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Pietro Mengoli and the Six-Square Problem

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The aim of this paper is to analyze a little known aspect of Pietro Mengoli's (1625–1686) mathematical activity: the difficulties he faced in trying to solve some problems in Diophantine analysis suggested by J. Ozanam. Mengoli's recently published correspondence reveals how he cherished his prestige as a scholar. At the same time, however, it also shows that his insufficient familiarity with algebraic methods prevented him, as well as other Italian mathematicians of his time, from solving the so-called "French" problems. Quite different was the approach used for the same problems by Leibniz, who, although likewise partially unsuccessful, demonstrated a deeper mathematical insight which led him to look for general algebraic methods. © 1994 Academic Press, Inc.

Le but de ce papier est l'analyse d'un aspect peu connu de l'activité de Pietro Mengoli (1625–1686). Plus précisément, nous nous occuperons des difficultés rencontrées par ce mathématicien bolonais pour résoudre quelques problèmes d'analyse diophantine proposés par J. Ozanam. La correspondance de Mengoli dernièrement publiée nous présente l'occasion d'illustrer comment Mengoli tenait à défendre son prestige de savant; mais, en même temps, elle révèle que les méthodes algébriques utiles à ce but ne lui étaient pas familières. Cela lui empêchera de résoudre les problèmes "français." L'approche et la sensibilité mathématique de Leibniz à l'égard des mêmes problèmes étaient différentes, quoique il n'eût que du succès partiel. © 1994 Academic Press, Inc.

In questo articolo si esamina un aspetto poco noto dell'attività matematica di Pietro Mengoli (1625–1686), cioè le sue difficoltà a risolvere alcuni problemi diofantei proposti da Ozanam. La recente pubblicazione della corrispondenza mengoliana offre qualche spunto relativo al suo rammarico per lo scacco subito e chiarisce a nostro parere la scarsa familiarità che i matematici italiani del periodo avevano con i metodi algebrici. Nelle conclusioni si sottolinea la diversa sensibilità matematica di Leibniz di fronte a questi problemi e la sua maggiore attenzione per la ricerca di metodi generali di soluzione, sebbene anch'egli abbia ripetutamente tentato di risolvere quei problemi. © 1994 Academic Press, Inc.

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1. INTRODUCTION

Pietro Mengoli (1625–1686) studied in Bologna with Bonaventura Cavalieri, whom he succeeded in the chair of mathematics. He took a degree in philosophy in 1650 and another in both civil and canon law in 1653, and he was in addition ordained into the priesthood. From 1660 until his death he served the parish of Santa Maria Maddalena in Bologna.

We can conveniently divide Mengoli's scientific activity into two main periods. During the first period, from 1649 to 1659, he worked, often with original insight,

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at the development of some themes originated by the Galilean school (such as sums of infinite series and the logical arrangement of the concepts of limit and definite integral). In the second period, from 1670 to 1682, which began after a silence of 10 years, Mengoli was more attentive to what he felt were the duties and responsibilities of a Catholic intellectual, especially with respect to the relationship between faith and reason.

Mengoli's works enjoyed wide circulation in the 17th century and were known to Collins, Wallis, and Leibniz, although they were soon forgotten because they were set out in a rather obscure Latin (especially those of his second period).

The so-called six-square problem was very famous among the mathematicians of the 17th and 18th centuries (among them Leibniz, Gregory, Landen, and above all Euler) and its history is well known [Dickson 1920, 443–447; Cassinet 1987; Hofmann 1958, 1969].

In this paper we focus on the contributions to this problem made by Mengoli. Our work has been motivated by the fact that, notwithstanding the recent publication of Mengoli's correspondence [Baroncini & Cavazza 1986, 41–59], one does not find a satisfactory analysis of this subject in the existing literature. This correspondence is rich in many themes, ranging from Mengoli's isolation in his Bolognese environment to the mathematical disputes that interested the Italian mathematical community. Mengoli shares with the Italian mathematicians of his time an inability to undergo a deep renewal or to escape from the provincial isolation into which they were falling. Both with the six-square problem and the so-called "Dutch Problems" the Italian mathematicians encountered severe difficulties.

We begin by discussing how the Bolognese mathematician Mengoli became acquainted with the Diophantine "French" problems (so named by Mengoli himself) proposed by Jacques Ozanam (1640–1616).

(I) It is known that the first "French" problem, the so-called six-square problem, had been formulated in different ways. In Mengoli's papers it appears in the two following equivalent formulations:

> (i) Find three numbers such that their differences are squares, & the differences of their squares are squares. [Mengoli 1674a]
>
> (ii) Find three squares, such that the difference of any two of them is a square, & the difference of the sides of any two of them is also a square. [Mengoli 1674b]

In modern notation, denoting the desired numbers by x, y, and z, this problem requires that the following relations be satisfied:

$$\begin{array}{ll} x - y = \square & x^2 - y^2 = \square \\ x - z = \square & x^2 - z^2 = \square \\ y - z = \square & y^2 - z^2 = \square \end{array}$$

(The symbol $\square$ stands for a square number whose value is irrelevant. It may differ from one equation to another.)

