The index of merit of \( k \)-th-copy integration lattices

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Abstract

\( r2d2lri \) is an automatic two-dimensional cubature algorithm that demonstrates the practical value of using an augmentation sequence consisting of \( (2^k)^2 \)-copy lattices as a basis for numerical integration. This paper investigates use of similar embedded augmentation sequences in higher dimensions by developing theoretical results relating to the index of merit of \( s \)-dimensional \( (2^k)^s \)-copy lattices generated from rank-1 simple lattices. The theoretical results can be used to guide the search for good augmentation sequences in \( s \) dimensions in the sense of high index of merit.

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1. Introduction

Rank-1 integration lattices, and higher rank copy lattices produced from them, have proved to be very useful in multidimensional integration (see [13]). An augmentation sequence is defined as a sequence of \( s \)-dimensional \( (2^k)^s \)-copy lattices of a given seed lattice (see Section 1.3). If the seed lattice has \( N \) points, \( N \) odd, then successive members of the sequence have \( (2^k)^s N \), \( k = 0, 1, 2, \ldots \) points. Such a sequence of lattices can provide a basis for an automatic integration algorithm, with the embedded nature of the sequence providing full re-use of points. These successive cubatures can also be used in a practical method of error estimation. A two-dimensional realization of such an algorithm called \( r2d2lri \) is given by Robinson and Hill [12].

A possible disadvantage of using such an augmentation sequence is the exponential growth in the number of lattice points in successive sequence members. For this reason, the use of such sequences is likely to be limited to relatively low dimensions.

Two important measures of the quality of a lattice for use in a cubature rule are the trigonometric degree of precision (TDOP) and the index of merit (IOM). If the TDOP of the seed lattice of an augmentation sequence is known, then it is a simple matter to compute the TDOP of each of the member lattices in the sequence. However, there is no corresponding simple method of determining the IOM of members of an augmentation sequence. In this paper, we investigate the growth of the IOM in such sequences. We begin with some preliminary definitions.

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1.1. Lattices, dual lattices and index of merit

**Definition 1.** A lattice in $\mathbb{R}^s$ is a discrete subset of $\mathbb{R}^s$ which is closed under addition and subtraction.

**Definition 2.** An integration lattice $L_N$ is a lattice in $\mathbb{R}^s$ such that $\mathbb{Z}^s \subseteq L_N$, where $N$ = the number of points in $L_N \cap [0, 1)^s$. We refer to $N$ as the order of $L_N$.

**Definition 3.** A cubature rule for a given integration lattice $L_N$ is a rule of the form

$$Q_f = \frac{1}{N} \sum_{j=0}^{N-1} f(x_j),$$

where $x_j$, $j = 0, 1, \ldots, N – 1$, are the distinct lattice points in $L_N \cap [0, 1)^s$.

It is assumed here that $f(x)$ is continuous on $[0, 1)^s$ and has a smooth 1-periodic extension beyond $[0, 1)^s$ in all $s$ components.

**Definition 4.** $L_N^\perp$ is the dual lattice of $L_N$. Let $u \in \mathbb{Z}^s$. Then,

$$u \in L_N^\perp \iff \forall q \in L_N, \quad u \cdot q \in \mathbb{Z}^s.$$

The dual lattice $L_N^\perp$ is central to the standard error analysis associated with lattice cubature rules. The points in the dual lattice indicate which Fourier coefficients from the Fourier expansion of the integrand remain in the error term when a cubature rule based on $L_N$ is used to estimate the integral.

**Definition 5.** The trigonometric degree of precision (TDOP) of the lattice $L_N$ is given by

$$\text{TDOP}(L_N) = \min_{p \in L_N^\perp \setminus \{0\}} \left( \sum_{i=1}^{s} |p_i| \right) - 1.$$

**Definition 6.** The index of merit (IOM) of the lattice $L_N$ is given by

$$\rho(L_N) = \min_{p \in L_N^\perp \setminus \{0\}} \left( \prod_{i=1}^{s} |p_i| \right), \quad \text{where } \overline{v} = \begin{cases} 1, & v = 0, \\ |v|, & v \neq 0. \end{cases}$$

The index of merit is also known as the Zaremba index (see [8]).

TDOP and IOM are indicators of the goodness of $L_N$ when this lattice is used to construct the corresponding cubature rule $Q$.

**Definition 7.** If the set $\{g_1, g_2, \ldots, g_s\}$ of linearly independent vectors is such that $g_i \in L_N$, $1 \leq i \leq s$ and $\forall x \in L_N$, $\exists x_i \in \mathbb{Z}$ such that

$$x = \sum_{i=1}^{s} x_i g_i,$$

then

$$G = (g_1, g_2, \ldots, g_s)^T$$

is called a generator matrix for $L_N$. 
Definition 8. A rank-1 lattice is of the form
\[ L_N = \bigcup_{w \in \mathbb{Z}^j} \bigcup_{j=0}^{N-1} \left( jz_iN + w \right), \]  
(1)
where \( z = (z_1, z_2, \ldots, z_s) \in \mathbb{Z}^s \). We will assume that \( z_1, z_2, \ldots, z_s \) and \( N \) are relative prime, \( 0 < z_i < N, i = 1, 2, \ldots, s \). \( L_N \) is called a rank-1 simple lattice if \( z_1 = 1 \).

