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Analytic mappings between noncommutative pencil balls

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ABSTRACT

In this paper, we analyze problems involving matrix variables for which we use a noncommutative algebra setting. To be more specific, we use a class of functions (called NC analytic functions) defined by power series in noncommuting variables and evaluate these functions on sets of matrices of all dimensions; we call such situations dimension-free. These types of functions have recently been used in the study of dimension-free linear system engineering problems (Helton et al. (2009) [10], de Oliviera et al. (2009) [8]). In the earlier paper (Helton et al. (2009) [9]) we characterized NC analytic maps that send dimension-free matrix balls to dimension-free matrix balls and carry the boundary to the boundary; such maps we call "NC ball maps". In this paper we turn to a more general dimension-free ball $\mathcal{B}_L$, called a "pencil ball", associated with a homogeneous linear pencil

$$L(x) := A_1 x_1 + \cdots + A_g x_g, \quad A_j \in \mathbb{C}^{d' \times d}.$$

For $X = \operatorname{col}(X_1, \dots, X_g) \in (\mathbb{C}^{n \times n})^g$, define $L(X) := \sum A_j \otimes X_j$ and let

$$\mathcal{B}_L := \left(\left\{X \in \left(\mathbb{C}^{n \times n}\right)^g \colon \left\|L(X)\right\| < 1\right\}\right)_{n \in \mathbb{N}}.$$

We study the generalization of NC ball maps to these pencil balls $\mathcal{B}_L$, and call them "pencil ball maps". We show that every $\mathcal{B}_L$ has a minimal dimensional (in a certain sense) defining pencil $\tilde{L}$. Up to normalization, a pencil ball map is the direct sum of $\tilde{L}$ with an NC analytic map of the pencil ball into the ball. That is, pencil ball maps are simple, in contrast to the classical result of D'Angelo (1993) [7, Chapter 5] showing there is a great variety of such analytic maps from $\mathbb{C}^g$ to $\mathbb{C}^m$ when $g \ll m$. To prove our main theorem, this paper uses the results of our previous paper (Helton et al. (2009) [9]) plus entirely different techniques, namely, those of completely contractive maps. What we do here is a small piece of the bigger puzzle of understanding how Linear Matrix Inequalities (LMIs) behave with respect to noncommutative change of variables.

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1. Introduction

Given positive integers n, d, d' and g, let $\mathbb{C}^{d'\times d}$ denote the $d'\times d$ matrices with complex coefficients and $(\mathbb{C}^{n\times n})^g$ the set of g-tuples of $n\times n$ matrices. For $A_1,\dots,A_g\in\mathbb{C}^{d'\times d}$, the expression

$$L(x)=\sum_{j=1}^{g}A_jx_j, \tag{1.1}$$

is a **homogeneous linear pencil**. (Often the term linear pencil refers to an *affine* linear function; i.e., a sum of a constant term plus a homogeneous linear pencil.) Given $X=\operatorname{col}(X_1,\dots,X_g)\in(\mathbb{C}^{n\times n})^g$, define

$$L(X)=\sum_{j=1}^{g}A_j\otimes X_j. \tag{1.2}$$

Let $\mathcal{B}_L(n)=\{X\in(\mathbb{C}^{n\times n})^g\colon\ \|L(X)\|<1\}$ and let $\mathcal{B}_L$ denote the sequence $(\mathcal{B}_L(n))_{n\in\mathbb{N}}$. Similarly, let $\mathcal{M}_{\ell',\ell}=((\mathbb{C}^{n\times n})^{\ell'\times\ell})_{n\in\mathbb{N}}$. The main result of this paper describes analytic mappings f from the **pencil ball** $\mathcal{B}_L$ to $\mathcal{M}_{\ell',\ell}$ that preserve the boundary in the sense described at the end of Section 1.1 below.

In the remainder of this introduction, we give the definitions and background necessary for a precise statement of the result, and provide a guide to the body of the paper.

1.1. Formal power series

Let $x=(x_1,\dots,x_g)$ be a g-tuple of noncommuting indeterminates and let $\langle x\rangle$ denote the set of all **words** in x. This includes the empty word denoted by 1. The length of a word $w\in\langle x\rangle$ will be denoted $|w|$. For an abelian group R we use $R\langle x\rangle$ to denote the abelian group of all (finite) sums of **monomials** (these are elements of the form rw for $r\in R$ and $w\in\langle x\rangle$).

Given positive integers ℓ, ℓ', a **formal power series** f in x with $\mathbb{C}^{\ell'\times\ell}$ coefficients is an expression of the form

$$f=\sum_{m=0}^{\infty}\sum_{\substack{w\in\langle x\rangle\\|w|=m}}f_ww=\sum_{m=0}^{\infty}f^{(m)}, \tag{1.3}$$

where $f_w\in\mathbb{C}^{\ell'\times\ell}$ and $f^{(m)}\in\mathbb{C}^{\ell'\times\ell}\langle x\rangle$ is the **homogeneous component** of degree m of f, that is, the sum of all monomials in f of degree m. For $X\in(\mathbb{C}^{n\times n})^g$, $X=\operatorname{col}(X_1,\dots,X_g)$ and a word

$$w=x_{j_1}x_{j_2}\cdots x_{j_m}\in\langle x\rangle,$$

let

$$w(X)=X_{j_1}X_{j_2}\cdots X_{j_m}\in\mathbb{C}^{n\times n}.$$

Define

$$f(X)=\sum_{m=0}^{\infty}\sum_{\substack{w\in\langle x\rangle\\|w|=m}}f_w\otimes w(X),$$

provided the series converges (summed in the indicated order). The function f is **analytic** on $\mathcal{B}_L$ if for each n and $X\in\mathcal{B}_L(n)$ the series $f(X)$ converges. Thus, in this case, the formal power series f determines a mapping from $\mathcal{B}_L(n)$ to $(\mathbb{C}^{n\times n})^{\ell'\times\ell}$ for each n which is expressed by writing $f:\mathcal{B}_L\to\mathcal{M}_{\ell',\ell}$.

The analytic function $f:\mathcal{B}_L\to\mathcal{M}_{\ell',\ell}$ is **contraction-valued** if $\|f(X)\|\leqslant 1$ for each $X\in\mathcal{B}_L$; i.e., if the values of f are contractions. Let $\partial\mathcal{B}_L(n)$ denote the set of all $X\in(\mathbb{C}^{n\times n})^g$ with $\|L(X)\|=1$. If $f:\mathcal{B}_L\to\mathcal{M}_{\ell',\ell}$ is contraction-valued and $X\in\partial\mathcal{B}_L(n)$, then, by Fatou's theorem, the analytic function $f_X:\mathbb{D}\to(\mathbb{C}^{n\times n})^{\ell'\times\ell}$ defined by $f_X(z)=f(zX)$ has boundary values almost everywhere; i.e., $f(\exp(it)X)$ is defined for almost every t. (We use $\mathbb{D}$ to denote the unit disc $\{z\in\mathbb{C}\colon\ |z|<1\}$.) The contraction-valued function f is a **pencil ball map** if $\|f(\exp(it)X)\|=1$ a.e. for every $X\in\partial\mathcal{B}_L$. Here the boundary $\partial\mathcal{B}_L$ of the pencil ball $\mathcal{B}_L$ is the sequence $(\partial\mathcal{B}_L(n))_{n\in\mathbb{N}}$.

1.2. The main result

The homogeneous linear pencil L is **nondegenerate**, if it is one–one in the sense that

$$\forall X\in\big(\mathbb{C}^{n\times n}\big)^g\colon\quad\big(L(X)=0\ \Rightarrow\ X=0\big) \tag{1.4}$$

for all $n\in\mathbb{N}$.

Lemma 1.1. *For a homogeneous linear pencil $L(x)=\sum_{j=1}^{g} A_j x_j$ the following are equivalent:*

(i) *L is nondegenerate;*
(ii) *$L(X)=0$ implies $X=0$ for all $X\in\mathbb{C}^g$, i.e., condition (1.4) holds for $n=1$;*
(iii) *the set $\{A_j\colon\ j=1,\dots,g\}$ is linearly independent.*

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious. For the remaining implication (iii) $\Rightarrow$ (i), note that $L(X)$ equals $\sum X_j\otimes A_j$ modulo the canonical shuffle. If this expression equals 0, then the linear dependence of the A_j (applied entrywise) implies $X=0$. □

The homogeneous linear pencil

$$\tilde{L}(x)=\sum_{1}^{g}\tilde{A}_j x_j,$$

is **equivalent** to L if $\|\tilde{L}(X)\|=\|L(X)\|$ for every n and $X\in(\mathbb{C}^{n\times n})^g$, i.e., $\mathcal{B}_L=\mathcal{B}_{\tilde{L}}$.

An $\ell'\times\ell$ nondegenerate homogeneous linear pencil $\tilde{L}$ is called **a minimal dimensional defining pencil for a ball** if $\mathcal{B}_L=\mathcal{B}_{\tilde{L}}$ for some $d'\times d$ homogeneous linear nondegenerate pencil L implies $\ell'\ell\leqslant d'd$. Equivalently: if L is equivalent to $\tilde{L}$, then $d'd\geqslant\ell'\ell$. Two equivalent minimal dimensional defining pencils for a ball are the same up to normalization, cf. Corollary 5.3 and Theorem 5.2. Thus, while there are potentially many ways of defining minimal, all will agree with the current usage. Indeed, heuristically any condition which eliminates redundant or simply irrelevant summands from the pencil L should do.

Lemma 1.2. *Suppose L is a nondegenerate homogeneous linear pencil and $\tilde{L}$ is a minimal dimensional defining pencil for $\mathcal{B}_L$. Then there is a homogeneous linear pencil J and unitaries Q, G satisfying*

$$L=Q(\tilde{L}\oplus J)G^*.$$

The proof of this lemma can be found in Section 5.1; it is an immediate consequence of Theorem 5.2 and Corollary 5.3.

The following theorem is the main result of this paper.

Theorem 1.3. *Suppose L is a nondegenerate homogeneous linear pencil and $\tilde{L}$ is a minimal dimensional defining pencil for $\mathcal{B}_L$. If $f:\mathcal{B}_L\to\mathcal{M}_{\ell',\ell}$ is a pencil ball map with $f(0)=0$, then there is a contraction-valued analytic $\tilde{f}:\mathcal{B}_L\to\mathcal{M}_{m',m}$ such that*

$$f(x)=U\begin{pmatrix}\tilde{L}(x) & 0\\ 0 & \tilde{f}(x)\end{pmatrix}V^*$$

for some $m',m\in\mathbb{N}_0$ and unitaries $U\in\mathbb{C}^{\ell'\times\ell'}$ and $V\in\mathbb{C}^{\ell\times\ell}$.

Corollary 1.4. *Suppose L is a nondegenerate homogeneous linear pencil and $\tilde{L}$ is an $\ell'\times\ell$ minimal dimensional defining pencil for $\mathcal{B}_L$. If $f:\mathcal{B}_L\to\mathcal{M}_{\ell',\ell}$ is a pencil ball map with $f(0)=0$, then*

$$f(x)=U\tilde{L}(x)V^*$$

for some unitaries $U\in\mathbb{C}^{\ell'\times\ell'}$ and $V\in\mathbb{C}^{\ell\times\ell}$.

Theorem 1.3 and its corollary indicates that proper noncommutative analytic mappings are simpler than their classical counterparts [7].

1.3. Related work

An elegant theory of noncommutative analytic functions is developed in the articles [11,20,21]; see also [14–19]. What we use in this article are specializations of definitions of these papers. Also there are results on various classes of noncommutative functions on several NC domains as we now describe.

(a) For the special case of the NC ball of row contractions, Corollary 1.4 is the same as Corollary 1.3 in [9] and appears in a weaker form in Popescu's paper [19]. Namely, the assumption that the map takes the boundary of the NC ball into the boundary of the NC ball is replaced in [19] by the stronger assumption that the NC ball map is biholomorphic.

(b) Also related to the current paper is [1] which studies the special cases of noncommutative polydiscs (and noncommutative poly-halfplanes) rather than general pencil balls. With this caveat in mind, the part of [1] most closely related the study of noncommutative analytic functions given by a formal power series with $\mathbb{C}^{q\times q}$ coefficients mapping an N-tuple of $n\times n$ unitary matrices to a $qn\times qn$ unitary matrix for each $n\in\mathbb{N}$. This class generalizes the notion of inner function (e.g., for the case $g=d=d'=1$ and $L(x)=x$ any scalar inner function is in the [1] class). Our pencil ball maps have the additional property that $f(B)$ is an $n\times n$ matrix of norm 1 for every $n\times n$ matrix B of norm 1 and as we shall see this forces f to be only a single Blaschke factor.

