is analytic over the whole interval \([0, T]\).

Proof: Preserving the notations of the proof of Theorem 4.1, we wish to show that the analytic continuations of the solution \( x \) on \( D_l \) and \( D_{l+1} \) agree on \( D_l \cap D_{l+1} \) for \( l = 0, 1, \ldots, q - 1 \). Let us demonstrate that first for \( l = 0 \): As we have said above, the function \( F(\varphi, \psi, t) \) admits an analytic continuation on a neighborhood of the origin. The function \( \psi(t) = (d\varphi/dt)(t) \) is clearly analytic there. Thus, once the two functions coincide on an interval \([-\varepsilon, 0]\), they do agree also on the whole neighborhood. In particular, the analytic continuation of \( \varphi \) is a solution of Eq. (3) on some interval \([0, \eta]\), which corresponds to the initial data \( \varphi, \psi \) and \( x_0 \). But, we have already seen that the solution is unique. Hence, the functions \( x \) and \( \varphi \) coincide on \([0, \eta]\). Since both \( \varphi \) and \( x \) are analytic, their analytic continuations should agree on \( D_0 \cap D_1 \) just as well.

Now assume that the solution is analytic on \([-\tau, \theta_i]\) for some \( i \) between 1 and \( q - 1 \). Then we solve Eq. (3) on \([\theta_i, \theta_{i+1}]\) with compatible initial function and derivative \( x \) and \( \dot{x} \) on \([\theta_i - \tau, \theta_i]\). The arguments above imply that the solution on \([\theta_i, \theta_{i+1}]\) is an analytic continuation of the solution on \([-\tau, \theta_i]\).

Q.E.D.

In the following we give an example of a differential difference equation of neutral type, where the analyticity of certain solutions breaks as a result of the non-compatibility of the initial data.

Example 5.2. Consider the delay equation

\[
\dot{x}(t) = x(t - \tau) + \dot{x}(t - \tau)
\]

on some large interval \([0, T]\) with an initial value \( x_0 \), and let the initial function and derivative \( \varphi \) and \( \psi \) be any non-zero constant-valued functions.
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\( \varphi \equiv c \) and \( \psi \equiv d \) on \([-\tau, 0]\). Clearly \( \varphi \) does not satisfy Eq. (15), neither to the left nor to the right of the origin, hence \( \varphi \) and \( \psi \) are not compatible. Indeed, the solution of Eq. (15) on \([0, \tau]\) is not an analytic continuation of \( \varphi \) and is given by \( x(t) = x_0 + (c + d)t \). Now, at time \( t = \tau \), the right-hand side of Eq. (15) "remembers" the analyticity breaking at \( t = 0 \) and the analyticity of the solution breaks once more. This latter analyticity breaking is the result of the non-compatibility of the initial data.

**Example 5.3.** A non-compatible initial function may still generate an analytic solution, if the equation manages to "forget" the analyticity breaking at the origin, as happens in the next case,

\[ \dot{x}(t) = x(t-1) - 2x\left(\frac{1}{2}(t-1)\right), \]  

when the prescribed initial function is \( \varphi(t) = t \) and \( x(0) = \varphi(0) = 0 \). Again, \( \varphi \) is not compatible and the corresponding solution \( x \equiv 0 \) is not an analytic continuation of \( \varphi \). However, at \( t \geq 1 \), the value of \( x(t) \) remains zero and the analyticity of the zero solution is not affected.

**Example 5.4.** An example of compatible initial data. We look for a solution of Eq. (15) of an exponential form: Let \( \lambda_0 \) be a solution of the associated characteristic equation (see Hale [5, p. 27]):

\[ \lambda = e^{-\lambda \tau}(1 + \lambda). \]

Then the function \( x(t) = e^{\lambda t} \) solves Eq. (15) throughout. In particular it is a compatible initial function.

6. **Analytic Solutions of Volterra Equations**

We consider now the integro-differential equation

\[ \dot{x}(t) = \int_{-\tau}^{t} A(t, s) x(s) \, ds \]  

\( t \in [0, T] \). Here no lags are involved and the integrations on the right-hand side are performed over intervals of a varying length \( t + \tau \) instead of over intervals of a fixed length \( \tau \), as in Eq. (3). The impact of these changes, as we shall immediately show, is that the solution is analytic regardless of the analyticity properties of the initial function.

Our hypotheses are the following:

(i) The kernel \( A(t, s) \) is analytic over the triangle \( \{(t, s) \in \mathbb{R}^2: 0 \leq s \leq t \leq T\} \).
(ii) $A$ is measurable and bounded on $[0, T] \times [-\tau, 0]$.

(iii) For each fixed $s$ in $[-\tau, 0]$, the function $A(t, s)$ is analytic in $t$. Moreover, the neighborhood of $[0, T]$ in $\mathbb{C}$ on which the analytic continuation exists is independent of $s$.

(iv) The initial function $\varphi$ is bounded and measurable.

**Theorem 6.1.** Under the hypotheses (i)-(iv) above, the solution of Eq. (17) is analytic over the whole interval $[0, T]$.

