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A two-person game on graphs where each player tries to encircle his opponent's men

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Abstract

We present results on a combinatorial game which was proposed to one of the authors by Ingo Althöfer (personal communication). Let G be an undirected finite graph without loops and multiple edges and let k be a positive integer with $k \leqslant \frac{1}{2}(|G|-1)$. There are two players, called *white* and *black*, both having k men of their color. In turn, beginning with white, the players position their men one at a time on unoccupied vertices of G. When all men are placed, the players take turns moving a man of their color along an edge to an unoccupied adjacent vertex (again beginning with white). A player *wins* if his opponent cannot carry out his next move since none of his men has an unoccupied neighbor. If the game does not stop, then the outcome is a *draw*. We always assume that both players play optimal. Among other questions, we deal with the following ones: 1. Is it true that, for all G and k, white cannot win the game? 2. Does there exist a tree T and a positive integer k for which the outcome is a draw? Let $\tau(G)$ denote the *covering number* of G, i.e., $\tau(G)$ is the minimum number of vertices covering all edges of G. We prove that black wins the game if $\tau(G) \leqslant k$. We use this result to show that white never wins the game if G is bipartite, thus providing a partial answer to the first question. We answer the second question in the affirmative by constructing an infinite series of trees for which the outcome is a draw (for some k). Moreover, we present results on extremal problems arising in the context of the game. We also completely solve the cases when G is a path or a cycle. Further, we completely settle the case $k \leqslant 2$. In the proofs of our results, matchings and cycles in graphs play a predominant role. © 1999—Elsevier Science B.V. All rights reserved

Keywords: Combinatioral games; Matchings and cycles in graphs; Extremal graph theory; Discrete pursuit-evasion games

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1. Introduction

In recent years, combinatorial games played on graphs have received increasing attention; see e.g. the bibliography of Fraenkel [13]. In the present paper, we investigate a two-person game which was proposed to one of the authors by Althöfer [1]. In this game two players, *white* and *black*, each place k men of their color onto the vertices of a graph G (the *board*) which, subsequently, are moved along edges trying to encircle their opponent's men thus making the opponent unable to carry out his next move. The precise rules are given in the abstract. In flavor, this game is related to the many pursuit and evasion games for graphs that appear in the literature; see e.g. [2, 3, 5, 6, 9, 13–16, 18–23] and the literature mentioned there.

Throughout, the symbol G denotes the board of the game and k denotes the number of men per player. We write $\omega_k(G)=1$ if white has a strategy to win the game, and $\omega_k(G)=2$ if black has a winning strategy; $\omega_k(G)=0$ means that neither white nor black possesses a winning strategy. In this paper, we focus our attention on the following questions.

Question 1. Is it true that $\omega_k(G)\neq 1$ for all G and $k\leqslant\frac{1}{2}(|G|-1)$?

Let $\tau(G)$ be defined as in the abstract. In Section 2, we show that $\tau(G)\leqslant k$ implies $\omega_k(G)=2$ (Theorem 1) and use this result to prove that the answer to Question 1 is yes if G is a bipartite graph (Theorem 2). We do not know whether or not the same holds in the general case and leave this as an open problem.

In Section 3, we consider the following extremal problems. Denote by $e(G)$ and $\delta(G)$ the number of edges and the minimum degree of G, respectively. Let n,k be positive integers with $n\geqslant 2k+1$. Clearly, if G is a complete graph on n vertices, then $\omega_k(G)=0$. This observation prompts us to pose the following question.

Question 2. What is the least positive integer $\alpha(n,k)$ such that $\delta(G)>\alpha(n,k)$ implies $\omega_k(G)=0$ for all G with $|G|=n$ and, similarly, what is the least positive integer $\beta(n,k)$ such that $e(G)>\beta(n,k)$ implies $\omega_k(G)=0$ *for all* G with $|G|=n$?

We show that $\alpha(n,k)=k$ (Theorem 3). Further, Theorem 4 states that $\beta(n,k)=\binom{k}{2}+k(n-k)$ provided that "n is not too close to $2k+1$" (for example, $n\geqslant\lceil 2,172k+0,5\rceil$ is sufficient); for the latter extremal problem, we also determine the corresponding (uniquely determined) extremal graph. The proofs of these results are based on results in extremal graph theory which were obtained in [4] as extensions of classical results due to Corrádi and Hajnal [10], Erdös and Gallai [11], and Erdös and Posa [12].

Question 3. What is the value of $\omega_k(G)$ when G is a path or a cycle?

In Section 4, we present the complete answer to this question. Let G be a path or a cycle of length n. Then Theorem 5 states that $\omega_k(G)=2$ unless G is a cycle and either $k=1$ or $k=\frac{1}{2}(n-1)$; for these exceptional cases we show that $\omega_k(G)=0$.

Question 4. Does there exist a tree T and a positive integer k such that $\omega_k(T)=0$?

In Section 5, we present an infinite series of examples showing that the answer is yes. The smallest of these examples is a tree T with 19 vertices and $k=6$; see Fig. 4. Finally, in Section 6, we present a solution of the case $k\leqslant 2$.

Our graph theoretic terminology is standard; for notions used but not defined here, we refer to [8]. For a graph H, we denote by $V(H)$ and $E(H)$ the set of vertices and edges, respectively. All considered graphs are undirected, finite and, if not stated otherwise, without loops and multiple edges. By $|H|$ we denote the number of vertices of H. For distinct vertices v,w, we denote by (v,w) (or just vw) the edge connecting v and w. For a graph H, if S,T are disjoint subsets of $V(H)$, then $H(S,T)$ denotes the bipartite graph with vertex set $S\cup T$ and edge set consisting of all edges of $E(H)$ which connect a vertex of S with a vertex of T. By $m(H)$, we denote the matching number of H. Let M be a matching of H. A path P of H is an *M-alternating path* if its edges are alternately members and non-members of M. A vertex $v\in V(H)$ is *M-exposed* if no edge of M is incident with v. An M-alternating path of H is *M-augmenting* if it connects a pair of distinct M-exposed vertices. A set $S\subseteq V(H)$ is *stable* if there exists no edge of H joining a pair of vertices of S. For $n\geqslant 0$, P_n denotes the path of length n and, for $n\geqslant 3$, C_n denotes the cycle of length n. For a path P, let $x,y\in V(P)$. Then $P[x,y]$ denotes the subpath of P connecting x and y. We define $P[x,y)$ as the path that results from $P[x,y]$ by deletion of y; $P(x,y]$ and $P(x,y)$ are defined analogously. (In these definitions, the case that $P[x,y)$, $P(x,y]$ or $P(x,y)$ is empty is not excluded.) For a path $P=[v_0,\dots,v_t]$, $t\geqslant 0$, the vertices v_0 and v_t are called *end vertices* (or *terminal vertices*) of P.