(c) The paper [2] obtains realization results for contraction-valued noncommutative analytic functions defined on pencil balls of a special form. In the multiplicity-one case, the coefficients of the pencil are all partial isometries whose set of initial spaces (overlaps of initial spaces for different coefficients allowed) form a direct sum decomposition for the whole domain space, and similarly for the final spaces.

Our interest in noncommutative analytic functions stems from noncommutative semialgebraic geometry and its applications to systems engineering [8,10].

1.4. The linear term and complete contractivity

The first step in the proof of Theorem 1.3 is an analysis of the linear term of f which is $f^{(1)}=\sum_{j=1}^{g} f_{x_j}x_j$ with f expressed as in Eq. (1.3). It involves heavily the theory of completely contractive and completely positive linear maps, the nature of which we illustrate later in this introduction by discussing Theorem 1.3 in the case of linear maps. The books [12,13,6] provide comprehensive introductions to the theory of operator systems, spaces, and algebras, and completely contractive and completely isometric mappings. The papers [3] and [4] treat very generally complete isometries into a C^*-algebra. Indeed, the results here in the linear case are very similar to those in [5,3,4] when specialized to our setting. Namely, the injective envelope of a subspace of $m\times n$ matrices is, up to equivalence, a direct sum of full matrix spaces. Rather than only quoting existing results from the literature, we have chosen to make the presentation self contained with an exposition of the operator space background at the level of generality needed for the present purposes. We believe this makes the article more accessible and may expose a wider audience to the utility of the extensive literature in operator spaces and systems.

Let F be a subspace of $\mathbb{C}^{d'\times d}$. For each n, there is the subspace $F_n=F\otimes\mathbb{C}^{n\times n}$ of $\mathbb{C}^{d'\times d}\otimes\mathbb{C}^{n\times n}$. A linear mapping $\varphi: F\to\mathbb{C}^{\ell'\times\ell}$ induces linear mappings $\varphi_n: F\otimes\mathbb{C}^{n\times n}\to\mathbb{C}^{\ell'\times\ell}\otimes\mathbb{C}^{n\times n}$ by $\varphi_n(e\otimes Y)=\varphi(e)\otimes Y$. Write $F\leqslant\mathcal{M}_{d',d}$ to indicate that we are identifying F with the sequence $(F_n)_{n\in\mathbb{N}}$. A **completely contractive** mapping $\varphi: F\to\mathcal{M}_{\ell',\ell}$ is a linear mapping $\varphi: F\to\mathbb{C}^{\ell'\times\ell}$ such that $\|\varphi_n(Z)\|\leqslant\|Z\|$ for every n and $Z\in F\otimes\mathbb{C}^{n\times n}$. If instead, $\|\varphi_n(Z)\|=\|Z\|$ for every n and $Z\in F\otimes\mathbb{C}^{n\times n}$, then φ is **completely isometric**. We shall be interested in completely isometric maps acting on the **range** $\mathcal{R}_L$ of a homogeneous linear pencil L. This range for a $d'\times d$ pencil in g variables is defined as follows. For each n, let $\mathcal{R}_L(n)$ denote the range of L applied to X in $(\mathbb{C}^{n\times n})^g$:

$$\mathcal{R}_L(n)=\{L(X):\ X\in(\mathbb{C}^{n\times n})^g\}.$$

Let $\mathcal{R}_L=(\mathcal{R}_L(n))_{n\in\mathbb{N}}$. In particular, we have $\mathcal{R}_L=F\leqslant\mathcal{M}_{d',d}$.

Theorem 1.5. *Let L be a nondegenerate homogeneous linear pencil and $\tilde{L}$ a minimal dimensional defining pencil for $\mathcal{B}_L$. If $\psi:\mathcal{R}_L\to\mathcal{M}_{\ell',\ell}$ is completely isometric, then there exist unitaries $U\in\mathbb{C}^{\ell'\times\ell'}$ and $V\in\mathbb{C}^{\ell\times\ell}$, positive integers m, m', and a completely contractive mapping $\phi:\mathcal{R}_{\tilde{L}}\to\mathcal{M}_{m',m}$ such that*

$$\psi\big(L(x)\big)=U\begin{pmatrix}\tilde{L}(x) & 0\\ 0 & \phi(\tilde{L}(x))\end{pmatrix}V^*.$$

If $f:\mathcal{B}_L\to\mathcal{M}_{\ell',\ell}$ is a pencil ball map with $f(0)=0$ and if L is a nondegenerate homogeneous linear pencil, then the linear part $f^{(1)}$ of f induces a completely isometric mapping

$$\psi:\mathcal{R}_L\to\mathcal{M}_{\ell',\ell},\quad L(x)\mapsto f^{(1)}(x)$$

(see Lemma 5.4) to which Theorem 1.5 applies.

1.5. Readers guide

The remainder of the paper is organized as follows. Section 2 provides background on completely contractive maps, operator spaces, and injective envelopes needed for the remainder of the paper. The point of departure is Arveson's extension theorem. In Section 3 a result from our previous paper [9] is used to characterize completely isometric mappings from $\bigoplus\mathcal{M}_{d_j',d_j}$ to $\mathcal{M}_{n',n}$ and in Section 4 the injective envelope of $\mathcal{R}_L$ is determined. The main results, Theorems 1.5 and 1.3 are proved in Section 5.

The main results naturally generalize to mappings $f:\mathcal{B}_L\to\mathcal{B}_{L'}$; i.e., analytic mappings between pencil balls determined by homogeneous linear (nondegenerate) pencils. The details are in Section 6.

2. The complete preliminaries

For the reader's convenience, in this section we have gathered the background material on completely positive, contractive, and isometric mappings and injective envelopes needed in the sequel. Since our attention is restricted to the case of finite dimensional subspaces of matrix algebras, the exposition here is considerably more concrete than that found in the literature, where the canonical level of generality involves arbitrary subspaces of C^*-algebras. We have followed the general outline, based upon the off-diagonal techniques of Paulsen, found in his excellent book, cf. [12, p. 98]. The reader with expertise in operator spaces and systems could likely skip or skim this section and proceed to Section 3. The reader familiar with the work of Blecher and Hay in [3,4] or of Blecher and Labuschagne in [5] might safely skip to Section 5. Indeed, this section and the next two establish the fact that the injective envelope of a subspace of $m \times n$ matrices is, up to equivalence, a direct sum of full matrix spaces – see Theorem 3.1.

2.1. Completely contractive and completely positive maps

Recall the notion of completely contractive defined in Section 1.4. We shall also need a related notion of positivity. A subspace $S \leqslant \mathcal{M}_{m,m}$ is an **operator system** if it is self-adjoint (that is, closed under $X \mapsto X^*$) and contains the identity. A **completely positive** mapping $\psi : S \to \mathcal{M}_{p,p}$ is a linear map $\psi : S \to \mathbb{C}^{p\times p}$ such that $\psi_n(Z)$ is a positive semi-definite for every n and every positive semi-definite matrix $Z \in S \otimes \mathbb{C}^{n\times n}$.

The significance of completely positive maps is that, while positive maps $S \to \mathbb{C}^{p\times p}$ do not necessarily extend to positive maps on all of $\mathbb{C}^{m\times m}$, *completely* positive maps on S *do* extend to completely positive maps on all of $\mathcal{M}_{m,m}$.

Theorem 2.1 *(Arveson's extension theorem). (Cf. [12, Theorem 7.5].) If $S \leqslant \mathcal{M}_{m,m}$ is an operator system and $\psi : S \to \mathcal{M}_{p,p}$ is completely positive, then there is a completely positive mapping $\Psi : \mathcal{M}_{m,m} \to \mathcal{M}_{p,p}$ such that $\psi = \Psi|_S$.*

There are numerous connections between completely contractive and completely positive maps of which the off-diagonal construction will be used repeatedly. Given $F \leqslant \mathcal{M}_{d',d}$, let

$$\mathcal{S}_F = \begin{pmatrix} \mathbb{C}I_d & F^* \\ F & \mathbb{C}I_{d'} \end{pmatrix} = \left\{ \begin{pmatrix} \lambda I_d & b^* \\ a & \eta I_{d'} \end{pmatrix} : \lambda, \eta \in \mathbb{C},\ a, b \in F \right\} \leqslant \mathcal{M}_{d+d',d+d'}.$$

Evidently $\mathcal{S}_F$ is an operator system.

Given a completely contractive $\varphi : F \to \mathcal{M}_{\ell',\ell}$, the mapping $\psi : \mathcal{S}_F \to \mathcal{M}_{\ell+\ell',\ell+\ell'}$ defined by

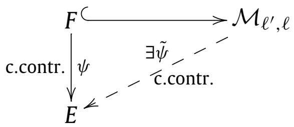
$$\psi\left(\begin{pmatrix} \lambda I_d & b^* \\ a & \eta I_{d'} \end{pmatrix}\right) = \begin{pmatrix} \lambda I_\ell & \varphi(b)^* \\ \varphi(a) & \eta I_{\ell'} \end{pmatrix} \tag{2.1}$$

is completely positive. It is also **unital** in that $\psi(I) = I$. The converse is true, too. The compression of a completely positive unital $\psi : \mathcal{S}_F \to \mathcal{M}_{\ell+\ell',\ell+\ell'}$ to its lower left-hand corner is completely contractive.

2.2. The injective envelope

Determining the structure of completely isometric mappings is intimately intertwined with the notion of the injective envelope.

A subspace $E \leqslant \mathcal{M}_{d',d}$ is **injective** if whenever $F \leqslant \mathcal{M}_{\ell',\ell}$ and $\psi : F \to E$ is completely contractive, then there exists a completely contractive $\tilde{\psi} : \mathcal{M}_{\ell',\ell} \to E$ such that $\tilde{\psi}|_F = \psi$:



The notion of injective envelope is categorical.

Lemma 2.2. *If $E \leqslant \mathcal{M}_{m',m}$ is injective and $\tau : E \to \mathcal{M}_{\ell',\ell}$ is completely isometric, then $\tau(E)$ is injective.*

Proof. It is enough to observe that the mapping $\tau^{-1} : \tau(E) \to E$ is completely isometric. For the details, suppose $F \leqslant \mathcal{M}_{d',d}$ and $\varphi : F \to \tau(E)$ is completely contractive. Then $\tau^{-1} \circ \varphi : F \to E$ is completely contractive and hence extends to a completely contractive $\tilde{\varphi} : \mathcal{M}_{d',d} \to E$. The mapping $\tau \circ \tilde{\varphi} : \mathcal{M}_{d',d} \to \tau(E)$ is then a completely contractive extension of φ. □

Lemma 2.3. *The spaces $\mathcal{M}_{\ell',\ell}$ are injective.*

Proof. View $\mathcal{M}_{\ell',\ell}$ as the lower left-hand corner of $\mathcal{M}_{\ell+\ell',\ell+\ell'}$ as in Eq. (2.1). Note that the mapping $\Psi:\mathcal{M}_{\ell+\ell',\ell+\ell'}\to\mathcal{M}_{\ell',\ell}$ given by

$$\Gamma\left(\begin{pmatrix} x_{11} & x_{12} \\ a & x_{22}\end{pmatrix}\right)=a \tag{2.2}$$

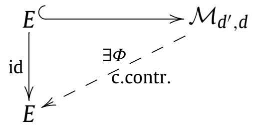
is completely contractive. Likewise, the inclusion ι of $\mathcal{M}_{d',d}$ into $\mathcal{M}_{d+d',d+d'}$ as the lower left-hand corner is completely contractive.

Suppose $F\leqslant\mathcal{M}_{d',d}$ and $\varphi:F\to\mathcal{M}_{\ell',\ell}$ is completely contractive. Then $\psi:F\to\mathcal{M}_{\ell+\ell',\ell+\ell'}$, as in Eq. (2.1), is completely positive. By Theorem 2.1, ψ extends to a completely positive $\Psi:\mathcal{M}_{d+d',d+d'}\to\mathcal{M}_{\ell+\ell',\ell+\ell'}$. The mapping $\Gamma\circ\Psi\circ\iota$ is a completely contractive extension of φ. □

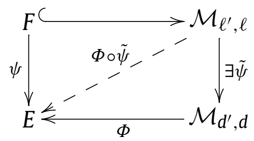
A mapping $\Phi:\mathcal{M}_{d',d}\to J\leqslant\mathcal{M}_{d',d}$ is a **projection** provided Φ is onto and $\Phi\circ\Phi=\Phi$.

Proposition 2.4. *A subspace $E\leqslant\mathcal{M}_{d',d}$ is injective if and only if there is a completely contractive projection $\Phi:\mathcal{M}_{d',d}\to E$.*

Proof. If E is injective, apply the definition with $F=E\leqslant\mathcal{M}_{d',d}$ and ψ the identity map. The conclusion is there exists a completely contractive Φ which extends ψ and maps onto (into) E. It follows that Φ is a projection (completely contractive).



