Pencils of Complex and Real Symmetric and Skew Matrices

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ABSTRACT

This expository paper establishes the canonical forms under congruence for pairs of complex or real symmetric or skew matrices. The treatment is in the spirit of the well-known book of Gantmacher on matrix theory, and may be regarded as a supplement to Gantmacher's chapters on pencils of matrices.

1. INTRODUCTION

Let matrices A and B have real or complex entries, with A symmetric or skew, and B symmetric or skew. There are classical results going back to Kronecker pertaining to the simultaneous reduction of A and B to a canonical form under a congruence transformation

$$A \to SAS^t, \qquad B \to SBS^t$$

(the superscript ^{t} denotes transposition) when the matrix entries are complex. Here S is nonsingular with complex entries. Less classical but equally important is the real case, in which A, B, and S have real entries. The objective of this paper is to provide a summary of the principal results in both the complex and real cases, with proofs. In the real cases the action of the law of inertia makes the study somewhat more intricate.

This is a revised version of a document the author prepared about 1973 but chose not to publish because of a perception of thin originality. However, there were requests for copies of it and invitations to publish it, and it has

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been cited in the literature, so an audience for it appears to exist, and it is therefore made public now. It still is true that no particular originality is claimed. In fact, this is an expository contribution, employing as technique only an unsophisticated partitioning of matrices. A very extensive bibliography covering many aspects of the study of pencils concludes this paper.

2. NOTATION

We assume a general familiarity with the chapter of Gantmacher's linear algebra text [8] on pencils, Chapter 12. We write a pencil as $A - \rho B$ in preference to the $A + \lambda B$ used by Gantmacher. Here A and B are symmetric or skew symmetric matrices with elements in a base field of characteristic not two, and λ is an indeterminate over the base field with $\rho = -\lambda$. Let μ be a second indeterminate over the base field.

Set

$$\mathscr{L}_{\mathscr{E}}(\rho) = \begin{bmatrix} -\rho & & \\ 1 & \ddots & \\ & \ddots & -\rho \\ & & 1 \end{bmatrix}, \qquad \mathscr{E}+1 \text{ rows, } \mathscr{E} \text{ columns.}$$

This matrix is the transpose of Gantmacher's $L_{\mathscr{E}}(\lambda)$. We use Gantmacher's notation, slightly modified, for the elementary nilpotent matrix

$$H_u = \begin{bmatrix} 0 & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & 0 \end{bmatrix}, \qquad u \text{ rows and columns.}$$

The subscript u will be dropped when convenient. The $u \times u$ identity matrix will be E_u, usually written as E.

The superscripts ^{t} and $*$ will respectively denote transposition and transposition combined with complex conjugation. We let $\mathscr{L}_{\mathscr{E}}(\rho)^{\pm t}$ denote the "natural" transpose of $\mathscr{L}_{\mathscr{E}}(\rho)$ induced by the symmetry or skew symmetry of A, B, defined by requiring

$$\mathscr{M}_{\mathscr{E}}(\rho) = \begin{bmatrix} 0_{\mathscr{E}+1} & \mathscr{L}_{\mathscr{E}}(\rho) \\ \mathscr{L}_{\mathscr{E}}(\rho)^{\pm t} & 0_{\mathscr{E}} \end{bmatrix}$$

to have the term not involving ρ symmetric or skew according as A is symmetric or skew, and the term in ρ symmetric or skew according as B is symmetric or skew. The symbol 0_x here denotes an $x \times x$ zero matrix. The matrix $\mathcal{M}_{\mathcal{E}}(\rho)$ has $2\mathcal{E}+1$ rows and columns, and if $\mathcal{E}=0$ it becomes the 1×1 zero matrix. We call $\mathcal{M}_{\mathcal{E}}(\rho)$ a *minimal index block*. Script letters will generally be used for quantities associated with minimal indices, and avoided for quantities associated with roots or elementary divisors.

A *Jordan block* belonging to an elementary divisor $(\alpha-\rho)^e$ of $A-\rho B$, where α is an element of the base field, is the $e \times e$ matrix

$$J_e(\alpha,\rho) = \begin{bmatrix} \alpha-\rho & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \alpha-\rho \end{bmatrix} = (\alpha-\rho)E_u + H_u.$$

If, however, α is infinite, then the Jordan block is

$$J_e(\alpha,\rho) = \begin{bmatrix} 1 & -\rho & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & -\rho \\ & & & & 1 \end{bmatrix} = E_u - \rho H_u,$$

and it belongs to an elementary divisor μ^e of $\mu A - B$. We say that $J_e(\alpha,\rho)$ belongs to the root α whether α is finite or infinite.