**Proof.** We define a function $z$ on $[0, T]$ by

$$z(t) = x_0 + \int_0^t \int_{-\tau}^0 A(r, s) \varphi(s) \, ds \, dr$$

and an integral operator on the continuous functions on $[0, T]$ by

$$\mathcal{F}y(t) = \int_0^t \int_{-\tau}^r A(r, s) y(s) \, ds \, dr$$

$t \in [0, T]$. In terms of $z$ and $\mathcal{F}$, Eq. (17) is equivalent to

$$(\mathcal{Y} - \mathcal{F}) x(t) = z(t)$$

$t \in [0, T]$. Successive applications of Lemmas 2.1 and 2.2 and of Corollary 2.3 imply that on a certain open neighborhood $D$ of $[0, T]$ in $\mathbb{C}$, the function $z$ admits a bounded analytic continuation and $\mathcal{F}$ maps the set of bounded analytic functions on $D$ into itself. We show in Proposition 8.1 (in Section 8 below) that the operator $\mathcal{Y} - \mathcal{F}$ is invertible. Hence, the solution of Eq. (17), $x = (\mathcal{Y} - \mathcal{F})^{-1} z$, admits an analytic continuation on $D$. Q.E.D.

**Remark 6.2.** Equation (17) is not a particular case of Eq. (3): Indeed, we can rewrite the integral $\int_{t-\tau}^t A(t, s) x(s) \, ds$, which appears in Eq. (3), in the form $\int_{t-\tau}^t B(t, s) x(s) \, ds$, setting

$$B(t, s) = \begin{cases} A(t, s), & s \in [t-\tau, t] \\ 0, & \text{otherwise.} \end{cases}$$

Then, however, even if $A$ were analytic, the kernel $B$ would not satisfy the analyticity requirements (i) and (iii).

**Remark 6.3.** If a discrete part is added on the right-hand side of Eq. (17), say, of a form similar to Eq. (1),

$$\dot{x}(t) = \int_{-\tau}^t A(t, s) x(s) \, ds + \sum_{j=1}^l \left[ A_{0j}(t) x(t - \tau_j) + A_{1j}(t) \dot{x}(t - \tau_j) \right]$$
\( t \in [0, T] \), or of the more general form,

\[
\dot{x}(t) - \int_{-\tau}^{t} A(t, s) x(s) \, ds + \sum_{j=1}^{\infty} \left[ A_{0j}(t) x(a_j(t)) + A_{1j}(t) \dot{x}(a_j(t)) \right]
\] (18)

\( t \in [0, T] \), then analyticity of the solution might no longer hold regardless of analyticity of the initial function. This property will, however, be maintained if each of the functions \( a_j \) maps the interval \([0, T]\) into itself (which is possible only in the case (18)). In general, sufficient conditions for analyticity and piecewise analyticity of solutions of Eq. (18) may be easily given in the nature of those specified for Eq. (3).

7. PIECEWISE ANALYTIC COEFFICIENTS AND INITIAL DATA

The ideas which served in the analysis in Section 4 apply to a more general situation, i.e., when the functions \( \phi, \psi, A, A_{0j}, A_{1j} \) and \( a_j, j = 1, 2, \ldots \), are piecewise analytic, instead of analytic on their whole domains. The conditions for the generalized version of Theorem 4.1 for such equations are, however, more complicated and we prefer to demonstrate the way one calculates the set \( \mathcal{E} \) of analyticity breaking points for a few particular equations, rather than presenting the theorem.

**Example 7.1.** Consider the differential-difference neutral equation

\[
\dot{x}(t) = \begin{cases} x(t), & t \in [0, 1] \\ x(t - \tau), & t \in (1, T] \end{cases}
\]

for some \( \tau \) in \((0, 1)\), \( T \gg 1 \) and given a non-zero initial value \( x(0) = x_0 \). (Since past values of the state and derivative prior to the initial time \( t = 0 \) are not taken into account, there is no need to specify \( \phi \) and \( \psi \).) The analyticity of the solution will break at the points \( t = 1 + j\tau, j = 1, 2, \ldots \), in \([0, T]\). Indeed, here the analyticity of the coefficients breaks at \( t = 1 \) and the solution “recalls” this analyticity breaking at \( t = 1 + \tau \) and again at \( t = 1 + 2\tau, 1 + 3\tau, \ldots \).

**Example 7.2.** Consider again Eq. (15)

\[
\dot{x}(t) = x(t - \tau) + \dot{x}(t - \tau)
\]
given an initial value \( x(0) = x_0 \) and with the initial function and derivative

\[
\phi(t) = \begin{cases} 1, & t \in [-\tau, -\frac{3}{2}\tau] \\ 0, & t \in (-\frac{3}{2}\tau, 0] \end{cases} \quad \psi(t) = \begin{cases} 0, & t \in [-\tau, -\frac{3}{2}\tau] \\ 1, & t \in [-\frac{3}{2}\tau, 0] \end{cases}
\]
Assume $T \gg \tau$. The solution "recalls" the analyticity breaking of $\phi$ and $\psi$ at the points $t = j\tau/3$, $j = 1, 2, \ldots$, and thus loses its own analyticity exactly at these points.