Sloan and Lyness [15] point out that the members of \( L_N \cap [0, 1)^s \) form a finite Abelian group under the operation \([x_i + x_j] = (x_i + x_j) \mod 1\). Lyness and Sorevik [9] also show that when \( N \) is prime, all lattices are rank-1 simple and when \( N \) has no square factor, all lattices are rank-1, with the overwhelming proportion of these being rank-1 simple.

1.2. Copy lattices

From this point in the paper, we will assume that \( L_N \) is a rank-1 simple integration lattice, where \( N > 1 \) is odd.

Following the definition of an \( n^s \)-copy cubature rule by Sloan and Joe [13], we define an \((n^k)^s\)-copy lattice of a rank-1 simple lattice as follows:

Definition 9. The \((n^k)^s\)-copy lattice of \( L_N \), \( L_{(n^k)^s} \) is given by
\[ L_{(n^k)^s} = \bigcup_{k=0}^{n^k-1} \bigcup_{k_1=0}^{n^k-1} \bigcup_{k_s=0}^{n^k-1} \left( x + \frac{(k_1, \ldots, k_s)}{n^k} \right) \]
\[ = \bigcup_{w \in \mathbb{Z}^j} \bigcup_{k=0}^{n^k-1} \bigcup_{k_1=0}^{n^k-1} \bigcup_{k_s=0}^{n^k-1} \left( jz_iN + \frac{(k_1, \ldots, k_s)}{n^k} + w \right). \]

Following Proposition 10.2 by Sloan and Joe [13], Definition 10 is an equivalent definition to Definition 9 for an \((n^k)^s\)-copy lattice of rank-1 simple lattice.

Definition 10.
\[ L_{(n^k)^s} = \bigcup_{k=0}^{n^k-1} \bigcup_{k_1=0}^{n^k-1} \bigcup_{k_s=0}^{n^k-1} \left( x + \frac{(k_1, \ldots, k_s)}{n^k} \right) \]
\[ = \bigcup_{w \in \mathbb{Z}^j} \bigcup_{k=0}^{n^k-1} \bigcup_{k_1=0}^{n^k-1} \bigcup_{k_s=0}^{n^k-1} \left( jz_iN + \frac{(k_1, \ldots, k_s)}{n^k} + w \right). \]

It is important to note that there are no duplicated points resulting from the copying process if \( N \) and \( n \) are co-prime. Disney and Sloan [4] point out that a maximal rank cubature rule obtained by \( n^s \)-copying can be as good as a rank-1 cubature rule of the same order. There are two advantages of using \( n^s \)-copy cubature rules derived from rank-1 cubature rules over constructing a rank-1 cubature rule directly. Firstly, the computation needed to search for a good \( n^s \)-copy lattice is much less than that needed to search for a good rank-1 lattice of the same order. Secondly, intermediate lattices obtained from copying a rank-1 lattice to a maximal rank lattice provide data that can be used for error estimation. Sloan and Joe [13] argue that on the grounds of both theory and practice, the best value to choose for \( n \) is \( n = 2 \). Accordingly, we restrict our discussion to sequences of \((2^k)^s\)-copy cubature rules in this paper.

1.3. Augmentation sequences

Definition 11. We call the lattice \( L_{(2^k)^s} \) the \( k \)th-copy lattice based on \( L_N \) and denote it by \( L_N(k), k = 0, 1, 2, \ldots, \) where \( L_N(0) = L_N \). Then \( \{L_N(0), L_N(1), L_N(2), \ldots, L_N(k), \ldots\} \) is an augmentation sequence, and the index of merit of \( L_N(k) \) is denoted by \( \rho_k = \rho(L_N(k)), k = 0, 1, 2, \ldots \). Note that \( \rho_0 = \rho(L_N) \).
From Definition 10 we know that the augmentation sequence \( \{ L_N(0), L_N(1), L_N(2), \ldots, L_N(k), \ldots \} \) is an embedded sequence, i.e. \( L_N(j) \subset L_N(k) \iff j < k \).

**Definition 12.** We denote the factor by which the IOM of the \( k \)th member of an augmentation sequence increases over that of the \((k-1)\)th member by

\[
\sigma_k = \frac{p_k}{p_{k-1}}, \quad k = 1, 2, \ldots .
\]

Lemma 6.5 in [13] shows that \( L_{\mathbb{N}}^{-1} = nL_N^{-1} \) (see also [11, Theorem 5.44]). A specific case of this result is given in the following lemma.

