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Quantum isometry groups of the Podles spheres

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Abstract

For $\mu \in (0, 1)$, $c \geqslant 0$, we identify the quantum group $SO_\mu(3)$ as the universal object in the category of compact quantum groups acting by 'orientation and volume preserving isometries' in the sense of Bhowmick and Goswami (2009) [4] on the natural spectral triple on the Podles sphere $S^2_{\mu,c}$ constructed by Dabrowski, D'Andrea, Landi and Wagner (2007) in [9].

Keywords: Quantum isometry groups; Podles spheres; Spectral triples

1. Introduction

In a series of articles initiated by [11] and followed by [3,4], we have formulated and studied a quantum group analogue of the group of Riemannian isometries of a classical or noncommutative manifold. This was motivated by previous work of a number of mathematicians including Wang, Banica, Bichon and others (see, e.g. [20,21,1,2,5,22] and the references therein), who have defined quantum automorphism and quantum isometry groups of finite spaces and finite dimensional algebras. Our theory of quantum isometry groups can be viewed as a natural generalization of such quantum automorphism or isometry groups of 'finite' or 'discrete' structures to the continuous or smooth set-up. Clearly, such a generalization is crucial to study the quantum

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symmetries in noncommutative geometry a la Connes [6], and in particular, for a good understanding of quantum group equivariant spectral triples.

The group of Riemannian isometries of a compact Riemannian manifold M can be viewed as the universal object in the category of all compact metrizable groups acting on M, with smooth and isometric action. Moreover, assume that the manifold has a spin structure (hence in particular orientable, so we can fix a choice of orientation) and D denotes the conventional Dirac operator acting as an unbounded self-adjoint operator on the Hilbert space $\mathcal{H}$ of square integrable spinors. Then, it can be proved that the action of a compact group G on the manifold lifts as a unitary representation (possibly of some group $\tilde{G}$ which is topologically a 2-cover of G, see [7] and [8] for more details) on the Hilbert space $\mathcal{H}$ which commutes with D if and only if the action on the manifold is an orientation preserving isometric action. Therefore, to define the quantum analogue of the group of orientation-preserving Riemannian isometry group of a possibly noncommutative manifold given by a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$, it is reasonable to consider a category $\mathbf{Q}'(D)$ of compact quantum groups having unitary (co-)representation, say U, on $\mathcal{H}$, which commutes with D, and the action on $\mathcal{B}(\mathcal{H})$ obtained via conjugation by U maps $\mathcal{A}^\infty$ into its weak closure. A universal object in this category, if it exists, should define the 'quantum group of orientation preserving Riemannian isometries' of the underlying spectral triple. Indeed (see [4]), if we consider a classical spectral triple, the subcategory of the category $\mathbf{Q}'(D)$ consisting of groups has the classical group of orientation preserving isometries as the universal object, which justifies our definition of the quantum analogue. Unfortunately, if we consider quantum group actions, even in the finite dimensional (but with noncommutative $\mathcal{A}$) situation the category $\mathbf{Q}'(D)$ may often fail to have a universal object. It turns out, however, that if we fix any suitable faithful functional τ_R on $\mathcal{B}(\mathcal{H})$ (to be interpreted as the choice of a 'volume form') then there exists a universal object in the subcategory $\mathbf{Q}'_R(D)$ of $\mathbf{Q}'(D)$ obtained by restricting the object-class to the quantum group actions which also preserve the given functional. The subtle point to note here is that unlike the classical group actions on $\mathcal{B}(\mathcal{H})$ which always preserve the usual trace, a quantum group action may not do so. In fact, it was proved by one of the authors in [10] that given an object $(\mathcal{Q}, U)$ of $\mathbf{Q}'(D)$ (where $\mathcal{Q}$ is the compact quantum group and U denotes its unitary co-representation on $\mathcal{H}$), we can find a suitable functional τ_R (which typically differs from the usual trace of $\mathcal{B}(\mathcal{H})$ and can have a nontrivial modularity) which is preserved by the action of $\mathcal{Q}$. This makes it quite natural to work in the setting of twisted spectral data (as defined in [10]). It may also be mentioned that in [4] we have actually worked in slightly bigger category $\mathbf{Q}_R(D)$ of the so-called 'quantum family of orientation and volume preserving isometries' and deduced that the universal object in $\mathbf{Q}_R(D)$ exists and coincides with that of $\mathbf{Q}'_R(D)$.

It is very important to explicitly compute the (orientation and volume preserving) quantum group of isometries for as many examples as possible. This programme has been successfully carried out for a number of spectral triples, including classical spheres and tori as well as their Rieffel deformations. The aim of the present article is to identify $SO_\mu(3)$ as the quantum group of orientation and volume preserving isometries for the spectral triples on the Podles spheres $S^2_{\mu,c}$, constructed by Dabrowski et al. in [9]. Let us mention here that although the quantum groups $SO_\mu(3)$ are 'deformations' of the classical $SO(3)$, these are not Rieffel deformations and so the results and techniques of [4] do not apply.

Our characterization of $SO_\mu(3)$ as the quantum isometry group of a noncommutative Riemannian manifold generalizes the classical description of the group $SO(3)$ as the group of orientation preserving isometries of the usual Riemannian structure on the 2-sphere. It may be mentioned here that in a very recent article [19], P.M. Soltan has characterized $SO_\mu(3)$ as the universal compact quantum group acting on the finite dimensional C^*-algebra $M_2(\mathbb{C})$ such that the action preserves a functional ω_μ defined in [19]. In the classical case, we have three equivalent descriptions of $SO(3)$: (a) as a quotient of $SU(2)$, (b) as the group of (orientation preserving) isometries of S^2, and (c) as the automorphism group of M_2. In the quantum case the definition of $SO_\mu(3)$ is an analogue of (a), so the characterization of $SO_\mu(3)$ obtained in this paper as the quantum isometry group, together with Soltan's characterization, completes the generalization of all three classical descriptions of $SO(3)$.

2. Notations and preliminaries

2.1. Basics of the theory of compact quantum groups

We begin by recalling the definition of compact quantum groups and their actions from [24,23]. A compact quantum group (to be abbreviated as CQG from now on) is given by a pair $(\mathcal{S}, \Delta)$, where $\mathcal{S}$ is a unital separable C^* algebra equipped with a unital C^*-homomorphism $\Delta : \mathcal{S} \to \mathcal{S} \otimes \mathcal{S}$ (where $\otimes$ denotes the injective tensor product) satisfying

(ai) $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ (co-associativity), and
(aii) the linear spans of $\Delta(\mathcal{S})(\mathcal{S} \otimes 1)$ and $\Delta(\mathcal{S})(1 \otimes \mathcal{S})$ are norm-dense in $\mathcal{S} \otimes \mathcal{S}$.

It is well known (see [24,23]) that there is a canonical dense $*$-subalgebra $\mathcal{S}_0$ of $\mathcal{S}$, consisting of the matrix coefficients of the finite dimensional unitary (co)-representations (to be defined below) of $\mathcal{S}$, and maps $\epsilon : \mathcal{S}_0 \to \mathbb{C}$ (co-unit) and $\kappa : \mathcal{S}_0 \to \mathcal{S}_0$ (antipode) defined on $\mathcal{S}_0$ which make $\mathcal{S}_0$ a Hopf $*$-algebra.

A CQG $(\mathcal{S}, \Delta)$ is said to (co)-act on a unital C^* algebra $\mathcal{B}$, if there is a unital C^*-homomorphism (called an action) $\alpha : \mathcal{B} \to \mathcal{B} \otimes \mathcal{S}$ satisfying the following:

(bi) $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$, and
(bii) the linear span of $\alpha(\mathcal{B})(1 \otimes \mathcal{S})$ is norm-dense in $\mathcal{B} \otimes \mathcal{S}$.

For a Hilbert $\mathcal{B}$-module E, (where $\mathcal{B}$ is a C^* algebra) we shall denote the set of adjointable $\mathcal{B}$-linear maps on E by $\mathcal{L}(E)$. The norm-closure of the linear span of the finite-rank $\mathcal{B}$-linear maps on E, to be called the set of compact operators on E, will be denoted by $\mathcal{K}(E)$. We note that $\mathcal{L}(E) = \mathcal{M}(\mathcal{K}(E))$, where $\mathcal{M}(\mathcal{C})$ denotes the multiplier algebra of a C^*-algebra $\mathcal{C}$. We shall also need the 'leg-numbering' notation: for an operator X in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, $X_{(12)}$ and $X_{(13)}$ will denote the operators $X \otimes I_{\mathcal{H}_2}$ in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2)$, and $\Sigma_{23} X_{12} \Sigma_{23}$, respectively, where Σ_{23} is the unitary on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2$ which flips the two copies of $\mathcal{H}_2$.

A unitary (co)-representation of a CQG $(\mathcal{S}, \Delta)$ on a Hilbert space $\mathcal{H}$ is given by a unitary element U of $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{S}) \equiv \mathcal{L}(\mathcal{H} \otimes \mathcal{S})$ satisfying

$$(\mathrm{id} \otimes \Delta)(U) = U_{(12)} U_{(13)}.$$

Given a unitary representation U we shall denote by α_U the $*$-homomorphism $\alpha_U(X) = U(X \otimes 1_{\mathcal{S}})U^*$ for X belonging to $\mathcal{B}(\mathcal{H})$. We shall sometimes identify U with the isometric map from the Hilbert space $\mathcal{H}$ to the Hilbert module $\mathcal{H} \otimes \mathcal{S}$ which sends a vector ξ of $\mathcal{H}$ to $U(\xi \otimes 1)$, and may even denote $U(\xi \otimes 1)$ by $U\xi$ by a slight abuse of notation. We say that a (possibly unbounded) operator T on $\mathcal{H}$ commutes with U if $T \otimes I$ (with the natural domain) commutes with U. Such an operator will also be called U-equivariant or $\mathcal{S}$-equivariant if U is understood.

2.2. The quantum group of orientation preserving Riemannian isometries

We briefly recall the definition of the quantum group of orientation preserving Riemannian isometries for a spectral triple (of compact type) $(\mathcal{A}^\infty, \mathcal{H}, D)$ as in [4]. We consider the category $\mathbf{Q}'(\mathcal{A}^\infty, \mathcal{H}, D) \equiv \mathbf{Q}'(D)$ whose objects (to be called orientation preserving isometries) are the triplets $(\mathcal{S}, \Delta, U)$, where $(\mathcal{S}, \Delta)$ is a CQG with a unitary representation U in $\mathcal{H}$, satisfying the following:

(i) U commutes with D,
(ii) for every state ω on $\mathcal{S}$, $(\mathrm{id} \otimes \omega) \circ \alpha_U(a)$ belongs to $(\mathcal{A}^\infty)''$ for all a in $\mathcal{A}^\infty$.

The category $\mathbf{Q}'(D)$ may not have a universal object in general, as pointed out in [4]. In case there is a universal object, we shall denote it by $\widetilde{QISO^+}(D)$, with the corresponding representation U, say, and we denote by $QISO^+(D)$ the Woronowicz subalgebra of $\widetilde{QISO^+}(D)$ generated by the elements of the form $\langle \xi \otimes 1, \alpha_U(a)(\eta \otimes 1)\rangle$, where ξ, η belong to $\mathcal{H}$, a belongs to $\mathcal{A}^\infty$ and $\langle \cdot, \cdot \rangle$ is the $\widetilde{QISO^+}(D)$-valued inner product of $\mathcal{H} \otimes \widetilde{QISO^+}(D)$. The quantum group $QISO^+(D)$ will be called the quantum group of orientation-preserving Riemannian isometries of the spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$.