For the above game, the *first phase* is the phase in which the players place their men on G; the subsequent phase, when the players move their men along edges, is called the *second phase.* We frequently use the expression *at time* t_i to indicate that, in the first phase, white has placed exactly $\lceil i/2\rceil$ men and black has placed exactly $\lfloor i/2\rfloor$ men ($i=0,\dots,2k$). Similarly, for $h\geqslant 1$, we use the term *at time* t_{2k+h} to indicate a point in time of the second phase. By w_i and b_i, we denote the vertices on which, in the first phase, white and black place their i-th man, respectively ($i=1,\dots,k$). If, in the first phase, a man is placed on $v\in V(G)$, then we denote this man by $\tilde{v}$. A vertex is *white* (*black*) if it is occupied by a white (black) man. A subgraph H of G is *completely occupied* if each of its vertices is white or black. For a path $P=[v_0,v_1,\dots,v_t]$ contained in G, assume that $\{v_{i_1},\dots,v_{i_s}\}$ is the set of occupied vertices of P, where $s\geqslant 0$ and $i_1<\cdots<i_s$. (Here $s=0$ means that no vertex of P is occupied.) Then P is called *alternately occupied* if the color of v_{i_j} does not equal the color of $v_{i_{j-1}}$ ($j=2,\dots,s$). Similarly, one defines when a cycle is alternately occupied.

For $s\geqslant 0$, let $V'\subseteq V(G)$ be a set of $2s$ vertices, s of which are white and the other s are black. We say that the men on the vertices of V' are *placed* (or *arranged*) *in pairs* if there exist disjoint subpaths $Q_1,\dots,Q_s$ of G each of which connects a white vertex of V' with a black one. For given paths $Q_1,\dots,Q_s$ with these properties and $j\in\{1,\dots,s\}$, the men on the end vertices of Q_j are said to form a *proper* (*odd*, *even*) *pair* if the length of Q_j is one (odd, even). Two men forming a proper pair are called *mates*.

2. Coverings, matchings, and the proof that white cannot win for bipartite graphs

Our first result states that black is the winner if the covering number of G is not greater than the number of men per player.

Theorem 1. *For a graph G and a positive integer $k \leqslant \frac{1}{2}(|G|-1)$, let $\tau(G) \leqslant k$. Then $\omega_k(G)=2$.*

Proof. Let $T \subseteq V(G)$ be a minimum set of vertices covering all edges of G and let $A=V(G)\setminus T$. In the following, for $i=0,\dots,k$, we inductively define subsets T_i and A_i of T and A, respectively, together with a matching M_i of the bipartite graph $G_i := G(T_i,A_i)$; simultaneously, we describe a winning strategy for black. We put

$$T_0 := T, \quad A_0 := A, \quad \text{and} \quad M_0 := \emptyset.$$

Assume that, for some $i \in \{0,\dots,k-1\}$, we have already defined subsets T_i and A_i of T and A, respectively, together with a matching M_i of G_i such that, at time t_{2i}, the following conditions hold:

(i) Each edge of M_i connects a white vertex of T_i with a black vertex of A_i;
(ii) T_i consists of the unoccupied and the white vertices of T;
(iii) A_i consists of the unoccupied vertices of A and those black vertices which are met by M_i;
(iv) For each white vertex $z \in T_i$ which is not met by M_i, there exists no M_i-augmenting path of G_i starting at z.

These conditions are illustrated in Fig. 1. Note also that, for $i=0$, (i)–(iv) trivially hold. Now, we consider the situation at time t_{2i+1}, i.e., white has just put his $(i+1)$-th man on the vertex w_{i+1}. We define the vertex b_{i+1} (i.e., the vertex on which black puts his $(i+1)$-th man) and, simultaneously, we define the sets T_{i+1}, A_{i+1}, and the matching M_{i+1} of $G_{i+1}=G(T_{i+1},A_{i+1})$.

If $w_{i+1} \in T$ and if there exists an M_i-augmenting path of G_i starting at w_{i+1}, then black picks one such path P and chooses b_{i+1} as the end vertex of P which is distinct from w_{i+1}; by assumption (iii), this vertex is unoccupied. We define $T_{i+1} := T_i$, $A_{i+1} := A_i$, and $M_{i+1} := (M_i\setminus E(P)) \cup (E(P)\setminus M_i)$.

On the other hand, if $w_{i+1} \in A$ or if $w_{i+1} \in T$ and there does not exist an M_i-augmenting path of G_i starting at w_{i+1}, then black picks b_{i+1} as some unoccupied member of T, provided that this is possible; if this is not possible, then black picks b_{i+1} as some unoccupied member of A. We define $T_{i+1} := T_i\setminus\{b_{i+1}\}$, $A_{i+1} := A_i\setminus\{w_{i+1},b_{i+1}\}$, and $M_{i+1} := M_i$.

We now show that the statements (i)–(iv) still hold if i is replaced by $i+1$. By the definitions this is immediately clear except for statement (iv) in the case when

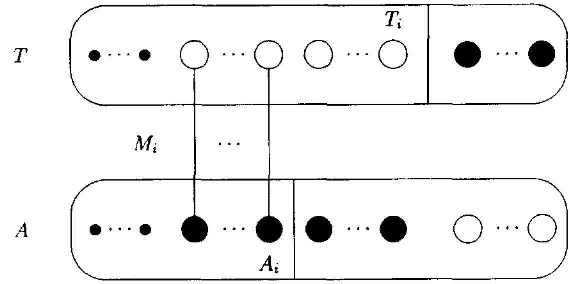

Fig. 1. An illustration of the conditions (i)–(iv) in the proof of Theorem 1.

$w_{i+1} \in T$ and w_{i+1}, b_{i+1} are the terminal verties of an M_i-augmenting path P of G_i. For the purpose of settling this case, let $z \in T_{i+1}(=T_i)$ be a white vertex which is not met by M_{i+1}. Then $z \neq w_{i+1}$ and z is not met by M_i, and thus we obtain from (iv) that there exists no M_i-augmenting path of G_i starting at z. It follows that there cannot exist an M_i-alternating path Q of G_i starting at z and having a vertex with P in common because, otherwise, an appropriate subpath of Q could be combined with an appropriate subpath of P to obtain an M_i-augmenting path of G_i starting at z. Now, suppose that there exists an M_{i+1}-augmenting path P' of $G_{i+1}(=G_i)$ starting at z. Since P' is not an M_i-augmenting path, we must have $V(P') \cap V(P) \neq \emptyset$. Let z' be the first vertex of P' which is on P. Then $P'[z,z']$ is an M_i-alternating path of G_i starting at z and having z' with P in common, in contradiction to what we have shown before. Hence (i)–(iv) hold for $i+1$ instead of i.

Note that $\{w_{i+1}, b_{i+1}\} \cap T \neq \emptyset$ if, at time t_{2i}, T still contains unoccupied vertices ($i=0,\ldots,k-1$). Hence, because $k \geqslant \tau(G) = |T|$, it follows that, at the end of the first phase,

$$\text{all vertices of } T \text{ are occupied.} \tag{1}$$

We next show that, in the second phase, black can play such that, when it is white's turn, white has no choice other than moving one of his men from a vertex of T to a vertex of A, and black can always answer with moving one of his men from A to the vertex which was just abandoned by white. Clearly this implies that, after at most k moves of the second phase, black wins the game. We now make this precise.

Denote by T' the set of vertices of T which are met by M_k. For some $j \in \{0,\ldots, k-1\}$, assume that, in his first j moves of the second phase, white has moved j of his men from T' to A and black has answered each of these moves by moving a man of his color along an edge of M_k to the vertex just left by white. We assume that white can still carry out his $(j+1)$-th move. Because A is a stable set and because, at time $t_{2(k+j)}$, the condition (1) still holds, white moves one of his men from a vertex z of T to a vertex of A. Suppose $z \notin T'$. Then one easily obtains from the way the first j moves of the second phase where carried out by the players that there exists an M_k-augmenting path of G_k starting at z. This contradicts the statement (iv) for $i=k$, and thus we have shown $z \in T'$. Hence black can use the edge of M_k which is adjacent to z to move a man of his color along this edge from A to z. □

We give examples of graphs G with $\omega_k(G)=0$ for $k=\tau(G)-1$, thus showing that, in a sense, Theorem 1 is sharp. Let G be a graph that consists of a cycle C of length four and a nontrivial path P of even length, where $C \cap P$ consists of an end vertex of P. Note that $\tau(G)=\frac{1}{2}|G|$ and let $k=\tau(G)-1$. It follows that G contains the disjoint union of C and a matching M consisting of $k-1$ edges. From this one easily obtains $\omega_k(G)=0$ (for example, by application of the forthcoming Lemma 1 in Section 3).