**Lemma 13.** Denote the dual lattice of \( L_N(k) \) by \( L_N^\perp(k) \). Then

\[
L_N^\perp(k) = 2^k L_N^\perp = \{ 2^k v \mid v \in L_N^\perp \}.
\]

The practical use of \( 2^s \)-copy cubature rules is demonstrated by Joe and Sloan [6,7]. Hill and Robinson [5] extend \( 2^2 \)-copying to \( (2^k)^2 \)-copying with the resulting algorithm, \( r2d2lri \) by Robinson and Hill [12], proving to be very effective in terms of accuracy and CPU time.

It is easy to show that if \( T_k \) is the TDOP of the \( k \)th member of an augmentation sequence, then

\[
T_k = 2T_{k-1} + 1, \quad k = 1, 2, \ldots .
\]

Thus, to find a good sequence based on maximizing the TDOP of sequence members, an \( N \)-point seed lattice with high TDOP should be chosen. Several searches for low-dimensional lattices with good TDOP have been reported in the literature (see e.g. [1,2,10]). In addition, Cools and Sloan [3] provide a number of theoretical results regarding the minimum number of points in a lattice needed to achieve a given TDOP.

Hill and Robinson [5] verified that IOM plays an important role in determining the goodness of a lattice. However, there is no simple general result corresponding to (3) for the IOM of members of an augmentation sequence. The aim of this paper is to investigate the growth of IOM within an augmentation sequence and henceforth, references to “good” lattices will mean good in the sense of high IOM, unless otherwise stated.

The theoretical results that we develop in this paper about the IOM of \( k \)th-copy lattices will be useful in the development of a goodness measure for an augmentation sequence which can subsequently be used to guide a computer search for good sequences.

### 2. Generator matrices for \( L_N(k) \) and \( L_N^\perp(k) \)

In this section, we characterize the generator matrices for \( L_N(k) \) and \( L_N^\perp(k) \). These results are used in subsequent sections to study the IOM of \( k \)th-copy lattices.

It is easy to show that

\[
M = \begin{pmatrix}
1 & \frac{z_2}{N} & \cdots & \frac{z_s}{N} \\
\frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]

(4)

is a generator matrix for \( L_N \) (see [14]). (Lyness and Sørevik [9] define a rank-1 lattice in terms of this generator matrix.) Furthermore, since \( L_N^{-1} = (L_N^T)^{-1} \), it follows that a generator matrix for \( L_N^{-1} \) is given by

\[
D = \begin{pmatrix}
N & 0 & \cdots & 0 \\
-z_2 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-z_s & 0 & \cdots & 1
\end{pmatrix}.
\]

(5)
The next two theorems describe the generator matrix for a \( k \)th-copy lattice and its dual lattice.

**Theorem 14.** A generator matrix for \( L_1^\perp_N(k) \) is

\[
2^k D = \begin{pmatrix} 2^k N & 0 & \cdots & 0 \\ -2^k z_2 & 2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -2^k z_s & 0 & \cdots & 2^k \end{pmatrix},
\]

where \( D \) is defined in (5).

**Proof.** This follows directly from the application of Lemma 13 to (5).

**Theorem 15.** A generator matrix for \( L_N(k) \) is

\[
2^{-k} M = \begin{pmatrix} 1 & & & z_s \\ 2^k N & 2^k N & \cdots & 2^k N \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2^k} \end{pmatrix},
\]

where \( M \) is defined in (4).

**Proof.** This follows directly from Theorem 14 and the fact \( G_1^\perp_k = (G_k^T)^{-1} \), where \( G_k \) is the generator matrix of \( L_N(k) \) and \( G_1^\perp_k \) is the generator matrix of \( L_1^\perp_N(k) \).

It should be noted that (4) and (5) and Theorems 14 and 15 hold for \( N \in \mathbb{Z}^+ \), not just for \( N \) odd or prime (which is the most common restriction for \( N \) in lattice rules).

3. **IOM of \( L_N(k) \)**

In this section we study the behavior of the IOM of lattices in an augmentation sequence based on \( L_N \). First, we need four important lemmas.

**Lemma 16.**

\[
L_1^\perp_N(k) = 2L_1^\perp_N(k - 1).
\]

**Proof.** The result follows by applying Lemma 13 to the \( k \)th-copy lattice and the \( (k - 1) \)th-copy lattice.

**Lemma 17** reveals the relationship between successive lattices in an augmentation sequence.

**Lemma 17.**

\[
L_N(k) = \bigcup_{k_1=0}^{1} \cdots \bigcup_{k_s=0}^{1} \bigcup_{x \in L_N(k-1)} \left( x + \frac{(k_1, \ldots, k_s)}{2^k} \right).
\]

**Proof.** Using Definition 10, \( y \in L_N(k) \) can be represented as

\[
y = j \frac{z}{N} + \frac{(k_1, \ldots, k_s)}{2^k} + w, \quad 0 \leq j < N, \quad k_i \in \mathbb{Z}, \quad 0 \leq k_i < 2^k, \quad i = 1, 2, \ldots, s.
\]
In particular, \( k_i \) is given by

\[ k_i = 2t_i + p_i, \quad p_i \in \{0, 1\}, \quad t_i \in \mathbb{Z}^+ \cup \{0\}, \]

then

\[ y = \frac{jz}{N} + \frac{(t_1, \ldots, t_s)}{2^{k-1}} + \frac{(p_1, \ldots, p_s)}{2^k} + w. \]