Conversely, suppose the completely contractive Φ exists and let F, $\mathcal{M}_{\ell',\ell}$ and ψ be as in the definition of an injective space. Since $\mathcal{M}_{d',d}$ is injective and $E\leqslant\mathcal{M}_{d',d}$, there exists a completely contractive extension $\tilde\psi$ of ψ mapping $\mathcal{M}_{\ell',\ell}$ into $\mathcal{M}_{d',d}$. The composition $\varphi=\Phi\circ\tilde\psi$ is completely contractive into E and extends ψ:



□

Suppose $F\leqslant E\leqslant\mathcal{M}_{d',d}$. A completely contractive mapping $\rho:E\to E$ is an F**-map** on E provided $\rho(f)=f$ for all $f\in F$. For E injective with $F\leqslant E\leqslant\mathcal{M}_{d',d}$, define a partial order $\leqslant_E$ on F-maps on E by $\sigma\leqslant_E\rho$ if $\|\sigma(x)\|\leqslant\|\rho(x)\|$ for each $x\in E$. An F-map on E which is minimal with respect to this ordering is an E**-minimal** F**-map**.

Lemma 2.5. *Given $F\leqslant E\leqslant\mathcal{M}_{d',d}$, the collection of F-maps on E is compact. Further, if E is injective, then there is an E-minimal F-map. Indeed, for any F-map ρ on E there exists an E-minimal F-map σ such that $\sigma\leqslant_E\rho$.*

Proof. Let $\mathcal{F}$ denote the collection of all F-maps on E. Since the inclusion $F\to\mathcal{M}_{d',d}$ is completely contractive and $\mathcal{M}_{d',d}$ is injective, the set of all F-maps is nonempty. The collection $\mathcal{F}$ is evidently closed and bounded and therefore compact.

To establish the existence of an E-minimal F-map, apply Zorn's lemma. Namely, given a decreasing net (ρ_λ) there exists, by compactness, a convergent subnet (ρ_η) converging to some κ (this does not depend upon decreasing). Because the original net was decreasing, it follows that $\kappa\leqslant\rho_\lambda$ for all λ. Hence every decreasing chain has a lower bound. An application of Zorn's lemma produces a minimal element. □

Lemma 2.6. *If σ is an E-minimal F-map, then $\sigma(\sigma(x))=\sigma(x)$ for $x\in E$.*

Proof. Let ψ_n be the average of the first n powers of σ,

$$\psi_n(x)=\frac{1}{n}\sum_{j=1}^{n}\sigma^j(x).$$

Then ψ_n is an F-map and $\|\psi_n(x)\|\leqslant\|\sigma(x)\|$ for all $x\in E$, so by minimality $\|\psi_n(x)\|=\|\sigma(x)\|$. The same argument shows $\|\sigma(\sigma(x))\|=\|\sigma(x)\|$. Thus,

$$\big\|\sigma\big(x-\sigma(x)\big)\big\|=\big\|\psi_n\big(x-\sigma(x)\big)\big\|=\frac{1}{n}\big\|\sigma(x)-\sigma^{n+1}(x)\big\|\leqslant\frac{2}{n}\|x\|$$

for all n. Hence $\sigma(x)=\sigma(\sigma(x))$. □

Let $F \leqslant \mathcal{M}_{d',d}$ be given. An E such that $F \leqslant E \leqslant \mathcal{M}_{d',d}$ is a **concrete injective envelope** of F if E is injective and if J is injective with $F \leqslant J \leqslant E$, then $J = E$.

Theorem 2.7. *Each $F \leqslant \mathcal{M}_{d',d}$ has a concrete injective envelope. In fact, the range of each $\mathcal{M}_{d',d}$-minimal F-map σ is a concrete injective envelope for F.*

Proof. Let σ be a given $\mathcal{M}_{d',d}$-minimal F-map and let E denote the range of σ. By Proposition 2.4 and Lemma 2.6, E is injective. Suppose J is also injective and $F \leqslant J \leqslant E$. There is a completely contractive projection $\psi : \mathcal{M}_{d',d} \to J$ (onto). It follows that $\|\psi(\sigma(x))\| \leqslant \|\sigma(x)\|$, and therefore $\|\psi(\sigma(x))\| = \|\sigma(x)\|$ by minimality of σ. Thus $\psi|_E : E \to J \subseteq E$ is isometric and, since the spaces are finite dimensional, $J = E$. □

Lemma 2.8. *Suppose E is a concrete injective envelope of $F \leqslant \mathcal{M}_{d',d}$. If σ is an E-minimal F-map, then σ is the identity.*

Proof. Let $J = \sigma(E)$. Then $F \leqslant J \leqslant E$. From Lemma 2.6 $\sigma : E \to J$ is a completely contractive projection. There is a completely contractive projection $\Phi : \mathcal{M}_{d',d} \to E$ and so $\sigma \circ \Phi$ is a completely contractive projection $\mathcal{M}_{d',d} \to J$. Hence J is injective. By minimality $J = E$. By Lemma 2.6, σ is the identity. □

Lemma 2.9. *Suppose E is a concrete injective envelope of $F \leqslant \mathcal{M}_{d',d}$. If ρ is an F-map on E, then ρ is the identity.*

Proof. Choose a minimal F-map $\sigma \leqslant \rho$. By the previous lemma, σ is the identity on E. Hence, for $y \in E$, we find $\|y\| = \|\sigma(y)\| \leqslant \|\rho(y)\| \leqslant \|y\|$ and therefore ρ is an E-minimal F-map. Another application of the previous lemma shows ρ is the identity. □

Corollary 2.10. *If $E_1, E_2 \leqslant \mathcal{M}_{d',d}$ are both concrete injective envelopes for F, then there exists a completely isometric isomorphism $\phi : E_1 \to E_2$.*

Proof. Since E_1 is injective, apply the definition of injective to $F \leqslant E_2$ and the inclusion mapping of F into E_1 produces a completely contractive $\varphi : E_2 \to E_1$ which is the identity on F. Similarly there exists a completely contractive $\psi : E_1 \to E_2$ which is the identity on F. The composition $\psi \circ \varphi$ is then an F-map on E_1 and is therefore the identity mapping by Lemma 2.9. □

Remark 2.11. The corollary allows one to define **the** concrete injective envelope of F; i.e., it is unique in the category whose objects are $F \leqslant \mathcal{M}_{d',d}$ and whose morphisms are completely contractive maps.

The following corollary plays an essential role in what follows.

Corollary 2.12. *Let E be a concrete injective envelope of $F \leqslant \mathcal{M}_{d',d}$. If $\psi : E \to \mathcal{M}_{\ell',\ell}$ is completely contractive and $\psi|_F$ is completely isometric, then ψ is completely isometric. Moreover, no proper super-space of E has this property.*

Proof. Let $F' = \psi(F)$. Because ψ is a complete isometry, $F' \leqslant \mathcal{M}_{\ell',\ell}$. Let $\phi : F' \to E$ denote the mapping $\psi(f) \mapsto f$ which is completely contractive (actually isometric). Since E is injective, it follows that there is a completely contractive $\tau : \mathcal{M}_{\ell',\ell} \to E$ extending ϕ. The composition $\rho = \tau \circ \psi$ is an F-map on E (completely contractive and the identity on F). Thus ρ is the identity on E by Lemma 2.9. It follows that ψ must be completely isometric.

For the second statement, given $E < J \leqslant \mathcal{M}_{\ell',\ell}$, by injectivity the identity map $E \to E$ extends to a completely contractive map $J \to E \leqslant J$ which is clearly not completely isometric. □

Remark 2.13. This corollary says that studying completely isometric mappings on F is the same as studying completely isometric mappings on an injective envelope E of F, since any completely isometric mapping $F \to \mathcal{M}_{\ell',\ell}$ extends to a completely isometric mapping $E \to \mathcal{M}_{\ell',\ell}$.

2.3. Operator systems and injectivity

Recall the notion of an operator system $\mathcal{S}$ defined in Section 2.1. Because a unital map $\Phi : \mathcal{M}_{m,m} \to \mathcal{S}$ is completely contractive if and only if it is completely positive, an operator system is injective (as an operator space) if and only if it is the range of a completely positive projection (which is automatically unital).

Proposition 2.14 *(Choi–Effros). (Cf. [12, Theorem 15.2].) An injective operator system $\mathcal{S} \leqslant \mathcal{M}_{m,m}$ is completely isometrically isomorphic to a C^*-algebra under the multiplication $a \circ b = \Phi(ab)$, where Φ is a given completely positive projection from $\mathcal{M}_{m,m}$ onto $\mathcal{S}$.*

2.4. Injective envelopes and corners of matrix algebras

A subspace $F \leqslant \mathcal{M}_{d',d}$ naturally embeds in the operator system

$$\mathcal{S}_F = \left\{ \begin{pmatrix} \lambda I_d & b^* \\ a & \eta I_{d'} \end{pmatrix} :\ \lambda, \eta \in \mathbb{C},\ a, b \in F \right\} \leqslant \mathcal{M}_{d+d',d+d'}.$$

Let $\Gamma : \mathcal{M}_{d+d',d+d'} \to \mathcal{M}_{d',d}$ denote the completely contractive projection onto the lower left-hand corner; i.e.,

$$\Gamma\left(\begin{pmatrix} x_{11} & x_{12} \\ a & x_{22} \end{pmatrix}\right) = a. \tag{2.3}$$

Proposition 2.15. *Suppose $E \leqslant \mathcal{M}_{d,d'}$ is an injective operator space. There exist subspaces $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{M}_{d,d}$ and $\mathcal{M}_{d',d'}$ respectively such that the concrete injective envelope $\mathcal{E}$ of S_E has the form*

$$\mathcal{E} = \left\{ \begin{pmatrix} a & f^* \\ e & b \end{pmatrix} :\ a \in \mathcal{A},\ b \in \mathcal{B},\ e, f \in E \right\}.$$

Proof. Let $n = d + d'$. Let $\mathcal{E} \leqslant \mathcal{M}_{n,n}$ be an injective envelope of S_E. There is a completely contractive unital projection $\Psi : \mathcal{M}_{n,n} \to \mathcal{E}$. By Stinespring's Theorem [12, Theorem 4.1], there exists a (finite dimensional) Hilbert space $\mathcal{H}$, a representation $\pi : \mathcal{M}_{n,n} \to \mathcal{B}(\mathcal{H})$, and an isometry $V : \mathbb{C}^n \to \mathcal{H}$ such that

$$\Psi(x) = V^* \pi(x) V.$$

Consider the $n \times n$ matrices

$$p_1 = \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix}, \qquad p_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_{d'} \end{pmatrix}.$$

It follows that $P_j = \pi(p_j)$ are projections with $P_1 + P_2 = I$. Hence we can decompose $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with $\mathcal{H}_j$ the range of P_j. With respect to this decomposition, and the natural decomposition of $\mathbb{C}^n = \mathbb{C}^d \oplus \mathbb{C}^{d'}$, express V as $V = (V_{j,k})$. Since $p_1 \in S_E \leqslant \mathcal{E}$ we have

$$\Psi(p_1) = \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} = V^* \begin{pmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & 0 \end{pmatrix} V.$$

It follows that $V_{12}^* V_{12} = 0$. Hence $V_{12} = 0$. A similar argument shows $V_{21} = 0$.

Let $V_j = P_j V$. At this point we have

$$\Psi(x) = (V_1 + V_2)^* \pi(x)(V_1 + V_2).$$

Let $\mathcal{A}_{j,k}$ denote the range of the mapping $x \mapsto V_j^* \pi(p_j x p_k) V_k$. Since $V_j^* \pi(p_j x p_k) V_k = \Psi(p_j x p_k) \in \mathcal{E}$, it follows that $\mathcal{A}_{j,k} \subset \mathcal{E}$. On the other hand, any x can be written as $\sum p_j x p_k$ and $V_\ell^* \pi(p_j x p_k) V_m = 0$ unless $(\ell, m) = (j, k)$. Hence $\mathcal{E}$ has the desired representation.