Although the category $\mathbf{Q}'(D)$ may fail to have a universal object, we can always get a universal object in suitable subcategories which will be described now. Suppose that we are given an invertible positive (possibly unbounded) operator R on $\mathcal{H}$ which commutes with D. Then we consider the full subcategory $\mathbf{Q}'_R(D)$ of $\mathbf{Q}'(D)$ by restricting the object class to those $(\mathcal{S}, \Delta, U)$ for which α_U satisfies $(\tau_R \otimes \mathrm{id})(\alpha_U(X)) = \tau_R(X)1$ for all X in the $*$-subalgebra generated by operators of the form $|\xi\rangle\langle\eta|$, where ξ, η are eigenvectors of the operator D which by assumption has discrete spectrum, and $\tau_R(X) = \mathrm{Tr}(RX) = \langle \eta, R\xi \rangle$ for $X = |\xi\rangle\langle\eta|$. We shall call the objects of $\mathbf{Q}'_R(D)$ orientation and (R-twisted) volume preserving isometries. It is clear (see Remark 2.9 in [4]) that when Re^{-tD^2} is trace-class for some $t > 0$, the above condition is equivalent to the condition that α_U preserves the bounded normal functional $\mathrm{Tr}(\cdot\, Re^{-tD^2})$ on the whole of $\mathcal{B}(\mathcal{H})$. It is shown in [4] that the category $\mathbf{Q}'_R(D)$ always admits a universal object, to be denoted by $\widetilde{QISO^+_R}(D)$, and the Woronowicz subalgebra generated by $\{\langle \xi \otimes 1, \alpha_W(a)(\eta \otimes 1)\rangle \colon\ \xi, \eta \in \mathcal{H},\ a \in \mathcal{A}^\infty\}$ (where W is the unitary representation of $\widetilde{QISO^+_R}(D)$ in $\mathcal{H}$) will be denoted by $QISO^+_R(D)$ and called the quantum group of orientation and (R-twisted) volume preserving Riemannian isometries of the spectral triple.

2.3. $SU_\mu(2)$ and the Podles spheres

Fix μ in $(0, 1)$. The C^* algebra underlying the CQG $SU_\mu(2)$ is defined as the universal unital C^* algebra generated by α, γ such that $\alpha^*\alpha + \gamma^*\gamma = 1$, $\alpha\alpha^* + \mu^2\gamma\gamma^* = 1$, $\gamma\gamma^* = \gamma^*\gamma$, $\mu\gamma\alpha = \alpha\gamma$, $\mu\gamma^*\alpha = \alpha\gamma^*$.

The CQG structure is given by the following fundamental representation: $\begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$.

The coproduct is defined by:

$$\Delta(\alpha) = \alpha \otimes \alpha - \mu\gamma^* \otimes \gamma,$$
$$\Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

The Haar state of $SU_\mu(2)$ will be denoted by h and the corresponding G.N.S. Hilbert space will be denoted by $L^2(SU_\mu(2))$. We will call the unital $*$-subalgebra of $SU_\mu(2)$ (without any norm-closure) generated by α, γ the 'co-ordinate Hopf $*$-algebra' of $SU_\mu(2)$ and denote it by $\mathcal{O}(SU_\mu(2))$ as in [17].

We now recall the definition of the Podles sphere from [9] (see also the original article [15] by Podles).

For $c \geqslant 0$, let t in $(0, 1]$ be given by $c = t^{-1} - t$. Let $[n] \equiv [n]_\mu = \frac{\mu^n - \mu^{-n}}{\mu - \mu^{-1}}$, $n \in \mathbb{N}$.

The Podles sphere $S^2_{\mu,c}$ is defined to be the universal unital C^* algebra generated by elements x_{-1}, x_0, x_1 satisfying the relations:

$$\begin{aligned} x_{-1}(x_0 - t) &= \mu^2(x_0 - t)x_{-1}, \\ x_1(x_0 - t) &= \mu^{-2}(x_0 - t)x_1, \\ -[2]x_{-1}x_1 + \big(\mu^2 x_0 + t\big)(x_0 - t) &= [2]^2(1 - t), \\ -[2]x_1x_{-1} + \big(\mu^{-2} x_0 + t\big)(x_0 - t) &= [2]^2(1 - t). \end{aligned}$$

The involution on $S^2_{\mu,c}$ is given by

$$x^*_{-1} = -\mu^{-1}x_1, \qquad x^*_0 = x_0.$$

We note that $S^2_{\mu,c}$ as defined above is the same as $\chi_{q,\alpha',\beta}$ in [13, p. 124] with $q = \mu$, $\alpha' = t$, $\beta = t^2 + \mu^{-2}(\mu^2 + 1)^2(1 - t)$.

Thus, from the expressions of x_{-1}, x_0, x_1 given in [13, p. 125], it follows that $S^2_{\mu,c}$ can be realized as a $*$-subalgebra of $SU_\mu(2)$ by setting:

$$x_{-1} = \frac{\mu\alpha^2 + \rho(1 + \mu^2)\alpha\gamma - \mu^2\gamma^2}{\mu(1 + \mu^2)^{\frac{1}{2}}}, \tag{1}$$

$$x_0 = -\mu\gamma^*\alpha + \rho\big(1 - \big(1 + \mu^2\big)\gamma^*\gamma\big) - \gamma\alpha^*, \tag{2}$$

$$x_1 = \frac{\mu^2\gamma^{*2} - \rho\mu(1 + \mu^2)\alpha^*\gamma^* - \mu\alpha^{*2}}{(1 + \mu^2)^{\frac{1}{2}}}, \tag{3}$$

where $\rho^2 = \frac{\mu^2 t^2}{(\mu^2+1)^2(1-t)}$.

Taking

$$A = \frac{1 - t^{-1}x_0}{1 + \mu^2}, \qquad B = \mu\big(1 + \mu^2\big)^{-\frac{1}{2}} t^{-1} x_{-1},$$

one obtains (see [9]) that the C^* algebra $S^2_{\mu,c}$ coincides with the original description given in [15], i.e., the universal C^* algebra generated by elements A and B satisfying the relations:

$$\begin{aligned} A^* = A, &\qquad AB = \mu^{-2}BA, \\ B^*B = A - A^2 + cI, &\qquad BB^* = \mu^2 A - \mu^4 A^2 + cI. \end{aligned}$$

We will denote by $\mathcal{O}(S^2_{\mu,c})$ the co-ordinate $*$-algebra of $S^2_{\mu,c}$, i.e. the unital $*$-subalgebra generated by A, B.

We recall from [17] the Hopf $*$-algebra $\mathcal{U}_\mu(su(2))$ which is generated by elements F, E, K, K^{-1} with defining relations:

$$KK^{-1} = K^{-1}K = 1, \qquad KE = \mu EK, \qquad FK = \mu KF,$$
$$EF - FE = \left(\mu - \mu^{-1}\right)^{-1}\left(K^2 - K^{-2}\right)$$

with involution $E^* = F$, $K^* = K$ and comultiplication:

$$\Delta(E) = E \otimes K + K^{-1} \otimes E, \qquad \Delta(F) = F \otimes K + K^{-1} \otimes F, \qquad \Delta(K) = K \otimes K.$$

The counit is given by $\epsilon(E) = \epsilon(F) = \epsilon(K - 1) = 0$ and antipode $S(K) = K^{-1}$, $S(E) = -\mu E$, $S(F) = -\mu^{-1}F$.

There is a dual pairing $\langle .,. \rangle$ of $\mathcal{U}_\mu(su(2))$ and $\mathcal{O}(SU_\mu(2))$, for which the nonzero values of the pairing among the generators are given below:

$$\left\langle K^{\pm 1}, \alpha^* \right\rangle = \left\langle K^{\mp 1}, \alpha \right\rangle = \mu^{\pm \frac{1}{2}}, \qquad \langle E, \gamma \rangle = \left\langle F, -\mu\gamma^* \right\rangle = 1.$$

The left action $\triangleright$ and right action $\triangleleft$ of $\mathcal{U}_\mu(su(2))$ on $SU_\mu(2)$ are given by:

$$f \triangleright x = \langle f, x_{(2)} \rangle x_{(1)}, \qquad x \triangleleft f = \langle f, x_{(1)} \rangle x_{(2)}, \quad x \in \mathcal{O}\left(SU_\mu(2)\right),\ f \in \mathcal{U}_\mu\left(su(2)\right),$$

where we have used the Sweedler notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$.

The actions satisfy the following:

$$(f \triangleright x)^* = S(f)^* \triangleright x^*, \qquad (x \triangleleft f)^* = x^* \triangleleft S(f)^*,$$
$$f \triangleright xy = (f_{(1)} \triangleright x)(f_{(2)} \triangleright y), \qquad xy \triangleleft f = (x \triangleleft f_{(1)})(y \triangleleft f_{(2)}).$$

The action on generators is given by:

$$E \triangleright \alpha = -\mu\gamma^*, \qquad E \triangleright \gamma = \alpha^*, \qquad E \triangleright \gamma^* = E \triangleright \alpha^* = 0,$$
$$F \triangleright \left(-\mu\gamma^*\right) = \alpha, \qquad F \triangleright \alpha^* = \gamma, \qquad F \triangleright \alpha = F \triangleright \gamma = 0,$$
$$K \triangleright \alpha = \mu^{-\frac{1}{2}}\alpha, \qquad K \triangleright \left(\gamma^*\right) = \mu^{\frac{1}{2}}\gamma^*, \qquad K \triangleright \gamma = \mu^{-\frac{1}{2}}\gamma, \qquad K \triangleright \alpha^* = \mu^{\frac{1}{2}}\alpha^*;$$
$$\gamma \triangleleft E = \alpha, \qquad \alpha^* \triangleleft E = -\mu\gamma^*, \qquad \alpha \triangleleft E = \gamma^* \triangleleft E = 0,$$
$$\alpha \triangleleft F = \gamma, \qquad -\mu\gamma^* \triangleleft F = \alpha^*, \qquad \gamma \triangleleft F = \alpha^* \triangleleft F = 0,$$
$$\alpha \triangleleft K = \mu^{-\frac{1}{2}}\alpha, \qquad \gamma^* \triangleleft K = \mu^{-\frac{1}{2}}\gamma^*, \qquad \gamma \triangleleft K = \mu^{\frac{1}{2}}\gamma, \qquad \alpha^* \triangleleft K = \mu^{\frac{1}{2}}\alpha^*.$$

We recall an alternative description of $S^2_{\mu,c}$ from [18] which we are going to need.

Let

$$X_c = \mu^{\frac{1}{2}}\left(\mu^{-1} - \mu\right)^{-1} c^{-\frac{1}{2}}\left(1 - K^2\right) + EK + \mu FK, \quad c > 0,$$
$$X_0 = 1 - K^2.$$

One has $\Delta(X_c) = 1 \otimes X_c + X_c \otimes K^2$. Moreover, we have the following [18, p. 9]:

Theorem 2.1. *We have,*

$$\mathcal{O}\left(S^2_{\mu,c}\right) = \left\{x \in \mathcal{O}\left(SU_\mu(2)\right)\colon\ x \triangleleft X_c = 0\right\}.$$

A basis of the vector space $\mathcal{O}(S^2_{\mu,c})$ is given by $\{A^k, A^k B^l, A^k B^{*m},\ k \geqslant 0,\ l, m > 0\}$.