In his *Theorema Arithmeticum* [Mengoli 1674a] Mengoli tried to prove that it was impossible to solve this problem. Mengoli's "Theorema" was reprinted in Paris on April 18, 1674 by Ozanam who, in an appendix, indicated the solution triplet $x = 2{,}288{,}168$, $y = 1{,}873{,}432$, and $z = 2{,}399{,}057$ without explaining how he had obtained it. This circumstance prompted Mengoli to tackle the problem again, trying to solve it by trial and error. As we will indicate below, he published the result as a "Problema arithmeticum" [Mengoli 1674b] in the version (ii).

It is known from several sources [Turnbull 1939, 430–433]; Hofmann 1958; Leibniz 1990, 229] that the six-square problem had a third formulation; in a late letter to Jacques De Billy dated 9 May 1676 Ozanam wrote: "So, being in doubt, I have not written the problem in my Diophantus in the form that I had first proposed, but I have proposed it as the problem of finding three numbers such that the sums and differences of any two of them are squares. The problem proposed in this form seems to me prettier" [1].

(II) The second "French" problem sent to Mengoli bore the title "Mathematicis Problema Unicum" [Mengoli 1675, 20 (page not numbered), Proposition 52] and had the following formulation: "Find three squares, such that their sum is a square; & the sum of their squares is a squared square" [2]. On the page mentioned above, Mengoli describes his unsuccessful attempts to find the sought squares, using successive powers of 2, 3, and 5. Curiously enough he ends up with three numbers (60, 4, and 105) which do not satisfy the problem at hand but rather another problem quoted in a letter by Leibniz to Oldenburg on 16 October 1674; Leibniz says that Ozanam had posed this problem some time after having suggested the six-square problem. Both in the quoted letter and in Leibniz's solution [Leibniz 1990, 270 ff.] we find the problem formulated in the following manner: "Find three numbers such that their sum is a square; & the sum of their squares is a squared square." Leibniz, after some trials, found the solution triplet 64, 152, and 409 using a method which he remarked could be used to solve a third problem [Leibniz 1990, 322]: "Find three numbers such that the sum of their squares is a square, and the sum of their squared squares is a squared square." Leibniz's statement is therefore to be interpreted in the sense that every triplet which solves the problem in this formulation also solves the Problema Unicum, but the contrary does not hold. Leibniz ventured even to state [Leibniz 1990, 328] that his method was also useful to solve "other countless problems [like]: Find three numbers, such that their sum is any power, i.e., a cube, a squared square, etc., and the sum of their squares is a squared square."

Even though the literature on the Diophantine problems has placed more emphasis on the work of Ozanam than that of De Billy [Cassinet 1987; Hofmann 1958, 1969], it was the second author who, in our opinion, inspired both of these problems and the methods for their solution. Our opinion is based on a brief analysis of De Billy's work, above all the unpublished manuscript of De Billy previously quoted, entitled *Novarum quaestionum libri tres,* and is further supported by the correspondence between the Jesuit priest, Jacques De Billy, and Ozanam. The manuscript *Novarum quaestionum libri tres* was certainly written after 1676, as indicated in a remark at the end of Question 32 of the second book. It consists of three books. Here Diophantine problems of various types and difficulty are solved by the so-called method of "simple, double, and triple equalities."

In our opinion, this manuscript should be considered as a kind of appendix of applications to *Diophantus redivivus* [De Billy 1670a], where De Billy had discussed his method for solving problems in Diophantine analysis.

While De Billy was collecting and organizing his large collection of Diophantine problems, Ozanam was writing his voluminous manuscript on *The Six Books of Diophantus' Arithmetic* [Cassinet 1987, 17]. One can assume that they were likely to have exchanged their problems, and that Ozanam may have submitted them to other European mathematicians in order to find out if others had a general method of solution, too.

In De Billy's manuscript one can find the six-square problem formulated in its classical form as well as in other equivalent forms. For the reader interested in his method we give an example concerning Question 75 of Book I from De Billy's manuscript:

> Find three squares such that the difference of any two of them is a square. Also find three numbers such that the sum of the difference of any two of them is a square.

For the first part of the problem, De Billy placed the three sought squares in the following form (we denote them by m^2, n^2, and p^2, respectively):

$$\begin{aligned} &\text{(a) } m^2 = x^4 + 8x^2 + 16 \\ &\text{(b) } n^2 = 16x^2 \\ &\text{(c) } p^2 + 16x^4 + 8x^2 + 1. \end{aligned} \tag{1}$$

One verifies that $m^2 - n^2 = \square$; $p^2 - n^2 = \square$.

Since $p^2 - m^2 = 15(x^4 - 1)$ must be a square, De Billy set

$$15(x^4 - 1) = 225, \tag{2}$$

which yields $x = 2$. If this value is substituted into the relations (1), one obtains a trivial solution of the problem. Then, in order to find another solution, De Billy substituted x with $(x - 2)$ into (2), obtaining again a polynomial of the fourth degree, complete and with the constant term equal to a square. He then was able to find a solution to the problem by applying the method explained in the *Inventum Novum* [De Billy 1670b, 378] [3].