3. MINIMAL INDICES

The row and column minimal indices of a matrix pencil $A-\rho B$ with A, B symmetric or skew must coincide. For example, if A is symmetric and B skew, and $x(\rho)$ is a nonzero row vector with polynomial entries of least possible degree satisfying $x(\rho)(A-\rho B)=0$, then $(A-\rho B)x(-\rho)^t=0$, and thus $A-\rho B$ has a column minimal index equal to its row minimal index, namely, the degree of $x(\rho)$. More generally, let row vectors $x_i(\rho)$ with polynomial entries satisfy $x_i(\rho)(A-\rho B)=0$, with $x_i(\rho)$ a lowest degree vector linearly independent of $x_1(\rho),\dots,x_{i-1}(\rho)$, for each i. Then $(A-\rho B)x_i(-\rho)^t=0$ for column vectors $x_i(-\rho)^t$ satisfying the same independence conditions. Thus the totality of row minimal indices [the degrees of the $x_i(\rho)$] of $A-\rho B$ coincides with the totality of its column minimal indices.

The block $\mathcal{M}_{\mathcal{E}}(\rho)$ has $\mathcal{E}$ as its only row minimal index, $\mathcal{E}$ as its only column minimal index, and no elementary divisors. On the other hand, a

Jordan block $J_e(\alpha,\rho)$ has a single elementary divisor and no minimal indices. Thus a suitable direct sum of blocks $\mathscr{M}_{\mathscr{E}}(\rho)$ and $J_e(\alpha,\rho)$, for various $\mathscr{E}$, α, e, will have the same row minimal indices, column minimal indices, roots, and elementary divisors belonging to the roots as $A-\rho B$ has.

4. CONSTRAINTS ON ELEMENTARY DIVISORS

Throughout this section A and B will be symmetric or skew matrices over an arbitrary algebraically closed field of characteristic not two. We wish to deduce properties of the elementary divisors of $A-\rho B$ when one of the matrices is symmetric and the other skew, or when both are skew. The pencil $A-\rho B$ may be singular, that is, $\det(A-\rho B)$ may be the zero polynomial. These properties were first noticed by Kronecker [51].

Let $M-\rho N$ be a direct sum of blocks $\mathscr{M}_{\mathscr{E}}(\rho)$ belonging to minimal indices, and blocks $J_e(\alpha,\rho)$ belonging to finite or infinite roots, such that $M-\rho N$ has the same minimal indices, roots, and elementary divisors as $A-\rho B$. Of course, M and N will then generally not be symmetric or skew, but $M-\rho N$ will be strictly equivalent to $A-\rho B$. This means $P(A-\rho B)Q = M-\rho N$ for certain nonsingular matrices P, Q with elements in the base field. Hence

$$A = P^{-1}MQ^{-1}, \qquad B = P^{-1}NQ^{-1}.$$

We write out the following discussion when A is symmetric and B skew; the changes to be made when A is skew and B symmetric, or both are skew, will be indicated later.

Using $A^t = A$, $B^t = -B$, we get $TM = M^tT^t$, $TN = -N^tT^t$, where $T = Q^tP^{-1}$ is a nonsingular matrix with elements in the base field. Hence

$$T(M-\rho N) = (M^t+\rho N^t)T^t. \tag{1}$$

Arrange the diagonal blocks in $M-\rho N$ so that

$$M-\rho N = \begin{bmatrix} M_m-\rho N_m & & & \\ & M_\infty-\rho N_\infty & & \\ & & M_0-\rho N_0 & \\ & & & M_f-\rho N_f \end{bmatrix}, \tag{2}$$

where:

(i) $M_m - \rho N_m$ incorporates all blocks $\mathscr{M}_{\mathscr{E}}(\rho)$ belonging to minimal indices;

(ii) $M_\infty - \rho N_\infty$ incorporates all Jordan blocks $J_e(\infty, \rho)$ belonging to the root ∞;

(iii) $M_0 - \rho N_0$ incorporates all Jordan blocks $J_e(0, \rho)$ belonging to the root 0; and

(iv) $M_f - \rho N_f$ incorporates all Jordan blocks belonging to finite nonzero roots.

Let there be s diagonal blocks in $M_m - \rho N_m$ belonging to minimal indices $\mathscr{E}_1, \dots, \mathscr{E}_s$, and let the blocks in $M_\infty - \rho N_\infty$ and in $M_0 - \rho N_0$ be arranged in order of increasing size, the smaller blocks higher up in the block diagonal. Suppose there are a total of k blocks in $M_\infty - \rho N_\infty$, $M_0 - \rho N_0$, $M_f - \rho N_f$.