**Example 7.3.** Consider the integro-differential equation

$$\dot{x}(t) = \int_{t-1}^{t} x(s) \, ds + x(t - \tau)$$

with $\tau > 1$. We can bring this equation to the form of Eq. (1), setting

$$A(t, s) = \begin{cases} 1, & s \in [t - 1, t] \\ 0, & s \in [t - \tau, t - 1). \end{cases}$$

Given an analytic initial function, say, $\varphi \equiv 1$ and an initial value $x(0) = x_0$, the corresponding solution recalls the analyticity breaking of $A$ at $t = 1, 2, \ldots$. Hence, the analyticity of the solution will break at $t = j + it$, $j = 1, 2, \ldots$, $i = 0, 1, 2, \ldots$. We can obtain the same result, in this particular case, taking the derivative of both sides of our equation, which yields

$$\dot{x}(t) = x(t) - x(t - 1) + \dot{x}(t - \tau).$$

Setting now $\dot{x}(t) = y(t)$ and $z(t) = (\dot{x}(t)_{j})$ we obtain a neutral difference-differential equation for $z$ and construct the set $\mathcal{E} = \{j + it\}$ as prescribed in Section 3.

**8. Convergence of Successive Approximations**

Here we consider the following setting: $D$ is a precompact convex and open domain in the complex plane and $\theta$ is a point in $D$. The sequences $A_{0j}$ and $A_{1j}$, $j = 1, 2, \ldots$, of bounded $n \times n$ matrix-valued functions are defined on $D$ and the corresponding series $\sum |A_{0j}|$ and $\sum |A_{1j}|$ converge uniformly on $D$ with $\sum |A_{1j}| \leq N$ for some constant $N$ in $(0, 1)$. Each of the scalar functions $a_j$, $j = 1, 2, \ldots$, maps the set $D$ into itself and satisfies the inequality $|a_j(t) - \theta| \leq |t - \theta|$ at each point $t$ of $D$. Finally, $A$ is a bounded and analytic, $n \times n$ matrix-valued function on the $C^2$ domain $D_{rs} = \{(t, s) \in D \times D: |s - \theta| \leq |t - \theta|\}$. Using these conventions we define an integral operator on the space $C^1(D)$, of analytic functions with bounded derivatives, by

$$\mathcal{F} y(t) = \int_{\theta}^{t} \left[ \int_{\theta}^{r} A(r, s) y(s) \, ds ight. \left. + \sum_{j=1}^{\infty} [A_{0j}(r) y(a_j(r)) + A_{1j}(r) y(a_j(r))] \right] dr$$

(19)
\( t \in D \) (the integration in Eq. (19) is performed along the segment joining \( \theta \) and \( t \)) and by \( \mathcal{F} \) we denote the identity operator on \( C'(D) \).

**PROPOSITION 8.1.** The operator \( \mathcal{F} \) maps \( C'(D) \) into itself, and \( \mathcal{Y} - \mathcal{F} \) is invertible on this space.

**Proof:** A key step in the proof of the Proposition is in Lemma 8.2 below, where we establish absolute convergence of the series \( \sum_{l=0}^{\infty} \mathcal{F}^l \) in the operator norm, subordinated to the \( C'(D) \) norm. (Here \( \mathcal{F}^0 = \mathcal{Y} \) and \( \mathcal{F}^{l+1}y = \mathcal{F}(\mathcal{F}^l y), \ l = 0, 1, 2, ..., y \in C'(D). \) The norm convergence of this series implies, for each \( y \) in \( C'(D) \), a uniform convergence of the series \( \sum \mathcal{F}^l y(t) \) and \( \sum (d/dt)(\mathcal{F}^l y)(t) \) on \( D \), to an analytic function and its bounded derivative. The inverse operator \( (\mathcal{Y} - \mathcal{F})^{-1} \) therefore exists and is given by \( \sum \mathcal{F}^l \). Q.E.D.

**LEMMA 8.2.** The series \( \sum_{l=0}^{\infty} \mathcal{F}^l \) converges absolutely in the operator norm, subordinated to the \( C'(D) \) norm. (Here \( \mathcal{F}^0 = \mathcal{Y} \) and \( \mathcal{F}^{l+1}y = \mathcal{F}(\mathcal{F}^l y), \ l = 0, 1, 2, ..., y \in C'(D). \)

**Proof:** For simplicity of notation we assume that \( \theta \) is the origin, and we denote by \( T \) and \( M \) the constants

\[
T = \sup \{ |t| : t \in D \} ,
\]

\[
M = \sup \{ T |A(t, s)| : (t, s) \in D_{TS} \} + \sup \left\{ \sum_{j=1}^{\infty} |A_{ij}(t)| : t \in D \right\}
\]

and set \( K = M + N \). We recall that the norm of a function in \( C'(D) \) is given by

\[
|y| = \sup_{t \in D} |y(t)| + \sup_{t \in D} \left| \frac{dy}{dt} (t) \right| .
\]

Hence, the norm of a bounded linear operator \( \mathcal{L} \) on \( C'(D) \) is given by

\[
|\mathcal{L}| = \sup \left\{ \sup_{t \in D} |\mathcal{L}y(t)| + \sup_{t \in D} \left| \frac{d}{dt} \mathcal{L}y(t) \right| : |y| = 1 \right\} .
\]

The proof is based on these next assertions:

**Claim a.** The following estimates hold: For each \( y \) in \( C'(D) \), for each point \( t \) in \( D \) and for each positive integer \( l = 1, 2, ..., \) we have
\begin{align*}
\left| \frac{d}{dt} \mathcal{F}' y(t) \right| & \leq K \sum_{k=0}^{l-1} \binom{l-1}{k} \frac{1}{k!} (M |t|)^k N^{l-k-1} |y| \\
& \overset{\text{def}}{=} \mathcal{H}_l(|t|) |y|
\end{align*}

and

\begin{align*}
| \mathcal{F}' y(t) | & \leq K |t| \sum_{k=0}^{l-1} \binom{l-1}{k} \frac{1}{(k+1)!} (M |t|)^k N^{l-k-1} |y| \\
& = \int_0^{1|t|} \mathcal{H}_l(\rho) \, d\rho |y|.
\end{align*}