After the substitution, one has from (2) $15x^4 - 120x^3 + 360x^2 - 480x + 225$.

Then De Billy set this polynomial equal to a square, that is

$$15x^4 - 120x^3 + 360x^2 - 480x + 225 = \square.$$

Now, he put [4]

$$\square = \left(15 - 16x + \frac{52}{15}x^2\right)^2$$

so that, from the equality

$$15x^4 - 120x^3 + 350x^2 - 480x + 225 = \left(15 - 16x + \frac{52}{15}x^2\right)^2,$$

he obtained

$$\left(15 - \frac{2704}{225}\right)x^4 - \left(120 - \frac{1664}{15}\right)x^3 = 0,$$

from which, apart from the trivial solution $x = 0$, one gets the values $x = 2040/671$ so $(x - 2) = 698/671$. Substituting this last value of $(x - 2)$ into (1), one has the three squares

$$m^2 = \frac{(2{,}288{,}168)^2}{671^4}; \qquad n^2 = \frac{(2792)^2}{671^2}; \qquad p^2 = \frac{(2{,}399{,}057)^2}{671^4}.$$

From these he obtained the three sought squares, which we denote also m^2, n^2, and p^2:

$$\begin{aligned} m^2 &= (2{,}288{,}168)^2 = 5{,}235{,}712{,}796{,}224 \\ n^2 &= (2792)^2(671)^2 = (1{,}873{,}432)^2 = 3{,}509{,}747{,}458{,}624 \\ p^2 &= (2{,}399{,}057)^2 = 5{,}755{,}474{,}489{,}249. \end{aligned}$$

These values satisfy the conditions of the problem; actually, one has

$$\begin{aligned} p^2 - m^2 &= 519{,}761{,}693{,}025 = (720{,}945)^2 \\ p^2 - n^2 &= 2{,}245{,}727{,}030{,}625 = (1{,}498{,}575)^2 \\ m^2 - n^2 &= 1{,}725{,}965{,}337{,}600 = (1{,}313{,}760)^2. \end{aligned}$$

The second part of the problem is satisfied by m, n, and p; that is,

$$2{,}288{,}168, \quad 1{,}873{,}432, \quad 2{,}399{,}057.$$

In fact, one has

$$\begin{aligned} m + n &= 4{,}687{,}225 = (2165)^2 \\ m + p &= 4{,}272{,}489 = (2067)^2 \\ n + p &= 4{,}161{,}600 = (2040)^2 \\ m - n &= 110{,}889 = (333)^2 \\ m - p &= 525{,}625 = (725)^2 \\ n - p &= 414{,}736 = (644)^2. \end{aligned}$$

In considering the method used by De Billy to solve all questions of this kind, we see that every possible approach to solving them, except for a few particular cases, utilizes in general the method of "simple, double, and triple equalities."

In this context, the six-square problem therefore appears as a particular case of De Billy's Question 75.

2. OZANAM'S SOLUTION

Ozanam published his method for solving the six-square problem several times [Leibniz 1990, 230] and also, in a very brief version, in [Ozanam 1691, 90–91]. In this work he formulated the six-square problem in the classical form: *Find three numbers such that the sum and the difference of any two of them is a square.*

Let u, v, and w indicate the three sought integers; Ozanam wrote them in the form

$$u = 2abxy \qquad \text{(a common side of two right-angled triangles)}$$

$$v = a^2x^2 + b^2y^2 \qquad \text{(the hypotenuse of the first triangle)}$$

$$w = b^2x^2 + a^2y^2 \qquad \text{(the hypotenuse of the second triangle),}$$

where a,b are indeterminates and x,y auxiliary variables.

Now, to solve the problem, it is sufficient to produce $v + w = \square$ and $v - w = \square$; that is,

$$(a^2 + b^2)(x^2 + y^2) = \square \tag{3}$$

$$(a^2 - b^2)(x^2 - y^2) = \square. \tag{4}$$

Thus, it was sufficient to find x and y, and, for this purpose, Ozanam had to solve the "doubles équations" (3) and (4). He used an algebraic method due to Fermat and described by De Billy in his *Inventum Novum*. That method required that the constant terms (not necessarily the same) of the two equations be squares. First, Ozanam substituted the following expression into (3) and (4):

$$x = z - \frac{a}{b}y.$$

Accordingly, (3) and (4) became

$$b^2(a^2 + b^2)z^2 - 2ab(a^2 + b^2)zy + (a^2 + b^2)^2y^2 = \square \tag{5}$$

$$b^2(a^2 - b^2)z^2 - 2ab(a^2 - b^2)zy + (a^2 - b^2)^2y^2 = \square. \tag{6}$$

He then multiplied (5) by $(a^2 - b^2)^2$ and (6) by $(a^2 + b^2)^2$, obtaining

$$b^2(a^2 + b^2)(a^2 - b^2)^2z^2 - 2ab(a^2 + b^2)(a^2 - b^2)^2zy + (a^4 - y^4)^2y^2 = \square \tag{7}$$

$$b^2(a^2 - b^2)(a^2 + b^2)^2z^2 - 2ab(a^2 - b^2)(a^2 + b^2)^2zy + (a^4 - b^4)^2y^2 = \square. \tag{8}$$