Partition $T = [\mathbf{T}^{uv}]_{1 \leqslant u, v \leqslant 4}$ conformally with the block diagonal partitioning just displayed of $M - \rho N$; then refine this partitioning to $T = [T_{ij}]_{1 \leqslant i,\, j \leqslant s+k}$ conforming to the decomposition of $M - \rho N$ as a direct sum of blocks $\mathscr{M}_{\mathscr{E}}(\rho)$ and $J_e(\alpha, \rho)$. Thus

$$\mathbf{T}^{11} = [T_{ij}]_{1 \leqslant i, j \leqslant s}, \qquad T_{ij} \text{ is } (2\mathscr{E}_i + 1) \times (2\mathscr{E}_j + 1),$$

$$[\mathbf{T}^{12}, \mathbf{T}^{13}, \mathbf{T}^{14}] = [T_{ij}]_{1 \leqslant i \leqslant s,\, s < j \leqslant s+k}, \qquad T_{ij} \text{ has } 2\mathscr{E}_i + 1 \text{ rows},$$

$$\begin{bmatrix} \mathbf{T}^{21} \\ \mathbf{T}^{31} \\ \mathbf{T}^{41} \end{bmatrix} = [T_{ij}]_{s < i \leqslant s+k,\, 1 \leqslant j \leqslant s}, \qquad T_{ij} \text{ has } 2\mathscr{E}_j + 1 \text{ columns}.$$

The number of columns (rows) in a block T_{ij} in the second (third) of these formulas is that for the block $J_e(\alpha, \rho)$ in the same block column (row, respectively) of $M - \rho N$. For the blocks T_{ij} in $\mathbf{T}_{11}$, introduce a further partitioning,

$$T_{ij} = \begin{bmatrix} U_{ij} & V_{ij} \\ W_{ij} & X_{ij} \end{bmatrix}, \qquad 1 \leqslant i, j \leqslant s,$$

where U_{ij} is $(\mathscr{E}_i + 1) \times (\mathscr{E}_j + 1)$ and X_{ij} is $\mathscr{E}_i \times \mathscr{E}_j$. Also partition further the blocks T_{ij} in $\mathbf{T}^{12},\mathbf{T}^{13},\mathbf{T}^{14}$ and in $\mathbf{T}^{21},\mathbf{T}^{31},\mathbf{T}^{41}$, so that

$$T_{ij}=\begin{bmatrix} U_{ij} \\ W_{ij} \end{bmatrix}, \qquad 1\leqslant i\leqslant s, \quad s<j\leqslant s+k,$$

$$T_{ij}=\begin{bmatrix} U_{ij}, V_{ij} \end{bmatrix}, \qquad s<i\leqslant s+k, \quad 1\leqslant j\leqslant s.$$

In the first of these two formulas U_{ij} has $\mathscr{E}_i+1$ rows, and in the second it has $\mathscr{E}_j+1$ columns.

The relation (1) induces relations involving the T_{ij}.

First, let i, j satisfy $1\leqslant i,j\leqslant s$. From (1), using $M_m^t=M_m$ and $N_m^t=-N_m$, we get

$$T_{ij}\mathscr{M}_{\mathscr{E}_j}(\rho)=\mathscr{M}_{\mathscr{E}_i}(\rho)T_{ji}^t,$$

and therefore

$$V_{ij}\mathscr{L}_{\mathscr{E}_j}(\rho)^{\pm t}=\mathscr{L}_{\mathscr{E}_i}(\rho)V_{ji}^t.$$

Comparing first the ρ term on each side [noting that ρ appears in $\mathscr{L}_{\mathscr{E}_j}(\rho)^{\pm t}$ as $+\rho$], and afterwards the constant term, we get

$$\begin{bmatrix} V_{ij}, 0 \end{bmatrix}=-\begin{bmatrix} V_{ji}^t \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0, V_{ij} \end{bmatrix}=\begin{bmatrix} 0 \\ V_{ji}^t \end{bmatrix},$$

where the symbol 0 denotes either a single row of zeros or a single column of zeros. Recursive comparison of the columns on each side of this pair of equations, beginning with the first column, yields $V_{ij}=0$, $V_{ji}=0$. Thus we actually have

$$T_{ij}=\begin{bmatrix} U_{ij} & 0 \\ W_{ij} & X_{ij} \end{bmatrix}, \qquad 1\leqslant i,j\leqslant s. \tag{3}$$

Next, let i and j satisfy $1\leqslant i\leqslant s$, $s<j\leqslant s+k$. From (1),

$$T_{ij}J_e(\alpha,\rho)=\mathscr{M}_{\mathscr{E}}(\rho)T_{ji}^t$$

for certain α, e, $\mathscr{E}$, and therefore

$$U_{ij}J_e(\alpha,\rho) = \mathscr{L}_{\mathscr{E}}(\rho)V_{ji}^t.$$

Assume first that α is finite. From the last equation we obtain

$$-U_{ij} = -\begin{bmatrix} V_{ji}^t \\ 0 \end{bmatrix}, \qquad U_{ij}(\alpha E + H) = \begin{bmatrix} 0 \\ V_{ji}^t \end{bmatrix},$$

yielding

$$\begin{bmatrix} V_{ji}^t \\ 0 \end{bmatrix}(\alpha E + H) = \begin{bmatrix} 0 \\ V_{ji}^t \end{bmatrix}.$$