Now, with Γ as in (2.3), the mapping $\Delta = \Gamma|_{\mathcal{E}} : \mathcal{E} \to \mathcal{A}_{2,1}$ is a completely contractive projection onto $\mathcal{A}_{2,1}$ and thus $\mathcal{A}_{2,1}$ is an injective operator space. Since $S_E \subset \mathcal{E}$, it follows that $E \subset \mathcal{A}_{2,1}$. To see that $E = \mathcal{A}_{2,1}$, note that there is a completely contractive projection $\varphi : \mathcal{A}_{2,1} \to E$. Hence the mapping $\Phi : S_{\mathcal{A}_{2,1}} \to S_{\mathcal{A}_{2,1}}$ defined by

$$\begin{pmatrix} \lambda & f^* \\ e & \mu \end{pmatrix} \mapsto \begin{pmatrix} \lambda & \varphi(f)^* \\ \varphi(e) & \mu \end{pmatrix}$$

is unital and completely positive. Thus, because $\mathcal{E} \supseteq S_{\mathcal{A}_{2,1}}$ is a concrete injective envelope of S_E, the map Φ extends uniquely as a completely contractive map on $S_{\mathcal{A}_{2,1}}$. Since Φ is also the identity on S_E it must be the identity on $\mathcal{A}_{2,1}$. The conclusion $\mathcal{A}_{2,1} = E$ follows. □

3. Completely isometric maps $\bigoplus \mathcal{M}_{d'_j,d_j} \to \mathcal{M}_{n',n}$

In this section we classify completely isometric maps Ψ from a direct sum of matrix spaces into matrices, which is a special case of our main result appearing in Section 5 below. More precisely, we consider a completely isometric $\Psi : \bigoplus_j \mathcal{M}_{d'_j,d_j} \to \mathcal{M}_{n',n}$ for $d_j, d'_j, n, n' \in \mathbb{N}$.

We note that the results here, and in the next section, are very much in the spirit of, and flow from the same considerations, and can be made to follow from the results, as in [3–5] when specialized to our concrete setting.

Theorem 3.1. *Let $d = \sum d_j$, $d' = \sum d'_j$. If $\Psi : \bigoplus_j \mathcal{M}_{d'_j,d_j} \to \mathcal{M}_{n',n}$ is completely isometric, then there exists a completely contractive mapping $\psi : \bigoplus_j \mathcal{M}_{d'_j,d_j} \to \mathcal{M}_{n'-d',n-d}$ and unitaries $U : \mathbb{C}^{n'} \to \mathbb{C}^{n'}$ and $V : \mathbb{C}^n \to \mathbb{C}^n$ such that*

$$\Psi(x) = U \begin{pmatrix} x & 0 \\ 0 & \psi(x) \end{pmatrix} V^*. \tag{3.1}$$

Two operator systems $\mathcal{S}, \mathcal{S}' \leqslant \mathcal{M}_{m,m}$ are equal, up to unitary equivalence, if there exists an $m \times m$ unitary matrix U such that

$$\mathcal{S} = \{UTU^*:\ T \in \mathcal{S}'\}.$$

Corollary 3.2. *If $\mathcal{S} \leqslant \mathcal{M}_{m,m}$ is an injective operator system, then there exist integers n_j and a completely contractive unital mapping $\Phi_0 : \bigoplus_j \mathcal{M}_{n_j,n_j} \to \mathcal{M}_{m-\sum n_j, m-\sum n_j}$ so that, up to unitary equivalence,*

$$\mathcal{S} = \left\{ \begin{pmatrix} a & 0 \\ 0 & \Phi_0(a) \end{pmatrix} :\ a \in \bigoplus_j \mathcal{M}_{n_j,n_j} \right\}. \tag{3.2}$$

Further, if $\tau : \mathcal{S} \to \mathcal{M}_{p,p}$ is completely isometric and unital, then the range of τ is also an injective operator system and τ is, up to unitary equivalence, of the form,

$$\tau\left(\begin{pmatrix} a & 0 \\ 0 & \Phi_0(a) \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & \tau_0(a) \end{pmatrix}, \tag{3.3}$$

for some completely contractive unital τ_0.

Proof. Since $\mathcal{S}$, being injective, is completely isometrically isomorphic to a finite dimensional C^*-algebra, there exist (finitely many) integers n_j and a completely isometric unital mapping $\Psi : \bigoplus_j \mathcal{M}_{n_j,n_j} \to \mathcal{S} \leqslant \mathcal{M}_{m,m}$. By Theorem 3.1, there are unitaries U and V and a completely contractive mapping ψ such that

$$\Psi(x) = U \begin{pmatrix} x & 0 \\ 0 & \psi(x) \end{pmatrix} V^*.$$

Since Ψ is unital, we may assume $V = U$. Hence, up to unitary equivalence, $\mathcal{S}$ being the image of Ψ, is of the desired form (3.2).

If τ is completely isometric and $\mathcal{S}$ is injective, then $\tau(\mathcal{S})$ is injective by Lemma 2.2.

To prove the last part, observe that the mapping $\tilde{\tau} : \bigoplus_j \mathcal{M}_{n_j,n_j} \to \mathcal{M}_{p,p}$ defined by

$$\tilde{\tau}(a) = \tau(\Psi(a))$$

is completely isometric. Hence, by the first part of the corollary there exists a unitary W and completely contractive τ_0 such that

$$\tau(\Psi(a)) = W \begin{pmatrix} a & 0 \\ 0 & \tau_0(a) \end{pmatrix} W^*. \qquad \square$$

Proof of Theorem 3.1. Let Ψ_j denote the restriction of Ψ to the j-th coordinate. Thus, $\Psi_j : \mathcal{M}_{d'_j,d_j} \to \mathcal{M}_{n',n}$ is completely isometric. From our earlier results (cf. [9, Theorem 1.3]), there exist unitaries $U_j : \mathbb{C}^{n'} \to \mathbb{C}^{n'}$ and $V_j : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\tau_j(x) := U_j^* \Psi_j(x) V_j = \begin{pmatrix} x & 0 \\ 0 & \psi_j(x) \end{pmatrix} \tag{3.4}$$

for some completely contractive $\psi_j : \mathcal{M}_{d'_j,d_j} \to \mathcal{M}_{n'-d'_j,n-d_j}$. These Ψ_j need to fit together in a way which keeps $\Psi(x) = \sum \Psi_j(x_j)$ completely isometric.

We now decompose each of $\mathbb{C}^n$ and $\mathbb{C}^{n'}$ compatibly with the Ψ_j and τ_j as follows using the notations and orthogonal decomposition of Eq. (3.4). In particular, $\tau_j(x)$ decomposes as a direct sum mapping $\mathbb{C}^{d_j} \oplus \mathbb{C}^{n-d_j} \to \mathbb{C}^{d'_j} \oplus \mathbb{C}^{n'-d'_j}$. We let $\mathcal{I}_j$ and $\mathcal{I}'_j$ denote the first summands, $\mathbb{C}^{d_j}$ and $\mathbb{C}^{d'_j}$, respectively. We further decompose the second summands, $\mathbb{C}^{n-d_j}$ and $\mathbb{C}^{n'-d'_j}$ as follows. Let $\mathcal{Z}'_j$ denote the subspace $\{z \in \mathbb{C}^{n'-d'_j}:\ z^*\psi_j(x) = 0 \text{ for all } x\}$. Let $\mathcal{C}'_j$ denote the orthogonal complement of $\mathcal{Z}'_j$ in $\mathbb{C}^{n'-d'_j}$. Similarly, let $\mathcal{Z}_j$ denote $\{z \in \mathbb{C}^{n-d_j}:\ \psi_j(x)z = 0 \text{ for all } x\}$ and let $\mathcal{C}_j$ denote the complement of $\mathcal{Z}_j$ in $\mathbb{C}^{n-d_j}$. Thus, we have $\mathbb{C}^n = \mathcal{I}_j \oplus \mathcal{C}_j \oplus \mathcal{Z}_j$ and likewise for the primes.

The claim is:

(a) $V_j\mathcal{I}_j \perp V_k\mathcal{C}_k$;
(b) $V_j\mathcal{I}_j \perp V_k\mathcal{I}_k$ for $j \neq k$;
(c) $U_j\mathcal{I}'_j \perp U_k\mathcal{C}'_k$;
(d) $U_j\mathcal{I}'_j \perp U_k\mathcal{I}'_k$ for $j \neq k$.

Clearly, (a) and (c) hold for $j = k$. For the rest of the proof we assume, without loss of generality, that $j = 1$ and $k = 2$.

(a) $\wedge$ (b): Let $v \in \mathcal{I}_1$ and $u \in \mathcal{I}'_1$ be given unit vectors. Let $x_1 = u_1 v^*$ and let $x_2 \in \mathcal{M}_{d'_2,d_2}$ of norm one be given. Both x_1 and x_2 have norm one and hence $x = x_1 \oplus \exp(it)x_2 \oplus 0 \in \bigoplus_j \mathcal{M}_{d'_j,d_j}$ also has norm one. Thus,

$$\begin{aligned}1 &\geqslant \|\Psi(x)V_1 v\| = \|\Psi_1(x_1)V_1 v + \exp(it)\Psi_2(x_2)V_1 v\| \\ &= \left\| U_1 \begin{pmatrix} x_1 & 0 \\ 0 & \psi_1(x_1)\end{pmatrix}\begin{pmatrix} v \\ 0\end{pmatrix} + \exp(it)U_2\begin{pmatrix} x_2 & 0 \\ 0 & \psi_2(x_2)\end{pmatrix} V_2^* V_1 v\right\| \\ &= \left\| U_1 \begin{pmatrix} u_1 \\ 0\end{pmatrix} + \exp(it)U_2\begin{pmatrix} x_2 & 0 \\ 0 & \psi_2(x_2)\end{pmatrix} V_2^* V_1 v\right\|.\end{aligned}$$

Since

$$1 = \left\| U_1 \begin{pmatrix} u_1 \\ 0\end{pmatrix}\right\|,$$

it follows that

$$\begin{pmatrix} x_2 & 0 \\ 0 & \psi_2(x_2)\end{pmatrix} V_2^* V_1 v = 0$$

for all x_2. $V_2^* V_1 v \in \mathcal{Z}_2$. Equivalently, $V_2^* V_1 v \in (\mathcal{I}_2 \oplus \mathcal{C}_2)^\perp$ which is equivalent to the claim.

(c) $\wedge$ (d): This argument is similar to that above. We claim that $U_1\mathcal{I}'_1$ is orthogonal to $U_2(\mathcal{I}'_2 \oplus \mathcal{C}'_2)$. Note, if $0 \neq \gamma \in \mathcal{I}'_2 \oplus \mathcal{C}'_2$, then there is an x_2 so that

$$\begin{pmatrix} x_2^* & 0 \\ 0 & \psi_2(x_2)^*\end{pmatrix}\gamma \neq 0.$$

Given a unit vector $v \in \mathcal{I}'_1$, let γ denote the projection of $U_2^* U_1 v$ onto $\mathcal{I}'_2 \oplus \mathcal{C}'_2$. Choose a unit vector $u_1 \in \mathbb{C}^{d_1}$, let $x_1 = v u_1^*$, and let $x_2 \in \mathcal{M}_{d'_2,d_2}$ of norm one be given. Let $x = x_1 \oplus \exp(it)x_2 \oplus 0$ and estimate,

$$\begin{aligned}1 &\geqslant \|\Psi(x)^* U_1 v\| = \|\Psi_1(x_1)^* v + \exp(it)\Psi_2(x_2)^* U_1 v\| \\ &= \left\| V_1 \begin{pmatrix} x_1^* & 0 \\ 0 & \psi_1(x_1)^*\end{pmatrix}\begin{pmatrix} v \\ 0\end{pmatrix} + \exp(it)V_2\begin{pmatrix} x_2^* & 0 \\ 0 & \psi_2(x_2)^*\end{pmatrix} U_2^* U_1 v\right\| \\ &= \left\| V_1 \begin{pmatrix} u_1 \\ 0\end{pmatrix} + \exp(it)V_2\begin{pmatrix} x_2^* & 0 \\ 0 & \psi_2(x_2)^*\end{pmatrix}\gamma\right\|.\end{aligned}$$

It follows that $\Psi_2(x_2)^*\gamma = 0$ over all choices of x_2. Hence $\gamma \in \mathcal{Z}'_2$; but also $\gamma \in \mathcal{I}'_2 \oplus \mathcal{C}'_2$. Thus, $\gamma = 0$. Since this is true for all v, the conclusion follows and the claim is proved.

Let $\mathcal{M} = \bigoplus V_j\mathcal{I}_j$. Note that $\mathcal{M}$ has dimension $d = \sum d_j$ and $\mathcal{M}^\perp$ has dimension $n - d$. Before proceeding, we make the following observations. If $j \neq k$ and $\gamma_j \in \mathcal{I}_j$, then $V_k^* V_j \gamma_j = 0$. Further, since $V_k\mathcal{I}_k \subset \mathcal{M}$, if $\Gamma \in \mathcal{M}^\perp$, then there is a $\delta_k \in \mathcal{I}_k^\perp$ such that $\gamma = V_k\delta_k$. Thus, for $x_k \in \mathcal{M}_{d'_k,d_k}$,

$$U_k\tau_k(x_k)V_k^*\Gamma = U_k\psi_k(x_k)\delta_k \in \left(\mathcal{I}'_k\right)^\perp.$$

On the other hand, $\psi_k(x_k)\delta_k$ is also orthogonal to $(\mathcal{Z}'_k)^\perp$ (by definition). We conclude that $U_k\tau_k(x_k)V_k^*\Gamma$ is in $U_k\mathcal{C}'_k$ and thus

$$U_k\tau_k(x_k)V_k^*\Gamma \in \left(\mathcal{M}'\right)^\perp. \tag{3.5}$$

Define $V^* : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\begin{aligned}&V^* : \mathcal{M} \oplus \mathcal{M}^\perp \to \bigoplus \mathbb{C}^{d_j} \oplus \mathbb{C}^{n-d}, \\ &\begin{pmatrix} \bigoplus V_j\gamma_j \\ V_0\gamma\end{pmatrix} \mapsto \begin{pmatrix} \bigoplus \gamma_j \\ V\gamma\end{pmatrix},\end{aligned}$$

where $V_0^* : \mathcal{M}^\perp \to \mathbb{C}^{n-d}$ is (any) unitary.