He could now apply Fermat's method. The difference between (7) and (8) is

$$2a^4b^4z^2 - 2b^8z^2 + 4ab^7zy - 4a^5b^3zy = -2ab^3z\left(\frac{b^5z}{a} - a^3bz + 2a^4y - 2b^4y\right). \tag{9}$$

The half-sum of the two factors on the right side of (9) is

$$\frac{b^5 z}{2a} - \frac{a^3 bz}{2} - ab^3 z + (a^4 - b^4)y. \tag{10}$$

By squaring (10) and by comparing the resulting expression with the first member of (8), he obtained

$$\frac{z}{y} = \frac{4ab^8 - 4a^9}{b^9 - 3a^8 b + 6a^4 b^5},$$

from which follows

$$\begin{aligned} z &= 4ab^8 - 4a^9 \\ y &= b^9 - 3a^8 b + 6a^4 b^5 \\ x &= z - \frac{a}{b} y = 3ab^8 - a^9 - 6a^5 b^4. \end{aligned}$$

Finally, the triplet of the sought numbers is given by

$$\begin{aligned} 2abxy &= 6a^2 b^{18} + 6a^{18} b^2 + 24a^6 b^{19} - 92a^{10} b^{10} + 24a^{14} b^6 \\ a^2 y^2 + b^2 x^2 &= 10a^2 b^{18} - 24a^6 b^{14} + 60a^{10} b^{10} - 24a^{14} b^6 + 10a^{18} b^2 \\ a^2 x^2 + b^2 y^2 &= a^{20} + 21a^{16} b^{14} - 6a^{12} b^8 - 6a^8 b^{12} + 21a^4 b^{16} + b^{20}. \end{aligned}$$

In particular, for $a = 1$ and $b = 2$ (or symmetrically for $a = 2$, $b = 1$) one gets the solution published in the appendix to Mengoli's *Theorema*:

1,873,432, 2,399,057, 2,288,168.

3. MENGOLI'S "THEOREMA ARITHMETICUM"

> It is not possible to find three different numbers [such that] their differences are three squares, and the differences of their squares are also three squares. [5]

Mengoli's proof of this "theorem" is discussed in seven pages, but it is false, as we have already indicated.

In attempting to prove his theorem, Mengoli utilized the following two properties:

(1) When the difference of the squares of two integers x, y is a square, then $(x + y)$ may be expressed as the product of the prime factors of the difference $(x - y)$ and another square. Mengoli reasoned as follows: if

$$x^2 - y^2 = (x + y)(x - y) = \square$$

and if, moreover, $x - y = ab$ (a, b primes), then since $(x + y) > (x - y)$, one must obtain

$$x + y = abc^2. \tag{11}$$

The last statement cannot be true, in general. In fact, if one takes $x = 73$, $y = 48$, then one has

$$x^2 - y^2 = 73^2 - 48^2 = 3025 = 55^2 = 5^2 \cdot 11^2$$
$$x - y = 73 - 48 = 25 = 5^2$$
$$x + y = 73 + 48 = 121 = 11^2$$

Indeed, this first property employed by Mengoli is correct only for particular Pythagorean triplets, such as

$$x = 3n^2; \qquad y = 4n^2; \qquad z = 5n^2.$$

The second property was the following:

(2) It is always possible to find two integers such that the difference of their squares is an even or an odd square. The two integers, as one can easily verify, are given respectively by the formulas

$$\begin{aligned} x &= b(a^2 + 1) \\ y &= b(a^2 - 1), \end{aligned} \tag{12}$$

where a and b are any integers, and

$$\begin{aligned} x &= b(2a^2 + 2a + 1) \\ y &= 2ab(a + 1), \end{aligned} \tag{13}$$

where a is any integer and b any odd integer. Mengoli tried to prove that if the difference of the squares of two integers is a square (even or odd), then the two integers can be expressed, respectively, by formulas (12) and (13).

This assertion is, unfortunately, also incorrect, and invalidates the final part of the "proof" of Mengoli's theorem.

A counterexample is provided by the same values, $x = 73$ and $y = 48$, considered above.

Since one has

$$73^2 - 48^2 = 3025 = 55^2,$$

73 and 48 ought to be expressed by (13); i.e.,

$$\begin{aligned} 73 &= b(2a^2 + 2a + 1) \\ 48 &= 2ab(a + 1) \end{aligned} \tag{14}$$

(where a is any integer and b any odd integer). But, as is easily verified, the system (14) is unsolvable.

Making use of the above properties, Mengoli stated his proof of the theorem according to the following reductio ad absurdum argument.