Claim b. The series \( \sum_{l=1}^{\infty} \mathcal{H}_l(\rho) \) converges uniformly on compact sets. Indeed the Lemma follows from the two claims: From Claim a we get

\[ |\mathcal{F}'| \leq \mathcal{H}_l(T) + \int_0^T \mathcal{H}_l(\rho) \, d\rho, \]

for \( l = 1, 2, \ldots \). These inequalities together with Claim b guarantee the convergence of \( \sum |\mathcal{F}'| \).

Proof of Claim a. For \( l = 1 \) the inequalities (20) follow from the definition (19) of the operator \( \mathcal{F} \). We proceed by induction: Assume that the claim is true for some \( l \geq 1 \). Then, the inequalities \( |a_j(t)| \leq |t|, j = 1, 2, \ldots \), yield

\[ \left| \left( \frac{d}{dt} \mathcal{F}' y \right)(a_j(t)) \right| \leq \mathcal{H}_l(|t|) |y| \]

and

\[ |\mathcal{F}' y(a_j(t))| \leq \int_0^{1|t|} \mathcal{H}_l(\rho) \, d\rho |y|. \]

Applying the induction hypotheses we obtain

\[ |\mathcal{F}'^{l+1} y(t)| = |\mathcal{F}(\mathcal{F}' y(t))| \]

\[ = \left| \int_0^t \left[ \int_0^r A(r, s) \mathcal{F}' y(s) \, ds \right. \right. \]

\[ \left. \left. + \sum_{j=1}^{\infty} \left[ A_{0j}(r) \mathcal{F}' y(a_j(r)) + A_{1j}(r) \left( \frac{d}{dr} \mathcal{F}' y \right)(a_j(r)) \right] \right| dr \right| \]

\[ = \left| \int_0^t \left[ \int_s^r A(r, s) \, dr \mathcal{F}' y(s) \right. \right. \]

\[ \left. \left. + \sum_{j=1}^{\infty} \left[ A_{0j}(s) \mathcal{F}' y(a_j(s)) + A_{1j}(s) \left( \frac{d}{ds} \mathcal{F}' y \right)(a_j(s)) \right] \right| ds \right|. \]
\[ \begin{align*}
&= \left| t \int_0^1 \left[ t \int_\sigma^1 A(\rho t, \sigma t) \, d\rho \right. \mathcal{F} y(\sigma t) \\
&\quad + \sum_{j=1}^{\infty} \left[ A_{0j}(\sigma t) \mathcal{F} y(a_j(\sigma t)) + A_{1j}(\sigma t) \left( \frac{d}{ds} \mathcal{F} y \right)(a_j(\sigma t)) \right] \, d\sigma \right| \\
&\leq |t| \int_0^1 \left[ t \int_\sigma^1 |A(\rho t, \sigma t)| \, d\rho + \sum_{j=1}^{\infty} |A_{0j}(\sigma t)| \int_0^\sigma \mathcal{F}_i(\omega) \, d\omega \\
&\quad + \sum_{j=1}^{\infty} |A_{1j}(\sigma t)| \mathcal{F}_i(\sigma |t|) \right] \, d\sigma \right| d\sigma |y| \\
&\leq |t| \int_0^1 \left[ M \int_0^\sigma \mathcal{F}_i(\omega) \, d\omega + N \mathcal{F}_i(\sigma |t|) \right] \, d\sigma |y|,
\end{align*} \]

and it follows in a straightforward way that the integrand in the last term is \( \mathcal{F}_i(\sigma |t|) \). This completes the proof of Claim a.

**Proof of Claim b.** Let us rewrite the terms \( \mathcal{F}_i(\rho) \) in their explicit form: We then obtain the series

\[ K \sum_{l=1}^{\infty} \sum_{k=0}^{l-1} \left( \frac{l-1}{k} \right) \frac{1}{k!} (Mp)^k N^{l-k-1} = K \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{k+j}{j} \right) \frac{1}{k!} (Mp)^k N^j. \]

Now, there exists an integer \( e \) such that for each \( k = e, e+1, e+2,... \), we have

\[ (1/k)(Mp) < \frac{1}{2}(1 - N). \]

(Recall that the constant \( N \) is smaller than one!) Given \( e \), we consider first the subseries

\[ K \sum_{j=0}^{\infty} \sum_{k=e}^{\infty} \left( \frac{k+j}{j} \right) \frac{1}{k!} (Mp)^k N^j, \]

and this subseries is bounded by

\[ K \frac{1}{e!} \left[ \frac{2Mp}{1-N} \right]^e \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{k+j}{j} \right) \left[ \frac{1}{2} (1 - N) \right]^k N^j \]

and since \( N \) is smaller than one, the geometric series on the right-hand side converges.