Similarly, define $U : \mathbb{C}^{n'} \to \mathbb{C}^{n'}$ by

$$U : \bigoplus \mathbb{C}^{d'_j} \oplus \mathbb{C}^{n'-d'} \to \mathcal{M}' \oplus (\mathcal{M}')^{\perp},$$
$$\begin{pmatrix} \bigoplus \delta_j \\ \delta \end{pmatrix} \mapsto \begin{pmatrix} \bigoplus U_j \delta_j \\ U_0 \delta \end{pmatrix},$$

where U_0 is (any) unitary from $\mathbb{C}^{n'-d'}$ to $(\mathcal{M}')^{\perp}$ and $\mathcal{M}' = \bigoplus U_j \mathcal{I}'_j$.

We record the following observation, which follows from (3.5). Given $x_k \in \mathcal{M}_{d'_k,d_k}$, $\bigoplus \gamma_j \in \bigoplus \mathbb{C}^{d_j}$ and $\gamma \in \mathbb{C}^{n-d}$,

$$U_k \tau_k(x_k) V_k^* \begin{pmatrix} \bigoplus V_j \gamma_j \\ V_0 \gamma \end{pmatrix} = U_k \tau_k(x_k) V_k^* \left(\sum_j V_j \gamma_j + V_0 \gamma \right) = U_k x_k \gamma_k \oplus \psi_k(x_k) V_k^* V_0 \gamma,$$

where the orthogonality respects the decomposition $\mathcal{M}' \oplus (\mathcal{M}')^{\perp}$.

To finish the proof, let $\bigoplus x_j \in \bigoplus \mathcal{M}_{d'_j,d_j}$ and $\bigoplus \gamma_j \in \bigoplus \mathbb{C}^{d_j}$ and $\gamma \in \mathbb{C}^{n-d}$ be given. Then,

$$\begin{aligned} U^* \Psi\Big(\bigoplus x_k\Big) V V^* \begin{pmatrix} \bigoplus V_j \gamma_j \\ V_0 \gamma \end{pmatrix} &= U^* \sum_k \Psi_k(x_k) \begin{pmatrix} \bigoplus V_j \gamma_j \\ V_0 \gamma \end{pmatrix} = U^* \sum_k U_k \tau_k(x_k) V_k^* \Big(\sum_j V_j \gamma_j + V_0 \gamma \Big) \\ &= U^* \sum U_k \big(x_k \gamma_k + \psi_k(x_k) V_k^* V_0 \gamma \big) = \begin{pmatrix} \bigoplus x_k \gamma_k \\ U_0^* \sum U_k \psi_k(x_k) V_k^* V_0 \gamma \end{pmatrix} \\ &= \begin{pmatrix} x & 0 \\ 0 & \psi(x) \end{pmatrix} \begin{pmatrix} \bigoplus \gamma_j \\ \gamma \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & \psi(x) \end{pmatrix} V^* \begin{pmatrix} \bigoplus V_j \gamma_j \\ V_0 \gamma \end{pmatrix} \end{aligned}$$

for the completely contractive $\psi(x) = U_0^* \sum U_k \psi_k(x_k) V_k^* V_0$. □

4. The injective envelope of $\mathcal{R}_L$

The following theorem exposes the structure of a concrete injective envelope of $\mathcal{R}_L$, for a nondegenerate homogeneous linear pencil L. It will be applied, along with Theorem 3.1, to prove Theorem 1.5 and then Theorem 1.3 in Section 5 below.

Theorem 4.1. *Let $L : \mathbb{C}^g \to \mathbb{C}^{d' \times d}$ be a nondegenerate homogeneous linear pencil. Then there is a concrete injective envelope $E \leqslant \mathcal{M}_{d',d}$ of $\mathcal{R}_L$ and unitaries V, W such that*

$$E = \left\{ W^* \begin{pmatrix} x & 0 \\ 0 & \phi(x) \end{pmatrix} V : x \in \bigoplus_1^N \mathcal{M}_{d'_j,d_j} \right\}$$

for some choice of integers (d_j, d'_j) and a completely contractive mapping $\phi : \bigoplus_1^N \mathcal{M}_{d'_j,d_j} \to \mathcal{M}_{s',s}$. (Here $s, s' \in \mathbb{N}_0$ are such that $s + \sum_1^N d_j = d$ and $s' + \sum_1^N d'_j = d'$.)

Proof. Let E denote a concrete injective envelope of $\mathcal{R}_L$ and let $\mathcal{E}$ denote a concrete injective envelope of the operator system S_E. From Proposition 2.15,

$$\mathcal{E} = \left\{ \begin{pmatrix} \zeta & b^* \\ a & \xi \end{pmatrix} : \zeta \in \mathcal{A},\ \xi \in \mathcal{B},\ a, b \in E \right\},$$

for some unital subspaces $\mathcal{A} \subseteq \mathbb{C}^{d \times d}$ and $\mathcal{B} \subseteq \mathbb{C}^{d' \times d'}$.

On the other hand, from Corollary 3.2, the injective operator system $\mathcal{E}$ has, up to unitary equivalence, the form,

$$\left\{ \begin{pmatrix} x & 0 \\ 0 & \Phi(x) \end{pmatrix} : x \in \bigoplus \mathcal{M}_{n_j,n_j} \right\},$$

where $\Phi : \bigoplus \mathcal{M}_{n_j,n_j} \to \mathcal{M}_{m,m}$ is a unital completely contractive mapping. With respect to $\mathcal{M}_{n_j,n_j}$ acting on $\bigoplus \mathbb{C}^{n_j}$, let $\mathcal{Q}_j$ denote the j-th coordinate $\mathbb{C}^{n_j}$.

Thus there is a unitary (block) matrix $U = (u_{ij})$ such that

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} \zeta & b^* \\ a & \xi \end{pmatrix} \begin{pmatrix} u_{11}^* & u_{21}^* \\ u_{12}^* & u_{22}^* \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & \Phi(x) \end{pmatrix}. \tag{4.1}$$

In particular,

$$\begin{aligned}
&u_{11}\zeta u_{11}^* + u_{12}\xi u_{12}^* + u_{12}au_{11}^* + u_{11}b^*u_{12}^* = x \in \bigoplus \mathcal{M}_{n_j,n_j},\\
&u_{11}\zeta u_{21}^* + u_{12}\xi u_{22}^* + u_{12}au_{21}^* + u_{11}b^*u_{22}^* = 0,\\
&u_{21}\zeta u_{21}^* + u_{22}\xi u_{22}^* + u_{22}au_{21}^* + u_{21}b^*u_{22}^* = \Phi(x).
\end{aligned} \tag{4.2}$$

Choosing $\zeta = 1$, $\xi = 0$ and $a = b = 0$ in Eq. (4.2) gives,

$$u_{11}u_{21}^* = 0 \quad \text{and} \quad u_{21}u_{11}^* = 0. \tag{4.3}$$

Further since in this case the right-hand side of Eq. (4.1) is a projection, it also follows that $u_{11}u_{11}^*$ and $u_{21}u_{21}^*$ are both projections. Equivalently, u_{11} and u_{21} are partial isometries.

Choosing $\xi = 1$, $\zeta = 0$ and $a = b = 0$ in Eq. (4.2) gives,

$$u_{12}u_{22}^* = 0 \quad \text{and} \quad u_{22}u_{12}^* = 0. \tag{4.4}$$

Moreover, $u_{12}u_{12}^*$ and $u_{22}u_{22}^*$ are projections and u_{12} and u_{22} are partial isometries.

Next, choosing $\zeta = 1$, $\xi = 1$ and $a = b = 0$ (or using that U is unitary) it follows that

$$u_{11}u_{11}^* + u_{12}u_{12}^* = I \quad \text{and} \quad u_{21}u_{21}^* + u_{22}u_{22}^* = I. \tag{4.5}$$

Using the fact that all the entries of U are partial isometries, it now follows that $u_{11}u_{11}^*$ and $u_{12}u_{12}^*$ are orthogonal projections and $u_{11}^*u_{12} = 0$.

Let $\mathcal{L} \subset \bigoplus \mathbb{C}^{n_j}$ denote the range of u_{11} so that, by the above relations, the range of u_{12} is $\mathcal{L}^\perp = (\bigoplus \mathbb{C}^{n_j}) \ominus \mathcal{L}$. Similarly, let $\mathcal{K} \subset \mathbb{C}^m$ denote the range of u_{21} so that, in view of the above relations, the range of u_{22} is $\mathcal{K}^\perp = \mathbb{C}^m \ominus \mathcal{K}$.

With these notations, we have

$$\begin{aligned}
W &= \begin{pmatrix} P_{\mathcal{L}^\perp}u_{12} \\ P_{\mathcal{K}^\perp}u_{22} \end{pmatrix} : \mathbb{C}^{d'} \to \mathcal{L}^\perp \oplus \mathcal{K}^\perp,\\
V &= \begin{pmatrix} P_{\mathcal{L}}u_{11} \\ P_{\mathcal{K}}u_{21} \end{pmatrix} : \mathbb{C}^{d} \to \mathcal{L} \oplus \mathcal{K}
\end{aligned}$$

are unitaries as is verified by computing V^*V and W^*W and noting that each is the identity (on the appropriate space).

We now turn to proving that W and V satisfy the conclusion of the theorem. For future reference, observe,

$$WaV^* = \begin{pmatrix} P_{\mathcal{L}^\perp}u_{12}au_{11}^*P_{\mathcal{L}} & P_{\mathcal{L}^\perp}u_{12}au_{21}^*P_{\mathcal{K}} \\ P_{\mathcal{K}^\perp}u_{22}au_{11}^*P_{\mathcal{L}} & P_{\mathcal{K}^\perp}u_{22}au_{21}^*P_{\mathcal{K}} \end{pmatrix}.$$

In view of the second equality in Eq. (4.2) (choose ζ, ξ and b equal 0 to deduce the $(1,2)$ term is 0 and let ζ, ξ, a be 0 to deduce the $(2,1)$ term is 0) the off-diagonal terms above are 0. From the first and third equalities in Eq. (4.2), there is an $x \in \bigoplus \mathcal{M}_{n_j,n_j}$ such that $u_{12}au_{11}^* = x$ and $u_{22}au_{21}^* = \Phi(x)$ (again choose ζ, ξ and b equal 0). Thus, for each $a \in E$ there is an $x \in \bigoplus \mathcal{M}_{n_j,n_j}$ such that

$$WaV^* = \begin{pmatrix} P_{\mathcal{L}^\perp}xP_{\mathcal{L}} & 0 \\ 0 & P_{\mathcal{K}^\perp}\Phi(x)P_{\mathcal{K}} \end{pmatrix}. \tag{4.6}$$

Thus WaV^* has a certain amount of block diagonal structure.

We now turn to proving that the upper left-hand corner of WaV^* has the additional block diagonal structure claimed in the theorem; i.e., that $P_{\mathcal{L}^\perp}xP_{\mathcal{L}} \in \bigoplus \mathcal{M}_{d'_j,d_j}$ for some choice of d_j and d'_j. Observe that $P_{\mathcal{L}} = u_{11}u_{11}^*$, the projection onto $\mathcal{L}$ is contained in $\bigoplus \mathcal{M}_{n_j,n_j}$ (choose $\zeta = 1$ and a, b, ξ equal 0 in (4.2)). Hence, with respect to this decomposition,

$$P_{\mathcal{L}} = \begin{pmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & P_N \end{pmatrix}.$$

It follows that each P_j is a projection which commutes with the projection Q_j onto $\mathcal{Q}_j$ equal to the $\mathbb{C}^{n_j}$ summand of $\bigoplus \mathbb{C}^{n_k}$. Letting $\mathcal{L}_j$ denote the range of P_j it follows that

$$\mathcal{L} = \bigoplus \mathcal{L}_j.$$

Similarly, $P_{\mathcal{L}^\perp} = u_{12}u_{12}^*$ is in $\bigoplus \mathcal{M}_{n_j,n_j}$ and thus commutes with each Q_j. Consequently,

$$\mathcal{L}^\perp = \bigoplus \mathcal{L}_j^\perp,$$

where $\mathcal{L}_j^\perp$ is the orthogonal complement of $\mathcal{L}_j$ in $\mathbb{C}^{n_j} = \mathcal{Q}_j$.