Thus, any element of $\mathcal{O}(S^2_{\mu,c})$ can be written as a *finite* linear combination of elements of the form A^k, $A^k B^l$, $A^k B^{*l}$.

Let ψ be the densely defined linear map on $L^2(SU_\mu(2))$ defined by $\psi(x) = x \triangleleft X_c$.

Lemma 2.2. *The map ψ is closable and we have $\overline{S^2_{\mu,c}} \subseteq \mathrm{Ker}(\overline{\psi})$ where $\overline{\psi}$ is the closed extension of ψ and $\overline{S^2_{\mu,c}}$ denotes the Hilbert subspace generated by $S^2_{\mu,c}$ in $L^2(SU_\mu(2))$. Moreover, $\mathcal{O}(S^2_{\mu,c}) = \mathcal{O}(SU_\mu(2)) \cap \mathrm{Ker}(\overline{\psi}) = \mathcal{O}(SU_\mu(2)) \cap \mathrm{Ker}(\psi)$.*

Proof. From the expression of X_c, it is clear that $\mathcal{O}(SU_\mu(2)) \subseteq \mathrm{Dom}(\psi^*)$ implying that ψ is closable, hence $\mathrm{Ker}(\overline{\psi})$ is closed. The lemma now follows from the observation that $\mathcal{O}(S^2_{\mu,c}) = \mathrm{Ker}(\psi) \subseteq \mathrm{Ker}(\overline{\psi})$. □

We end this subsection with a discussion on the CQG $SO_\mu(3)$ as described in [16].

It is the universal unital C^* algebra generated by elements M, N, G, C, L satisfying:

$$L^*L = (I - N)\left(I - \mu^{-2}N\right), \qquad LL^* = \left(I - \mu^2 N\right)\left(I - \mu^4 N\right), \qquad G^*G = GG^* = N^2,$$
$$M^*M = N - N^2, \qquad MM^* = \mu^2 N - \mu^4 N^2, \qquad C^*C = N - N^2,$$
$$CC^* = \mu^2 N - \mu^4 N^2, \qquad LN = \mu^4 NL, \qquad GN = NG, \qquad MN = \mu^2 NM,$$
$$CN = \mu^2 NC, \qquad LG = \mu^4 GL, \qquad LM = \mu^2 ML, \qquad MG = \mu^2 GM, \qquad CM = MC,$$
$$LG^* = \mu^4 G^* L, \qquad M^2 = \mu^{-1} LG, \qquad M^*L = \mu^{-1}(I - N)C, \qquad N^* = N.$$

This CQG can be identified with a Woronowicz subalgebra of $SU_\mu(2)$ by taking:

$$N = \gamma^*\gamma, \qquad M = \alpha\gamma, \qquad C = \alpha\gamma^*, \qquad G = \gamma^2, \qquad L = \alpha^2.$$

The canonical action of $SU_\mu(2)$ on $S^2_{\mu,c}$, i.e. the action obtained by restricting the coproduct of $SU_\mu(2)$ to the subalgebra $S^2_{\mu,c}$, is actually a faithful action of $SO_\mu(3)$. On the subspace spanned by $\{x_{-1}, x_0, x_1\}$ this action is given by the following $SO_\mu(3)$-valued 3×3-matrix:

$$Z' := \begin{pmatrix} \alpha^2 & -\mu(1+\mu^{-2})^{\frac{1}{2}}\alpha\gamma^* & \mu^2\gamma^{*2} \\ (1+\mu^{-2})^{\frac{1}{2}}\alpha\gamma & I-\mu(\mu+\mu^{-1})\gamma^*\gamma & -\mu(1+\mu^{-2})^{\frac{1}{2}}\gamma^*\alpha^* \\ \gamma^2 & (1+\mu^{-2})^{\frac{1}{2}}\gamma\alpha^* & \alpha^{*2} \end{pmatrix}.$$

3. Spectral triples on the Podles spheres and their quantum isometry groups

3.1. Description of the spectral triples

We now recall the spectral triples on $S^2_{\mu,c}$ discussed in [9] (see also [17] for the case $c = 0$).

Let $s = -c^{-\frac{1}{2}}\lambda_-$, $\lambda_\pm = \frac{1}{2} \pm (c + \frac{1}{4})^{\frac{1}{2}}$.

For all j in $\frac{1}{2}\mathbb{N}$,

$$\begin{aligned} u_j &= (\alpha^* - s\gamma^*)(\alpha^* - \mu^{-1}s\gamma^*)\ldots(\alpha^* - \mu^{-2j+1}s\gamma^*), \\ w_j &= (\alpha - \mu s\gamma)(\alpha - \mu^2 s\gamma)\ldots(\alpha - \mu^{2j}s\gamma), \\ u_{-j} &= E^{2j} \triangleright w_j, \\ u_0 &= w_0 = 1, \\ y_1 &= (1+\mu^{-2})^{\frac{1}{2}}(c^{\frac{1}{2}}\mu^2\gamma^{*2} - \mu\gamma^*\alpha^* - \mu c^{\frac{1}{2}}\alpha^{*2}), \\ N^l_{kj} &= \| F^{l-k} \triangleright (y_1{}^{l-|j|}u_j)\|^{-1}. \end{aligned}$$

Define $v^l_{k,j} = N^l_{k,j} F^{l-k} \triangleright (y_1^{l-|j|}u_j)$, $l \in \frac{1}{2}\mathbb{N}_0$, $j, k = -l, -l+1, \ldots, l$.

Let $\mathcal{M}_N$ be the Hilbert subspace of $L^2(SU_\mu(2))$ with the orthonormal basis $\{v^l_{m,N}$: $l = |N|, |N|+1, \ldots,\ m = -l, \ldots, l\}$.

Set

$$\mathcal{H} = \mathcal{M}_{-\frac{1}{2}} \oplus \mathcal{M}_{\frac{1}{2}},$$

and define a representation π of $S^2_{\mu,c}$ on $\mathcal{H}$ by

$$\pi(x_i)v^l_{m,N} = \alpha^-_i(l,m;N)v^{l-1}_{m+i,N} + \alpha^0_i(l,m;N)v^l_{m+i,N} + \alpha^+_i(l,m;N)v^{l+1}_{m+i,N},$$

where α^-_i, α^0_i, α^+_i are as defined in [9].

We will often identify $\pi(S^2_{\mu,c})$ with $S^2_{\mu,c}$.

Finally by Proposition 7.2 of [9], the following Dirac operator D gives a spectral triple $(\mathcal{O}(S^2_{\mu,c}), \mathcal{H}, D)$ which we are going to work with:

$$D(v^l_{m,\pm\frac{1}{2}}) = (c_1 l + c_2)v^l_{m,\mp\frac{1}{2}}$$

where c_1, c_2 belong to $\mathbb{R}$, $c_1 \neq 0$.

It is easy to see that the action of $SU_\mu(2)$ on itself keeps the subspace $\mathcal{H}$ invariant and so induces a unitary representation, say U_0 on $\mathcal{H}$.

We define a positive, unbounded operator R on $\mathcal{H}$ by $R(v^n_{i,\pm\frac{1}{2}}) = \mu^{-2i} v^n_{i,\pm\frac{1}{2}}$.

Proposition 3.1. *α_{U_0} preserves the R-twisted volume. In particular, for x belonging to $\pi(S^2_{\mu,c})$ and $t > 0$, we have $h(x) = \frac{\tau_R(x)}{\tau_R(1)}$, where $\tau_R(x) := \mathrm{Tr}(xRe^{-tD^2})$, and h denotes the restriction of the Haar state of $SU_\mu(2)$ to the subalgebra $S^2_{\mu,c}$, which is the unique $SU_\mu(2)$-invariant state on $S^2_{\mu,c}$.*

Proof. It is enough to prove that τ_R is α_{U_0}-invariant. Define $R_0(v^n_{i,\pm\frac{1}{2}}) = \mu^{-2i\mp 1} v^n_{i,\pm\frac{1}{2}}$, and note that it has been observed in [12] that $\mathrm{Tr}(R_0 e^{-tD^2}) < \infty$ (for all $t > 0$) and one has

$$(\tau_{R_0} \otimes \mathrm{id})\big(U_0(x \otimes 1)U_0{}^*\big) = \tau_{R_0}(x).1,$$

for all x in $\mathcal{B}(\mathcal{H})$, where $\tau_{R_0}(x) = \mathrm{Tr}(xR_0 e^{-tD^2})$.

Let us denote by $P_{\frac{1}{2}}, P_{-\frac{1}{2}}$ the projections onto the closed subspaces generated by $\{v^l_{i,\frac{1}{2}}\}$ and $\{v^l_{i,-\frac{1}{2}}\}$ respectively. Moreover, let $\tau_\pm$ be the functionals defined by $\tau_\pm(x) = \mathrm{Tr}(xR_0 P_{\pm\frac{1}{2}} e^{-tD^2})$. We observe that R_0, e^{-tD^2} and U_0 commute with $P_{\pm\frac{1}{2}}$ so that for x belonging to $\mathcal{B}(\mathcal{H})$,

$$(\tau_\pm \otimes \mathrm{id})\big(\alpha_{U_0}(x)\big) = (\tau_{R_0} \otimes \mathrm{id})\big(\alpha_{U_0}(xP_{\pm\frac{1}{2}})\big) = \tau_{R_0}(xP_{\pm\frac{1}{2}})1 = \tau_\pm(x)1,$$

i.e. $\tau_\pm$ are α_{U_0}-invariant. Moreover, since we have $RP_{\pm\frac{1}{2}} = \mu^{\pm} R_0 P_{\pm\frac{1}{2}}$, the functional τ_R coincides with $\mu^{-1}\tau_+ + \mu\tau_-$, hence is α_{U_0}-invariant. □

Theorem 3.2. *$(SU_\mu(2), \Delta, U_0)$ is an object in $\mathbf{Q}'_R(D)$.*

Proof. The above spectral triple is equivariant with respect to this representation (see [9]) and it preserves τ_R by Proposition 3.1, which completes the proof. □

We now note down some useful facts for later use.

Remark 3.3. Using the definition of $v^l_{i,j}$ and $\triangleright$, we observe:

1. The eigenspaces of D corresponding to $(c_1 l + c_2)$ and $-(c_1 l + c_2)$ are $\mathrm{span}\{v^l_{m,\frac{1}{2}} + v^l_{m,-\frac{1}{2}}: -l \leqslant m \leqslant l\}$ and $\mathrm{span}\{v^l_{m,\frac{1}{2}} - v^l_{m,-\frac{1}{2}}: -l \leqslant m \leqslant l\}$ respectively.
2. The eigenspace of $|D|$ corresponding to the eigenvalue $(c_1.\frac{1}{2} + c_2)$ is $\mathrm{span}\{\alpha, \gamma, \alpha^*, \gamma^*\}$.

Remark 3.4.

1. $\pi(A)v^l_{m,N}$ belongs to $\mathrm{Span}\{v^{l-1}_{m,N}, v^l_{m,N}, v^{l+1}_{m,N}\}$,
   $\pi(B)v^l_{m,N}$ belongs to $\mathrm{Span}\{v^{l-1}_{m-1,N}, v^l_{m-1,N}, v^{l+1}_{m-1,N}\}$,
   $\pi(B^*)v^l_{m,N}$ belongs to $\mathrm{Span}\{v^{l-1}_{m+1,N}, v^l_{m+1,N}, v^{l+1}_{m+1,N}\}$.