Theorem 2. *Let G be a bipartite graph. Then $\omega_k(G) \neq 1$ for each positive integer $k \leqslant \frac{1}{2}(|G|-1)$.*

Proof. If $m(G) \leqslant k$, then (by a well-known theorem of König [7, 8, 17]) $\tau(G) \leqslant k$ and the assertion follows from Theorem 1. Thus we may assume $m(G) \geqslant k$. Let M be a matching of G with $|M|=k$ and let further $V(G)=A \cup B$ be a partition of $V(G)$ such that all edges of G connect a vertex of A with a vertex of B.

We now describe a strategy for black which guarantees that white cannot win. Let $i \in \{0,\dots,k-1\}$ and assume that, at time t_{2i}, the following conditions hold:

(i) Each edge of M either connects a black vertex with a white one or a black vertex with an unoccupied vertex or it connects a pair of unoccupied vertices,

(ii) each black vertex is incident with an edge of M,

(iii) the number of white vertices of A and the number of black vertices of B are of the same parity.

These conditions trivially hold for $i=0$. Denote by M' the subset of edges of M which, at time t_{2i}, connect a pair of unoccupied vertices. By (i), in conjunction with the fact that $i<|M|$, we have $M' \neq \emptyset$. Now, if white picks w_{i+1} such that w_{i+1} is on an edge $e \in M'$, then black picks b_{i+1} such that $(w_{i+1}, b_{i+1})=e$; otherwise, black picks b_{i+1} on an arbitrary edge of M' such that the condition (iii) is maintained. In any case, the conditions (i)–(iii) still hold at time $t_{2(i+1)}$ and, consequently, these conditions hold at the end of the first phase.

Now, let $j \geqslant 0$ be an integer and assume that, at time $t_{2(k+j)}$, the conditions (i)–(iii) hold. Then, at time $t_{2(k+j)+1}$, the numbers mentioned in (iii) are of distinct parities and thus (because (i) and (ii) hold at time $t_{2(k+j)}$) there exists an edge of M connecting a black vertex b with an unoccupied vertex u. Now, in his $(j+1)$-th move of the second phase, black moves his man from b to u. Then, clearly, the conditions (i)–(iii) are maintained, and thus we have shown that white cannot win. □

The following proposition provides another class of graphs for which white cannot win.

Proposition 1. *Let G be a graph with a perfect matching. Then $\omega_k(G) \neq 1$ for each positive integer $k \leqslant \frac{1}{2}(|G|-1)$.*

Proof. Since G has a perfect matching, black can place his men such that, at the end of the first phase, the white and black men are arranged in proper pairs. Thus, in the second phase, each black man can follow his mate wherever he goes and thus black cannot loose. □

The general question whether or not there exist graphs G for which white has a winning strategy for some $k \leqslant \frac{1}{2}(|G|-1)$ remains unanswered and we pose it as an open problem. We remark that, if loops are allowed, than it is easy to give an example of a graph G for which white is the winner: just take G as the graph with $V(G)=\{a,b,c\}$ and $E(G)=\{(a,a),(a,b),(a,c)\}$ and let $k=1$. Then, clearly, white wins the game.

3. Extremal problems

We introduce some additional notations. For nonnegative integers s,t, we write $G \supseteq O^s \uplus e^t$ to indicate that G contains the disjoint union of $s+t$ graphs, s of which are cycles and t of which are complete graphs on two vertices. By $\langle r,s\rangle$ we denote the complete bipartite graph with color classes of cardinality r and s, respectively. By $\langle\langle r\rangle,s\rangle$, we denote the graph that results from $\langle r,s\rangle$ by adding all possible $\binom{r}{2}$ edges connecting the vertices of a color class of cardinality r. By $\langle n\rangle$, we denote the complete graph on n vertices, and $\{n\}$ denotes the graph consisting of n isolated vertices.

Lemma 1. *For integers $s \geqslant 1$ and $t \geqslant 0$, let G be a graph with $G \supseteq O^s \uplus e^t$. Assume that $|G| \geqslant 5s+2t-2$ and let $k=2s+t-1$. Then $\omega_k(G)=0$.*

Proof. Denote one of the players (white or black) by P and the other by Q. Let $H_1,\ldots,H_{s+t}$ be a system of disjoint subgraphs of G, where $H_1,\ldots,H_s$ are cycles and $H_{s+1},\ldots,H_{s+t}$ are complete graphs on two vertices. For each $i \in \{1,\ldots,s\}$, pick arbitrarily $|H_i|-3$ vertices of H_i and call these vertices *extra vertices*; moreover, call all vertices of G which are not lying on any of the H_i $(i=1,\ldots,s+t)$ *extra vertices.* Then, because $|G| \geqslant 5s+2t-2$, the number of extra vertices is at least $2(s-1)$. From this one easily concludes that player P can manage to place his men such that exactly $s-1$ of his men are on extra vertices and such that his remaining $s+t$ men are on non-extra vertices of the subgraphs H_i $(i=1,\ldots,s+t)$, exactly one of these men on each of these subgraphs. Then, for each H_i with $i \in \{1\ldots,s\}$, the number of player $P's$ men placed on H_i is at least one and at most $|H_i|-2$; and for each H_i with $i \in \{s+1,\ldots,s+t\}$, there is exactly one vertex of H_i which is occupied by P. Hence, in order to win the game, Q must have at least two of his men on each of the graphs H_i $(1 \leqslant i \leqslant s)$ and one of his men on each of the graphs H_i $(s+1 \leqslant i \leqslant s+t)$. Because $k<2s+t$ this is impossible, and thus we have shown that Q cannot win the game. Hence $\omega_k(G)=0$. □

For positive integers n,k with $n \geqslant 2k+1$, let $\alpha(n,k)$ and $\beta(n,k)$ be defined as in the introduction. We start with the problem of determining $\alpha(n,k)$. Clearly,

$$\alpha(n,k) \geqslant k$$

since the graph $\langle k, n-k \rangle$ has minimum degree k and $\omega_k(\langle k, n-k \rangle) = 2$ by Theorem 1. We will see that in fact $\alpha(n,k) = k$. For the purpose of proving this, we need the following result which was obtained in [4] as a corollary of the famous Corrádi/Hajnal theorem [10]. (In [4], Theorem A was not explicitly stated as a theorem, but it was proved in passing in the course of the proof of [4, Theorem 2].)

Theorem A. *For nonnegative integers s,t, let G be a graph with $|G| \geqslant 3s + 2t$ and $\delta(G) \geqslant 2s + t$. Then $G \supseteq O^s \cup e^t$.*

The next theorem is an immediate consequence of Theorem A and Lemma 1. It shows that $\alpha(n,k) = k$.