Let d_j and d'_j denote the dimensions of $\mathcal{L}_j$ and $\mathcal{L}_j^\perp$ respectively (thus $d_j + d'_j = n_j$). Identifying $\mathcal{L}_j$ with $\mathbb{C}^{d_j}$ and $\mathcal{L}_j^\perp$ with $\mathbb{C}^{d'_j}$, it follows, for $x \in \bigoplus \mathcal{M}_{n_j,n_j}$, that

$$P_{\mathcal{L}^\perp} x P_{\mathcal{L}} \in \bigoplus \mathcal{M}_{d'_j,d_j}. \tag{4.7}$$

As explained above (4.6), for every $a \in E$ we have

$$a = W^* \begin{pmatrix} P_{\mathcal{L}^\perp} x P_{\mathcal{L}} & 0 \\ 0 & P_{\mathcal{K}^\perp}\Phi(x)P_{\mathcal{K}} \end{pmatrix} V = W^* \begin{pmatrix} x & 0 \\ 0 & P_{\mathcal{K}^\perp}\Phi(x)P_{\mathcal{K}} \end{pmatrix} V \in W^*\Big(\bigoplus \mathcal{M}_{d'_j,d_j} \oplus \mathcal{M}_{s',s}\Big)V. \tag{4.8}$$

Here x denotes $u_{12}au_{11}^*$ and $s = \dim \mathcal{K}$, $s' = \dim \mathcal{K}'$. The second equality uses that $\mathcal{L}$ is the range of u_{11} and $\mathcal{L}^\perp$ is the range of u_{12}. In this way, since $\bigoplus \mathcal{M}_{n_j,n_j}$ is the set of all linear maps $\mathcal{L} \oplus \mathcal{L}^\perp \to \mathcal{L} \oplus \mathcal{L}^\perp$, we identify $x \in \bigoplus \mathcal{M}_{d'_j,d_j}$ with $\left(\begin{smallmatrix} 0 & 0 \\ x & 0 \end{smallmatrix}\right) : \mathcal{L} \oplus \mathcal{L}^\perp \to \mathcal{L} \oplus \mathcal{L}^\perp$. Defining

$$\phi : \bigoplus \mathcal{M}_{d'_j,d_j} \to \mathcal{M}_{s',s}, \quad x \mapsto P_{\mathcal{K}^\perp}\Phi(x)P_{\mathcal{K}},$$

we thus obtain $E \subset F$ with F defined by

$$F = \left\{ W^* \begin{pmatrix} x & 0 \\ 0 & \phi(x) \end{pmatrix} V : x \in \bigoplus \mathcal{M}_{d'_j,d_j} \right\}. \tag{4.9}$$

Note ϕ is completely contractive.

To prove the reverse inclusion, $E \supseteq F$, pick any f in F. This f corresponds to an $x \in \bigoplus \mathcal{M}_{d'_j,d_j}$ in the definition (4.9) of F. By Eq. (4.1), there exist ζ, ξ, a, b such that

$$U \begin{pmatrix} \zeta & b^* \\ a & \xi \end{pmatrix} U^* = \begin{pmatrix} x & 0 \\ 0 & \Phi(x) \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ u_{12}^* & u_{22}^* \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & \Phi(x) \end{pmatrix} \begin{pmatrix} u_{11} & 0 \\ u_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ u_{12}^* x u_{11} + u_{22}^*\Phi(x)u_{21} & 0 \end{pmatrix}.$$

We conclude

$$a = u_{12}^* x u_{11} + u_{22}^* \Phi(x) u_{21}.$$

Multiplying this identity on the left by u_{12} and the right by u_{11}^* and using various orthogonality relations gives

$$u_{12} a u_{11}^* = P_{\mathcal{L}^\perp} x P_{\mathcal{L}} = x.$$

There is a y such that

$$\begin{aligned} \begin{pmatrix} y & 0 \\ 0 & \Phi(y) \end{pmatrix} &= U \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} U^* \\ &= WaV^* \\ &= \begin{pmatrix} u_{12}au_{11}^* & \star \\ \star & u_{22}au_{21}^* \end{pmatrix}. \end{aligned}$$

It follows that $y = x$ and thus $\Phi(x) = u_{22}au_{21}^* = P_{\mathcal{K}^\perp}\Phi(x)P_{\mathcal{K}} = \phi(x)$. In conclusion,

$$f = W^* \begin{pmatrix} x & 0 \\ 0 & \phi(x) \end{pmatrix} V = a \in E,$$

for an $x \in \bigoplus \mathcal{M}_{d'_j,d_j}$. □

We point out that the L in Theorem 4.1 plays no role beyond defining a subspace Z of rectangular matrices. The theorem is a characterization of injective envelopes of a space Z of rectangular matrices, or equivalently of complete isometries from a direct sum of rectangular matrices into Z.

5. The main results

In this section we prove our main result on completely isometric maps $\mathcal{R}_L \to \mathcal{M}_{\ell',\ell}$, Theorem 1.5, and then Theorem 1.3.

5.1. Completely isometric maps $\mathcal{R}_L \to \mathcal{M}_{\ell',\ell}$

Remark 5.1. Given a completely isometric $\Phi : \mathcal{R}_L \to \mathcal{M}_{\ell',\ell}$, it is tempting to guess that, up to unitaries,

$$\Phi\big(L(x)\big) = \begin{pmatrix} L(x) & 0 \\ 0 & \phi(L(x)) \end{pmatrix},$$

for some completely contractive ϕ. However, if $L(x) = x \oplus \frac{1}{2}x$, then the mapping $\Phi(L(x)) = x$ is completely isometric, but not of the form above. This prompts us to decompose $L = \tilde{L} \oplus J$ in the theorem below.

We now rephrase Theorem 1.5 (together with Lemma 1.2). Then we set about to prove it. Theorem 1.5 will follow as soon as the equivalence between "minimal pencil" and "minimal dimensional defining pencil for a ball" is established in Corollary 5.3.

Theorem 5.2. *Given a nondegenerate homogeneous linear pencil $L : \mathbb{C}^g \to \mathbb{C}^{d' \times d}$, there are homogeneous linear pencils $\tilde{L}$ and J and unitaries Q and G such that*

(1) $L = Q(\tilde{L} \oplus J)G^*$;
(2) $\|L(X)\| = \|\tilde{L}(X)\|$ *for all* $n \in \mathbb{N}$ *and* $X \in (\mathbb{C}^{n\times n})^g$, *i.e.,* $\tilde{L}$ *is equivalent to* L;
(3) *there are* $d_j, d'_j \in \mathbb{N}$ *such that* $\tilde{L} : \mathbb{C}^g \to \bigoplus_1^N \mathbb{C}^{d'_j \times d_j}$ *and the injective envelope of* $\mathcal{R}_{\tilde{L}}$ *is* $\bigoplus_1^N \mathcal{M}_{d'_j,d_j}$.

If $\Phi : \mathcal{R}_L \to \mathcal{M}_{\ell',\ell}$ *is completely isometric, then*

$$\Phi\big(L(x)\big) = U \begin{pmatrix} \tilde{L}(x) & 0 \\ 0 & \phi(\tilde{L}(x)) \end{pmatrix} V^*,$$

for some completely contractive $\phi : \bigoplus \mathcal{M}_{d'_j,d_j} \to \mathcal{M}_{\ell' - \sum g'_j, \ell - \sum d_j}$ *and unitaries* U, V.

Proof. Let $E \leqslant \mathcal{M}_{d',d}$ denote a concrete injective envelope of $\mathcal{R}_L$. By Theorem 4.1 there exist unitaries Q, G such that

$$E = \left\{ Q \begin{pmatrix} a & 0 \\ 0 & \psi(a) \end{pmatrix} G^* \colon a \in \bigoplus_1^N \mathcal{M}_{d'_j,d_j} \right\}$$

for some choice of integers (d_j, d'_j) and a completely contractive mapping $\psi : \bigoplus_1^N \mathcal{M}_{d'_j,d_j} \to \mathcal{M}_{s',s}$, where $s, s' \in \mathbb{N}_0$ with $s + \sum_1^N d_j = d$ and $s' + \sum_1^N d'_j = d'$.

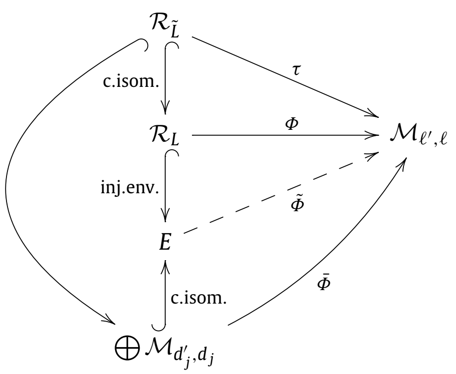
The mapping $\mathcal{R}_L \to \bigoplus \mathcal{M}_{d'_j,d_j}$ defined on $L(x) \in \mathcal{R}_L$ by

$$L(x) = Q \begin{pmatrix} a & 0 \\ 0 & \psi(a) \end{pmatrix} G^* \mapsto a$$

is linear and completely isometric, and so $x \mapsto L(x) \mapsto a$ defines a linear map $\tilde{L} : \mathbb{C}^g \to \bigoplus \mathcal{M}_{d'_j,d_j}$ satisfying $\|L(X)\| = \|\tilde{L}(X)\|$ for all X. Similarly, $J : \mathbb{C}^g \to \bigoplus \mathcal{M}_{d'_j,d_j}$ is constructed by mapping $x \mapsto L(x) \mapsto \psi(a)$. By construction,

$$L(x) = Q \begin{pmatrix} \tilde{L}(x) & 0 \\ 0 & J(x) \end{pmatrix} G^*.$$

Now if $\Phi : \mathcal{R}_L \to \mathcal{M}_{\ell',\ell}$ is completely isometric, then, $\tau : \mathcal{R}_{\tilde{L}} \to \mathcal{M}_{\ell',\ell}$ given by $\tau(\tilde{L}(x)) = \Phi(L(x))$ is well defined and completely isometric. Consider the following commutative diagram:



Since E is the injective envelope of $\mathcal{R}_L$, Φ extends to a completely isometric $\tilde{\Phi}: E \to \mathcal{M}_{\ell',\ell}$ (cf. Corollary 2.12 and Remark 2.13). Hence $\bar{\Phi}: \bigoplus \mathcal{M}_{d'_j,d_j} \to \mathcal{M}_{\ell',\ell}$, being the composite of two completely isometric maps, is completely isometric. Thus by Theorem 3.1, there are unitaries U, V such that

$$\bar{\Phi}(a) = U \begin{pmatrix} a & 0 \\ 0 & \phi(a) \end{pmatrix} V^*$$

for some completely contractive ϕ and all $a \in \bigoplus \mathcal{M}_{d'_j,d_j}$. In particular, this holds for all $a \in \mathcal{R}_{\tilde{L}}$, that is,

$$\Phi\big(L(x)\big) = \tau\big(\tilde{L}(x)\big) = U \begin{pmatrix} \tilde{L}(x) & 0 \\ 0 & \phi(\tilde{L}(x)) \end{pmatrix} V^*,$$

finishing the proof. (Note: along the way we have shown that $\bigoplus \mathcal{M}_{d'_j,d_j}$ is the injective envelope of $\mathcal{R}_{\tilde{L}}$.) □

A nondegenerate linear pencil L is called **minimal** if the concrete injective envelope of $\mathcal{R}_L$ is $\bigoplus \mathcal{M}_{d'_i,d_i}$ for some d_i, d'_i. From Theorem 5.2 it follows that every homogeneous linear nondegenerate pencil is equivalent to a minimal one. Recall the notion of a minimal dimensional defining pencil for a ball from Section 1.2.

Corollary 5.3. *A homogeneous linear pencil L is minimal if and only if it is a minimal dimensional defining pencil for a ball.*

Proof. Suppose $\mathcal{B}_{L'} = \mathcal{B}_L$ for some $d' \times d$ homogeneous linear nondegenerate pencil L'. Then

$$\Psi: \mathcal{R}_L \to \mathcal{M}_{d',d}, \quad L(x) \mapsto L'(x)$$

is completely isometric. Since L is minimal, the injective envelope of $\mathcal{R}_L$ is $\bigoplus \mathcal{M}_{d'_j,d_j}$. Hence Ψ extends to a completely isometric $\Psi: \bigoplus \mathcal{M}_{d'_j,d_j} \to \mathcal{M}_{d',d}$ and is thus described by Theorem 3.1. There exist unitaries U, V and a completely contractive ψ such that

$$\Psi(y) = U \begin{pmatrix} y & 0 \\ 0 & \psi(y) \end{pmatrix} V^*$$

for $y \in \bigoplus \mathcal{M}_{d'_j,d_j}$. Applying this to $y = L(x)$ yields

$$L'(x) = \Psi\big(L(x)\big) = U \begin{pmatrix} L(x) & 0 \\ 0 & \psi(L(x)) \end{pmatrix} V^*.$$

Thus if L' is a minimal dimensional defining pencil for a ball, $\psi = 0$ and $L'(x) = UL(x)V^*$, so L is a minimal dimensional defining pencil for a ball, too.