Thus, if \( x = \frac{jz}{N} + \frac{(t_1, \ldots, t_s)}{2^{k-1}} + w \), then \( x \in L_N(k-1) \) and we have \( y = x + \frac{(p_1, \ldots, p_s)}{2^k} \). It follows that

\[ L_N(k) \subseteq \bigcup_{k_1=0}^{1} \cdots \bigcup_{k_s=0}^{1} \bigcup_{x \in L_N(k-1)} \left( x + \frac{(k_1, \ldots, k_s)}{2^k} \right). \]  

(9)

Conversely, since \( x \in L_N(k-1) \subset L_N(k) \) and by Definition 9, \( (k_1, \ldots, k_s)/2^k \in L_N(k) \), \( k_i = 0 \) or 1, \( i = 1, \ldots, s \), then

\[ \bigcup_{k_1=0}^{1} \cdots \bigcup_{k_s=0}^{1} \bigcup_{x \in L_N(k-1)} \left( x + \frac{(k_1, \ldots, k_s)}{2^k} \right) \subseteq L_N(k). \]  

(10)

The result follows by combining (9) and (10). □

**Definition 18.** Let

\[ P_k = \left\{ p = (p_1, p_2, \ldots, p_s) \in L_N^+(k) \mid \prod_{i=1}^{s} p_i = \rho_k \right\}, \]

\[ Q_k = \{ x \mid x = \text{number of non-zero components in} \ p, \ p \in P_k \} \]

and

\[ q_{\max} = \max \{q \mid q \in Q_k\}, \]

\[ q_{\min} = \min \{q \mid q \in Q_k\}. \]

**Lemma 19.** Let \( p \in P_k \). If there are \( q > 0 \) non-zero components in \( p \), then \( \rho_k \geq 2^{q_k} \rho_0 \).

**Proof.** Since \( p = (p_1, p_2, \ldots, p_s) \in L_N^+(k) \), \( \bar{p} = (p_1/2^k, p_2/2^k, \ldots, p_s/2^k) \in L_N^+(k) \) (Lemma 13). Therefore,

\[ \rho_k = \prod_{i=1}^{s} \frac{p_i}{2^k} \]

\[ = (2^k)^q \prod_{i=1}^{s} \frac{p_i}{2^k} \]

\[ \geq 2^{q_k} \rho_0. \]  

□

**Lemma 20.** Let \( p \in P_k \). If there are \( q > 0 \) non-zero components in \( p \), then

\[ \sigma_k \geq 2^q \geq \sigma_{k+1}, \]

where \( \sigma_k \) is defined in (2).
Proof. Since \( p = (p_1, p_2, \ldots, p_s) \in L_{N/2}^+(k) \), \( \tilde{p} = (p_1/2, p_2/2, \ldots, p_s/2) \in L_{N/2}^+(k-1) \) (Lemma 16). Therefore

\[
\rho_k = \prod_{i=1}^s \frac{p_i}{2} \geq 2^q \rho_{k-1},
\]

thus establishing inequality \( \sigma_k \geq 2^q \).

Conversely, since \( p \in L_{N/2}^+(k) \), \( 2p \in L_{N/2}^+(k+1) \), and so

\[
\rho_{k+1} \leq \prod_{i=1}^s 2p_i = 2^q \prod_{i=1}^s p_i = 2^q \rho_k.
\]

Thus, inequality \( \sigma_{k+1} \leq 2^q \) holds. \( \square \)

With Definition 18 and Lemma 20, we have the following corollary.

**Corollary 21.**

\( \sigma_k \geq 2^{\text{max}} \geq 2^{\text{min}} \geq \sigma_{k+1}. \)

It follows that the IOM at least doubles between successive lattices in an augmentation sequence.

**Corollary 22.** \( \sigma_k \geq 2 \), \( k = 1, 2, \ldots. \)

When one considers the exponential increase in the number of points as \( k \) increases, a simple doubling in the IOM is not of great practical value unless the dimension is low (e.g. \( s = 2 \) as used in the algorithm \( r2d2lri \) by Robinson and Hill [12]). In what follows, we establish conditions under which the IOM is guaranteed to increase by a factor more than 2 (and by as much as \( 2^s \)), and also conditions under which the increase is limited to a factor of 2.

The next theorem shows that an increase by a factor of at least 4 is achieved on the first augmentation.