Theorem 3. *For a positive integer k, let G be a graph with $|G| \geqslant 2k+1$ and $\delta(G) \geqslant k+1$. Then $\omega_k(G) = 0$.*

Proof. Apply Theorem A and Lemma 1 for the special case $s=1$ and $t=k-1$. □

We now turn to the problem of determining $\beta(n,k)$. Clearly

$$\beta(n,k) \geqslant \binom{k}{2} + k(n-k)$$

since the graph $\langle\langle k \rangle, n-k \rangle$ has $\binom{k}{2} + k(n-k)$ edges and $\omega_k(\langle\langle k \rangle, n-k \rangle) = 2$ by Theorem 1. The next theorem shows that $\beta(n,k) = \binom{k}{2} + k(n-k)$ provided that "n is not too close to $2k+1$". More precisely, we assume

$$n \geqslant \lceil \tfrac{1}{2}(10k - \sqrt{32k^2+1} + 1) \rceil. \tag{2}$$

Note that (2) implies $n \geqslant 2k+1$, which can be verified by an easy computation.

Theorem 4. *For integers n,k with $k \geqslant 2$, assume that* (2) *holds. Let G be a graph with $|G| = n$ and*

$$e(G) \geqslant \binom{k}{2} + k(n-k). \tag{3}$$

Then $\omega_k(G) = 0$ unless $G \cong \langle\langle k \rangle, n-k \rangle$.

For the proof of Theorem 4, we need the following Theorem B which is an immediate consequence of results obtained in [4] (namely, Theorem 2 and statement (6) of [4]).

Theorem B. *For positive integers s, t put*

$$\varphi(s,t) := \left\lceil 3s + 2t - 1 + \frac{(s+t)(s+t-1)}{2(2s+t-1)} \right\rceil \tag{4}$$

and let G be a graph with $|G| = n \geqslant \varphi(s,t)$. Assume that

$$e(G) \geqslant \binom{2s+t-1}{2} + (2s+t-1)(n-2s-t+1). \tag{5}$$

Then $G \supseteq O^s \cup e^t$ or $G \cong \langle\langle 2s+t-1\rangle, n-2s-t+1\rangle$ or $G \cong \langle 3s+2t-1\rangle \cup \{n-3s-2t+1\}$.

Before we prove Theorem 4, we derive the following Theorem 4′ which can be considered as a preliminary version of Theorem 4.

Theorem 4′. *For positive integers s, t, let $\varphi(s,t)$ be as in* (4) *and let G be a graph with*

$$|G| = n \geqslant \max\{5s+2t-2, \varphi(s,t)\}. \tag{6}$$

Assume that (5) *holds and let $k = 2s+t-1$. Then $\omega_k(G) = 0$ unless $G \cong \langle\langle k\rangle, n-k\rangle$.*

Proof. From Theorem B one concludes that $G \supseteq O^s \cup e^t$ or $G \cong \langle\langle 2s+t-1\rangle, n-2s-t+1\rangle$ or $G \cong \langle 3s+2t-1\rangle \cup \{n-3s-2t+1\}$. If $G \supseteq O^s \cup e^t$ or $G \cong \langle\langle 2s+t-1\rangle, n-2s-t+1\rangle$, then the assertion follows from Lemma 1. Hence assume $G \cong \langle 3s+2t-1\rangle \cup \{n-3s-2t+1\}$. We show that in this case $\omega_k(G) = 0$. Denote by P one of the players (white or black) and by Q the other one. Denote by K the complete subgraph of G with $3s+2t-1$ vertices. Player P employs the following strategy: P puts his first man on a vertex of K and, thereafter, P puts as many men on isolated vertices as possible. We show that Q cannot win. Consider the situation at time t_{2k}. If all isolated vertices of G are occupied, then at least one vertex of K is unoccupied and thus (because at least one vertex of K is occupied by P) Q cannot win. On the other hand, if not all isolated vertices of G are occupied, then it follows from P's strategy, that P has placed exactly $k-1$ men onto isolated vertices. Hence at most $k+1$ men are placed on K, which implies that at least $|K| - k - 1 = s+t-1 \geqslant 1$ vertices of K are unoccupied and thus Q cannot win in either case. □

Proof of Theorem 4. Let $k \geqslant 2$ be a fixed integer. For the purpose of proving Theorem 4 with the aid of Theorem 4′, we want to determine integers $s, t \geqslant 1$ such that $k = 2s+t-1$ and such that $\max\{5s+2t-2, \varphi(s,t)\}$ is minimal. We put $m = s+t$. Then

$$5s + 2t - 2 = 3k - m + 1$$

and

$$\varphi(s,t)=\left\lceil k+m+\frac{m(m-1)}{2k}\right\rceil.$$

Hence our task is to determine an integer m, $2\leqslant m\leqslant k$, such that

$$\max\left\{3k-m+1,\left\lceil k+m+\frac{m(m-1)}{2k}\right\rceil\right\}$$

is minimal. By elementary considerations (which are left to the reader) one finds that a solution of this task is

$$m_0=\lfloor\tfrac{1}{2}(\sqrt{32k^2+1}-4k+1)\rfloor. \tag{7}$$

Moreover, one obtains

$$\max\left\{3k-m_0+1,\left\lceil k+m_0+\frac{m_0(m_0-1)}{2k}\right\rceil\right\}$$
$$=3k-m_0+1=\lceil\tfrac{1}{2}(10k-\sqrt{32k^2+1}+1)\rceil. \tag{8}$$

Now, let $s:=k-m_0+1$ and $t:=2m_0-k-1$. Then s,t are integers satisfying the equations $k=2s+t-1$ and $m_0=s+t$. Because $m_0\leqslant k$, we have $s\geqslant 1$; moreover, $t\geqslant 1$ follows from (7) by an elementary computation. Hence we have determined integers s,t as desired. Moreover, one obtains from (8) that

$$\max\{5s+2t-2,\varphi(s,t)\}=\lceil\tfrac{1}{2}(10k-\sqrt{32k^2+1}+1)\rceil.$$

Hence Theorem 4′ can be employed to obtain Theorem 4. □

In the context of Theorem 4 it is interesting to observe that

$$2k+1\leqslant\lceil\tfrac{1}{2}(10k-\sqrt{32k^2+1}+1)\rceil\leqslant\lceil 2,172k+0,5\rceil.$$

4. Paths and cycles

In this section, we solve the problem of determining $\omega_k(G)$ when G is a path or a cycle. For settling the case when G is a cycle, we need the following lemma. This lemma also shows that $\omega_k(G)=2$ when G is a path. (We mention that there exist other more direct and shorter ways to settle the case when G is a path.)

Lemma 2. *Assume that the board is a path P and that, in the first phase, black plays according to the following rule.*

> *Black chooses b_i such that, at time t_{2i}, $P[w_i,b_i]$ is completely and, if possible, alternately occupied $(i=1,\dots,k)$.* (9)

Then, for the second phase, black has a strategy to win the game.

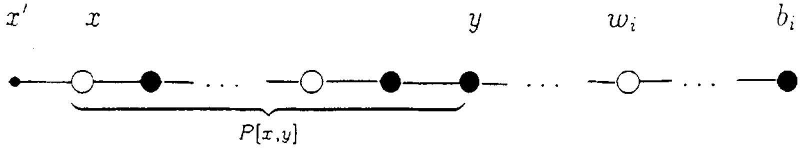

Fig. 2. The path $P[x', b_i]$ in the proof of Lemma 2.