It remains to check the convergence of

\[ K \sum_{j=0}^{\infty} N^j \sum_{k=0}^{e-1} \left( \frac{k+j}{j} \right) \frac{1}{k!} (Mp)^k. \]
Setting $S_j = \sum_{k=0}^{j-1} \binom{k+j}{j} \frac{1}{j!} (M\rho)^k$ we observe that

$$S_{j+1} \leq \frac{e + j}{1 + j} S_j,$$

$j = 1, 2, \ldots$. Thus, since $\lim_{j \to \infty} (e + j)/(1 + j) = 1$, it follows that the series $\sum N^j S_j$ converges too. This completes the proof of Claim b. Q.E.D.

9. **Existence of Piecewise Continuous Controls**

In this part we briefly demonstrate an application of our previous results to a control problem. We consider the control system

$$\dot{x}(t) = \int_{t-\tau}^{t^*} A(t, s) x(s) \, ds$$

$$+ \sum_{j=1}^{\infty} [A_{0j}(t) x(a_j(t)) + A_{1j}(t) \dot{x}(a_j(t))] + f_u(t) \tag{21}$$

$t \in [0, T]$, when the control element $f_u$ is of either of the following two forms: A control element of a retarded type

$$f_u(t) = \int_{-\tau}^{t} B(t, s) h(s, u(s)) \, ds + \sum_{j=1}^{\infty} B_j(t) u(b_j(t)), \tag{22}$$

or of the form

$$f_u(t) = \int_{\mathbb{I}} B(t, s) h(s, u(s)) \, d\mu(s) \tag{23}$$

where $I$ is a real interval, where $\mu$ is a given non-atomic positive, finite measure on $I$ and where (in both cases) $u$ is the control function. (Note that neither of these two forms generalizes the other: In (22) we allow atomic dependence of $f_u$ on the control function $u$, whereas in (23) we allow dependence on advanced values of $u$, in case $I$ is interpreted as the time interval.)

We investigate the possibility of using only piecewise continuous control functions (instead of measurable controls, in general) without affecting the attainable set. This problem has already been encountered for equations of retarded type, by Banks and Jacobs in [3]. They applied their analyticity results [3, Section 5], Banks' representation formula [2] and the theory of Halkin [7] and of Halkin and Hendricks [8] on subintegrals of set-valued functions to obtain existence of almost piecewise continuous controls. Using similar techniques, our contribution is in two respects. Knowing that solutions of Eq. (3) are piecewise analytic (instead of quasi-piecewise
analytic) we verify the existence of piecewise continuous control functions (rather than almost piecewise continuous). We also refer to a larger class of equations.

For the sake of clarity we address ourselves not to the general form of Eq. (21), but rather demonstrate the main ideas for the particular cases where only finitely many constant lags are involved (i.e., Eqs. (4) and (5)).

The terminology used is as follows: We assume that certain initial function and derivative, \( \varphi \) and \( \psi \), and an initial value \( x_0 \), are specified. Focusing on the relations between a given control function \( u \) and the respective solution of Eqs. (4) and (5), we denote that solution by \( x(t, u) \).

Given a bounded set-valued function \( U \) from \([-\tau, T]\) (resp. from \( I \), in case of Eq. (5)) into the set of non-empty subsets of \( \mathbb{R}^m \), we say that a measurable control function \( u \) is admissible if it satisfies the constraint \( u(s) \subseteq U(s) \) almost everywhere. The attainable set at time \( t \) is the collection of points in \( \mathbb{R}^n \) which can be reached by the system, at that time, using admissible control functions. (Namely, the set \( \{x(t, u) : u \text{ is an admissible control}\} \).

Our hypothesis on the constraint set-valued function \( U \) stands for both systems, (4) and (5).

Assumption 1. The set \( U(s) \) is compact for each \( s \) in \([-\tau, T]\) (resp. in the interval \( I \), in case of Eq. (5)) and \( \text{Graph } U \) is a bounded semianalytic set (see Halkin and Hendricks [B] and Banks and Jacobs [3]).

As for the system itself, we first treat Eq. (4)

\[
\dot{x}(t) = \int_{-\tau}^{T} A(t, s) x(s) \, ds + \sum_{j=1}^{l} \left[ A_{0j}(t) x(t - \tau_j) + A_{1j}(t) \dot{x}(t - \tau_j) \right] \\
+ \int_{-\tau}^{T} B(t, s) h(s, u(s)) \, ds + \sum_{j=1}^{k} B_j(t) u(t - \sigma_j).
\]

Assumption 2. The coefficients \( A, B, A_{0j}, A_{1j}, \) \( j = 1, \ldots, l \), and \( B_j, \) \( j = 1, \ldots, k \), and the function \( h \) are analytic. (Note that it suffices to define \( h \) on the subset \( \text{Graph } U \) of \([-\tau, T] \times \mathbb{R}^m \).)

Theorem 9.1. Given a terminal time \( t \in [0, T] \) and assuming Assumptions 1 and 2, then any point of the attainable set at \( t \) can be reached, at that time, using a piecewise continuous control function.

Proof: As shown in Corollary 10.2 in Section 10 below, the solutions is given by

\[
x(t, u) = c(t) + \int_{0}^{t} K(t, r) f_u(r) \, dr
\]
where the kernel $K$ is given in terms of the solution of the advanced equation

$$
\frac{\partial}{\partial s} Y(s, t) = - \int_s^{s+\tau} Y(r, t) A(r, s) \, dr \\
+ \sum_{j=1}^{l} \left[ Y(s + \tau_j, t) [A_{1j}(s + \tau_j) - A_{0j}(s + \tau_j)] \\
+ \frac{\partial}{\partial s} Y(s + \tau_j, t) A_{1j}(s + \tau_j) \right]
$$

(25)

$T \geq t \geq s \geq 0$, with terminal conditions $Y(t, t) = E$ (the $n \times n$ identity matrix) and $Y(s, t) = 0$ for $s > t$.