Recursive comparison of the rows on each side of this equation produces $V_{ji} = 0$, and hence $U_{ij} = 0$. If, however, α is infinite, then

$$-\left[0, U_{ij}^{\hat{}}\right] = -\begin{bmatrix} V_{ji}^t \\ 0 \end{bmatrix}, \qquad U_{ij} = \begin{bmatrix} 0 \\ V_{ji}^t \end{bmatrix},$$

(the symbol $\hat{}$ here denotes deletion of the last column of U_{ij}), and recursive comparison of the columns beginning with the first again leads to $U_{ij} = 0$, $V_{ji} = 0$. Therefore we actually have

$$T_{ij} = \begin{bmatrix} 0 \\ W_{ij} \end{bmatrix}, \qquad 1 \leqslant i \leqslant s, \quad s < j \leqslant s + k, \tag{4}$$

$$T_{ij} = \left[U_{ij}, 0\right], \qquad s < i \leqslant s + k, \quad 1 \leqslant j \leqslant s. \tag{5}$$

It follows from the partitionings so far obtained that the submatrix $[X_{ij}]_{1 \leqslant i,j \leqslant s}$ of T, comprising the blocks X_{ij}, is nonsingular. To see this, first note that this submatrix is square, having $\mathscr{E}_1 + \cdots + \mathscr{E}_s$ rows and columns. If its columns are dependent, then the columns of T passing through it will also be dependent, since outside this submatrix these columns have only zero entries [by (3) and (5)].

Now take $i > s$ and $j > s$. From (1) we obtain

$$T_{ij}J_e(\alpha,\rho) = J_f(\alpha', -\rho)^t T_{ji}^t, \tag{6}$$

where e, f are certain block sizes and α, α' certain roots. We consider several cases. If α is infinite and α' finite, from (6) we get

$$-T_{ij}H = T_{ji}^t, \qquad T_{ij} = J_f(\alpha',0)^t T_{ji}^t.$$

Hence $T_{ij} = -J_f(\alpha',0)^t T_{ij}H$. Iterating this equation yields $T_{ij} = 0$, since H is nilpotent. Then also $T_{ji} = 0$. Thus in the lower right portion of T there is a direct sum splitting, blocks associated with root ∞ splitting away from blocks associated with finite roots. Next let α be zero, α' finite but nonzero. Then (6) yields

$$-T_{ij} = T_{ji}^t, \qquad T_{ij}H = J_f(\alpha',0)^t T_{ji}^t,$$

and thus

$$T_{ij}H = -J_f(\alpha',0)^t T_{ij}.$$

This yields

$$T_{ij}H^p = \left[-J_f(\alpha',0)^t\right]^p T_{ij}$$

for $p = 1, 2, \ldots$. For sufficiently great p we deduce $T_{ij} = 0$ since H is nilpotent and $J_f(\alpha',0)$ nonsingular. This proves that a further splitting occurs in the lower right portion of T: blocks associated with root zero are split away from blocks associated with nonzero roots.

Thus T actually has the form

$$T = \begin{bmatrix} \mathbf{T}^{11} & \mathbf{T}^{12} & \mathbf{T}^{13} & \mathbf{T}^{14} \\ \mathbf{T}^{21} & \mathbf{T}^{22} & 0 & 0 \\ \mathbf{T}^{31} & 0 & \mathbf{T}^{33} & 0 \\ \mathbf{T}^{41} & 0 & 0 & \mathbf{T}^{44} \end{bmatrix}.$$

We now consider the blocks T_{ij} contained in $\mathbf{T}^{22}$. From (6) with $\alpha = \alpha' = \infty$ we get

$$T_{ij} = T_{ji}^t, \qquad -T_{ij}H = H^t T_{ji}^t.$$

Thus $\mathbf{T}^{22}$ is symmetric, and its submatrix T_{ij} lies in the kernel of the operator $T_{ij} \to T_{ij}H + H^t T_{ij}$. Hence T_{ij} has the structure

$$T_{ij} = \begin{bmatrix} & & \cdot \\ \cdot & \cdot & \\ \cdot & \cdot & \cdot \end{bmatrix} \quad \text{or} \quad T_{ij} = \begin{bmatrix} & & \cdot \\ & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}. \tag{7}$$

Here each cross diagonal (orthogonal to the main diagonal) not meeting the lower or right edges is zero, and the remaining cross diagonals are alternating, that is, have the form $x, -x, x, -x, \ldots$.