Assuming that it is possible to find the three sought numbers, they could be relatively prime or not. First, Mengoli considered the case in which they are relatively prime (otherwise they would be divisible by their greatest common divisor).

If a, b, c are relatively prime, they cannot all be even, so one of them must be odd. But it is impossible that all three of them are odd [6], so that only one of them is odd and the other two are even. Note that Mengoli wanted to emphasize that it is not possible for a to be an even integer.

Let the three numbers be taken, for example, so that a is odd, and b and c are even.

Mengoli considered the couples (a,b) and (b,c), and proved that it is always possible to find a and b in such a way that the relations

$$a - b = \square \qquad a^2 - b^2 = \square$$
$$b - c = \square \qquad b^2 - c^2 = \square$$

are satisfied. In fact, one has

$$\begin{aligned} a &= 2e^2d^2 + 2ed^2 + d^2 \\ b &= 2e^2d^2 + 2ed^2, \end{aligned} \tag{15}$$

which is equivalent to (12), where d^2 and e are replaced respectively by b (odd) and a; moreover

$$\begin{aligned} b &= 2f^2g^2 + 2f^2 \\ c &= 2f^2g^2 - 2f^2, \end{aligned} \tag{16}$$

which is equivalent to (13), where g and $2f^2$ are replaced respectively by a and b. Thus, from (15) and (16), the two expressions for b are equivalent. Finally, Mengoli considered the couple (a, c), proving that one gets the absurd relationship $a + c = a - c$ [7]!

It is clear that the "proof" is invalidated by the fact that in order to derive (15) and (16), Mengoli repeatedly uses the relation

$$\sqrt{x^2 - y^2} = c(x - y)$$

which follows from (11) and from $(x - y) = a \cdot b$, which is, in general, incorrect, as we have seen.

4. MENGOLI'S "PROBLEMA ARITHMETICUM"

Mengoli felt that this incorrect proof of his "theorem" dealt a serious blow to his mathematical reputation, and this "French" problem continued to plague him for some time. He tried to remedy his error in solving the six-square problem, and presented this instead as "Problema Arithmeticum." The following "Praemonitio ad Lectorem" [Mengoli 1674b, 3] was placed before the proof, perhaps in order to justify himself by accusing others of ignorance and arrogance:

> Few are those who understand our reasonings, very many those who do not understand them. Among the latter some, being more intelligent, no doubt would understand if only they could judge without arrogance, if they could read not superficially and quickly but with attention, humility, patience; above all, those who are used to studying mathematics have acquired a method of learning: in fact for all of us the great extent of the subject is an

indisputable sign of intelligibility. Those who do not want to understand because they consider what we are saying obscure, nevertheless hold us responsible for that. In fact they will likewise find obscure the words in texts about law or medicine if they have no familiarity with them. Moreover, let them not forget, that they obtained the books from us with very little or no change at all. In fact, we did not think we should sell them to commoners, because only few are able to understand them. For those who are not able to understand our new and so important endeavours (it being impossible to prevent some books from reaching their hands) and who hoped they could, by mere chance, attracted by the title of the book, learn something about the arithmetic of numbers, I will try to satisfy them by means of a little appendix of no little value.

While Mengoli herein emphasized his mathematical work in general, his contemptuous tone in the last part of this "Praemonitio" betrays his disappointment over his earlier incorrect proof of the six-square problem.

In order to solve this "Problema arithmeticum," Mengoli first solved the following problem:

Find four numbers such that the sum of the squares of the first two numbers is a square, the sum of the squares of the third and the fourth numbers is a square, the product of all four is a square, and the ratio of the first to the second is greater than the ratio of the third to the fourth.

If we denote the sought four numbers with $a, b, c, d,$ then

$$\left.\begin{aligned} a^2 + b^2 &= \square \\ c^2 + d^2 &= \square \\ abcd &= \square \end{aligned}\right\} \quad \text{with} \frac{a}{b} > \frac{c}{d}. \tag{17}$$

Mengoli solved this problem by an empirical method working with Pythagorean triplets. He began by listing, in two columns, the first two terms of particular Pythagorean triplets:

I	II
3	4
4	3
5	12
6	8
7	24
8	15
.	.
.	.
.	.

(18)

Then he noted that, in the second column, starting with 4, all other numbers are, alternatively, in the ratio of 1 to 3, 2 to 4, 3 to 5, 4 to 6, etc.; namely $4/x = 1/3$ so that $x = 12$; $12/x = 2/4$ so that $x = 24$; $24/x = 3/5$ so that $x = 40$; and so on.

In the first column, starting with the second term 3, all other numbers are, alternatively, in the ratio of 1 to 5, 3 to 7, 5 to 9, 7 to 11, and so on; namely $3/x = 1/5$ so that $x = 15$; $15/x = 3/7$ so that $x = 35$; and so on.

Mengoli gave, moreover, some rules to establish the relationships between the numbers of the first and the second columns.

If, in the first column, a number n is odd, $n = 2k + 1$, then its corresponding value in the second column is $n' = (2k)\,(2k + 2)/2 = 2k(k + 1)$.