2. $\pi(A^k)(v^l_{m,N})$ belongs to $\mathrm{Span}\{v^{l-k}_{m,N}, v^{l-k+1}_{m,N}, \ldots, v^{l+k}_{m,N}\}$.
3. $\pi(A^{m'}B^{n'})(v^l_{m,N})$ belongs to $\mathrm{Span}\{v^{l-m'-n'}_{m-n',N}, v^{l-(n'+m'-1)}_{m-n',N}, \ldots, v^{l+n'+m'}_{m-n',N}\}$.
4. $\pi(A^rB^{*s})(v^l_{m,N})$ belongs to $\mathrm{Span}\{v^{l-s-r}_{m+s,N}, v^{l-s-r+1}_{m+s,N}, \ldots, v^{l+s+r}_{m+s,N}\}$.

We shall now proceed to show that $QISO^+_R(D)$ is isomorphic with $SO_\mu(3)$. Let $(\tilde{\mathcal{Q}}, U)$ be an object in the category $\mathbf{Q}'_R(D)$ of CQG s acting by orientation and R-twisted volume preserving isometries on this spectral triple and $\mathcal{Q}$ be the Woronowicz C^* subalgebra of $\tilde{\mathcal{Q}}$ generated by $\langle(\xi \otimes 1), \alpha_U(a)(\eta \otimes 1)\rangle_{\tilde{\mathcal{Q}}}$, for ξ, η in $\mathcal{H}$, a in $S^2_{\mu,c}$ (where $\langle\cdot,\cdot\rangle_{\tilde{\mathcal{Q}}}$ is the $\tilde{\mathcal{Q}}$-valued inner product of $\mathcal{H} \otimes \tilde{\mathcal{Q}}$). We shall denote α_U by ϕ from now on.

The proof has two main steps: first, we prove that ϕ is 'linear', in the sense that it keeps the span of $\{1, A, B, B^*\}$ invariant, and then we shall exploit the facts that ϕ is a $*$-homomorphism and preserves the canonical volume form on $S^2_{\mu,c}$, i.e. the restriction of the Haar state of $SU_\mu(2)$.

Remark 3.5. The first step does not make use of the fact that ϕ preserves the R-twisted volume, so linearity of the action follows for any object in the bigger category $\mathbf{Q}'(D)$.

3.2. Linearity of the action

For a vector v in $\mathcal{H}$, we shall denote by T_v the map from $\mathcal{B}(\mathcal{H})$ to $L^2(SU_\mu(2))$ given by $T_v(x) = xv \in \mathcal{H} \subset L^2(SU_\mu(2))$. It is clearly a continuous map with respect to the strong operator topology on $\mathcal{B}(\mathcal{H})$ and the Hilbert space topology of $L^2(SU_\mu(2))$.

For an element a in $SU_\mu(2)$, we consider the right multiplication R_a as a bounded linear map on $L^2(SU_\mu(2))$. Clearly the composition R_aT_v is a continuous linear map from $\mathcal{B}(\mathcal{H})$ (with the strong operator topology) to the Hilbert space $L^2(SU_\mu(2))$. We now define

$$T = R_{\alpha^*}T_\alpha + \mu^2 R_\gamma T_{\gamma^*}.$$

Lemma 3.6. *For any state ω on $\tilde{\mathcal{Q}}$ and x in $S^2_{\mu,c}$, we have $T(\phi_\omega(x)) = \phi_\omega(x) \equiv R_1(\phi_\omega(x))$ belonging to $\overline{S^2_{\mu,c}} \subseteq L^2(SU_\mu(2))$, where $\phi_\omega(x) = (\mathrm{id} \otimes \omega)(\phi(x))$.*

Proof. It is clear from the definition of T (using $\alpha\alpha^* + \mu^2\gamma\gamma^* = 1$) that $T(x) = x \equiv R_1(x)$ for x in $S^2_{\mu,c} \subset \mathcal{B}(\mathcal{H})$, where x in the right-hand side of the above denotes the identification of $x \in S^2_{\mu,c}$ as a vector in $L^2(SU_\mu(2))$. Now, the lemma follows by noting that for x in $S^2_{\mu,c}$, $\phi_\omega(x)$ belongs to $(S^2_{\mu,c})''$, which is the closure of $S^2_{\mu,c}$ in the strong operator topology, and the continuity of T in this topology discussed before. □

Let

$$\mathcal{V}^l = \mathrm{Span}\left\{v^{l'}_{i,\pm\frac{1}{2}},\ -l' \leqslant i \leqslant l',\ l' \leqslant l\right\}.$$

Since $\mathrm{Span}\{v^l_{i,\pm\frac{1}{2}}, -l \leqslant i \leqslant l\}$ is the eigenspace of $|D|$ corresponding to the eigenvalue $c_1l + c_2$, U and U^* must keep $\mathcal{V}^l$ invariant for all l.

Lemma 3.7. *There is some finite dimensional subspace $\mathcal{V}$ of $\mathcal{O}(SU_\mu(2))$ such that $R_{\alpha^*}(\phi_\omega(A)v^{\frac{1}{2}}_{j,\pm\frac{1}{2}})$, $R_\gamma(\phi_\omega(A)v^{\frac{1}{2}}_{j,\pm\frac{1}{2}})$ belong to $\mathcal{V}$ for all states ω on $\tilde{\mathcal{Q}}$.*

The same holds when A is replaced by B or B^*.

Proof. We prove the result for A only, since a similar argument will work for B and B^*.

We have $\phi(A)(v^{\frac{1}{2}}_{j,\pm\frac{1}{2}} \otimes 1) = U(\pi(A) \otimes 1)U^*(v^{\frac{1}{2}}_{j,\pm\frac{1}{2}} \otimes 1)$.

Now, $U^*(v^{\frac{1}{2}}_{j,\pm\frac{1}{2}} \otimes 1)$ belongs to $\mathcal{V}^{\frac{1}{2}} \otimes \tilde{\mathcal{Q}}$, and then using the definition of π as well as Remark 3.4, we observe that $(\pi(A) \otimes 1)U^*(v^{\frac{1}{2}}_{j,\pm\frac{1}{2}} \otimes 1)$ belongs to $\mathrm{Span}\{v^{l'}_{j,\pm\frac{1}{2}}:\ -l' \leqslant j \leqslant l',\ l' \leqslant \frac{3}{2}\} \otimes \tilde{\mathcal{Q}} = \mathcal{V}^{\frac{3}{2}} \otimes \tilde{\mathcal{Q}}$. Again, U keeps $\mathcal{V}^{\frac{3}{2}} \otimes \tilde{\mathcal{Q}}$ invariant, so $R_{\alpha^*}(\phi_\omega(A)v^{\frac{1}{2}}_{\pm\frac{1}{2}})$ belongs to $\mathrm{Span}\{v\alpha^*:\ v \in \mathcal{V}^{\frac{3}{2}}\}$. Similarly, $R_\gamma(\phi_\omega(A)(v^{\frac{1}{2}}_{\pm\frac{1}{2}}))$ belongs to $\mathrm{Span}\{v\gamma:\ v \in \mathcal{V}^{\frac{3}{2}}\}$. So, the lemma follows for A by taking $\mathcal{V} = \mathrm{Span}\{v\alpha^*, v\gamma:\ v \in \mathcal{V}^{\frac{3}{2}}\} \subset \mathcal{O}(SU_\mu(2))$. $\square$

Since α, γ^* belong to $\mathrm{Span}\{v^{\frac{1}{2}}_{j,\pm\frac{1}{2}}\}$, we have the following immediate corollary:

Corollary 3.8. *There is a finite dimensional subspace $\mathcal{V}$ of $\mathcal{O}(SU_\mu(2))$ such that for every state* (*hence for every bounded linear functional*) *ω on $\tilde{\mathcal{Q}}$, we have $T(\phi_\omega(A))$ belongs to $\mathcal{V}$. A similar conclusion holds for B and B^* as well.*

Proposition 3.9. *$\phi(A)$, $\phi(B)$, $\phi(B^*)$ belong to $\mathcal{O}(S^2_{\mu,c}) \otimes_{\mathrm{alg}} \mathcal{Q}$.*

Proof. We give the proof for $\phi(A)$ only, the proof for B, B^* being similar. From Corollary 3.8 and Lemma 3.6 it follows that for every bounded linear functional ω on $\tilde{\mathcal{Q}}$, $T(\phi_\omega(A))$ belongs to $\mathcal{V} \cap \overline{S^2_{\mu,c}} \subset \mathcal{O}(SU_\mu(2)) \cap \mathrm{Ker}(\psi)$ and hence $\mathcal{V} \cap \overline{S^2_{\mu,c}} = \mathcal{V} \cap \mathcal{O}(S^2_{\mu,c})$, where $\mathcal{V}$ is the finite dimensional subspace mentioned in Corollary 3.8. Clearly, $\mathcal{V} \cap \mathcal{O}(S^2_{\mu,c})$ is a finite dimensional subspace of $\mathcal{O}(S^2_{\mu,c})$ implying that there must be finite m, say, such that for every ω, $T(\phi_\omega(A))$ belongs to $\mathrm{Span}\{A^k, A^kB^l, A^kB^{*l}:\ 0 \leqslant k, l \leqslant m\}$. Denote by $\mathcal{W}$ the (finite dimensional) subspace of $\mathcal{B}(\mathcal{H})$ spanned by $\{A^k, A^kB^l, A^kB^{*l}:\ 0 \leqslant k, l \leqslant m\}$. Since for every state (and hence for every bounded linear functional) ω on $\tilde{\mathcal{Q}}$, we have $T(\phi_\omega(A)) = R_1(\phi_\omega(A)) \equiv \phi_\omega(A)1$, it is clear that $\phi_\omega(A)$ belongs to $\mathcal{W}$ for every ω in $\tilde{\mathcal{Q}}^*$. Now, let us fix any faithful state ω on the separable unital C^*-algebra $\tilde{\mathcal{Q}}$ and embed $\tilde{\mathcal{Q}}$ in $\mathcal{B}(L^2(\mathcal{Q}, \omega)) \equiv \mathcal{B}(\mathcal{K})$. Thus, we get a canonical embedding of $\mathcal{L}(\mathcal{H} \otimes \tilde{\mathcal{Q}})$ in $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. Let us thus identify $\phi(A)$ as an element of $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$, and then by choosing a countable family of elements $\{q_1, q_2, \ldots\}$ of $\tilde{\mathcal{Q}}$ which is an orthonormal basis in $\mathcal{K} = L^2(\omega)$, we can write $\phi(A)$ as a weakly convergent series of the form $\sum_{i,j=1}^{\infty} \phi^{ij}(A) \otimes |q_i\rangle\langle q_j|$. But $\phi^{ij}(A) = (\mathrm{id} \otimes \omega_{ij})(\phi(A))$, where $\omega_{ij}(\cdot) = \omega(q_i^* \cdot q_j)$. Thus, $\phi^{ij}(A)$ belongs to $\mathcal{W}$ for all i, j, and hence the sequence $\sum_{i,j=1}^{n} \phi^{ij}(A) \otimes |q_i\rangle\langle q_j| \in \mathcal{W} \otimes \mathcal{B}(\mathcal{K})$ converges weakly, and $\mathcal{W}$ being finite dimensional (hence weakly closed), the limit, i.e. $\phi(A)$, must belong to $\mathcal{W} \otimes \mathcal{B}(\mathcal{K})$. In other words, if $A_1, \ldots, A_k$ denotes a basis of $\mathcal{W}$, we can write $\phi(A) = \sum_{i=1}^{k} A_i \otimes B_i$ for some $B_i \in \mathcal{B}(\mathcal{K})$.