We now prove the following result, due to Kronecker:

When A is symmetric and B skew, each elementary divisor of even degree belonging to root ∞ occurs with even multiplicity.

Proof. If this is not the case, we shall prove that $\mathbf{T}^{22}$ is singular, and the singularity of $\mathbf{T}^{22}$ will lead to the singularity of T, a contradiction. Recall that the diagonal blocks in $M_\infty - \rho N_\infty$ are arranged in order of increasing size; thus as one moves downward or to the right in $\mathbf{T}^{22}$, the blocks T_{ij} become no smaller.

Suppose there exists an elementary divisor μ^e of fixed even degree e belonging to root ∞ occurring with an odd multiplicity r. Consider the blocks T_{ij} in $\mathbf{T}^{22}$ having not less than e rows and e columns. These blocks constitute a lower right section of $\mathbf{T}^{22}$, so that $\mathbf{T}^{22}$ partitions as

$$\mathbf{T}^{22} = \begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{bmatrix},$$

where $\mathbf{S}$ comprises all blocks T_{ij} in $\mathbf{T}^{22}$ having e or more rows and columns. All blocks T_{ij} in $\mathbf{Q}$ then have, by (7), a zero initial column. Form a new matrix $\tilde{\mathbf{S}}$ by extracting the extreme lower left element from each block T_{ij} in $\mathbf{S}$. By the structure (7) of the blocks T_{ij}, this matrix $\tilde{\mathbf{S}}$ is itself block triangular, the leading block in $\tilde{\mathbf{S}}$ being $r \times r$. Because $\mathbf{T}^{22}$ is symmetric, with the principal cross diagonal in each T_{ij} of alternating character, and using the evenness of the block size e, it follows that the leading $r \times r$ section in $\tilde{\mathbf{S}}$ is skew symmetric. Since r is odd, this leading segment is singular, and as $\tilde{\mathbf{S}}$ is block triangular, it follows that $\tilde{\mathbf{S}}$ is singular. From this meager fact we shall deduce the singularity of T.

We have $\tilde{\mathbf{S}}\tilde{x} = 0$ for some nonzero column vector $\tilde{x} = [x_1, x_2, \ldots]^t$. Expand $\tilde{x}$ to a partitioned vector x, the partitioning of x conforming with the partitioning of $\mathbf{S}$ into blocks T_{ij}, by inserting zero components such that the elements $x_1, x_2, \ldots$ lead within each segment of x:

$$x = [x_1, 0, \ldots, 0; x_2, 0, \ldots, 0; \ldots]^t.$$

Using the structure (7) of the T_{ij}, we get $\mathbf{S}x = 0$. Also $\mathbf{Q}x = 0$, since the initial column is zero in each block T_{ij} in $\mathbf{Q}$. Now expand x to

$$\xi = \begin{bmatrix} 0 \\ x \end{bmatrix}$$

by adding initial zero components to form a vector with the same number of rows as $\mathbf{T}^{22}$. Then $\mathbf{T}^{22}\xi = 0$. Now augment ξ to a vector

$$z = \begin{bmatrix} y \\ \xi \\ 0 \\ 0 \end{bmatrix}$$

with the same number of rows as T. The two zeros here are column vectors with the same number of rows as $\mathbf{T}^{33}$ and $\mathbf{T}^{44}$ respectively, and the column vector y has the form

$$y = \left[0_{\mathscr{E}_1+1}, y_1; 0_{\mathscr{E}_2+1}, y_2; \ldots; 0_{\mathscr{E}_s+1}, y_s\right]^t$$

with temporarily unknown row vectors $y_1, \ldots, y_s$ with $\mathscr{E}_1, \ldots, \mathscr{E}_s$ components, respectively. We wish to choose $y_1, \ldots, y_s$ such that $Tz = 0$, and this requires [see (3) and (4)] that

$$\begin{bmatrix} X_{11} & \cdots & X_{1s} \\ \vdots & & \vdots \\ X_{s1} & \cdots & X_{ss} \end{bmatrix} \begin{bmatrix} y_1^t \\ \vdots \\ y_s^t \end{bmatrix} + \begin{bmatrix} W_{1,s+1} & \cdots & W_{1,s+k} \\ \vdots & & \vdots \\ W_{s,s+1} & \cdots & W_{s,s+k} \end{bmatrix} \begin{bmatrix} \xi \\ 0 \\ 0 \end{bmatrix} = 0.$$

This is because of the structures of the various blocks T_{ij}. Owing to the nonsingularity of the matrix $[X_{ij}]$, it is possible to choose $y_1, \ldots, y_s$. But then $Tz = 0$ with $z \neq 0$, implying that T is singular. The desired contradiction has been obtained. ■

In the same way we prove that

When A is symmetric and B skew, each elementary divisor of odd degree belonging to root 0 occurs with even multiplicity.