If, on the other hand n is even, $n = 2k$, then its corresponding value in the second column is

$$n' = \frac{2k-2}{2}\,\frac{2k+2}{2} = (k-1)(k+1) = k^2 - 1.$$

Thus one has

$$n^2 + n'^2 = (2k + 1)^2 + [2k(k + 1)]^2 = [2k(k + 1) + 1]^2 = (n' + 1)^2$$

and

$$n^2 + n'^2 = k^4 + 2k^2 + 1 = (k^2 + 1)^2 = (n' + 2)^2.$$

Actually, these "laws and rules" governing the two columns are nothing but the description of the form of the general terms of the two following kinds of Pythagorean triplets:

Pythagorean Triplets

n	I $a = 2n + 1$ $b = 2n^2 + 2n$ $c = 2n^2 + 2n + 1$	II $a = 2n$ $b = n^2 - 1$ $c = n^2 + 1$
1	3 − 4 − 5	
2	5 − 12 − 13	4 − 3 − 5
3	7 − 24 − 25	6 − 8 − 10
4	9 − 40 − 41	8 − 15 − 17
5	11 − 60 − 61	10 − 24 − 26
6	13 − 84 − 85	12 − 35 − 37
7	15 − 112 − 113	14 − 48 − 50
8	17 − 144 − 145	16 − 63 − 65
9	19 − 180 − 181	18 − 80 − 82
. . .	. . .	. . .

(19)

At this stage, Mengoli again rewrote the two columns (18), factoring the composite numbers into prime factors:

3	2, 2
2, 2	3
5	2, 2, 3
2, 3	2, 2, 2
7	2, 2, 2, 3

$$\begin{array}{ll} 2,2,2 & 3,5 \\ 3,3 & 2,2,2,5 \\ 2,5 & 2,2,2,3 \\ 11 & 2,2,3,5 \\ 2,2,3 & 5,7 \\ 13 & 2,2,3,7 \\ 2,7 & 2,2,2,2,3 \\ 3,5 & 2,2,2,2,7 \\ \ldots & \ldots \end{array} \tag{20}$$

He then asserted that from (20) one can determine, by trial and error, four numbers, with no pair of proportional numbers among them, and such that the sum of their squares is a square. For instance, from (2) one deduces that four of the sought numbers are 112, 15, 35, and 12. In fact, these fulfill the conditions posed in problem (17). All numbers proportional to 112, 15, 35, and 12 will also solve this problem. Another quartet of numbers solving the problem is 364, 27, 84 and 13, and likewise all numbers proportional to these.

Mengoli used these solutions of (17) to solve the "Problema Arithmeticum," or six-square problem. By translating Mengoli's rhetorical language and setting

$$\begin{aligned} u &= \frac{1}{2}[(p^2t^2 + s^2q^2) - (p^2q^2 + s^2t^2)] \\ v &= u + (pq - st)^2 \\ w &= v + 4pstq \end{aligned} \tag{21}$$

one has

$$\begin{aligned} w - v &= 4pstq & u + v &= (pt - sq)^2 \\ w - u &= (pq + st)^2 & w + u &= (pt + sq)^2 \\ v - u &= (pq - st)^2 & w + v &= (p^2 + s^2)(q^2 + t^2); \end{aligned} \tag{22}$$

so that

$$\begin{aligned} w^2 - v^2 &= (w - v)(w + v) = 4pstq(p^2 + s^2)(q^2 + t^2) = \square \\ w^2 - u^2 &= (w - u)(w + u) = (pq + st)^2(pt + sq)^2 = \square \\ v^2 - u^2 &= (v - u)(v + u) = (pq - st)^2(pt - sq)^2 = \square. \end{aligned} \tag{23}$$

Utilizing the numerical solution to (17) given by

$$p = 112; \quad s = 15; \quad t = 35; \quad q = 12$$

and the identities (21), Mengoli obtained

$$\begin{aligned} u &= 6{,}658{,}419.5 \\ v &= 7{,}329{,}180.5 \\ w &= 10{,}151{,}580.5, \end{aligned}$$

from which, by multiplying by 4, one has

$$u' = 26{,}633{,}678$$
$$v' = 29{,}316{,}722$$
$$w' = 40{,}606{,}322.$$

He produced a second solution by taking

$$p = 364; \qquad s = 27; \qquad t = 84; \qquad q = 13,$$

from which he obtained

$$u' = 1{,}814{,}958{,}658$$
$$v' = 1{,}839{,}243{,}842$$
$$w' = 2{,}010{,}958{,}658.$$

5. CONCLUSIONS

When Ozanam communicated a numerical solution of the six-square problem as a final remark on [Mengoli 1674a], Mengoli's pride was deeply hurt. This was all the more so, because he had just published *Arithmetica Realis* [Mengoli 1675] where he had corrected his own mistake, even if by trial and error. In a letter to Marchetti on June 2, 1674 he wrote painfully [8]:

> I am enclosing the true solution of the French Problem: the one I gave previously, wrong because of a paralogism used in it, allowed me to understand the true reason behind the reaction to it, that is to hurt the reputation of my person, already pursued in France. Hence, I have discovered the true purpose behind the inopportune proposal of the precise Problem suggested to me by the Frenchman [Ozanam] several times through the years and by different means. God forgive him.