Changing the order of integration in (24) we obtain $x(t, u) = x(t, 0) + \int_{-\tau}^{t} g(s, u(s)) \, ds$, where $g$ is a function defined on Graph $U$ by

$$
g(s, u) = \chi_{[-\tau, 0]}(s) \int_{0}^{t} K(t, r) B(r, s) \, dr h(s, u) \\
+ \chi_{[0, t]}(s) \int_{s}^{t} K(t, r) B(r, s) \, dr h(s, u) \\
+ \left[ \sum_{j=1}^{k} \chi_{[-\sigma_j, t-\sigma_j]}(s) K(t, s + \sigma_j) B_j(s + \sigma_j) \right] u
$$

$s \in [-\tau, t]$ and $u \in U(s)$. ($\chi_{I}$ stands for the characteristic function of the set $I$.)

Now, as is borne out from Corollary 10.2 and Assumption 2, the kernel $K(t, s)$ is piecewise analytic in the variable $s$. Due to the analyticity of $B, B_j, j = 1, ..., k,$ and of $h$, there exists a finite partition of the interval $[-\tau, t]$, say,

$$
\theta_{-1} = -\tau < \theta_0 = 0 < \theta_1 < \cdots < t = \theta_q,
$$

such that $g(s, u)$ is analytic on the restriction of Graph $U$ to each of the subintervals $[\theta_{j-1}, \theta_j], j = 0, 1, ..., q$.

We conclude the proof, applying Theorem 2 of Halkin and Hendricks [8] (in fact the generalized version of this theorem, as appears in Banks and Jacobs [3, Section 6]). Subject to Assumptions 1, Halkin and Hendricks' theorem states that the sets

$$
\left\{ \int_{-\tau}^{t} g(s, u(s)) \, ds : u \text{ is an admissible control} \right\}
$$

and

$$
\left\{ \int_{-\tau}^{t} g(s, u(s)) \, ds : u \text{ is a piecewise continuous admissible control} \right\}
$$

are equal. Q.E.D.
Treating Eq. (5)

\[ \dot{x}(t) = \int_{t-\tau}^{t} A(t, s) x(s) \, ds + \sum_{j=1}^{l} [A_{0j}(t) x(t - \tau_j) + A_{1j}(t) \dot{x}(t - \tau_j)] \]

\[ + \int_{I} B(t, s) h(s, u(s)) \, d\mu(s), \]

analyticity properties of the kernel \( Y(s, t) \) do not play any role. Our hypotheses considering this system are eased accordingly.

Assumption 2°. The function \( h \) is analytic on Graph \( U \), the kernel \( B(t, s) \) is bounded, measurable and—for fixed \( t \)—analytic in \( s \), whereas the coefficients \( A \) and \( A_{0j}, j = 1, \ldots, l \), are essentially bounded measurable functions, and \( A_{1j}, j = 1, \ldots, l \), are absolutely continuous functions with essentially bounded derivates.

Theorem 9.2. Given a terminal time \( t \in [0, T] \) and assuming Assumptions 1 and 2°, any point of the attainable set at \( t \) can be reached, at that time, using a piecewise continuous control function.

Proof: By Corollary 10.2, the solution is of the form

\[ x(t, u) = x(t, 0) + \int_{0}^{t} K(t, r) \int_{I} B(r, s) h(s, u(s)) \, d\mu(s) \, dr \]

\[ = x(t, 0) + \int_{0}^{t} K(t, r) B(r, s) \, dr h(s, u(s)) \, d\mu(s) \]

\[ = x(t, 0) + \int_{I} g(s, u(s)) \, d\mu(s) \]

where the function \( g \) is defined by

\[ g(s, u) = \int_{0}^{t} K(t, r) B(r, s) \, dr h(s, u) \]

\( s \in I, u \in U(s) \). Following from the analyticity of \( B(t, s) \) in the variable \( s \), from Lemma 2.2 and from the analyticity of \( h \), the function \( g \) is analytic on Graph \( U \).

The situation here differs from that which fits into the framework of Halkin and Hendricks' theorem, as given in Banks and Jacobs [3, Section 6], since some arbitrary measure \( \mu \) replaces Lebesgue's measure on \( I \). This difference, however, may take effect only in the proof of Theorem 1* in Halkin [7], and the argumentation there holds when the Lebesgue measure is substituted by a non-negative Borel measure which satisfies the following:
For each subinterval $I_1$ of $I$ there exists a chain $\{I_\tau : \tau \in [0, 1]\}$ of subsets such that

(i) each set $I_\tau$ consists of a finite union of subintervals,

(ii) for each $\tau$ in $[0, 1]$, $\mu(I_\tau) = \tau \mu(I_1) < \infty$, and

(iii) the inclusion $I_\sigma \subset I_\tau$ holds whenever $\sigma < \tau$.

These three conditions are met exactly when $\mu$ is a finite atomless measure. We can therefore conclude, applying the version of Halkin and Hendricks [8, Theorem 2] which relates to integrals of the form $\int_I g(s, u(s)) \, d\mu(s)$. Q.E.D.