We claim that each B_i must belong to $\tilde{\mathcal{Q}}$. For any trace-class positive operator ρ in $\mathcal{H}$, say of the form $\rho = \sum_j \lambda_j |e_j\rangle\langle e_j|$, where $\{e_1, e_2, \ldots\}$ is an orthonormal basis of $\mathcal{H}$ and $\lambda_j \geqslant 0$,

$\sum_j \lambda_j < \infty$, let us denote by ψ_ρ the normal functional on $\mathcal{B}(\mathcal{H})$ given by $x \mapsto \mathrm{Tr}(\rho x)$, and it is easy to see that it has a canonical extension $\tilde{\psi}_\rho := (\psi_\rho \otimes \mathrm{id})$ on $\mathcal{L}(\mathcal{H} \otimes \tilde{\mathcal{Q}})$ given by $\tilde{\psi}_\rho(X) = \sum_j \lambda_j \langle e_j \otimes 1, X(e_j \otimes 1)\rangle_{\tilde{\mathcal{Q}}}$, where X belongs to $\mathcal{L}(\mathcal{H} \otimes \tilde{\mathcal{Q}})$ and $\langle \cdot,\cdot \rangle_{\tilde{\mathcal{Q}}}$ denotes the $\tilde{\mathcal{Q}}$-valued inner product of $\mathcal{H} \otimes \tilde{\mathcal{Q}}$. Clearly, $\tilde{\psi}_\rho$ is a bounded linear map from $\mathcal{L}(\mathcal{H} \otimes \tilde{\mathcal{Q}})$ to $\tilde{\mathcal{Q}}$. Now, since $A_1, \ldots, A_k$ in the expression of $\phi(A)$ are linearly independent, we can choose trace class operators $\rho_1, \ldots, \rho_k$ such that $\psi_{\rho_i}(A_i) = 1$ and $\psi_{\rho_i}(A_j) = 0$ for $j \neq i$. Then, by applying $\tilde{\psi}_{\rho_i}$ on $\phi(A)$ we conclude that B_i belongs to $\tilde{\mathcal{Q}}$. But by definition, $\mathcal{Q}$ is the Woronowicz subalgebra of $\tilde{\mathcal{Q}}$ generated by $\langle \xi \otimes 1, \phi(x)(\eta \otimes 1)\rangle_{\tilde{\mathcal{Q}}}$, with η, ξ belonging to $\mathcal{H}$ and x in $\mathcal{O}(S^2_{\mu,c})$, and hence it follows that B_i belongs to $\mathcal{Q}$. □

Proposition 3.10. *ϕ keeps the span of $1, A, B, B^*$ invariant.*

Proof. We prove the result for $\phi(A)$ only, the proof for the other cases being quite similar.

Using Proposition 3.9, we can write $\phi(A)$ as a finite sum of the form:

$$\sum_{k \geqslant 0} A^k \otimes Q_k + \sum_{m',n',n' \neq 0} A^{m'} B^{n'} \otimes R_{m',n'} + \sum_{r,s,s \neq 0} A^r B^{*s} \otimes R'_{r,s}.$$

Let $\xi = v^l_{m_0,N_0}$.

We have that $U(\xi)$ belongs to $\mathrm{Span}\{v^l_{m,N},\ m = -l, \ldots, l,\ N = \pm\frac{1}{2}\}$. Let us write

$$U(\xi \otimes 1) = \sum_{m=-l,\ldots,l,\ N=\pm\frac{1}{2}} v^l_{m,N} \otimes q^l_{(m,N),(m_0,N_0)},$$

where $q^l_{(m,N),(m_0,N_0)}$ belong to $\mathcal{Q}$. Since α_U preserves the R-twisted volume, we have:

$$\sum_{m',N'} q^l_{(m,N),(m',N')} q^{l*}_{(m,N),(m',N')} = 1. \tag{4}$$

It also follows that $U(A\xi)$ belongs to $\mathrm{Span}\{v^{l'}_{m,N},\ m = -l', \ldots, l',\ l' = l-1, l, l+1,\ N = \pm\frac{1}{2}\}$.

Recalling Remark 3.4, we have

$$\begin{aligned}\phi(A)U(\xi \otimes 1) = & \sum_{k,m=-l,\ldots,l,\ N=\pm\frac{1}{2}} A^k v^l_{m,N} \otimes Q_k q^l_{(m,N),(m_0,N_0)} \\ & + \sum_{m',n',\ n' \neq 0,\ m=-l,\ldots,l,\ N=\pm\frac{1}{2}} A^{m'} B^{n'} v^l_{m,N} \otimes R_{m',n'} q^l_{(m,N),(m_0,N_0)} \\ & + \sum_{r,s,\ s \neq 0,\ m=-l,\ldots l,\ N=\pm\frac{1}{2}} A^r B^{*s} v^l_{m,N} \otimes R'_{r,s} q^l_{(m,N),(m_0,N_0)}.\end{aligned}$$

Let m'_0 denote the largest integer m' such that there is a nonzero coefficient of $A^{m'} B^{n'}$, $n' \geqslant 1$ in the expression of $\phi(A)$. We claim that the coefficient of $v^{l-m'_0-n'}_{m-n',N}$ in $\phi(A)U(\xi \otimes 1)$ is $R_{m'_0,n'} q^l_{(m,N),(m_0,N_0)}$.

Indeed, the term $v_{m-n',N}^{l-m_0'-n'}$ can arise in three ways: it can come from a term of the form $A^{m''}B^{n''}v_{m,N}^l$ or $A^k v_{m,N}^l$ or $A^r B^{*s} v_{m.N}^l$ for some m'', n'', k, r, s.

In the first case, we must have $l-m_0'-n' = l-m''-n''+t$, $0 \leqslant t \leqslant 2m''$ and $m-n' = m-n''$ implying $m'' = m_0'+t$, and since m_0' is the largest integer such that $A^{m_0'}B^{n'}$ appears in $\phi(A)$, we only have the possibility $t=0$, i.e. $v_{m-n',N}^{l-m_0'-n'}$ appears only in $A^{m_0'}B^{n'}$.

In the second case, we have $m-n' = m$ implying $n' = 0$ – a contradiction. In the last case, we have $m-n' = m+s$ so that $-n' = s$ which is only possible when $n' = s = 0$ which is again a contradiction.

It now follows from the above claim, using Remark 3.4 and comparing coefficients in the equality $U(A\xi \otimes 1) = \phi(A)U(\xi \otimes 1)$, that $R_{m_0',n'} q^l_{(m,N),(m_0,N_0)} = 0$ for all $n' \geqslant 1$, for all m, N when $m_0' \geqslant 1$. Now varying (m_0, N_0), we conclude that the above holds for all (m_0, N_0). Using (4), we conclude that

$$R_{m_0',n'} \sum_{m',N'} q^l_{(m,N),(m',N')} q^{l*}_{(m,N),(m',N')} = 0 \quad \text{for all } n' \geqslant 1,$$

that is, $R_{m_0',n'} = 0$ for all $n' \geqslant 1$ if $m_0' \geqslant 1$. Proceeding by induction on m_0', we deduce $R_{m',n'} = 0$ for all $m' \geqslant 1$, $n' \geqslant 1$.

Similarly, we have $Q_k = 0$ for all $k \geqslant 2$ and $R'_{r,s} = 0$ for all $r \geqslant 1$, $s \geqslant 1$.

Thus, $\phi(A)$ belongs to $\mathrm{Span}\{1, A, B, B^*, B^2, \ldots, B^n, B^{*2}, \ldots, B^{*m}\}$. But the coefficient of $v_{m-n',N}^{l-n'}$ in $\phi(A)U(\xi \otimes 1)$ is $R_{0,n'}$. Arguing as before, we conclude that $R_{0,n'} = 0$ for all $n' \geqslant 2$. In a similar way, we can prove $R'_{0,n'} = 0$ for all $n' \geqslant 2$. □

In view of the above, let us write:

$$\phi(A) = 1 \otimes T_1 + A \otimes T_2 + B \otimes T_3 + B^* \otimes T_4, \tag{5}$$

$$\phi(B) = 1 \otimes S_1 + A \otimes S_2 + B \otimes S_3 + B^* \otimes S_4, \tag{6}$$

for some T_i, S_i in $\mathcal{Q}$.

3.3. Identification of $SO_\mu(3)$ as the quantum isometry group

In this subsection, we shall use the facts that ϕ is a $*$-homomorphism and it preserves the R-twisted volume to derive relations among T_i, S_i in (5), (6).

Lemma 3.11.

$$T_1 = \frac{1-T_2}{1+\mu^2},$$

$$S_1 = \frac{-S_2}{1+\mu^2}.$$

Proof. We have the expressions of A and B in terms of the $SU_\mu(2)$ elements from Eqs. (1), (2) and (3). From these, we note that $h(A) = (1+\mu^2)^{-1}$ and $h(B) = 0$. By recalling Proposition 3.1, we use $(h \otimes \mathrm{id})\phi(A) = h(A).1$ and $(h \otimes \mathrm{id})\phi(B) = h(B).1$ to have the above two equations. □

Lemma 3.12.

$$T_1^* = T_1, \qquad T_2^* = T_2, \qquad T_4^* = T_3.$$

Proof. It follows by comparing the coefficients of $1, A$ and B respectively in the equation $\phi(A^*) = \phi(A)$. □

Lemma 3.13.

$$\begin{aligned} &S_2^* S_2 + c\big(1+\mu^2\big)^2 S_3^* S_3 + c\big(1+\mu^2\big)^2 S_4^* S_4 \\ &\quad = (1-T_2)\big(\mu^2+T_2\big) - c\big(1+\mu^2\big)^2 T_3 T_3^* - c\big(1+\mu^2\big)^2 T_3^* T_3 + c\big(1+\mu^2\big)^2.1, \end{aligned} \tag{7}$$

$$\begin{aligned} &-2S_2^* S_2 + \big(1+\mu^2\big) S_3^* S_3 + \mu^2\big(1+\mu^2\big) S_4^* S_4 \\ &\quad = \big(\mu^2 + 2T_2 - 1\big)T_2 - \mu^2\big(1+\mu^2\big) T_3 T_3^* - \big(1+\mu^2\big) T_3^* T_3, \end{aligned} \tag{8}$$

$$S_2^* S_2 - S_3^* S_3 - \mu^4 S_4^* S_4 = -T_2^2 + \mu^4 T_3 T_3^* + T_3^* T_3, \tag{9}$$

$$S_2^* S_4 + S_3^* S_2 = -\big(\mu^2 + T_2\big) T_3^* + T_3^*(1-T_2), \tag{10}$$

$$S_2^* S_3 + \mu^2 S_4^* S_2 = -T_2 T_3 - \mu^2 T_3 T_2, \tag{11}$$

$$S_4^* S_3 = -T_3^2. \tag{12}$$

Proof. It follows by comparing the coefficients of $1, A, A^2, B^*, AB$ and B^2 in the equation $\phi(B^*B) = \phi(A) - \phi(A^2) + c\phi(I)$ and then using Lemmas 3.11 and 3.12. □