Conversely, if L is a minimal dimensional defining pencil for a ball, then by Theorem 5.2, $\mathcal{B}_L = \mathcal{B}_{\tilde{L}}$ and the size of $\tilde{L}$ is at most that of L. By the minimality of L, $J = 0$ and so L equals the minimal pencil $\tilde{L}$ up to unitaries. Hence L is minimal. □

The proof shows that a minimal dimensional defining pencil for a ball is also minimal with respect to ℓ and ℓ', respectively.

5.2. Pencil ball maps

In this subsection we present the proof of our main result, Theorem 1.3.

Lemma 5.4. *Suppose L is a minimal linear pencil and let $f: \mathcal{B}_L \to \mathcal{M}_{\ell',\ell}$ be a pencil ball map with $f(0) = 0$. Then*

$$f^{(1)}(x) = U \begin{pmatrix} L(x) & 0 \\ 0 & \phi(L(x)) \end{pmatrix} V^*$$

for some completely contractive ϕ and unitaries U, V.

Proof. By definition,

$$\big\|L(X)\big\| < 1 \quad \Rightarrow \quad \big\|f(X)\big\| \leqslant 1, \qquad \big\|L(X)\big\| = 1 \quad \Rightarrow \quad \big\|f\big(\exp(it)X\big)\big\| = 1 \quad \text{for a.e. } t \in \mathbb{R}. \tag{5.1}$$

For $X \in \mathcal{B}_L$, $\left(\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{B}_L$ so

$$f\left(\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & f^{(1)}(X) \\ 0 & 0 \end{pmatrix}$$

has norm at most 1 and is of norm 1 (a.e.) if $X \in \partial\mathcal{B}_L$. By linearity, (5.1) implies $\|f^{(1)}(X)\| = \|L(X)\|$ for all $X \in \mathcal{B}_L$.

Since L is minimal, f induces

$$h:\mathcal{R}_L\to\mathcal{M}_{\ell',\ell},\quad h\big(L(x)\big)=f(x).$$

Moreover, $h^{(1)}(L(x))=f^{(1)}(x)$ is a complete isometry $\mathcal{R}_L\to\mathcal{M}_{\ell',\ell}$. Theorem 5.2 implies

$$f^{(1)}(x)=U\begin{pmatrix} L(x) & 0\\ 0 & \phi(L(x))\end{pmatrix}V^*$$

for unitaries U, V and a completely contractive ϕ. □

Before moving on to the general situation we explain the main idea for the case of the quadratic homogeneous component of a pencil ball map.

Lemma 5.5. *Suppose L is a minimal linear pencil and let $f:\mathcal{B}_L\to\mathcal{M}_{\ell',\ell}$ be a pencil ball map with $f(0)=0$. Suppose $f^{(1)}$ is as in Lemma* 5.4. *Then $f^{(2)}$ has the form*

$$f^{(2)}(x)=U\begin{pmatrix} 0 & 0\\ 0 & \star\end{pmatrix}V^*.$$

Proof. Write the homogeneous linear pencil L as

$$L(x)=\sum_{j=1}^{g}A_jx_j.$$

Given a tuple $X\in\mathcal{B}_L(n)$, define,

$$T_1=\begin{pmatrix} 0 & \lambda I & 0\\ 0 & 0 & X_1\\ 0 & 0 & 0\end{pmatrix},\qquad T_j=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & X_j\\ 0 & 0 & 0\end{pmatrix}\quad\text{for } j\geqslant 2,$$

where λ will be chosen later. For now, it suffices to note

$$L(T)=\begin{pmatrix} 0 & A_1\otimes\lambda I & 0\\ 0 & 0 & L(X)\\ 0 & 0 & 0\end{pmatrix},$$

and thus $\|L(T)\|\leqslant 1$ if $|\lambda|$ is sufficiently small. For use below, observe that

$$T_1T_j=\begin{pmatrix} 0 & 0 & \lambda X_j\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\qquad T_kT_j=0\quad\text{for } k\geqslant 2.$$

Write $f^{(2)}$, the quadratic part of f as

$$f^{(2)}=\sum_{a,b=1}^{g}f_{a,b}x_ax_b.$$

With this notation,

$$f^{(2)}(T)=\begin{pmatrix} 0 & 0 & \sum f_{1,j}\otimes\lambda X_j\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

and

$$f(T)=\begin{pmatrix} 0 & f^{(1)}(\lambda I,0,\dots,0) & \sum f_{1,j}\otimes\lambda X_j\\ 0 & 0 & f^{(1)}(X)\\ 0 & 0 & 0\end{pmatrix}.$$

And further, for λ of sufficiently small norm, $f(T)$ is a contraction and $\|L(X)\|=1$ implies $\|f(T)\|=1$. Hence,

$$\Lambda(X)=\begin{pmatrix} \sum f_{1,j}\otimes\lambda X_j\\ f^{(1)}(X)\end{pmatrix}$$

for small enough λ satisfies

$$1 = \|L(X)\| = \|f(T)\| \geqslant \|\Lambda(X)\| \geqslant \|f^{(1)}(X)\| = \|L(X)\|.$$

Hence by linearity, $\|\Lambda(X)\| = \|f^{(1)}(X)\| = \|L(X)\|$ for all X.

Since L is nondegenerate,

$$\Delta : \mathcal{R}_L \to \mathcal{M}_{2\ell',\ell}, \quad L(x) \mapsto \Lambda(x)$$

is a well-defined linear map. Also,

$$\Delta\big(L(x)\big) = \begin{pmatrix} J(L(x)) \\ f^{(1)}(x) \end{pmatrix}$$

for some linear $J : \mathcal{R}_L \to \mathcal{M}_{\ell',\ell}$. In the coordinates given by U, V:

$$\Delta\big(L(x)\big) = \begin{pmatrix} J_{11}(L(x)) & J_{12}(L(x)) \\ J_{21}(L(x)) & J_{22}(L(x)) \\ L(x) & 0 \\ 0 & \phi(L(x)) \end{pmatrix},$$

where ϕ is a complete contraction.

Obviously, $\Delta : \mathcal{R}_L \to \mathcal{M}_{2\ell',\ell}$ is completely isometric. It thus has a completely isometric extension to $\bigoplus \mathcal{M}_{d'_j,d_j}$. Moreover, since the $(3,1)$ term, $L(x)$, is the identity on $\mathcal{R}_L$, the only extension of this term to all of $\bigoplus \mathcal{M}_{d'_j,d_j}$ is the identity (cf. Lemma 2.9). It now follows that the extension of J_{j1} must be zero (cf. Theorem 5.2). Now repeat the argument with k replacing 1 to get

$$f_{j,k} = U \begin{pmatrix} 0 & f_{j,k}^{1,2} \\ 0 & f_{j,k}^{2,2} \end{pmatrix} V^*$$

for all j, k.

To show that $f_{j,k}^{1,2} = 0$ for all j, k we simply repeat the entire argument using

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ \lambda I & 0 & 0 \\ 0 & X_1 & 0 \end{pmatrix}, \qquad T_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & X_j & 0 \end{pmatrix} \quad \text{for } j \geqslant 2. \qquad \square$$

Proof of Theorem 1.3. We may replace L by a minimal pencil equivalent to it and thus assume that L is minimal. Also, by Lemma 5.4,

$$f^{(1)}(x) = U \begin{pmatrix} L(x) & 0 \\ 0 & \phi(L(x)) \end{pmatrix} V^*$$

for unitaries U, V and a complete contraction ϕ. We will prove that for $m \geqslant 2$,

$$f^{(m)}(x) = U \begin{pmatrix} 0 & 0 \\ 0 & \phi^{(m)}(L(x)) \end{pmatrix} V^*. \tag{5.2}$$

This holds for $m = 2$ by Lemma 5.5. Let $m \geqslant 2$, assume (5.2) holds up to $m-1$ and write

$$f^{(m)} = \sum_{\substack{w \in \langle x \rangle \\ |w| = m}} f_w w, \quad f_w \in \mathbb{C}^{\ell' \times \ell}.$$

Fix $i_1, \ldots, i_{m-1} \in \{1, \ldots, g\}$ and consider the coefficient $f_{x_{i_1} \cdots x_{i_{m-1}} x_j}$ of f. To "isolate" this coefficient we construct block $(m+1) \times (m+1)$ matrices $T_1, \ldots, T_g$ as follows. Each T_i has X_i as its $(m, m+1)$ entry. In addition to that, we put a λI as the $(k, k+1)$ entry of T_{i_k}. All the other entries not explicitly given above are set to 0. (Note that T_i might have several nonzero superdiagonal entries as the i_k are not necessarily pairwise distinct.)

For example, if $m = 4$ and $x_{i_1} x_{i_2} x_{i_3} = x_1 x_2 x_1$, then

$$T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

but $i_1 = i_3 = 1$, so λI occurs twice in T_1, once as the $(1,2)$ entry and also as the $(3,4)$ entry:

$$T_1 = \begin{pmatrix} 0 & \lambda I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda I & 0 \\ 0 & 0 & 0 & 0 & X_1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus

$$T_1 T_2 T_1 = \begin{pmatrix} 0 & 0 & 0 & \lambda^3 I & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

More generally, if $u \in \langle x \rangle$ is of degree k, then all the nonzero entries of $u(T)$ are on the k-th superdiagonal.

Let us consider a product $T_{j_1} \cdots T_{j_{m-1}}$ of the T_j's of length $m-1$. Its $(1,m)$ entry is

$$(T_{j_1})_{1,2} (T_{j_2})_{2,3} \cdots (T_{j_{m-1}})_{m-1,m}$$

and is nonzero if and only if $(T_{j_k})_{k,k+1} \neq 0$ for all $k = 1, \ldots, m-1$. So the only product of the T_j's of length $m-1$ with a nonzero entry in the m-th column is

$$T_{i_1} \cdots T_{i_{m-1}} = \begin{pmatrix} 0 & \cdots & 0 & \lambda^{m-1} I & 0 \\ \vdots & \ddots & & 0 & \star \\ \vdots & & \ddots & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Likewise the only words that produce a nonzero top right entry are those of length m that start with $x_{i_1} \cdots x_{i_{m-1}}$, e.g. $x_{i_1} \cdots x_{i_{m-1}} x_j$ produces $\lambda^{m-1} T_j$ in the top right corner. Hence the top right entry in $f(T)$ is

$$\delta = \sum_j f_{x_{i_1} \cdots x_{i_{m-1}} x_j} \otimes \lambda^{m-1} X_j,$$

so

$$f(T) = \begin{pmatrix} 0 & \star & \cdots & \cdots & \delta \\ 0 & 0 & \star & \cdots & \star \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & f^{(1)}(X) \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Here all the entries denoted by $\star$ come from homogeneous components of f of degree $< m$ and are of degree $\leqslant 1$ in the X_i. The entries in the last column are all of degree $= 1$ in the X_i. Since L is nondegenerate, $\Delta : \mathcal{R}_L \to \mathcal{M}_{m\ell',\ell}$ which maps $L(X)$ to the last column of $f(T)$ (without the bottom entry 0) is a well-defined linear map. Furthermore, $\delta = J(L(X))$ for some linear $J : \mathcal{R}_L \to \mathcal{M}_{\ell',\ell}$.

Assume $X \in \mathcal{B}_L$ and $|\lambda|$ is small. Then all the entries $\star$ have norm < 1 and $\|f(T)\| \leqslant 1$. Furthermore, $\|L(X)\| = 1$ implies $\|f(T)\| = 1$. So

$$1 = \|L(X)\| = \|f(T)\| \geqslant \|\Delta(L(X))\| \geqslant \|f^{(1)}(X)\| = \|L(X)\|.$$

Hence by linearity, $\|\Delta(L(X))\| = \|f^{(1)}(X)\| = \|L(X)\|$ for all X.