Remark 9.3. Theorem 9.2 is true under weaker conditions. In particular, we assume that the coefficients $A_{ij}$ are absolutely continuous in order to be able to apply our representation formula of Theorem 10.1. More sophisticated techniques would enable us to drop this assumption. This will not be done here.

10. A Representation Formula for Solutions

We consider the inhomogeneous form of Eq. (1)

$$\dot{x}(t) = \int_{t-\tau}^t A(t, s) x(s) \, ds$$

$$+ \sum_{j=1}^l \left[ A_{0j}(t) x(t - \tau_j) + A_{1j}(t) \dot{x}(t - \tau_j) \right] + f(t)$$

(26)

$t \in [0, T]$, under the following hypotheses: The $n \times n$ matrix coefficients $A$ and $A_{0j}$, $j = 1,..., l$, and the inhomogeneous element $f$ are measurable essentially bounded functions, whereas the coefficients $A_{1j}$, $j = 1,..., l$, are absolutely continuous with essentially bounded derivatives. We seek a representation of the solution in terms of the initial data $\phi, \psi$ and $x_0$, and of the function $f$. To this end we consider the unique $n \times n$ matrix solution of Eq. (25)

$$\frac{\partial}{\partial s} Y(s, t) = -\int_s^{s+\tau} Y(r, t) A(r, s) \, dr$$

$$+ \sum_{j=1}^l \left[ Y(s + \tau_j, t)[A_{1j}(s + \tau_j) - A_{0j}(s + \tau_j)] \right]$$

$$+ \frac{\partial}{\partial s} Y(s + \tau_j, t) A_{1j}(s + \tau_j)$$

$0 \leq s \leq t \leq T$, under the terminal conditions $Y(t, t) = E$ ($=$ the $n \times n$ identity.
matrix) and $Y(s, t) \equiv 0$ for $s > t$. (The values of $A_{0j}(s)$ and of $A_{1j}(s)$ for $s > T$
may be arbitrarily specified since they are not taken into account.)

Now let $\varphi$ and $\psi$ be any bounded measurable functions on $[-\tau, 0]$ and let
$x_0$ be a vector in $\mathbb{R}^n$.

**Theorem 10.1.** The solution of Eq. (26) satisfies the equality

$$x(\varphi, \psi, x_0, f, t) = \left[ Y(0, t) - \sum_{j=1}^l Y(\tau_j, t) A_{1j}(\tau_j) \right] x_0
+ \sum_{j=1}^l \chi_{[0, \tau_j]}(\tau_j) A_{1j}(t) x(t - \tau_j)
+ \int_0^t Y(s, t) \chi_{[0, \tau_j]}(s) \int_{s-\tau}^0 A(s, t) \varphi(r) \, dr
+ \sum_{j=1}^l \chi_{[0, \tau_j]}(s)[A_{0j}(s) \psi(s - \tau_j) + A_{1j}(s) \psi(s - \tau_j)] + f(s) \right] ds
$$

(where $\chi_j(s)$ is the characteristic function of the interval $I$).

**Proof.** Let $x(t)$ be the solution of Eq. (26). Integration by parts yields

$$x(t) = Y(0, t) x(0) + \int_0^t \frac{\partial}{\partial s} Y(s, t) x(s) \, ds + \int_0^t Y(s, t) \dot{x}(s) \, ds.
$$

(28)

Consider now the last term on the right-hand side of (28):

$$\int_0^t Y(s, t) \dot{x}(s) \, ds
= \int_0^t Y(s, t) \chi_{[0, \tau_j]}(s) \int_{s-\tau}^0 A(s, r) \varphi(r) \, dr
+ \sum_{j=1}^l \chi_{[0, \tau_j]}(s)[A_{0j}(s) \psi(s - \tau_j) + A_{1j}(s) \psi(s - \tau_j)] + f(s) \right] ds
+ \int_0^t Y(s, t) \left[ \int_0^s \chi_{[s-\tau, \tau_j]}(r) A(s, r) \dot{x}(r) \, dr
+ \sum_{j=1}^l \chi_{[0, \tau_j]}(s) \chi_{[0, \tau_j]}(\tau_j) [A_{0j}(s) \dot{x}(s - \tau_j) + A_{1j}(s) \dot{x}(s - \tau_j)] \right] ds
$$