Proof. Assuming that black plays according to (9), we claim that, for each $i \in \{1, \ldots, k\}$, the following holds. At time t_{2i}, there exists a collection $\mathscr{P}_i$ of disjoint subpaths of P such that the vertices covered by the paths of $\mathscr{P}_i$ are exactly the occupied vertices and such that the following holds:

For each $P' \in \mathscr{P}_i$, if P' has a white end vertex which is not an end vertex of P, then P' is alternately occupied and has an even number of vertices. (10)

By (9), this holds at time t_2. For some $i \in \{2, \ldots, k\}$, assume that the claim holds at time $t_{2(i-1)}$. We are considering the situation at time t_{2i} and show that the claim still holds.

If $P[w_i, b_i]$ is alternately occupied, then we choose $P[w_i, b_i]$ as one of the paths of $\mathscr{P}_i$ and, as the remaining paths of $\mathscr{P}_i$, we take those members of $\mathscr{P}_{i-1}$ which are disjoint to $P[w_i, b_i]$. Then, clearly, $\mathscr{P}_i$ has the required properties, and thus we may assume that $P[w_i, b_i]$ is not alternately occupied.

Let x be the uniquely determined vertex of P such that $P[x, w_i]$ is completely occupied, $w_i \in P[x, b_i]$, and x is either an end vertex of P or x has an unoccupied neighbor x'. If x is black or if x is an end vertex of P, then we are done since we can choose $P[x, b_i]$ as one member of $\mathscr{P}_i$ and, as the remaining paths of $\mathscr{P}_i$, we can pick those members of $\mathscr{P}_{i-1}$ which are disjoint to $P[x, b_i]$. Hence we can assume that x is white and that there exists an unoccupied neighbor x' of x.

Let $y \in P[x, w_i]$ be the uniquely determined vertex such that $P[x, y)$ is alternately occupied, $|P[x, y)|$ is even, and $P[x, y)$ is maximal with these properties. We claim that y is black. (See also Fig. 2.) For the proof of this claim, suppose that y is white. Note that $y \neq w_i$ since, otherwise, by rule (9), black would have put his i-th man onto x'. Let $y' \in P[y, w_i]$ be adjacent to y. By the induction-hypothesis, there are disjoint paths $Q_1, \ldots, Q_r \in \mathscr{P}_{i-1}$ such that

$$\bigcup_{i=1}^{r} V(Q_i) = V(P[x, w_i]).$$

Because $Q_1, \ldots, Q_r$ are members of $\mathscr{P}_{i-1}$, the following holds. If Q_j has a white end-vertex, then Q_j is alternately occupied and has an even number of vertices ($j = 1, \ldots, r$). However, this is only possible if y' is black since $P[x, y]$ is alternately occupied and because x, y are white. This contradicts the maximality of $P[x, y)$. Hence we have proved that y is black.

Now, we can choose $P[x, y)$ and $P[y, b_i]$ as paths of $\mathscr{P}_i$ and select the other paths of $\mathscr{P}_i$ as those members of $\mathscr{P}_{i-1}$ which are disjoint to both $P[x, y)$ and $P[y, b_i]$. This yields $\mathscr{P}_i$ as desired. Hence we have proved that our claim holds at time t_{2i}, and thus it holds for all $i = 1, \ldots, k$.

For $P' \in \mathscr{P}_k$, denote by $V_0(P')$ the set of end vertices of P' which are not end vertices of P. At time t_{2k}, if all members of $V_0(P')$ are black, then black never moves his men on P', and so white cannot move his men on P'. On the other hand, if $V_0(P')$ contains a white vertex, then (10) implies that the white and black vertices of P' are arranged in proper pairs and each black man on P' can follow his mate wherever he goes. This clearly ends up with a win for black. □

Theorem 5. *Let n, k be positive integers with $n \geqslant 3$ and $k \leqslant \frac{1}{2}(n-1)$. Then $\omega_k(P_n) = 2$ and*

$$\omega_k(C_n) = \begin{cases} 0 & \text{if } k = 1 \quad \text{or } k = \frac{1}{2}(n-1), \\ 2 & \text{otherwise.} \end{cases}$$

Proof. The case $k = 1$ is trivial and the case that the board is a path is covered by Lemma 2. Hence let $k \geqslant 2$ and assume that C_n is the board. We first present a strategy for black to win the game if $k < \frac{1}{2}(n-1)$ and to achieve a draw if $k = \frac{1}{2}(n-1)$; thereafter, we show that white can achieve a draw if $k = \frac{1}{2}(n-1)$.

Let $i \in \{1, \ldots, k\}$. Assume that, at time $t_{2(i-1)}$, the players have placed $2(i-1)$ men in proper pairs and assume further that white places his i-th man on a neighbor of a white vertex. Then, clearly, black can place his i-th man such that there are disjoint paths $Q_1, Q_2 \subseteq C_n$ covering all vertices of C_n and having the following properties:

(i) Q_1 is non-trivial, the first and the last vertex of Q_1 is black, and all vertices of Q_1 are occupied,

(ii) the men on Q_2 are arranged in proper pairs.

Hence, in order to win the game, black never moves his men on Q_1 and imagines that the game is played on Q_2 (rather than C_n). Moreover, because of (ii), black can also imagine that the men placed so far on Q_2 were placed according to rule (9). Now, black continues to play on Q_2 according to (9) and, by Lemma 2, wins the game. Hence we may assume that the following holds.

At time $t_{2(i-1)}$, if the players have placed $2(i-1)$ men in proper pairs, then white never places his i-th man on a neighbor of a white vertex $(i = 2, \ldots, k)$. (11)

We may assume that the vertices of C_n are denoted by $0, 1, \ldots, n-1$ where $E(C_n) = \{(j, j+1)\colon j = 0, \ldots, n-2\} \cup \{(n-1, 0)\}$. Now, for $i = 1, \ldots, k-2$, black chooses b_i such that

$$b_i \equiv w_i - 1 \pmod{n}. \tag{12}$$

Note that, because of (11), this choice is possible. If $k = \frac{1}{2}(n-1)$, then black also picks b_{k-1} and b_k according to (12) and thus achieves a draw. Now, let $k < \frac{1}{2}(n-1)$. Consider the situation at time t_{2k-3}. We may assume $w_{k-1} = 0$. Then, for some $s \geqslant 1$,

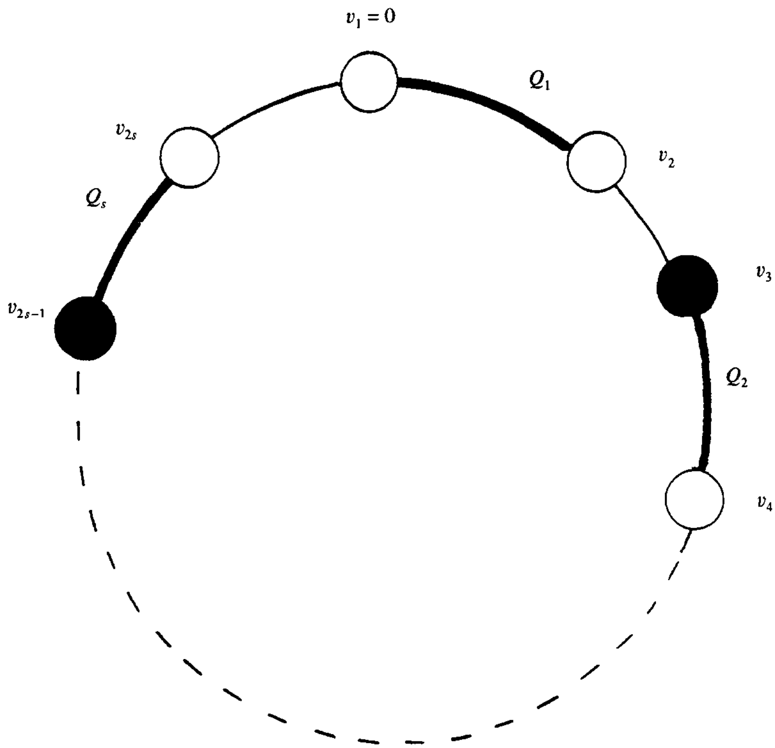

Fig. 3. The situation at time t_{2k-3}; $v_1=v_2$ is possible.

there exist vertices $v_1, v_2, \dots, v_{2s-1}, v_{2s}$ of C_n for which the following conditions (a)–(c) hold. (This immediately follows from (11) together with our assumption (12) on black's first $k-2$ moves; for an illustration, see Fig. 3.)