**Theorem 23.** \( \sigma_1 \geq 4. \)

Proof. From (5), we know

\( v = (-z_2, 1, \ldots, 0) + k(N, 0, \ldots, 0) \in L_{N/2}^+, \ k \in \mathbb{Z}. \)

Thus,

\( (-z_2, 1, \ldots, 0) \in L_{N/2}^+, \)

\( (N - z_2, 1, 0, \ldots, 0) \in L_{N/2}^+. \)

Since \( N \) and \( z_2 \) are co-prime, either \( 0 < N - z_2 < N/2 \) or \( 0 < z_2 < N/2 \) holds. Accordingly, it must be that

\[
\rho_0 \leq \min(z_2, N - z_2) < \frac{N}{2}.
\]

(11)

Assume \( p = (p_1, \ldots, p_s) \in P_1 \). Then \( \tilde{p} = (p_1/2, \ldots, p_s/2) \in L_{N/2}^+(\text{Lemma 16}) \). There are two cases to consider:

Case 1: \( p \) has only one non-zero component, say \( p_i \). We know that \( \tilde{p} \cdot z/N = (p_i/2)z_i/N \) is an integer. Thus, since \( z_i \) and \( N \) are co-prime, it follows that \( p_i/2 = mN, m \in \mathbb{Z} \) and \( m \neq 0 \). Using (11) we conclude that \( \rho_1 = |p_i|/2m|N \geq 4\rho_0. \)

Case 2: \( p \) has \( q > 1 \) non-zero components. Then using Lemma 20, we have \( \rho_1 \geq 2^q \rho_0 \geq 4\rho_0. \) \( \square \)
Theorem 24 demonstrates that the increase in IOM, by comparison with \( \rho_0 \), can be quite significant for the first several augmentations.

**Theorem 24.** Let \( K = \lfloor \log_2(N/\rho_0) \rfloor \). Then \( \rho_k \geq 4^k \rho_0 \), for \( 1 \leq k \leq K \).

Note

\[
K = \lfloor \log_2 \left( \frac{N}{\rho_0} \right) \rfloor \Rightarrow N \geq 2^K \rho_0. \tag{12}
\]

**Proof.** The proof for this theorem is quite similar to the proof of Theorem 23. If \( \mathbf{p} = (p_1, \ldots, p_k) \in P_k \), then \( \tilde{\mathbf{p}} = (p_1/2^k, \ldots, p_k/2^k) \in L_{N/2^k}^1 \). There are two cases to consider:

Case 1: \( \mathbf{p} \) has only one non-zero component, say \( p_{i_0} \). We know that \( \tilde{\mathbf{p}} \cdot \mathbf{z}/N = (p_{i_0}/2^k)z_{i_0}/N \) is an integer. Thus, since \( z_{i_0} \) and \( N \) are co-prime, it follows that \( p_{i_0}/2^k = mN, \) \( m \in \mathbb{Z} \) and \( m \neq 0 \). Using (12), we can conclude that \( \rho_k = |p_{i_0}| = 2^k |m| \). \( \rho_k \geq 2^{k+K} |m| \rho_0 \geq 4^k \rho_0 \).

Case 2: \( \mathbf{p} \) has \( q > 1 \) non-zero components. Then, it follows from Lemma 19 \( \rho_k \geq 2^{kq} \rho_0 \geq 4^k \rho_0 \). \( \square \)

**Example 25.** For \( L_{101} \) in Table 1, since \( K = 4, \rho_1 > 4 \rho_0, \rho_2 > 4^2 \rho_0, \rho_3 > 4^3 \rho_0 \) and \( \rho_4 > 4^4 \rho_0 \). Nevertheless, \( \rho_5 < 4^5 \rho_0 \). For \( L_{103} \), \( K = 5 \).

However, after several augmentations in which the increase in IOM may be significant, it is always the case that a point is reached where the relative increase thereafter is exactly 2.

**Theorem 26.** There exists \( K' > 0 \) such that \( \sigma_k = 2 \) for all \( k > K' \).

**Proof.** Assume \( \mathbf{p}_{k'} = (p_{k'_1}, \ldots, p_{k'_s}) \in P_{k'} \). If there is only one non-zero component in \( \mathbf{p}_{k'} \), it is known from Lemma 20 and Corollary 22 that \( \sigma_k = 2 \), \( k > k' \). Suppose there are \( q \geq 2 \) non-zero components in \( \mathbf{p}_{k'} \). Let \( m \in \mathbb{Z} \) such that \( (2^m)^{q-1} > N \). Assume \( \mathbf{p}_m = (p_{m_1}, \ldots, p_{m_s}) \in P_m \). Then \( \mathbf{p}_m \) has at most \( q - 1 \) non-zero components. Otherwise, we have contradictory inequalities (13) and (14)

\[
\rho_m = \prod_{i=1}^{s} \left( \frac{p_{m_i}}{N} \right) \geq (2^m)^{q-1}. \tag{13}
\]

Conversely, \( (2^m N, 0, \ldots, 0) \in L_{1/N}^1(k) \), thus

\[
\rho_m \leq 2^m N < (2^m)^q. \tag{14}
\]

Therefore, \( \mathbf{p}_m \) has at most \( q - 1 \) non-zero components. Doing this repeatedly, there must exist a \( K' \in \mathbb{Z} \) and \( \mathbf{p}_{k'} \in P_{k'} \) such that there is only one non-zero component in \( \mathbf{p}_{k'} \). It is known from Lemma 20 and Corollary 22 that \( \sigma_k = 2 \) for all \( k > K' \). \( \square \)