Lemma 3.14.

$$\begin{aligned} &-S_2(1-T_2) + c\big(1+\mu^2\big)^2 S_3 T_3^* + c\big(1+\mu^2\big)^2 S_4 T_3 \\ &\quad = -\mu^2(1-T_2) S_2 + c\mu^2\big(1+\mu^2\big)^2 T_3 S_4 + c\mu^2\big(1+\mu^2\big)^2 T_3^* S_3, \end{aligned} \tag{13}$$

$$\begin{aligned} &S_2 - 2S_2T_2 + \big(1+\mu^2\big)\big(\mu^2 S_3 T_3^* + S_4 T_3\big) \\ &\quad = \mu^2 S_2 - 2\mu^2 T_2 S_2 + \mu^4\big(1+\mu^2\big) T_3 S_4 + \mu^2\big(1+\mu^2\big) T_3^* S_3, \end{aligned} \tag{14}$$

$$-S_2 T_3 + S_3(1-T_2) = -\mu^2 T_3 S_2 + \mu^2(1-T_2) S_3, \tag{15}$$

$$-S_2 T_3^* + S_4(1-T_2) = \mu^2(1-T_2) S_4 - \mu^2 T_3^* S_2, \tag{16}$$

$$S_2 T_3 + \mu^2 S_3 T_2 = \mu^2\big(T_2 S_3 + \mu^2 T_3 S_2\big), \tag{17}$$

$$S_3 T_3 = \mu^2 T_3 S_3, \tag{18}$$

$$S_4 T_3^* = \mu^2 T_3^* S_4. \tag{19}$$

Proof. It follows by equating the coefficients of $1, A, B, B^*, AB, B^2$ and B^{*2} in the equation $\phi(BA) = \mu^2\phi(AB)$ and then using Lemmas 3.11 and 3.12. □

Lemma 3.15.

$$-S_2S_4^* - S_3S_2^* = \mu^2(1+\mu^2)T_3 - \mu^4(1-T_2)T_3 - \mu^4T_3(1-T_2), \tag{20}$$

$$S_2S_4^* + \mu^2S_3S_2^* = -\mu^4T_2T_3 - \mu^6T_3T_2, \tag{21}$$

$$S_3S_4^* = -\mu^4T_3^2. \tag{22}$$

Proof. The lemma is proved by equating the coefficient of B, AB, B^2 in the equation $\phi(BB^*) = \mu^2\phi(A) - \mu^4\phi(A^2) + c\phi(I)$ and then using Lemmas 3.11 and 3.12. □

Now, we compute the antipode, say κ of $\mathcal{Q}$.

To begin with, we note that $\{x_{-1}, x_0, x_1\}$ is a set of orthogonal vectors. Moreover, they have the same norm. The first assertion being easier, we prove below the second one.

Lemma 3.16.

$$\begin{aligned} h(x_{-1}^*x_{-1}) &= h(x_0^*x_0) = h(x_1^*x_1) \\ &= t^2(1-\mu^2)(1-\mu^6)^{-1}\left[\mu^2 + t^{-1}(\mu^4+2\mu^2+1) + t(-\mu^4-2\mu^2-1)\right]. \end{aligned}$$

Proof. We have $x_{-1}^*x_{-1} = t^2\mu^{-2}(1+\mu^2)(A - A^2 + cI)$, $x_0^*x_0 = t^2(1-2(1+\mu^2)A + (1+\mu^2)^2A^2)$, $x_1^*x_1 = t^2(1+\mu^2)(\mu^2A - \mu^4A^2 + cI)$.

We recall from [18] that for all bounded Borel function f on $\sigma(A)$,

$$h(f(A)) = \gamma_+\sum_{n=0}^{\infty} f(\lambda_+\mu^{2n})\mu^{2n} + \gamma_-\sum_{n=0}^{\infty} f(\lambda_-\mu^{2n})\mu^{2n},$$

where $\lambda_+ = \frac{1}{2} + (c+\frac{1}{4})^{\frac{1}{2}}$, $\lambda_- = \frac{1}{2} - (c+\frac{1}{4})^{\frac{1}{2}}$, $\gamma_+ = (1-\mu^2)\lambda_+(\lambda_+ - \lambda_-)^{-1}$, $\gamma_- = (1-\mu^2)\lambda_-(\lambda_- - \lambda_+)^{-1}$.

The lemma follows by applying this relation to the above expressions of $x_{-1}^*x_{-1}$, $x_0^*x_0$, $x_1^*x_1$. □

If x'_{-1}, x'_0, x'_1 is the normalized basis corresponding to $\{x_{-1}, x_0, x_1\}$, then from (5) and (6) along with the fact that each of the vectors x_{-1}, x_0, x_1 has the same norm, it follows that

$$\begin{aligned} \phi(x'_{-1}) &= x'_{-1}\otimes S_3 + x'_0 \otimes -\mu^{-1}(1+\mu^2)^{-\frac{1}{2}}S_2 + x'_1 \otimes -\mu^{-1}S_4, \\ \phi(x'_0) &= x'_{-1}\otimes -\mu(1+\mu^2)^{\frac{1}{2}}T_3 + x'_0\otimes T_2 + x'_1\otimes (1+\mu^2)^{\frac{1}{2}}T_4, \\ \phi(x'_1) &= x'_{-1}\otimes -\mu S_4^* + x'_0\otimes (1+\mu^2)^{-\frac{1}{2}}S_2^* + x'_1\otimes S_3^*. \end{aligned}$$

Since ϕ is kept invariant by the Haar state h of $SU_\mu(2)$ and ϕ keeps the span of the orthonormal set $\{x'_{-1}, x'_0, x'_1\}$ invariant too, we get a unitary representation of the CQG $\mathcal{Q}$ on the span of $\{x'_{-1}, x'_0, x'_1\}$. If we denote by Z the $M_3(\mathcal{Q})$-valued unitary corresponding to this unitary representation with respect to the ordered basis $\{x'_{-1}, x'_0, x'_1\}$, we get by using $T_4 = T_3^*$ from

Lemma 3.12 the following:

$$Z = \begin{pmatrix} S_3 & -\mu\sqrt{1+\mu^2}T_3 & -\mu S_4^* \\ \frac{-S_2}{\mu\sqrt{1+\mu^2}} & T_2 & \frac{S_2^*}{\sqrt{1+\mu^2}} \\ -\mu^{-1}S_4 & \sqrt{1+\mu^2}T_3^* & S_3^* \end{pmatrix}.$$

Recall that (see, for example, [14]), the antipode κ on the matrix elements of a finite dimensional unitary representation $U^\alpha \equiv (u^\alpha_{pq})$ is given by $\kappa(u^\alpha_{pq}) = (u^\alpha_{qp})^*$. Thus, the antipode κ is given by:

$$\kappa(T_2) = T_2, \qquad \kappa(T_3) = \frac{S_2^*}{\mu^2(1+\mu^2)}, \qquad \kappa(S_2) = \mu^2\big(1+\mu^2\big)T_3^*,$$
$$\kappa(S_3) = S_3^*, \qquad \kappa(S_4) = \mu^2 S_4, \qquad \kappa\big(T_3^*\big) = \frac{S_2}{1+\mu^2},$$
$$\kappa\big(S_2^*\big) = \big(1+\mu^2\big)T_3, \qquad \kappa\big(S_3^*\big) = S_3, \qquad \kappa\big(S_4^*\big) = \mu^{-2}S_4^*.$$

Now we derive some more relations by applying the anti-homomorphism κ on the relations obtained earlier.

Lemma 3.17.

$$-2\mu^4\big(1+\mu^2\big)^3 T_3^*T_3 + \mu^2\big(1+\mu^2\big)^2 S_3^*S_3 + \mu^4\big(1+\mu^2\big)^2 S_4S_4^* = \mu^2\big(1+\mu^2\big)T_2\big(\mu^2+2T_2-1\big) - \mu^2 S_2S_2^* - S_2^*S_2, \tag{23}$$

$$\mu^4\big(1+\mu^2\big)^4 T_3^*T_3 - \mu^2\big(1+\mu^2\big)^2 S_3^*S_3 - \mu^6\big(1+\mu^2\big)^2 S_4S_4^* = -\mu^2\big(1+\mu^2\big)^2 T_2^2 + \mu^4 S_2S_2^* + S_2^*S_2, \tag{24}$$

$$\mu^2\big(1+\mu^2\big)^2 S_4T_3 + \mu^2\big(1+\mu^2\big)^2 T_3^*S_3 = -S_2\big(\mu^2+T_2\big) + (1-T_2)S_2, \tag{25}$$

$$S_4S_3 = -\frac{-S_2^2}{\mu^2(1+\mu^2)^2}. \tag{26}$$

Proof. The relations follow by applying κ on (8), (9), (10) and (12) respectively. □

Lemma 3.18.

$$-\mu^2(1-T_2)T_3^* + cS_2S_3^* + cS_2^*S_4 = -\mu^4 T_3^*(1-T_2) + c\mu^2 S_4S_2^* + c\mu^2 S_3^*S_2, \tag{27}$$

$$S_3S_2 = \mu^2 S_2S_3, \tag{28}$$

$$S_2S_4 = \mu^2 S_4S_2, \tag{29}$$

$$-S_2^*T_3^* + (1-T_2)S_3^* = -\mu^2 T_3^*S_2^* + \mu^2 S_3^*(1-T_2), \tag{30}$$

$$-S_2T_3^* + (1-T_2)S_4 = \mu^2 S_4(1-T_2) - \mu^2 T_3^*S_2. \tag{31}$$

Proof. The relations follow by applying κ on (13), (18), (19), (15) and (16) respectively. □

Lemma 3.19.

$$S_3S_4 = -\frac{\mu^2 S_2^2}{(1+\mu^2)^2}, \tag{32}$$

$$\begin{aligned}
&-\mu^2\big(1+\mu^2\big)^2 S_4^* T_3^* - \mu^2\big(1+\mu^2\big)^2 T_3 S_3^* \\
&\quad = \mu^2\big(1+\mu^2\big)S_2^* - \mu^4 S_2^*(1-T_2) - \mu^4(1-T_2)S_2^*,
\end{aligned} \tag{33}$$

$$\big(1+\mu^2\big)^2 S_4^* T_3^* + \mu^2\big(1+\mu^2\big)^2 T_3 S_3^* = -\mu^2 S_2^* T_2 - \mu^4 T_2 S_2^*. \tag{34}$$

Proof. The relations follow by applying κ on (22), (20) and (21) respectively. □

Remark 3.20. It follows from (26) and (32) that $\mu^4 S_4 S_3 = S_3 S_4$.