Proof. The proof imitates that just given, focusing attention on $\mathbf{T}^{33}$ instead of $\mathbf{T}^{22}$. From (6) with $\alpha = \alpha' = 0$, we deduce that the blocks T_{ij} in $\mathbf{T}^{33}$ satisfy $T_{ij} = -T_{ji}^t$ and $T_{ij}H = H^t T_{ji}^t$. Thus $\mathbf{T}^{33}$ is skew symmetric, and each block T_{ij} has the structure shown in (7), the nonzero cross diagonals again being alternating. We suppose that an elementary divisor λ^e of fixed odd degree e occurs with an odd multiplicity r. Partition

$$\mathbf{T}^{33} = \begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{bmatrix}$$

where $\mathbf{S}$ comprises all blocks T_{ij} in $\mathbf{T}^{33}$ having e or more rows and columns. We form $\tilde{\mathbf{S}}$ by extracting the extreme lower left element of each T_{ij} in $\mathbf{S}$. By the structure (7) of the blocks T_{ij}, the matrix $\tilde{\mathbf{S}}$ is block triangular, there being a leading $r \times r$ block. The skew symmetry of $\mathbf{T}^{33}$, combined with the oddness of e and the alternating character of the principal cross diagonals in the T_{ij}, implies that the leading $r \times r$ segment of $\tilde{\mathbf{S}}$ is skew symmetric, and hence singular, since r is odd. The proof now continues almost precisely as before to prove that T is singular, a contradiction. ∎

We also have:

When A is symmetric and B skew, the elementary divisors belonging to nonzero finite roots occur in pairs $(\alpha - \rho)^e, (\alpha + \rho)^e$.

To see this, note that $(A - \rho B)^t = A + \rho B$. If $(\alpha - \rho)^e$ is an elementary divisor of $A - \rho B$, then $(\alpha + \rho)^e$ is an elementary divisor of $A + \rho B$, hence of $(A - \rho B)^t$, and therefore also of $A - \rho B$.

The three italicized statements above apply to the elementary divisors of $A - \rho B$ when A is symmetric and B skew.

Similarly,

When A is skew and B symmetric,

(i) *an elementary divisor for root ∞ of a given odd degree occurs with even multiplicity;*

(ii) *an elementary divisor for root 0 of a given even degree occurs with even multiplicity; and*

(iii) *the elementary divisors for nonzero finite roots occur in pairs $(\alpha - \rho)^e, (\alpha + \rho)^e$.*

We prove this most simply by noting that interchanging A and B causes μ and ρ to interchange, thereby causing each root α to be replaced with α^{-1}. Or the proof given above may be imitated.

Now suppose A and B are both skew. Then (1) is replaced with $T(M-\rho N)=-(M^t-\rho N^t)T^t$. This time we take $M-\rho N$ in the more refined block diagonal form

$$M-\rho N=\operatorname{diag}\left(M_m-\rho N_m, M_{\alpha_1}-\rho N_{\alpha_1}, M_{\alpha_2}-\rho N_{\alpha_2},\dots\right),$$

where $M_m-\rho N_m$ comprises all blocks $\mathscr{M}_{\mathscr{E}}(\rho)$ belonging to minimal indices, and $M_{\alpha_i}-\rho N_{\alpha_i}$ comprises all Jordan blocks $J_e(\alpha_i,\rho)$ belonging to root α_i, with $\alpha_i\neq\alpha_j$ if $i\neq j$. The possibility that an α_i is ∞ or 0 is allowed. Within $M_{\alpha_i}-\rho N_{\alpha_i}$ take the Jordan blocks in order of increasing size. Partition $T=[\mathbf{T}^{uv}]$ conformally with the partitioning just displayed of $M-\rho N$, then refine this partitioning to $T=[T_{ij}]$ where each $\mathbf{T}^{uv}$ contains perhaps several T_{ij}. We follow the previous argument. For each block T_{ij} in $\mathbf{T}^{11}$ we obtain the decomposition (3); for each block T_{ij} in any of $\mathbf{T}^{12},\mathbf{T}^{13},\dots$ we obtain the decomposition (4); and for each block T_{ij} in any of $\mathbf{T}^{21},\mathbf{T}^{31},\dots$ we obtain (5). For the "lower right" T_{ij} we obtain, in place of (6),

$$T_{ij}J_e(\alpha,\rho)=-J_f(\alpha',\rho)^tT_{ji}^t.$$

If $\alpha=\alpha'=\infty$, this yields $T_{ij}=-T_{ji}^t$. If $\alpha=\infty$, $\alpha'\neq\infty$, we get $T_{ij}=0$, $T_{ji}=0$, as before. If α and α' are both finite, we obtain first $T_{ij}=-T_{ji}^t$, then

$$T_{ij}J_e(\alpha,0)=J_f(\alpha',0)^tT_{ij}.$$

From this last equation we deduce

$$T_{ij}[J_e(\alpha,0)]^d=\left[J_f(\alpha',0)\right]^dT_{ij},\qquad d=0,1,2,\dots,$$

and hence

$$T_{ij}F(J_e(\alpha,0))=\left[F(J_f(\alpha',0))\right]^tT_{ij}$$

for any polynomial $F(\rho)$. If $\alpha\neq\alpha'$, we may choose $F(\rho)$ such that