**Example 27.** For \( L_{101} \) and \( L_{103} \) in Table 1, \( K' = 3 \) and 4, respectively.

Because different seed lattices may have different \( K' \) in Theorem 26, a seed lattice with high IOM may not lead to much improvement in IOM after a few augmentations while a seed lattice with small IOM can lead to a large number

<table>
<thead>
<tr>
<th>Seed lattice</th>
<th>( \rho_0 )</th>
<th>( \rho_1(\sigma_1) )</th>
<th>( \rho_2(\sigma_2) )</th>
<th>( \rho_3(\sigma_3) )</th>
<th>( \rho_4(\sigma_4) )</th>
<th>( \rho_5(\sigma_5) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_{101} )</td>
<td>6</td>
<td>48 (8)</td>
<td>256 (5.3)</td>
<td>808 (3.2)</td>
<td>1616 (2)</td>
<td>3232 (2)</td>
</tr>
<tr>
<td>( L_{103} )</td>
<td>2</td>
<td>16 (8)</td>
<td>128 (8)</td>
<td>640 (5)</td>
<td>1648 (2.6)</td>
<td>3296 (2)</td>
</tr>
</tbody>
</table>
of significant improvements during copying, resulting in some copied lattices having high IOM. In other words, a high IOM in a \( k \)th-copy lattice does not depend on the original seed lattice having high IOM. This indicates that the search for a good augmentation sequence is not simply a search for a good seed lattice. For example, in Table 1, \( \rho(L_{103}) < \rho(L_{101}) \), but \( \rho(L_{103}(4)) > \rho(L_{101}(4)) \).

The following corollary is straightforward.

**Corollary 28.** \( \lim_{k \to \infty} \sigma_k = 2. \)

Based on Theorem 26 and Corollary 28, we focus our interest on the first few augmentations in a sequence when seeking the criteria for a good augmentation sequence.

**Theorem 29.** Let \( k_c = \min\{k \mid \sigma_k < 4\} \), then

\[
\begin{align*}
\sigma_k \geq 4, & \quad k < k_c, \\
\sigma_k = 2, & \quad k > k_c.
\end{align*}
\]

**Proof.** Since \( \sigma_{k_c - 1} \geq 4 \) and \( \sigma_k \geq \sigma_{k+1} \) (Lemma 20), (15) is correct.

Assume \( p = (p_1, \ldots, p_s) \in P_{k_c} \). Since \( k_c \) is the minimum number of the value \( k \) for which \( \sigma_k < 4 \), we can assert that there is only one non-zero component in \( p \). If there were more than one non-zero components in \( p \), then from Lemma 20, it would follow that

\[
\sigma_{k_c} \geq 4,
\]

which is a contradiction. Because there is only one non-zero component in \( p \), (16) follows from Lemma 20 and Corollary 22. \( \square \)

**Example 30.** For \( L_{101} \) in Table 1, \( \sigma_3 = 3.2 \) and so \( k_c = 3 \). Observe that \( \sigma_1 \geq \sigma_2 \geq 4 \) and \( \sigma_4 = \sigma_5 = 2 \).

**Definition 31.** For a given augmentation sequence, the \( k_c \)th-copy lattice satisfying (15) and (16) is called the critical copy lattice.

Theorem 29 indicates that a critical copy lattice only occurs once in the sequence. Before the critical copy lattice occurs, \( \sigma_k \geq 4 \) and \( \sigma_k = 2 \) afterwards. In other words, cubature rules constructed from lattices before the critical copy lattice are more effective, relative to the number of points in the lattice, than the ones constructed from lattices after the critical copy lattice. So, if maximizing the IOM of the later members of a sequence is the aim, then for a given \( N \), one should seek a seed lattice for which the critical lattice appears as early as possible in the sequence. On the other hand, if the aim is to achieve a better relative separation of the IOM of successive sequence members (perhaps to improve the reliability of an error estimate based on differences between sequence members), then a seed lattice leading to the appearance of the critical lattice as late as possible might be sought. These and other competing aims need to be taken into account in the determination of the criteria for a good sequence.

Theorem 32 relates the IOM of the critical copy lattice to \( N \), the number of points in the seed lattice.

**Theorem 32.** \( k_c = \min\{k \mid \rho_k \geq 2^k N\} \). Furthermore,

\[
\rho_{k+1} = 2^{k+1} N, \quad k \geq k_c.
\]

**Proof.** \( (2^{k_c} N, 0, \ldots, 0) \in L_N^{\perp}(k_c) \Rightarrow \rho_{k_c} \leq 2^{k_c} N \). But by definition \( \rho_{k_c} \geq 2^{k_c} N \) and so \( \rho_{k_c} = 2^{k_c} N \). Using Corollary 22, we know that

\[
\rho_{k_c+1} \geq 2 \rho_{k_c} = 2^{k_c+1} N.
\]

Conversely, from Theorem 29, \( (2^{k_c+1} N, 0, \ldots, 0) \) is the first row of the generator matrix for \( L_N^{\perp}(k_c + 1) \) and so

\[
\rho_{k_c+1} \leq 2^{k_c+1} N.
\]
Thus, \( \rho_{k+1} = 2^{k+1}N \) holds. Repeating this process for \( k_c + 2, k_c + 3, \ldots \), we prove that \( \sigma_k = 2, k > k_c \) and (17) follows.