(a) $0=v_1 \leqslant v_2$, $\quad v_{2i}+1<v_{2i+1}<v_{2i+2} (i=1,\dots,s-1)$, $\quad v_{2s}<n-1$,

(b) the subpaths $Q_i := [v_{2i-1},\dots,v_{2i}]$ of C_n are completely and alternately occupied $(i=1,\dots,s)$ and all vertices outside $Q_1 \cup \dots \cup Q_s$ are unoccupied,

(c) v_i is black if and only if $i \geqslant 3$ and $i \equiv 1 \pmod 2$ $(i=1,\dots,2s)$.

We now describe rules for black for choosing b_{k-1} and b_k. These rules ensure that, at time t_{2k}, the following condition (13) holds. (One easily finds that, in the second phase, this results into a win for black.)

$$\begin{gathered}\text{The } 2k \text{ men are placed in pairs on } C_n, \text{ but not alternately;}\\ k-2 \text{ of the pairs are proper and the remaining two pairs are of the same parity.}\end{gathered} \tag{13}$$

Case 1: $s \geqslant 2$ and $v_4 \neq n-2$.

Black chooses $b_{k-1}=n-1$. Then the so far placed $2(k-1)$ men are not on a single path $P \subseteq C_n$ with $|P|=2(k-1)$. From this one easily concludes that black can answer the k-th move of white such that, at time t_{2k}, the $2k$ men are arranged in proper pairs on C_n, but not alternately. Hence (13).

Case 2: $s \geqslant 2$ and $v_4 = n-2$.

Note that $v_4 = n-2$ implies $s=2$. Black chooses $b_{k-1} = v_2 + 1$. By (11), $w_k \neq n-1$ and thus black can choose b_k as a neighbor of w_k. It follows that the $2k$ men are arranged in proper pairs on C_n, but not alternately. Hence (13).

Case 3: $s=1$.

Black chooses $b_{k-1} = v_2 + 3$. If $w_k = v_2 + 1$, then black chooses $b_k = n-2$. (Because $k < \frac{1}{2}(n-1)$, the vertex $n-2$ is unoccupied.) If $w_k = v_2 + 2$, then black's choice is $b_k = n-1$. If $w_k = v_2 + 4$, then black chooses $b_k = v_2 + 1$. Finally, if $w_k \neq v_2 + i$ for $i = 1,2,4$, then black chooses $b_k = w_k - 1$. One easily checks that, in each of the subcases, condition (13) is met.

Now, let $k = \frac{1}{2}(n-1)$. It remains to show that white has a strategy which prevents a win of black. Suppose that this is not the case and choose n minimal with this property. If black picks b_1 such that b_1 is a neighbor of w_1, then white picks w_2 as the unoccupied neighbor of b_1. In the second phase, there is only one unoccupied vertex and thus the three men $\tilde{w}_1, \tilde{b}_1, \tilde{w}_2$ "behave like a single white man". More precisely, this means the following. Assume that, at some point in time of the second phase, the white men $\tilde{w}_1$ and $\tilde{w}_2$ are placed on the neighbors of $\tilde{b}_1$. For some $i \in \{1,2\}$, assume further that white moves $\tilde{w}_i$ to an unoccupied neighbor. Then black has no choice but moving $\tilde{b}_1$ to the vertex just left by $\tilde{w}_i$, and subsequently white has to move $\tilde{w}_j$ to the vertex just left by $\tilde{b}_1$ (for $j \in \{1,2\}$ with $j \neq i$). It follows that, in the obvious way, one can simulate the game on C_n with k men per player by the game on C_{n-2} with $k-1$ men per player. Hence one concludes from the minimal choice of n, in conjunction with the trivial statement $\omega_1(C_3) = 0$, that white can prevent a win of black. This contradiction settles the case that b_1 is a neighbor of w_1.

Hence assume that black picks b_1 such that b_1 is not a neighbor of w_1. Then white picks w_2 as a neighbor of w_1 such that the w_1, b_1-path of C_n which does not contain w_2 has even length; this is possible because n is odd. For $j = 1,2$, denote by Q_j the w_j, b_1-path of C_n which contains w_{3-j}. Note that both Q_1 and Q_2 have odd length. In the sequel, whenever necessary, we consider the paths Q_j to be oriented from w_j to b_1. For example, we use expressions like "the i-th unoccupied vertex of Q_j" thereby meaning the i-th unoccupied vertex of Q_j when Q_j is traversed from w_j to b_1. By $U_{j,l}$, we denote the set of vertices of Q_j which are unoccupied at time t_l; by $u_{j,l,i}$, we denote the i-th unoccupied vertex of Q_j at time t_l $(3 \leqslant l \leqslant 2k, 1 \leqslant j \leqslant 2)$.

In the sequel, we describe rules for white for the choice of $w_3, \ldots, w_k$ and, subsequently, show that these rules ensure a win for white, which contradicts the supposition that white does not have a strategy preventing a win of black. Let $i \in \{2, \ldots, k-1\}$. We assume that, at time t_{2i-1}, the following holds.

$$\begin{aligned} &\text{If } U_{1,2i-1} \text{ and } U_{2,2i-1} \text{ both are nonempty,} \\ &\quad \text{then } |U_{1,2i-1}| \text{ and } |U_{2,2i-1}| \text{ both are odd.} \end{aligned} \tag{14}$$

We now consider the situation at time t_{2i}. By symmetry, we may assume $b_i \in Q_1$. (If $b_i \in Q_2$, then white employs rules for the choice of w_{i+1} which are analogous to the forthcoming rules for the case $b_i \in Q_1$.) If $U_{1,2i} = \emptyset$, then white chooses $w_{i+1} = u_{2,2i,2}$.

Let $U_{1,2i} \neq \emptyset$. If $U_{2,2i} = \emptyset$ and $b_i = u_{1,2i-1,1}$, then white's choice is $w_{i+1} = u_{1,2i,2}$. Now assume $U_{2,2i} \neq \emptyset$ or $b_i \neq u_{1,2i-1,1}$. If there exists an unoccupied vertex on $Q_1[b_i, b_1]$, then white chooses w_{i+1} as the first such vertex; otherwise, white chooses $w_{i+1} \in U_{1,2i} \setminus \{u_{1,2i,1}\}$. (The latter choice is possible for the following reason. From $U_{1,2i} \neq \emptyset$ and $b_i \in Q_1$, it follows that $|U_{1,2i-1}| \geqslant 2$. If $U_{2,2i-1} \neq \emptyset$, then $|U_{1,2i-1}| \geqslant 3$ by (14) and thus the claimed choice of w_{i+1} is possible. If $U_{2,2i-1} = \emptyset$, then the claimed choice of w_{i+1} is clearly possible.)