Actually Mengoli's reputation in Europe at that time does not warrant these feelings of persecution. Collins, for instance, in a letter to Newton of some years before referred to him as "an excellent mathematician and musician" [9]. The fact that Mengoli had a respectable reputation in the European scientific community of his time is confirmed by the attention paid him in Oldenburg's correspondence, which aroused Leibniz's interest in his work. In conclusion, it seems there is enough evidence to conclude that Ozanam's publication of his counterexample was not meant as a blow to Mengoli's reputation. It is more likely that Ozanam proposed the "French" problems to Mengoli, as well as to other mathematicians, simply to learn to what extent algebraic methods were being used in Europe.

At the time Mengoli tried to solve the six-square problem, only a few European mathematicians were aware of systematic treatments of such problems. Such methods were barely known in Italy, as we can deduce from correspondence between Carlo Renaldini (1615–1698) and the Czechoslovak Jesuit Adam Adamand Kochanski, both of whom were interested in Diophantine problems [Renaldini 1684, 53–58]. In fact, on April 27, 1675, almost a year after Ozanam's communication of the numerical counterexample in an appendix to the theorem of Mengoli, Renaldini, who knew Ozanam's numerical counterexample, sent the Jesuit mathematician a letter concerning this problem and pointing out Mengoli's incorrect proof. The answer of Kochanski, full of mistakes, clearly shows that, like Mengoli, he relied on empirical methods based on the "Scholia" on Pythagorean triplets that Christopher Clavius (1537–1612) added to Proposition I,47 in his edition of Euclid's *Elements* (1574). Thus it is quite likely that, being almost entirely unaware of Fermat's researches, Kochanski also tried to solve the problem by trial and error, using certain sequences of Pythagorean triplets. The Jesuit mathematician even asked Renaldini to suggest to him "some other way of finding the Pythagorean numbers, different from the one suggested by Clavius in the above-mentioned place."

Needless to say, Renaldini was completely unable to satisfy the curiosity of Kochanski, and in his answer (January 1675) he referred only to Ozanam's solution triplet, adding the following comment: "Here is the solution of that famous Problem which, throughout all Europe, has challenged the mathematicians' brains for a long time." Promising to send him his solution in the future (which he never did), he added: "It is evident that the capability of the human mind to conceive such an extraordinary Problem indicates that it is of a sublime nature; in fact, to find numbers like those required by the Problem is certainly not a human endeavor, at least until a divine light will shine on it; this, however, would not bring a great glory to the Mathematical disciplines."

In our opinion, the fact that both Leibniz and Mengoli encountered such severe difficulties in trying to solve Diophantine problems was due to their lack of a general method. At the time when these problems were formulated, De Billy had already set out a general algebraic method for solving them, a method that Ozanam subsequently learned from the Jesuit priest, as their correspondence reveals.

Yet there was an essential difference between Leibniz's approach to the six-square problem and Mengoli's. Although neither possessed a general algebraic method for solving such problems, their respective approaches reveal different mathematical sensibilities.

In Mengoli's case, when he was unable to solve the problem, he tried to prove its impossibility. Only when he became aware of Ozanam's numerical counterexample did he succeed in solving it by trial and error. He never looked for a general method of solution nor did he seek Ozanam's advice; in fact, it seems he knew nothing at all about the European literature concerning such questions. These weaknesses go far to explain the pathetic tone of his "Praemonitio" to the second attempt to solve the six-square problem, as well as the answer he gave to Rudolph Christian Bodenhausen in response to a problem proposed by him [Baroncini & Cavazza 1986, 123]:

> As far as the problem is concerned, Your Excellency will remember that I have already said that I do not practice Algebra anymore . . . and at my age being over sixty years old it is not appropriate for me to seek the company of Apollo and the Muses.

It seems clear that here he was using his old age as a screen to hide the true reason for his refusal to discuss the problem, i.e., his lack of algebraic proficiency.

Mengoli's case, in the final period of his mathematical activity, seems to confirm Pepe's thesis [Pepe 1982, 271], according to which, in spite of the important work of Angeli, Borelli, Mengoli, Viviani, and others, from about 1650 onward, one finds frequent signs of decline in Italian mathematics. Whereas algebraic methods became important in other European countries, Italian mathematicians began to lose contact with the most active centers of European scientific culture.

Another mathematical challenge, the so-called "Dutch Problems," serves as another indicator of the weakness of Italian mathematics at this time. Most of the Italian scholars interested in these problems, which concerned the construction of triangles satisfying given conditions, used the synthetic geometrical method and thereby ran into many difficulties. Mengoli seems to have had no interest in the mathematics underlying these "Dutch Problems." Leibniz, on the other hand, went well beyond the mere effort of finding particular solutions, and tried to develop methods of more general validity [Robinet 1988, 35]; [Gatto & Palladino 1992]. Indeed, Michelangelo Ricci had good reason to tell the Italian mathematicians they should stop worrying about the "Dutch Problems," since the scholars on the other side of the Alps "who master well the Algebra are able to solve them in a short time and with no great effort" [10].