$F(J_e(\alpha,0))=0$ and $F(J_f(\alpha',0))$ is nonsingular, yielding $T_{ij}=0$. Thus there is a splitting of T into blocks associated with distinct roots:

$$T=\begin{bmatrix} \mathbf{T}^{11} & \mathbf{T}^{12} & \mathbf{T}^{13} & \cdots \\ \mathbf{T}^{21} & \mathbf{T}^{22} & & \\ \mathbf{T}^{31} & & \mathbf{T}^{33} & \\ \vdots & & & \ddots \end{bmatrix}.$$

Furthermore, since $T_{ij}=-T_{ji}^t$ when $\alpha=\alpha'$, each of $\mathbf{T}^{22},\mathbf{T}^{33},\ldots$ is skew symmetric. For each T_{ij} in any of $\mathbf{T}^{22},\mathbf{T}^{33},\ldots$ we obtain (7), with the difference that the nontrivial cross diagonals are now constant (instead of alternating). Imitating the argument used in the symmetric–skew-symmetric case for roots ∞ and 0, but now applying it to each root α_i whether infinite, zero, or finite nonzero, we reach this conclusion:

When A and B are both skew, the elementary divisors belonging to each fixed root occur in pairs $(\alpha-\rho)^e,(\alpha-\rho)^e$ (if $\alpha\neq\infty$) or μ^e,μ^e (if $\alpha=\infty$).

The italicized statement in this section for the case when A is symmetric and B skew goes back to Kronecker's 1874 paper [51]. See the historical remarks in Turnbull and Aitken's book [22, p. 142].

5. THE COMPLEX SYMMETRIC AND SKEW CASES

Let A and B be complex symmetric or skew matrices. Changing slightly the use of the symbols M, N from Section 4, construct a canonical matrix $M-\rho N$ as a direct sum of blocks as follows. The symbols m, ∞, 0, α with subscripts attached will be used to denote blocks of various types belonging to minimal indices, root ∞, root 0, or finite nonzero root α, respectively.

In order to describe the various matrix forms concisely, let

$$\Delta_e=\begin{bmatrix} & & & 1\\ & & 1 & \\ & \cdot & & \\ \cdot & & & \end{bmatrix},\qquad \Lambda_e=\begin{bmatrix} & & & & 0\\ & & & 0 & 1\\ & & \cdot & 1 & \\ & \cdot & \cdot & & \\ 0 & 1 & & & \end{bmatrix},$$

be $e\times e$ with all entries zero except for an all one principal secondary diagonal in Δ_e and an all one adjacent secondary diagonal in Λ_e, as shown. If e is even, let the skew version of Δ_e be

$$S\Delta_e = \begin{bmatrix} 0 & \Delta_{\frac{1}{2}e} \\ -\Delta_{\frac{1}{2}e} & 0 \end{bmatrix},$$

and if e is odd, let the skew version of Λ_e be

$$S\Lambda_e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta_{\frac{1}{2}(e-1)} \\ 0 & -\Delta_{\frac{1}{2}(e-1)} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & S\Delta_{e-1} \end{bmatrix},$$

in which

$$\begin{bmatrix} 0 & \Delta_{\frac{1}{2}(e-1)} \\ -\Delta_{\frac{1}{2}(e-1)} & 0 \end{bmatrix}$$

is bordered with a row and a column of zeros.

Now take $M - \rho N$ to be a direct sum of blocks as follows.

(a) *When A and B are both symmetric*:

(ai) A block $m_1 = \mathscr{M}_{\mathscr{E}}(\rho)$ belonging to a minimal index $\mathscr{E}$ of $A - \rho B$.

(aii) For root ∞, belonging to an elementary divisor μ^e of $\mu A - B$, an $e \times e$ block

$$\infty_1 = \Delta_e - \rho\Lambda_e.$$

(aiii) For a finite root α (possibly zero), belonging to an elementary divisor $(\alpha - \rho)^e$ of $A - \rho B$, an $e \times e$ block

$$\alpha_1 = (\alpha - \rho)\Delta_e + \Lambda_e.$$

(b) *Where A is symmetric and B skew*:

(bi) A block $m_2 = \mathscr{M}_{\mathscr{E}}(\rho)$ belonging to a minimal index $\mathscr{E}$ of $A - \rho B$.