The last column of $f(T)$ in the coordinates given by U, V is

$$\Delta(L(x)) = \begin{pmatrix} J_{11}(L(x)) & J_{12}(L(x)) \\ J_{21}(L(x)) & J_{22}(L(x)) \\ 0 & 0 \\ 0 & \star \\ 0 & 0 \\ 0 & \star \\ \vdots & \vdots \\ L(x) & 0 \\ 0 & \phi(L(x)) \end{pmatrix},$$

where ϕ is completely contractive. (Here we have used the induction hypothesis (5.2) on all the terms of degree between 2 and $n-1$.) We can now proceed as in the proof of Lemma 5.5 to conclude that $J_{j1}(L(x)) = 0$. Similarly one obtains $J_{12}(L(x)) = 0$. All this proves (5.2) and finishes the proof. □

Alternative proof of Theorem 1.3. We may replace L by the minimal pencil $\tilde{L}$ equivalent to it and thus assume that L is minimal. Also, by Lemma 5.4,

$$f^{(1)}(x) = U \begin{pmatrix} L(x) & 0 \\ 0 & \phi(L(x)) \end{pmatrix} V^*$$

for unitaries U, V and a complete contraction ϕ. We will prove that for $m \geqslant 2$,

$$f^{(m)}(x) = U \begin{pmatrix} 0 & 0 \\ 0 & \phi^{(m)}(L(x)) \end{pmatrix} V^*. \tag{5.3}$$

This holds for $m = 2$ by Lemma 5.5. Consider $f^{(m)}$,

$$f^{(m)} = \sum_{\substack{w \in \langle x \rangle \\ |w| = m}} f_w w, \quad f_w \in \mathbb{C}^{\ell' \times \ell}.$$

For notational convenience we omit U, V in what follows.

Fix a word $v \in \langle x \rangle$ of length $m-1$. To show that $f_{vx_k} = 0$ for each k and hence $f_w = 0$ for each word w of length m, we shall use compressions of the creation operators on Fock space to build more elaborate versions of the block matrices in the proof of Lemma 5.5. To be specific, let $\mathcal{K} = \mathcal{K}_m$ denote the Hilbert space with orthonormal basis $\{u \in \langle x \rangle\colon\ 1 \leqslant |u| \leqslant m\}$. Define S_j on $\mathcal{K}$ by $S_j u = x_j u$ provided $|u| < m$ and $S_j u = 0$ if $|u| = m$. Note that $S_j^*(x_j u) = u$ and $S_j^*(w) = 0$ if the word $w \in \mathcal{K}$ does not begin with x_j.

For $n \in \mathbb{N}$ define $E_j^{(n)} : \mathbb{C}^n \to \mathcal{K} \otimes \mathbb{C}^n$ by $y \mapsto x_j \otimes y$.

Given n and a tuple $X \in (\mathbb{C}^{n\times n})^g$, define, with $\lambda > 0$ to be chosen later, matrices T_j mapping from $(\mathcal{K} \otimes \mathbb{C}^n) \oplus \mathbb{C}^n \oplus \mathbb{C}^n$ to itself by

$$T_j = \begin{pmatrix} \lambda S_j \otimes I & \lambda E_j & 0 \\ 0 & 0 & X_j \\ 0 & 0 & 0 \end{pmatrix}.$$

(Here $E_j = E_j^{(n)}$.) Note that

$$L(T)^* L(T) = \begin{pmatrix} \star & \star & 0 \\ \star & \star & 0 \\ 0 & 0 & L(X)^* L(X) \end{pmatrix},$$

where all the entries denoted by $\star$ are quadratic in λ. Thus we can choose $\lambda > 0$ small enough so that if $\|L(X)\| = 1$, then $\|L(T)\| = 1$ too.

As an example, let us compute $u(T)$ for $u = x_1 x_2 x_3$:

$$u(T) = T_1 T_2 T_3 = \begin{pmatrix} \lambda^3 (S_1 S_2 S_3 \otimes I) & \lambda^3 (S_1 S_2 \otimes I) E_3 & \lambda^2 (S_1 \otimes I) E_2 X_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and thus

$$u(T)\big((w \otimes y_1) \oplus y_2 \oplus y_3\big) = \lambda^3 (S_1 S_2 S_3 w \otimes y_1) + \lambda^3 (S_1 S_2 x_2 \otimes y_2) + \lambda^2 (S_1 x_2 \otimes X_3 y_3)$$

for $w \in \mathcal{K}$ and $y_i \in \mathbb{C}^n$. The general calculation along the same lines yields

$$f(T) = \begin{pmatrix} \star & \star & \delta \\ 0 & 0 & f^{(1)}(X) \\ 0 & 0 & 0 \end{pmatrix}$$

for

$$\delta = \sum_k \sum_j \sum_{|u| \leqslant m-2} \lambda^{|u|+1} f_{u x_j x_k} \otimes \big((u(S) \otimes I) E_j X_k\big).$$

By Lemma 5.4, $\|L(X)\| = \|f^{(1)}(X)\|$ for all X.

Let $P_v : \mathcal{K} \otimes \mathbb{C}^n \to \mathbb{C}^n$ denote the projection $P_v(\sum_u u \otimes y_u) = y_v$. For $y \in \mathbb{C}^n$, and $u \in \mathcal{K}$ with $|u| \leqslant m-2$, we have

$$P_v\big(\big(u(S) \otimes I\big)E_j X_k\big)y = P_v\big(u(S)x_j \otimes X_k y\big) = P_v(ux_j \otimes X_k y) = \begin{cases} X_k y & \text{if } ux_j = v, \\ 0 & \text{otherwise.} \end{cases}$$

We extend P_v to the projection $\Pi_v : \mathbb{C}^{d' \times d} \otimes \mathcal{K} \otimes \mathbb{C}^n \to \mathbb{C}^{d' \times d} \otimes \mathbb{C}^n$, $\Pi_v = I \otimes P_v$. With this notation,

$$\begin{pmatrix} \Pi_v & 0 & 0 \\ 0 & I & 0 \end{pmatrix} f(T) \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} = \begin{pmatrix} \sum_k \lambda^{m-1} f_{vx_k} \otimes X_k \\ f^{(1)}(X) \end{pmatrix}.$$

Now proceed as in the proof of Lemma 5.5 to conclude

$$f_{vx_k}(x) = \begin{pmatrix} 0 & 0 \\ 0 & \phi^{(m)}(L(x)) \end{pmatrix}$$

(in the coordinates given by U, V) for some completely contractive $\phi^{(m)}$. □

6. More generality

In this section we extend the main results presented so far in two directions. First of all, we use linear fractional transformations to classify pencil ball maps that do not preserve the origin. For the second generalization we study pencil ball maps mapping between two pencil balls and preserving the boundary.

6.1. Linear fractional transformations

We provide a cursory treatment of linear fractional maps and refer the reader to [9, Section 5] for details and proofs. Let $\mathcal{B}_{\ell',\ell} := \{X \in \mathbb{C}^{\ell' \times \ell}: \|X\| < 1\}$. For a given $\ell' \times \ell$ scalar matrix v with $\|v\| < 1$, define $\mathscr{F}_v : \mathcal{B}_{\ell',\ell} \to \mathcal{B}_{\ell',\ell}$ by

$$\mathscr{F}_v(u) := v - \big(I_{\ell'} - vv^*\big)^{1/2} u \big(I_\ell - v^* u\big)^{-1} \big(I_\ell - v^* v\big)^{1/2}. \tag{6.1}$$

Of course it must be shown that $\mathscr{F}_v$ actually takes values in $\mathcal{B}_{\ell',\ell}$; this is done in [9, Lemma 5.2].

Linear fractional transformations such as $\mathscr{F}_v$ are common in circuit and system theory, since they are associated with energy conserving pieces of a circuit, cf. [22].

Notice that if $\ell = \ell' = 1$, then v and u are scalars, hence

$$\mathscr{F}_v(u) = (v-u)(1-u\bar{v})^{-1} = (1-u\bar{v})^{-1}(v-u).$$

Now fix $v \in \mathbb{D}$ and consider the map $\mathbb{D} \to \mathbb{C}$, $u \mapsto \mathscr{F}_v(u)$. This map is a linear fractional map that maps the unit disc to the unit disc, maps the unit circle to the unit circle, and maps v to 0. The geometric interpretation of the map in (6.1) is similar:

Lemma 6.1. *(See Lemma 5.2 in [9].) Suppose that $N \in \mathbb{N}$ and $V \in \mathcal{B}_{\ell',\ell}(N)$.*

(1) *$U \mapsto \mathscr{F}_V(U)$ maps $\mathcal{B}_{\ell',\ell}(N)$ into itself with boundary to the boundary.*
(2) *If $U \in \mathcal{B}_{\ell',\ell}(N)$, then $\mathscr{F}_V(\mathscr{F}_V(U)) = U$.*
(3) *$\mathscr{F}_V(V) = 0$ and $\mathscr{F}_V(0) = V$.*

6.2. Classification of pencil ball maps

General pencil ball maps f – those where $f(0)$ is not necessarily 0 – are described using the linear fractional transformation $\mathscr{F}$.

Corollary 6.2. *Suppose L is a nondegenerate homogeneous linear pencil and $\tilde{L}$ is a minimal dimensional defining pencil for $\mathcal{B}_L$. If $f : \mathcal{B}_L \to \mathcal{M}_{\ell',\ell}$ is a pencil ball map with $\|f(0)\| < 1$, then there exists a contraction-valued analytic $\tilde{f} : \mathcal{B}_L \to \mathcal{M}_{m',m}$ such that*

$$f(x) = \mathscr{F}_{f(0)}\big(\varphi(x)\big), \tag{6.2}$$

where

$$\varphi(x) = \mathscr{F}_{f(0)}\big(f(x)\big) = U \begin{pmatrix} \tilde{L}(x) & 0 \\ 0 & \tilde{f}(x) \end{pmatrix} V^* \tag{6.3}$$

for some $m, m' \in \mathbb{N}_0$ and unitaries $U \in \mathbb{C}^{\ell' \times \ell'}$ and $V \in \mathbb{C}^{\ell \times \ell}$.

6.3. Pencil ball to pencil ball maps

Suppose L, L' are homogeneous linear pencils. An analytic map $f:\mathcal{B}_L \to \mathcal{B}_{L'}$ is a **pencil ball to pencil ball map** if $f(\partial\mathcal{B}_L) \subseteq \partial\mathcal{B}_{L'}$.

Corollary 6.3. *Suppose L is a nondegenerate homogeneous linear pencil, $\tilde{L}$ is a minimal dimensional defining pencil for $\mathcal{B}_L$, and let L' be an arbitrary homogeneous linear pencil with $\mathbb{C}^{\ell'\times\ell}$ coefficients. If $f:\mathcal{B}_L \to \mathcal{B}_{L'}$ is a pencil ball to pencil ball map with $f(0)=0$, there exists a contraction-valued analytic $\tilde{f}:\mathcal{B}_L \to \mathcal{M}_{m',m}$ such that*

$$\big(L' \circ f\big)(x) = U \begin{pmatrix} \tilde{L}(x) & 0 \\ 0 & \tilde{f}(x) \end{pmatrix} V^*$$

for some $m, m' \in \mathbb{N}_0$ and unitaries $U \in \mathbb{C}^{\ell'\times\ell'}$ and $V \in \mathbb{C}^{\ell\times\ell}$.

Corollary 6.4. *Suppose L, L' are nondegenerate homogeneous linear pencils. If $f:\mathcal{B}_L \to \mathcal{B}_{L'}$ is a pencil ball to pencil ball map, then*

$$\big(L' \circ f\big)(x) = \mathscr{F}_{L'\circ f(0)}\big(\varphi(x)\big), \tag{6.4}$$

where

$$\varphi(x) = \mathscr{F}_{L'\circ f(0)}\big(L' \circ f(x)\big) \tag{6.5}$$

is a pencil ball map $\mathcal{B}_L \to \mathcal{M}_{\ell',\ell}$ mapping 0 to 0 and is therefore completely described by Theorem 1.3.

It is clear that converses of Corollaries 6.3 and 6.4 hold as well.

As a last result we show that origin-preserving *scalar* analytic self-maps of $\mathcal{B}_L$ are trivial.

Corollary 6.5. *Suppose L is a nondegenerate homogeneous linear pencil. If $f:\mathcal{B}_L \to \mathcal{B}_L$ is a pencil ball to pencil ball map with scalar coefficients and $f(0)=0$, then f is linear.*

Proof. We may assume without loss of generality L is minimal. Then by Corollary 6.3,

$$(L \circ f)(x) = U \begin{pmatrix} L(x) & 0 \\ 0 & \tilde{f}(x) \end{pmatrix} V^*$$

for some unitaries U, V, and contraction-valued analytic $\tilde{f}:\mathcal{B}_L \to \mathcal{M}_{m',m}$. Comparing dimensions we see $m'=m=0$, i.e., there is no $\tilde{f}$. Hence

$$(L \circ f)(x) = UL(x)V^*.$$

Since L is nondegenerate this implies f is linear. □

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