Note that, since both Q_1 and Q_2 have odd length, (14) is true for $i = 2$. Further, if (14) holds for some $i \in \{2, \ldots, k-1\}$, then it clearly follows from the just described rules for the choice of w_{i+1} that (14) also holds for $i+1$ instead of i, and thus white can employ these rules for choosing all the vertices $w_3, \ldots, w_k$. In order to show that this results into a win for white, we need the following statement (15) which is a consequence of the rules for the choice of $w_3, \ldots, w_k$. The proof of (15) can be carried out by induction on i; we leave the easy (but somewhat lengthy) proof to the reader.

For $j \in \{1,2\}$ and $i \in \{2, \ldots, k\}$, let $u \in U_{j,2i-1}$ and $v \in V(Q_j[w_j, u))$. Then, at time t_{2i-1}, $Q_j[v, u]$ contains at least as many white vertices as black ones and, if $v = u_{j,2i-1,1}$, $u = u_{j,2i-1,2}$ and $U_{3-j,2i-1} = \emptyset$, then $Q_j[v, u]$ contains more white vertices than black ones. (15)

We define vertices x_i, y_i $(i = 2, \ldots, k)$ as follows. Let $i \in \{2, \ldots, k\}$. If both $U_{1,2i-1}$ and $U_{2,2i-1}$ are nonempty, then let $x_i = u_{1,2i-1,1}$ and $y_i = u_{2,2i-1,1}$; if $U_{j,2i-1} \neq \emptyset$ and $U_{3-j,2i-1} = \emptyset$ for some $j \in \{1,2\}$, then let $x_i = u_{j,2i-1,1}$ and $y_i = u_{j,2i-1,2}$. For $j \in \{1,2\}$, let $P = [v_0, v_1, \ldots, v_{2r}]$ be a subpath of Q_j, where $r \geqslant 1$ and $v_1 \in Q_j(v_0, b_1]$. For all $\rho \in \{1, \ldots, 2r-1\}$, assume that, at time t_{2i-1}, v_ρ is white if ρ is odd and black if ρ is even. Then P is called a *chain for* x_i if $v_{2r} = x_i$ and if v_0 is white; further, P is called a *chain for* y_i if $v_{2r} = y_i$ and if either v_0 is white or $v_0 = x_i$.

At time t_{2i-1}, there exist both a chain for x_i and a chain for y_i $(i = 2, \ldots, k)$. (16)

For the proof of (16), let x_i' be the neighbor of x_i for which $x_i \in Q_j[x_i', b_1]$ (where Q_j denotes the path with $x_i \in Q_j$). Similarly, we define y_i'. Then neither x_i' nor y_i' is black since this would contradict (15). Further, y_i' cannot be unoccupied since this would imply $y_i' = x_i$, in contradiction to (15). Hence both x_i' and y_i' are white. Let $x_i \in Q_j$ and suppose that there exists no chain for x_i. From this, together with the fact that x_i', w_1 and w_2 are white, one concludes that there exists a subpath $Q = [u_0, u_1, \ldots, u_{2s} = x_i']$ of Q_j such that u_σ is black if either $\sigma = 0$ or $\sigma \equiv 1 \pmod 2$ and such that u_σ is white, otherwise. But this contradicts (15). In a similar manner, the supposition that there exists no chain for y_i leads to a contradiction to (15). Hence (16).

Recall that, at time t_{2k-1}, x_k and y_k are the only unoccupied vertices. From this, together with (16) for $i = k$, one easily concludes that white wins the game. □

5. Trees for which the outcome is a draw

We frequently make use of the following simple lemma.

Lemma 3. *For $S \subseteq V(G)$, let H be a subgraph of G consisting of some of the components of $G-S$. Let M be a matching of H. At time t_{2k} or at some point in time of the second phase, assume that all vertices of S are white and, for each $e \in M$, there is exactly one white vertex incident with e. Assume further that one of the following conditions* (i), (ii) *holds.*

(i) *The number of black vertices of H is less than $|M|$.*

(ii) *For each $e \in M$, there is exactly one black vertex incident with e and there do not exist any other black vertices in H; further, it is black's turn to do the next move and black cannot move any men on vertices outside H.*

Then black cannot win the game.

Proof. Assume that (i) holds. Then, in order to prevent a win of black, white can employ the following strategy: white never moves any of his men except for those which are on vertices incident with edges of M, and these white men are just moved along edges of M. This is always possible because condition (i) is assumed and because the white men positioned on S do not allow black to move any of his men from $(G-S)-H$ to H. Thus black cannot win the game, i.e., the outcome is either a draw or a win of white.

Next assume that (ii) holds. Then the men on M are arranged in proper pairs and, in order to prevent a win of black, each white man on M just has to follow his mate whereever he goes. □

For each integer $m \geqslant 2$, we define a tree T_m as follows. (For an illustration, see Fig. 4.) The vertices of T_m are denoted by $\varepsilon, \alpha, \alpha', \alpha_1, \ldots, \alpha_m, \alpha'_1, \ldots, \alpha'_m, \beta, \beta', \beta_1, \ldots, \beta_m, \beta'_1, \ldots, \beta'_m, \gamma, \gamma', \gamma_1, \ldots, \gamma_m, \gamma'_1, \ldots, \gamma'_m$ and the edges of T_m are the following: $\varepsilon\alpha, \varepsilon\beta, \varepsilon\gamma, \alpha\alpha', \beta\beta', \gamma\gamma', \alpha\alpha_i, \beta\beta_i, \gamma\gamma_i, \alpha_i\alpha'_i, \beta_i\beta'_i, \gamma_i\gamma'_i$ $(i=1,\ldots,m)$.

The component of $T_m - \varepsilon$ which contains α is denoted by A; similarly, we denote by B and C the components of $T_m - \varepsilon$ containing β and γ, respectively. Let $M'_A := \{\alpha_i\alpha'_i\colon i=1,\ldots,m\}$ and $M_A := M'_A \cup \{\alpha\alpha'\}$; similarly, the matchings M'_B, M_B, M'_C, and M_C are defined.

Theorem 6. *For $m \geqslant 2$, let T_m be the above defined tree and let $k = 2(m+1)$. Then $\omega_k(T_m) = 0$.*

Proof. By Theorem 2, $\omega_k(T_m) \neq 1$. We show how white can prevent a win of black. White chooses $w_1 = \varepsilon$ and, in his next two moves, occupies as many vertices of $\{\alpha, \beta, \gamma\}$ as possible.