Lemma 3.21.

$$S_2^* S_2 = (1-T_2)\big(\mu^2 + T_2\big).$$

Proof. Subtracting the equation obtained by multiplying $c(1+\mu^2)$ with (8) from (7), we have

$$\begin{aligned}
&\big(1+2c\big(1+\mu^2\big)\big)S_2^*S_2 + c\big(1+\mu^2\big)^2\big(1-\mu^2\big)S_4^*S_4 \\
&\quad = (1-T_2)\big(\mu^2+T_2\big) - c\big(1+\mu^2\big)\big(\mu^2+2T_2-1\big)T_2 \\
&\qquad + c\big(1+\mu^2\big)^2\big(\mu^2-1\big)T_3T_3^* + c\big(1+\mu^2\big)^2.1.
\end{aligned} \tag{35}$$

Again, by adding (7) with $c(1+\mu^2)^2$ times (9) gives

$$\begin{aligned}
&\big(1+c\big(1+\mu^2\big)^2\big)S_2^*S_2 + c\big(1-\mu^4\big)\big(1+\mu^2\big)^2 S_4^*S_4 \\
&\quad = (1-T_2)\big(\mu^2+T_2\big) - c\big(1+\mu^2\big)^2 T_2^2 + c\big(1+\mu^2\big)^2\big(\mu^4-1\big)T_3T_3^* + c\big(1+\mu^2\big)^2.1.
\end{aligned} \tag{36}$$

Subtracting the equation obtained by multiplying (μ^2+1) with (35) from (36) we obtain

$$\begin{aligned}
&-\big(\mu^2 + c\big(1+\mu^2\big)^2\big)S_2^*S_2 \\
&\quad = (1-T_2)\big(\mu^2+T_2\big) - c\big(1+\mu^2\big)^2 T_2^2 \\
&\qquad - \big(1+\mu^2\big)(1-T_2)\big(\mu^2+T_2\big) - c\mu^2\big(1+\mu^2\big)^2.1 + c\big(1+\mu^2\big)^2\big(\mu^2+2T_2-1\big)T_2.
\end{aligned}$$

The right-hand side can be seen to equal $-(\mu^2 + c(1+\mu^2)^2)(1-T_2)(\mu^2+T_2)$.
Thus, $S_2^*S_2 = (1-T_2)(\mu^2+T_2)$. □

Lemma 3.22.

$$\mu^2\big(1+\mu^2\big)^2 T_3^* T_3 = (1-T_2)\big(\mu^2+T_2\big), \tag{37}$$

$$\big(1+\mu^2\big)^2 T_3 T_3^* = (1-T_2)\big(1+\mu^2 T_2\big), \tag{38}$$

$$S_2 S_2^* = \mu^2(1-T_2)\big(1+\mu^2 T_2\big). \tag{39}$$

Proof. Applying κ on Lemma 3.21, we obtain (37).

Unitarity of the matrix Z ((2, 2) position of the matrix Z^*Z) gives $\mu^2(1+\mu^2)T_3^*T_3 + T_2^2 + (1+\mu^2)T_3T_3^* = 1$.

Using (37) we deduce $-(1+\mu^2)^2T_3T_3^* = (T_2-1)(1+\mu^2T_2)$. Thus we obtain (38).

Applying κ on (38), we deduce (39). □

Lemma 3.23.

$$S_4^*S_4 = S_4S_4^* = \left(1+\mu^2\right)^{-2}\mu^2(1-T_2)^2.$$

Proof. Adding (23) and (24), we have:

$$\begin{aligned}&-\mu^4\left(1+\mu^2\right)^3\left(1-\mu^2\right)T_3^*T_3 + \mu^4\left(1+\mu^2\right)^2\left(1-\mu^2\right)S_4S_4^* \\ &\quad = -\mu^2\left(1+\mu^2\right)\left(1-\mu^2\right)T_2(1-T_2) - \mu^2\left(1-\mu^2\right)S_2S_2^*.\end{aligned}$$

Using $\mu^2 \neq 1$, we obtain,

$$-\mu^4\left(1+\mu^2\right)^3T_3^*T_3 + \mu^4\left(1+\mu^2\right)^2S_4S_4^* = -\mu^2\left(1+\mu^2\right)T_2(1-T_2) - \mu^2S_2S_2^*.$$

Now using (37) and (39), we reduce the above equation to

$$\begin{aligned}\mu^4\left(1+\mu^2\right)^2S_4S_4^* &= -\mu^2(1-T_2)\left(T_2+\mu^2T_2+\mu^2+\mu^4T_2\right) + \mu^2\left(1+\mu^2\right)(1-T_2)\left(\mu^2+T_2\right) \\ &= \mu^6(1-T_2)^2.\end{aligned}$$

Thus,

$$\begin{aligned}S_4S_4^* &= \frac{\mu^6}{\mu^4(1+\mu^2)^2}(1-T_2)^2 \\ &= \frac{\mu^2}{(1+\mu^2)^2}(1-T_2)^2.\end{aligned}$$

Applying κ, we have $S_4^*S_4 = \frac{\mu^2}{(1+\mu^2)^2}(1-T_2)^2$.

Thus, $S_4^*S_4 = S_4S_4^* = \frac{\mu^2}{(1+\mu^2)^2}(1-T_2)^2$. □

Lemma 3.24.

$$\mu^2\left(1+\mu^2\right)^2S_3^*S_3 = \left(\mu^2+T_2\right)\left[\mu^2\left(1+\mu^2\right)-(1-T_2)\right].$$

Proof. Using Lemma 3.21 in (7), we have

$$S_3^*S_3 + T_3^*T_3 + T_3T_3^* + S_4^*S_4 = 1. \tag{40}$$

The lemma is derived by substituting the expressions of $T_3^*T_3$, $T_3T_3^*$ and $S_4^*S_4$ from (37), (38) and Lemma 3.23 in Eq. (40). □

Lemma 3.25.

$$\left(1+\mu^2\right)^2 S_3 S_3^* = \left(1+\mu^2 T_2\right)\left(1+\mu^2-\mu^4(1-T_2)\right).$$

Proof. By unitarity of the matrix Z, in particular equating the $(1,1)$-th entry of ZZ^* to 1 we get $S_3 S_3^* + \mu^2(1+\mu^2)T_3 T_3^* + \mu^2 S_4^* S_4 = 1$. Then the lemma follows by using (38) and Lemma 3.23 in the above equation. □

Lemma 3.26.

$$-S_2^* S_3 = \left(\mu^2+T_2\right)T_3.$$

Proof. By applying the adjoint and then multiplying by μ^2 on (10) we have $\mu^2 S_2^* S_3 + \mu^2 S_4^* S_2 = -\mu^2 T_3(\mu^2+T_2) + \mu^2(1-T_2)T_3$. Subtracting this from (11) we have $(1-\mu^2)S_2^* S_3 = -T_2 T_3 - \mu^2 T_3 T_2 + \mu^2 T_3(\mu^2+T_2) - \mu^2(1-T_2)T_3$ which implies $-S_2^* S_3 = (\mu^2+T_2)T_3$ as $\mu^2 \neq 1$. □

Lemma 3.27.

$$S_2(1-T_2) = \mu^2(1-T_2)S_2.$$

Proof. Applying κ to Lemma 3.26 and then taking adjoint, we have

$$\mu^2\left(1+\mu^2\right)^2 T_3^* S_3 = -\left(\mu^2+T_2\right)S_2. \tag{41}$$

Adding (33) and (34) and then taking adjoint, we get (by using $\mu^2 \neq 1$)

$$\mu^2\left(1+\mu^2\right)^2 T_3 S_4 = \mu^4(1-T_2)S_2. \tag{42}$$

Moreover, (25) gives

$$\mu^2\left(1+\mu^2\right)^2 S_4 T_3 = -S_2\left(\mu^2+T_2\right) + (1-T_2)S_2 - \mu^2\left(1+\mu^2\right)^2 T_3^* S_3.$$

Using (41), the right-hand side of this equation turns out to be $S_2(1-T_2)$.

Thus,

$$\left(1+\mu^2\right)^2 S_4 T_3 = \mu^{-2} S_2(1-T_2). \tag{43}$$

Again, application of adjoint to Eq. (33) gives:

$$\mu^2\left(1+\mu^2\right)^2 S_3 T_3^* = -\mu^2\left(1+\mu^2\right)^2 T_3 S_4 - \mu^2\left(1+\mu^2\right)S_2 + \mu^4(1-T_2)S_2 + \mu^4 S_2(1-T_2).$$

Using (42), we get

$$\left(1+\mu^2\right)^2 S_3 T_3^* = -S_2\left(1+\mu^2 T_2\right). \tag{44}$$

Using (41)–(44) to Eq. (14), we obtain:

$$S_2 - 2S_2T_2 - \left(1+\mu^2\right)^{-1}\mu^2 S_2\left(1+\mu^2 T_2\right) + \mu^{-2}\left(1+\mu^2\right)^{-1} S_2(1-T_2)$$
$$= \mu^2 S_2 - 2\mu^2 T_2 S_2 + \left(1+\mu^2\right)^{-1}\mu^6(1-T_2)S_2 - \left(1+\mu^2\right)^{-1}\left(\mu^2+T_2\right)S_2.$$

This gives

$$\mu^2\left(1+\mu^2\right)\left[(S_2 - S_2T_2) - \left(\mu^2 S_2 - \mu^2 T_2 S_2\right)\right] - \mu^2\left(1+\mu^2\right)\left(S_2T_2 - \mu^2 T_2 S_2\right)$$
$$- \mu^4 S_2 - \mu^6 S_2 T_2 + S_2(1-T_2) - \mu^8(S_2 - T_2S_2) + \mu^4 S_2 + \mu^2 T_2 S_2 = 0.$$

Thus,

$$\mu^2\left(1+\mu^2\right)\left[S_2(1-T_2) - \mu^2(1-T_2)S_2\right] + S_2(1-T_2) - \mu^2(S_2 - T_2S_2)$$
$$+ \mu^6\left[S_2(1-T_2) - \mu^2(1-T_2)S_2\right] - \mu^6(1-T_2)S_2 + \mu^4 S_2(1-T_2)$$
$$+ \mu^2\left(S_2(1-T_2) - \mu^2(1-T_2)S_2\right) = 0.$$

On simplifying, $(\mu^6 + 2\mu^4 + 2\mu^2 + 1)(S_2(1-T_2) - \mu^2(1-T_2)S_2) = 0$, which proves the lemma as $0 < \mu < 1$. □

Lemma 3.28.

$$T_3(1-T_2) = \mu^2(1-T_2)T_3, \tag{45}$$

$$S_3 S_4^* = \mu^4 S_4^* S_3. \tag{46}$$

Proof. Eq. (45) follows by applying κ on Lemma 3.27 and then taking adjoint.

We have $S_4^* S_3 = -T_3^2$ from (12). On the other hand we have $S_3 S_4^* = -\mu^4 T_3^2$ from (22). Combining these two, we get (46). □

Lemma 3.29.

$$S_4 T_2 = T_2 S_4.$$

Proof. Subtracting (31) from (16) we get the required result. □

Lemma 3.30.

$$T_3 S_2 = S_2 T_3.$$

Proof. By applying adjoint on (30) and then subtracting it from (15) we obtain $S_2T_3 - T_3S_2 = 0$. □

Lemma 3.31.

$$S_3(1-T_2) = \mu^4(1-T_2)S_3.$$

Proof. By adding (15) with (17) we obtain

$$S_3(1-T_2)+\mu^2 S_3(T_2-1)=\mu^2\big(\mu^2-1\big)T_3S_2.$$

Thus, using $\mu^2\neq 1$,

$$S_3(1-T_2)=-\mu^2 T_3S_2. \tag{47}$$

Moreover, by taking adjoint of (30), we obtain $\mu^2(1-T_2)S_3=\mu^2S_2T_3-T_3S_2+S_3(1-T_2)$.