Leibniz's approach to the six-square problem was different from Mengoli's. His 30 attempted solutions [Leibniz 1990, 229–636], even if unsuccessful, show his scientific interest in finding and understanding a general algebraic method in order to solve the double and triple equalities. He used the same approach for the following problems described in two notes of Ozanam [Leibniz 1990, 334–336]:

> (1) [Find] three numbers in geometric progression such that the sum of their product and the product of any two of them is a square. [11]
>
> (2) [Find] three numbers such that their product is a cube, the sum of their product and the square of any one of them is a square, and the sums of any couple of them are third powers. [12]

Regarding the second problem Leibniz remarked that it was "the most difficult among those solved by Ozanam." Part of it had already been solved by De Billy in his *Novarum Quaestionum Libri Tres,* where it appears as the 33rd problem of Book I. Also, in this case it is interesting to note how Leibniz undertook repeated attempts to develop a general method of solution based on the double and triple equalities referred to in Ozanam's notes. Leibniz clearly understood that the algebraic method of solving the double and triple equalities was an essential tool in order to tackle such Diophantine problems.

NOTES

1. See the Manuscripts of the National French Library, Latin manuscript 8600, sheet 48 r°.

2. The Latin text is the following: "Invenire tres quadratos, quorum summa quadratus, & summa quadratorum ab ipsis quadrato-quadratus. Parisijs datam."

3. See Weil [1984, 105–107] for a modern reading of De Billy's and Fermat's method of solving the double and triple equalities.

4. De Billy's method of solving a polynomial of five terms into which the only known term is a square was the following. Let the equality be

$$ax^4 + bx^3 + cx^2 + dx + e^2 = \square,$$

then he puts

$$\square = \left(e + \frac{d}{2e}x + \frac{c - d^2/4e^2}{2e}x^2\right)^2 = e^2 + dx + cx^2 + \frac{d(4ce^2 - d^2)}{8e^4}x^3 + \frac{(4ce^2 - d^2)^2}{64e^6}x^4$$

so that the terms in x, x^2, and the constant term are eliminated. He obtains the rational solution

$$x = \frac{8e^2[d(4ce^2 - d^2) - 8be^4]}{64ae^6 - (4ce^2 - d^2)^2}.$$

5. The Latin text is: "Non est possibile, tres inequales numeros assignare: quorum differentiae, tres quadrati, et differentiae quadratorum, tres quadrati."

6. To confirm such a statement, Mengoli proves that a, b, c cannot be odd two by two. Suppose, he says, a and b are odd. So, their difference is even. It must be a square; hence it is an even square, hence divisible by 4. Therefore, $a - b = 4k^2$, hence $(a - b)/2 = 2k^2$. On the other hand (see the end of Section 2) a and b must have the form

$$a = x(y^2 + 1)$$
$$b = x(y^2 - 1)$$

(for arbitrary x, y), so that

$$(a - b)/2 = x = 2k^2.$$

Therefore, b should be even, against the hypothesis.

7. Mengoli's reasoning is the following: $b = 2e^2d^2 + 2ed^2 = 2f^2g^2 + 2f^2$ by (15) and (16). By (15) one has $a = 2f^2g^2 + 2f^2 + d^2$ so that, using the expression for c in (16), one has

$$a - c = 4f^2 + d^2$$
$$a^2 - c^2 = 16g^2f^4 + 4g^2f^2d^2 + 4f^2d^2 + d^4.$$

Now, Mengoli imprudently states that $a^2 - c^2 = (a - c)^2$, from which the absurd is deduced.

8. See G. B. Tondini, *Lettere di uomini illustri,* Macerata 1783, 128–129. The Italian text is the following: "Mando inclusa la vera soluzione del Problema franzese, avendo scoperta con la falsa, che mandai ultimamente per un paralogismo commessovi, la verità d'un Problema morale toccante la mia persona perseguitata in Francia, per iscreditarmi. Onde ho riconosciuta l'intenzione dell'importuna inchiesta del Problema secco propostomi dal Franzese più volte in alquanti anni, e per più mezzi. Il Sig. Iddio gli perdoni."

9. See I. H. W. Turnbull (Ed.) 1959. *The Correspondence of I. Newton,* pp. 32–33. Cambridge. The letter dates from 13th July, 1670.

10. See letter to Alessandro Marchetti on June 4, 1675, Florence, Bibl. Naz., Ms. Gal. 258, sheet 140 r.

11. The Latin text is: "Tres numeri proportionales geometrice, quorum solidus auctus plano duorum quorumlibet faciat quadratum."

12. The Latin text is: "Tres numeros, ita ut solidus sub ipsis tribus sit cubus, qui auctus quadrato cuiuslibet faciat quadratos, et quorum latera bina sumta faciunt cubum."

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