(27)
\[
\begin{align*}
&= \int_0^t Y(s, t) \left[ \chi_{[0, \tau]}(s) \int_{s-\tau}^0 A(s, r) \varphi(r) \, dr \\
&\quad + \sum_{j=1}^l \chi_{[0, \tau_j]}(s) \left[ A_{0j}(s) \psi(s - \tau_j) + A_{1j}(s) \psi(s - \tau_j) \right] + f(s) \right] \, ds \\
&\quad + \int_0^t \left[ \int_{s-\tau}^{s+\tau} Y(r, t) A(r, s) \, dr + \sum_{j=1}^l Y(s + \tau_j, t) A_{0j}(s + \tau_j) \right] x(s) \, ds \\
&\quad + \int_0^t \left[ \sum_{j=1}^l Y(s + \tau_j, t) A_{1j}(s + \tau_j) \right] \dot{x}(s) \, ds \\
&= \int_0^t Y(s, t) \left[ \chi_{[0, \tau]}(s) \int_{s-\tau}^0 A(s, r) \varphi(r) \, dr \\
&\quad + \sum_{j=1}^l \chi_{[0, \tau_j]}(s) \left[ A_{0j}(s) \psi(s - \tau_j) + A_{1j}(s) \psi(s - \tau_j) \right] + f(s) \right] \, ds \\
&\quad + \int_0^t \left[ \int_{s-\tau}^{s+\tau} Y(r, t) A(r, s) \, dr + \sum_{j=1}^l Y(s + \tau_j, t) A_{0j}(s + \tau_j) \right] x(s) \, ds \\
&\quad - \int_0^t \frac{\partial}{\partial s} \left[ \sum_{j=1}^l Y(s + \tau_j, t) A_{1j}(s + \tau_j) \right] x(s) \, ds \\
&= \int_0^t Y(s, t) \left[ \chi_{[0, \tau]}(s) \int_{s-\tau}^0 A(s, r) \varphi(r) \, dr \\
&\quad + \sum_{j=1}^l \chi_{[0, \tau_j]}(s) \left[ A_{0j}(s) \psi(s - \tau_j) + A_{1j}(s) \psi(s - \tau_j) \right] + f(s) \right] \, ds \\
&\quad + \int_0^t \left[ \int_{s-\tau}^{s+\tau} Y(r, t) A(r, s) \, dr + \sum_{j=1}^l Y(s + \tau_j, t) A_{0j}(s + \tau_j) \right] x(s) \, ds \\
&\quad - \int_0^t \sum_{j=1}^l \left[ \frac{\partial}{\partial s} Y(s + \tau_j, t) A_{1j}(s + \tau_j) + Y(s + \tau_j, t) A_{1j}(s + \tau_j) \right] x(s) \, ds \\
&= \int_0^t Y(s, t) \left[ \chi_{[0, \tau]}(s) \int_{s-\tau}^0 A(s, r) \varphi(r) \, dr \\
&\quad + \sum_{j=1}^l \chi_{[0, \tau_j]}(s) \left[ A_{0j}(s) \psi(s - \tau_j) + A_{1j}(s) \psi(s - \tau_j) \right] + f(s) \right] \, ds
\end{align*}
\]
\[-\sum_{j=1}^{l} Y(\tau_j, t) A_{ij}(\tau_j) x(0) + \sum_{i=1}^{l} x_{10,ij}(\tau_j) A_{ij}(t) x(t - \tau_j)\]

\[-\int_0^t \frac{\partial}{\partial s} Y(s, t) x(s) \, ds.\]

Substituting this last term into (28) we obtain

\[x(t) = \left[ Y(0, t) - \sum_{j=1}^{l} Y(\tau_j, t) A_{ij}(\tau_j) \right] x_0 + \sum_{j=1}^{l} x_{10,ij}(\tau_j) A_{ij}(t) x(t - \tau_j)\]

\[+ \int_0^t Y(s, t) \left[ x_{10,\tau_j}(s) \int_s^0 A(s, r) \varphi(r) \, dr + \sum_{j=1}^{l} x_{10,\tau_j}(s) [A_{0j}(s) \varphi(s - \tau_j) + A_{ij}(s) \psi(s - \tau_j)] + f(s) \right] \, ds.\]

Q.E.D.

**Corollary 10.2.** The solution of Eq. (26) admits a representation

\[x(t) = c(\varphi, \psi, x_0, t) + \int_0^t K(t, s) f(s) \, ds \quad (29)\]

where the function \(c\) is determined by the initial data and where \(K\) is a bounded kernel. If the coefficients \(A, A_{0j}\) and \(A_{ij}, j = 1, \ldots, l\), are analytic, then given \(t\), \(K(t, s)\) is a piecewise analytic function in \(s\) on \([0, t]\).

**Proof.** Substituting the right-hand side of (27) for the terms \(x(t - \tau_j)\) therein, we obtain, after a finite number of steps, a formula in which none of those terms appear, i.e., a representation of the form (29). In case the coefficients \(A, A_{0j}\) and \(A_{ij}\) are analytic, it is implied by Theorem 4.1 that for each \(t\), \(Y(s, t)\) is piecewise analytic in \(s\) on \([0, t]\). Since the kernel \(K(t, s)\) is of the form

\[K(t, s) = Y(s, t) + \sum_{j=1}^{l} x_{10,ij}(\tau_j) A_{ij}(t) Y(s, t - \tau_j)\]

\[+ \sum_{j,k=1}^{l} x_{10,ij}(\tau_j + \tau_k) A_{ij}(t) A_{1k}(t - \tau_j) Y(s, t - \tau_j - \tau_k) + \cdots\]

it too is piecewise analytic in \(s\).

Q.E.D.

**Remark 10.3.** Representation formulas for solutions of linear inhomogeneous neutral equations appear in the literature (see, e.g., Banks and Kent [4], Hale and Meyer [6] and Kolmanovski [10]). The advantage in our presentation is that by constructing the kernel \(Y(s, t)\) as a solution of
a neutral equation, we can derive analyticity properties of $Y$ as a function of the variable $s$. These properties are needed, e.g., in the analysis of the control system (4).

ACKNOWLEDGMENTS

I would like to thank Professor Zvi Artstein, who suggested the problem, for constructive criticism and valuable advice. This work addressed itself originally to problems concerning retarded equations and control systems. I also wish to thank the referee who proposed the generalization of the results to systems of neutral type.

REFERENCES