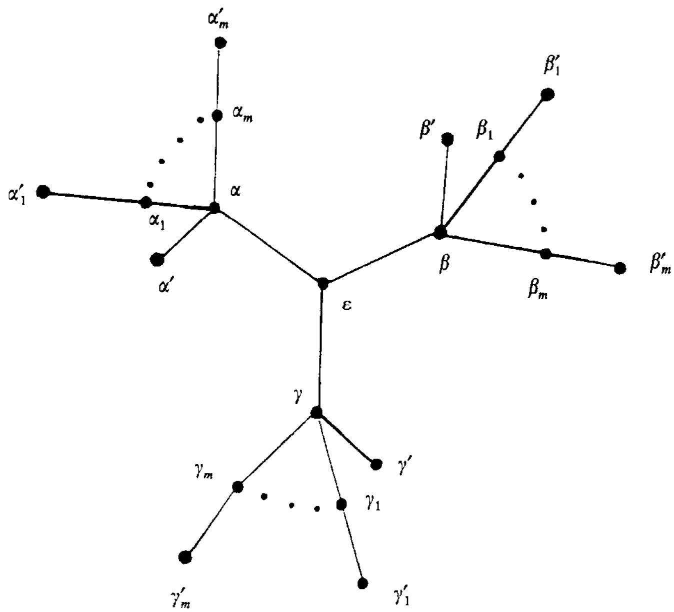

Fig. 4. The tree T_m $(m \geqslant 2)$.

Case 1: $\{w_2, w_3\} \subseteq \{\alpha, \beta, \gamma\}$.

We may assume $\{w_2, w_3\} = \{\beta, \gamma\}$. First assume that $b_i \in A$ for at most one $i \in \{1,2,3\}$. Then white picks $w_4, \ldots, w_{m+4}$ such that each edge of M_A contains exactly one of these vertices. (This choice is possible since $b_i \in A$ for at most one $i \in \{1,2,3\}$.) Further, white chooses $w_{m+5}, \ldots, w_{2m+2} \in B \cup C$ such that each edge of $M'_B \cup M'_C$ contains at most one of these vertices. (This choice is possible since $|M'_B \cup M'_C| = 2m$ and because the number of edges of $M'_B \cup M'_C$ for which both ends are black can be at most $m+1$.) Then, at time t_{2k}, the number of white vertices in $V(A) \cup \{\varepsilon\}$ is greater than the number of black vertices in this set and thus B or C must contain more black vertices than white ones; we may assume that this holds for B. Then application of Lemma 3(i) (with $S = \{\beta\}, H = T_m - B$, and M appropriately chosen) yields that black cannot win.

Now assume that at least two of the vertices b_1, b_2, b_3 are in A. Then white chooses $w_4, \ldots, w_{2m+2}$ such that these vertices are on $2m-1$ distinct edges of $M'_B \cup M'_C$, which clearly is possible. Application of Lemma 3(i) (with $S = \{\varepsilon\}, H = B \cup C$, and M appropriately chosen) yields the assertion.

Case 2: $\{w_2, w_3\} \not\subseteq \{\alpha, \beta, \gamma\}$.

We may assume $b_1 = \alpha, w_2 = \beta, b_2 = \gamma$. White picks $w_3, \ldots, w_{2m+2}$ such that, at time t_{2k}, each of the edges of $M'_A \cup M'_B$ is incident with exactly one of these vertices and such that the following holds. At time t_{2k}, if neither α_j nor α'_j is black, then α'_j is white $(j = 1, \ldots, m)$. It follows that, at time t_{2k}, there must be $m+1$ black vertices in B and m black vertices in A since, otherwise, Lemma 3(i) could be employed to find that black cannot win. Denote by F the set of vertices of A which, at time t_{2k}, are unoccupied. Clearly, $|F|=2$ and it follows from the choice of $w_3,\dots,w_{2k+2}$ that, up to symmetry, there are just two cases to be considered, namely, $F=\{\alpha_1,\alpha'\}$ and $F=\{\alpha_1,\alpha_2\}$.

In both of these cases, in his first move of the second phase, white moves $\widetilde{\alpha_1'}$ to α_1. If $F=\{\alpha_1,\alpha'\}$, then application of Lemma 3(ii) (with $S=\{\beta\}$, $H=T_m-B$, and M appropriately chosen) yields the assertion. Hence let $F=\{\alpha_1,\alpha_2\}$. Then $\tilde{\varepsilon}$ follows $\tilde{\gamma}$ wherever $\tilde{\gamma}$ goes and $\widetilde{\alpha_1'}$ follows $\tilde{\alpha}$ wherever $\tilde{\alpha}$ goes; further, for $j=3,\dots,m$, if $\widetilde{\alpha_j'}$ is white and $\tilde{\alpha}_j$ is black, then $\widetilde{\alpha_j'}$ follows $\tilde{\alpha}_j$ wherever $\tilde{\alpha}_j$ goes. It follows that, after a finite number of moves of the second phase, black must move $\widetilde{\alpha'}$ to α. Then white answers this move of black by moving $\widetilde{\alpha_2'}$ to α_2. It follows that we are in the situation of Lemma 3(ii) (with $S=\{\beta\}, H=T_m-B$, and M appropriately chosen). Hence black cannot win the game. □

6. The case $k\leqslant 2$

For $m\geqslant 3$, denote by $\mathscr{S}_m$ the class of graphs G having the following properties: G contains a chordless cycle C of length m, all vertices of $V(G)\setminus V(C)$ have degree at most one, and no two vertices of $V(G)\setminus V(C)$ are joined by an edge. With this notation, our result on the case $k\leqslant 2$ reads as follows.

Proposition 2. *(a) For each graph G with $|G|\geqslant 3$, $\omega_1(G)=0$ if G contains a cycle, and $\omega_1(G)=2$, otherwise.*

(b) For each graph G with $|G|\geqslant 5$, $\omega_2(G)=0$ if $G\supseteq O^1\cup e^1$ or $G\in\mathscr{S}_5$, and $\omega_2(G)=2$, otherwise.

Proof. We prove part (b), part (a) being trivial. By Lemma 1, we have $\omega_2(G)=0$ if $G\supseteq O^1\cup e^1$. If G is a forest, then one easily finds $\omega_2(G)=2$. (The proof is left to the reader.) Now assume $G\not\supseteq O^1\cup e^1$, G is not a forest, and $G\notin\mathscr{S}_m$ for all $m\geqslant 3$. From these assumptions, one easily concludes that $\tau(G)=2$ and thus $\omega_2(G)=2$ by Theorem 1.

Hence let $G\in\mathscr{S}_m$ for some $m\geqslant 3$ and let C be the uniquely determined cycle of G. For $v\in V(C)$, let $B(v)$ be the set of neighbors of v which are not on C.

We start with discussing the case $m=5$. The easy proof that black can prevent a win of white is left to the reader. We sketch the proof that white can prevent a win of black. White picks w_1 on C. Clearly, it may be assumed that b_1 is not an isolated vertex. If $b_1\in B(w_1)$, then white picks w_2 arbitrarily on C; if $b_1\in C$ and $(b_1,w_1)\notin E(C)$, then white chooses w_2 as the uniquely determined neighbor of w_1 on C which is not adjacent to b_1; in all remaining cases, white chooses w_2 as a neighbor of b_1 on C. One easily checks that, for all possible choices of b_2, white can force that, at the end of the first phase or after a few moves of the second phase, one of the following holds:

(i) White wins the game;

(ii) the four men are arranged alternately on C;

(iii) both white men are on C and there exists a white vertex $v \in C$ such that at least one vertex of $B(v)$ is black.

One easily finds that, if (ii) or (iii) holds, black cannot win the game.

Now let $m \neq 5$. If $w_1 \notin C$ or $m \in \{3,4\}$, then one easily checks that $w_2(G) = 2$. Let $m \geqslant 6$ and $w_1 \in C$. Then black chooses b_1 on C such that the distance of w_1 and b_1 is three. A similar discussion as in Case 3 in the proof of Theorem 5 yields $\omega_2(G) = 2$. (We leave the details to the reader.) □

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