Now, by definition, for \( 1 \leq k \leq k_c \),

\[
\rho_k < 2^k N. \tag{18}
\]

Assume \( p = (p_1, \ldots, p_s) \in P_k \). There must exist \( j_1, \ldots, j_s \in \mathbb{Z} \) such that

\[
p = j_1v_1 + j_2v_2 + \cdots + j_sv_s, \tag{19}
\]

where \( v_i \) is the \( i \)th row of the generator matrix \( (6) \) for \( L_N^2(k) \). We consider separately the case \( j_1 = 0 \) and \( j_1 \neq 0 \) in (19).

**Case 1**: \( j_1 = 0 \). There are two cases to consider:

1. There is exactly one \( i \), say \( i_1 \neq 1 \), such that \( j_{i_1} \neq 0 \). In this case, \( p_1 \neq 0 \) and \( p_{i_1} \neq 0 \), and so \( p \) has two non-zero components.
2. There is more than one \( i, i \neq 1 \), such that \( j_i \neq 0 \), and therefore \( p \) has at least two non-zero components.

Thus, in all cases when \( j_1 = 0 \), \( p \) has at least two non-zero components and so by Lemma 20, \( \sigma_k \geq 4, k < k_c \).

**Case 2**: \( j_1 \neq 0 \). There must be \( i \neq 1 \) such that \( j_i \neq 0 \). Otherwise, \( p = (j_12^kN, 0, \ldots, 0) \Rightarrow \rho_k \geq 2^kN \), contradicting (18). Again, there are two cases to consider:

1. There is only one \( i_1 \neq 1 \) in (19) such that \( j_{i_1} \neq 0 \). Then \( p = j_1v_1 + j_1v_{i_1} \). If \( p_1 = 0 \), then \( j_1N = j_1z_{i_1} \). Because \( z_{i_1} \) and \( N \) are relatively prime, \( j_1 \) must be divisible by \( N \). That means \( j_1 = cN \), where \( c \in \mathbb{Z} \) and \( c \neq 0 \). Subsequently \( \rho_k \geq |c|2^kN \), which contradicts (18). Thus, \( p \) has two non-zero components, \( p_1 \) and \( p_{i_1} \).
2. There are at least two values of \( i, i_1 \neq 0 \) and \( i_2 \neq 0 \) in (19) such that \( j_i \neq 0 \), and therefore \( p_{i_1} \neq 0 \) and \( p_{i_2} \neq 0 \).

Therefore, in all cases when \( j_1 \neq 0 \), \( p \) has at least two non-zero components. Using Lemma 20, \( \sigma_k \geq 4, k < k_c \).

Together with the proved inequality \( \sigma_k = 2, k > k_c \), we conclude that the \( k_c \)th-copy lattice is a critical copy lattice. \( \square \)

**Corollary 33.** \( \sigma_k < 4 \Rightarrow \rho_k = 2^kN. \)

**Proof.** From Theorem 32, \( k \geq k_c \) when \( \sigma_k < 4 \), and therefore \( \rho_k = 2^kN. \) \( \square \)

Thus, in an augmentation sequence, we know that after a certain number of augmentations, the IOM of \( k \)th-copy lattices is \( 2^kN \). Note that this depends entirely on the size of the seed lattice and not on the location of points in the lattice. Thus, for a given \( N \) and \( k, k \) large enough, the IOM of all \( k \)th-copy lattices is fixed, regardless of the choice of seed lattice. Accordingly, the search for a good augmentation sequence for a given \( N \) must focus on the IOM of sequence members before this point is reached.

From Theorem 32 and Corollary 33, one cannot predict the value of \( k \) for which \( \sigma_k = 2 \) first occurs until the IOM of \( L_N(1), L_N(2), \ldots, L_N(k - 1) \) have been calculated. The following theorems identify when \( \sigma_k \) first drops to 2 by only studying the seed lattice \( L_N \).

**Theorem 34.**

\[
k \geq \log_2 \left( \frac{N}{\rho_0} \right) \Rightarrow \sigma_k = 2.
\]

**Proof.** Firstly, the reader may note that because \( N \) is odd, \( k = \log_2(N/\rho_0) \) never happens. Assume \( p = (p_1, \ldots, p_s) \in P_k \). Then \( p/2^k \in L_N^2 \), and there must exist \( j_1, \ldots, j_s \in \mathbb{Z} \) such that

\[
p = j_1v_1 + j_2v_2 + \cdots + j_sv_s, \tag{20}
\]
where $v_i$ is the $i$th row of (6). If there are $q \geq 2$ non-zero coefficients $j_i$ in (20) other than $j_1$, then,

$$
\rho_k = \prod_{i=1}^{s} \frac{p_i}{k} = (2^k)^q \prod_{i=1}^{s} \frac{p_i}{2k} \geq 2^{2k} \prod_{i=1}^{s} \frac{p_i}{2k} > 2^k N.
$$

(21)

Conversely, since $v_1 = (2^k N, 0, \ldots, 0) \in L_{(2^k)^q N}$,

$$
\rho_k \leq 2^k N,
$$

(22)

which contradicts (21).