Thus,

$$\mu^4(1-T_2)S_3=\mu^4S_2T_3-\mu^2T_3S_2+\mu^2S_3(1-T_2).$$

Hence, to prove the lemma it suffices to prove:

$$S_3(1-T_2)=\mu^4S_2T_3-\mu^2T_3S_2+\mu^2S_3(1-T_2).$$

After using $T_3S_2=S_2T_3$ obtained from Lemma 3.30 we get this to be the same as $(1-\mu^2)S_3(1-T_2)=\mu^2(\mu^2-1)T_3S_2$. This is equivalent to $S_3(1-T_2)=-\mu^2T_3S_2$ (as $\mu^2\neq 1$) which follows from (47). □

Proposition 3.32. *The map $SO_\mu(3)\to\mathcal{Q}$ sending M, L, G, N, C to $-(1+\mu^2)^{-1}S_2$, S_3, $-\mu^{-1}S_4$, $(1+\mu^2)^{-1}(1-T_2)$, μT_3 respectively is a CQG homomorphism.*

Proof. It is enough to check that the map is $*$-homomorphic, since the coproducts on $SO_\mu(3)$ and $\mathcal{Q}$ are determined in terms of the fundamental unitaries Z' and Z respectively, and the map described in the statement of the proposition sends (ij)-th entry of Z' to the (ij)-th entry of Z for all (ij).

Now, it can easily be checked that the proof of the homomorphic property of the given map reduces to verification of the relations on $\mathcal{Q}$ as derived in Lemmas 3.21–3.31 along with the following equations:

$$S_3S_4=\mu^4S_4S_3, \tag{48}$$

$$S_3S_2=\mu^2S_2S_3, \tag{49}$$

$$S_2S_4=\mu^2S_4S_2, \tag{50}$$

$$S_3S_4=-\frac{\mu^2}{(1+\mu^2)^2}S_2^2, \tag{51}$$

which follow from Remark 3.20, (28), (29), (32) respectively. □

Theorem 3.33. *We have the isomorphism*:

$$QISO_R^+\big(\mathcal{O}\big(S^2_{\mu,c}\big),\mathcal{H},D\big)\cong SO_\mu(3).$$

Proof. $SU_\mu(2)$ is an object in $\widetilde{QISO_R^+}(D)$ as remarked before, and thus one gets a surjective morphism from $\widetilde{QISO_R^+}(D)$ to $SU_\mu(2)$ which clearly maps $QISO_R^+(D)$ onto $SO_\mu(3)$, identifying the latter as a quantum subgroup of $QISO_R^+(D)$. Let us denote the surjective map from $QISO_R^+(D)$ to $SO_\mu(3)$ by Π. On the other hand, Proposition 3.32 implies that $QISO_R^+(D)$ is a quantum subgroup of $SO_\mu(3)$, and the corresponding surjective CQG morphism from $SO_\mu(3)$ onto $QISO_R^+(D)$ is clearly seen to be the inverse of Π, thereby completing the proof. □

Remark 3.34. Theorem 3.33 shows that for a fixed μ, the quantum isometry group $QISO_R^+(D)$ of $S^2_{\mu,c}$ does not depend on c. This may appear somewhat surprising, but let us remark that in the classical situation (that is for $\mu = 1$), c corresponds to the radius of the sphere and $S^2_{1,c}$ are isomorphic as C^* algebras for all $c \geqslant 0$. We refer the reader to [13, p. 126], for the details regarding this. Although in the noncommutative case, that is, when $\mu \neq 1$, we do get non-isomorphic C^*-algebras $S^2_{\mu,c}$ for different choices of c, one may still think that the parameter c in some sense determines the 'radius' of the noncommutative sphere, and thus one should get the same (quantum) isometry group for different choices of c.

In view of the above, it seems impossible to 'reconstruct' the quantum homogeneous spaces $S^2_{\mu,c}$ from the quantum isometry groups $SO_\mu(3)$. In this context, it may be mentioned that for $\mu \neq 1$, although all $S^2_{\mu,c}$ are quantum homogeneous spaces corresponding to $SO_\mu(3)$, only $S^2_{\mu,0}$ arises as a quotient of $SO_\mu(3)$ by a quantum subgroup (see [16] for more details). Thus, it is perhaps possible to somehow 'reconstruct' $S^2_{\mu,0}$ from the quantum group $SO_\mu(3)$.

3.4. Existence of $\widetilde{QISO^+}(D)$

For the above spectral triple, we have been unable to settle the issue of the existence of $\widetilde{QISO^+}(D)$ which is the universal object (if it exists) in the category $\mathbf{Q}'(D)$ mentioned in Section 1. Nevertheless, we shall show that if a universal object in $\mathbf{Q}'(D)$ exists, then $QISO^+(D)$ must coincide with $SO_\mu(3)$.

Lemma 3.35. *If $\widetilde{QISO^+}(D)$ exists, its induced action on $S^2_{\mu,c}$, say α_0, must preserve the state h on the subspace spanned by* $\{1, A, B, B^*, AB, AB^*, A^2, B^2, {B^*}^2\}$.

Proof. Let $\mathcal{W}_0 = \mathbb{C}.1$, $\mathcal{W}_{\frac{1}{2}} = \mathrm{Span}\{1, A, B, B^*\}$,

$$\mathcal{W}_{\frac{3}{2}} = \mathrm{Span}\{1, A, B, B^*, AB, AB^*, A^2, B^2, B^{*2}\}.$$

We note that the proof of Proposition 3.10 and the lemmas preceding it do not use the assumption that the action is R-twisted volume preserving, so the proof of Proposition 3.10 goes through verbatim implying that α_0 keeps invariant the subspace spanned by $\{1, A, B, B^*\}$ and hence it preserves $\mathcal{W}_{\frac{3}{2}}$ as well. Let $\mathcal{W}_{\frac{3}{2}} = \mathcal{W}_{\frac{1}{2}} \oplus \mathcal{W}'$ be the orthogonal decomposition with respect to the Haar state (say h_0) of $QISO^+(D)$. Since $SO_\mu(3)$ is a sub-object of $QISO^+(D)$, there is a CQG morphism π from $QISO^+(D)$ onto $SO_\mu(3)$ satisfying $(\mathrm{id} \otimes \pi)\alpha_0 = \Delta$, where Δ is the $SO_\mu(3)$ action on $S^2_{\mu,c}$. It follows from this that any $QISO^+(D)$-invariant subspace (in particular $\mathcal{W}'$) is also $SO_\mu(3)$-invariant. On the other hand, it is easily seen that on $\mathcal{W}_{\frac{3}{2}}$, the $SO_\mu(3)$-action decomposes as $\mathcal{W}_{\frac{1}{2}} \oplus \mathcal{W}''$ (orthogonality with respect to h, the Haar state of $SO_\mu(3)$), where $\mathcal{W}''$ is a five-dimensional irreducible subspace.

We claim that $\mathcal{W}' = \mathcal{W}''$, which will prove that the $QISO^+(D)$-action α_0 has the same h-orthogonal decomposition as the $SO_\mu(3)$-action on $\mathcal{W}_{\frac{3}{2}}$, so preserves $\mathbb{C}.1$ and its h-orthogonal complements. This will prove that α_0 preserves the Haar state h on $\mathcal{W}_{\frac{3}{2}}$.

We now prove the claim. Observe that $\mathcal{V} := \mathcal{W}' \cap \mathcal{W}''$ is invariant under the $SO_\mu(3)$-action but due to the irreducibility of Δ on the vector space $\mathcal{W}'$ or $\mathcal{W}''$, it has to be zero or $\mathcal{W}' = \mathcal{W}''$. Now, $\dim(\mathcal{V}) = 0$ implies $\dim(\mathcal{W}') + \dim(\mathcal{W}'') = 5 + 5 > 9 = \dim(\mathcal{W}_{\frac{3}{2}})$ which is a contradiction unless $\mathcal{W}' = \mathcal{W}''$. □

Theorem 3.36. *If $\widetilde{QISO^+}(D)$ exists, then we must have that $QISO^+(D) \cong SO_\mu(3)$.*

Proof. In the proof of Lemma 3.35, it was noted that Proposition 3.10 follows under the assumption of the present theorem. To complete the proof of the theorem, we just need to observe that the other lemmas used for proving Theorem 3.33 require the conclusion of Lemma 3.35 as the only extra ingredient. □

Let us conclude the article with brief explanation of the technical difficulties regarding the issue of existence of $\widetilde{QISO^+}(D)$. Let R' be a positive, invertible operator commuting with D such that $\tau_{R'} \neq \tau_R$ and let ϕ' denote the action of $QISO^+_{R'}(D)$ on $S^2_{\mu,c}$. The problem of existence of $\widetilde{QISO^+}(D)$ is closely related to the question whether it is possible to identify $QISO^+_{R'}(D)$ as a quantum subgroup of $SO_\mu(3)$ for a general R'. By Theorem 3.36, a negative answer of this question will prove that $\mathbf{Q}'(D)$ does not have a universal object.

Now, as has been noted in Remark 3.5, ϕ' is linear, that is, it keeps the span of $\{1, A, B, B^*\}$ invariant and hence it is given by an expression similar to Eqs. (5) and (6) with T_i, S_i replaced by some T_i', S_i' which generate $QISO^+_{R'}(D)$ as a C^*-algebra. We can in principle write down all the relations satisfied by these generators, proceeding as in Section 3.3. These relations will be analogous to Eqs. (7)–(34), and in fact, the relations which make use of the homomorphism property only remain unchanged. However, the ones making use of the fact that ϕ' preserves $\tau_{R'}$ will change, since $\tau_{R'}$ is in general different from τ_R. In particular, the expression of the antipode will change, which will affect all the relations starting from (23). We need to have a deeper and systematic understanding of the relations satisfied by T_i', S_i' for a general R', and possibly study their representations in concrete Hilbert spaces, to decide whether $QISO^+_{R'}(D)$ is a quantum subgroup of $SO_\mu(3)$ or not. We are not yet able to do this.

Moreover, even if $\widetilde{QISO^+}(D)$ exists, although we can identify $\widetilde{QISO^+}(D)$ with the well-known quantum group $SO_\mu(3)$, it is not so easy to explicitly compute $\widetilde{QISO^+}(D)$. If U denotes the unitary representation corresponding to $\widetilde{QISO^+}(D)$, the fact that U commutes with D implies that U must preserve each of the two-dimensional eigenspaces $\mathrm{span}\{v^l_{m,\frac{1}{2}} + v^l_{m,-\frac{1}{2}}\colon m = \pm\frac{1}{2}\}$ and $\mathrm{span}\{v^l_{m,\frac{1}{2}} - v^l_{m,-\frac{1}{2}}\colon m = \pm\frac{1}{2}\}$ of D. Suppose that $(q_{ij})_{i,j=1,2}$ and $(r_{ij})_{i,j=1,2}$ are the matrices (with entries in $\widetilde{QISO^+}(D)$) of U corresponding to these two spaces respectively. Then it is clear that as a C^* algebra $\widetilde{QISO^+}(D)$ will be generated by q_{ij}, r_{ij}'s as well as the generators T_i, S_i of $SO_\mu(3)$. However, the mutual relations among these generating elements have to be determined from the fact that U preserves each of the eigenspaces of D. In principle one gets infinitely many such relations which are quite complicated and it is not clear how to simplify them.

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