If there is only one $j_{i_0}$ other than $j_1$ not equal to zero in (20), then the possible non-zero components of $p$ are $p_1$ and $p_{i_0}$. If $p_1 = 0$, then from Lemma 20, the theorem is proved. If $p_1 \neq 0$ then

$$
\rho_k = |p_1||p_{i_0}| = 2^{2k} \frac{p_1}{p_{i_0}} \frac{p_{i_0}}{2k} > 2^k (2^k \rho_0) > 2^k N,
$$

which contradicts (22). Therefore $\sigma_k = 2$. □

**Example 35.** For $L_{101}$ and $L_{103}$ in Table 1, $\lceil \log_2(N/\rho_0) \rceil = 4$. Observe that $\sigma_5 = 2$ in both cases. Note also that $\sigma_4 = 2$ in the case of $L_{101}$, indicating that the obverse of Theorem 34 is not true.

**Theorem 36.** Let $m = \min_{i=2,\ldots,s} \{|z_i|, |N - z_i|\}$. Then $k < \log_2(N/m) \Rightarrow \sigma_k \geq 4$.

**Proof.** Assume $p = (p_1, \ldots, p_s) \in P_k$. Denote the $i$th row of (6) as $v_i$. There must exist $j_1, \ldots, j_k \in \mathbb{Z}$ such that

$$
p = j_1 v_1 + j_2 v_2 + \cdots + j_k v_k.
$$

(23)

If only $j_1$ in (23) is not equal to 0, then without loss of generality, we can assume that $m = |z_2|$.

$$
\rho_k = \prod_{i=1}^{s} \frac{p_i}{k} = |j_1|2^k N > 2^k \cdot 2^k m = 2^k \cdot 2^k |z_2| = \prod_{i=1}^{s} \frac{p_i}{k}.
$$

(24)

Inequality (24) contradicts the definition of IOM, since $v_2 \in L_N(k)$. The result can be proved for the case, $m = N - z_i$, using the fact that

$$
2^k \begin{pmatrix}
N & 0 & \cdots & 0 \\
-(N - z_2) & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots \\
-(N - z_s) & 0 & \cdots & 1
\end{pmatrix}
$$

is also a generator matrix for $L_N(k)$.

We conclude that there is at least one $j_i$ other than $j_1$ in (23) such that $j_i \neq 0$. There are two cases for this situation:

1. There is more than one $j_i$ other than $j_1$ in (23) such that $j_i \neq 0$. It must be that $p$ has at least two non-zero components, in which case, by Lemma 20, $\sigma_k \geq 4$.
2. There is only one $j_{i_0} \neq 0$ in (23), where $i_0 \neq 1$. In this case,

$$
p = j_1 v_1 + j_{i_0} v_{i_0}
$$

$$
= j_1 (2^k N, 0, \ldots, 0) + j_{i_0} (-2^k z_{i_0}, 0, \ldots, 2^k, 0, \ldots, 0).
$$

If $p_1 = 0$, then

$$
j_1 N = j_{i_0} z_{i_0}.
$$

Since $N$ and $z_{i_0}$ are co-prime, $j_{i_0}$ is divisible by $N$ and thus $|j_{i_0}| \geq N$. Consequently,

$$
\prod_{i=1}^{s} \frac{p_i}{k} \geq |p_{i_0}| = |j_{i_0}|2^k \geq 2^k N > 2^{2k} m,
$$
again contradicting the definition of IOM. Therefore, there are two non-zero components, $p_1$ and $p_{i_0}$ in $p$. By Lemma 20, we conclude that $\sigma_k \geq 4$. □

4. Concluding remarks and future work

The theoretical results in this paper describe how IOM increases during $k$th-copying. The IOM increase factor $\sigma_k$ will drop to 2 after a certain number of augmentations depending on the seed lattice chosen. However, before this happens, IOM increases by a factor of at least 4 and as much as $2^s$ for each augmentation. The minimum factor 4 is optimal for $s = 2$, but increasingly less than optimal for larger values of $s$. The likelihood is that for $s > 3$, $k$th-copy lattices for moderate to large values of $k$ will be uncompetitive (in the sense of both IOM and TDOP) with well-chosen rank-1 lattices of the same order. Coupled with the exponential growth in the number of lattice points, this finding suggests that a search for good augmentation sequences should be limited to dimensionality $\leq 4$.

Based on these theoretical results, future work will focus on the development of criteria for good augmentation sequences in low dimensions, taking into account both the IOM and TDOP of constituent lattices. Using these criteria, a good augmentation sequence can be obtained by computer search and used in developing an automatic cubature algorithm.

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References