(bii) For root ∞, belonging to an elementary divisor μ^e of $\mu A - B$ with e odd, an $e \times e$ block

$$\infty_2 = \Delta_e - \rho S\Lambda_e,$$

and belonging to an elementary divisor pair μ^e, μ^e of $\mu A - B$ with e even, a $2e \times 2e$ block

$$\infty_3 = \begin{bmatrix} 0 & \Delta_e - \rho\Lambda_e \\ \Delta_e + \rho\Lambda_e & 0 \end{bmatrix}.$$

(b iii) For root zero, belonging to an elementary divisor ρ^e of $A - \rho B$ with e even, an $e \times e$ block

$$0_1 = -\rho S\Delta_e + \Lambda_e,$$

and belonging to an elementary divisor pair ρ^e, ρ^e of $A - \rho B$ with e odd, a $2e \times 2e$ block

$$0_2 = \alpha_2 \quad \text{(see below)} \qquad \text{with} \quad \alpha = 0.$$

(b iv) For a finite nonzero root α, belonging to an elementary divisor pair $(\alpha - \rho)^e, (\alpha + \rho)^e$ of $A - \rho B$, a $2e \times 2e$ block

$$\alpha_2 = \begin{bmatrix} 0 & (\alpha - \rho)\Delta_e + \Lambda_e \\ (\alpha + \rho)\Delta_e + \Lambda_e & 0 \end{bmatrix}.$$

(c) *When A is skew and B symmetric*:

(c i) A block $m_3 = \mathscr{M}_{\mathscr{E}}(\rho)$ belonging to a minimal index $\mathscr{E}$ of $A - \rho B$.

(c ii) For root ∞, belonging to an elementary divisor μ^e of $\mu A - B$ with e even, an $e \times e$ block

$$\infty_4 = S\Delta_e - \rho\Lambda_e,$$

and belonging to an elementary divisor pair μ^e, μ^e of $\mu A - B$ with e odd, a $2e \times 2e$ block

$$\infty_5 = \begin{bmatrix} 0 & \Delta_e - \rho\Lambda_e \\ -\Delta_e - \rho\Lambda_e & 0 \end{bmatrix}.$$

(ciii) For root zero, belonging to an elementary divisor ρ^e of $A-\rho B$ with e odd, an $e\times e$ block

$$0_3 = -\rho\Delta_e + S\Lambda_e,$$

and belonging to an elementary divisor pair ρ^e, ρ^e of $A-\rho B$ with e even, a $2e\times 2e$ block

$$0_4 = \alpha_3 \quad \text{(see below)} \qquad \text{with} \quad \alpha \text{ zero}.$$

(civ) For a finite nonzero root α, belonging to an elementary divisor pair $(\alpha-\rho)^e, (\alpha+\rho)^e$ of $A-\rho B$, a $2e\times 2e$ block

$$\alpha_3 = \begin{bmatrix} 0 & (\alpha-\rho)\Delta_e + \Lambda_e \\ (-\alpha-\rho)\Delta_e - \Lambda_e & 0 \end{bmatrix}.$$

(d) *When A and B are both skew*:

(di) A block $m_4 = \mathscr{M}_{\mathscr{E}}(\rho)$ belonging to a minimal index $\mathscr{E}$ of $A-\rho B$.

(dii) For root ∞, belonging to an elementary divisor pair μ^e, μ^e of $\mu A - B$, a $2e\times 2e$ block

$$\infty_6 = \begin{bmatrix} 0 & \Delta_e - \rho\Lambda_e \\ -\Delta_e + \rho\Lambda_e & 0 \end{bmatrix}.$$

(diii) For a finite root α (possibly zero), belonging to an elementary divisor pair $(\alpha-\rho)^e, (\alpha-\rho)^e$ of $A-\rho B$, a $2e\times 2e$ block

$$\alpha_4 = \begin{bmatrix} 0 & (\alpha-\rho)\Delta_e + \Lambda_e \\ (-\alpha+\rho)\Delta_e - \Lambda_e & 0 \end{bmatrix}.$$

Although m_1, m_2, m_3, m_4 each equal $\mathscr{M}_{\mathscr{E}}(\rho)$, in fact they have slightly different forms according as A and B are symmetric or skew; see the definition of $\mathscr{M}_{\mathscr{E}}(\rho)$.

Let $M-\rho N$ be constructed as a direct sum of blocks as described above such that $A-\rho B$ and $M-\rho N$ have the same minimal indices and the same sets of roots and elementary divisors. Then M and N are symmetric or skew according as A and B are symmetric or skew, respectively. Furthermore, $A - \rho B$ and $M - \rho N$ are strictly equivalent, so that

$$A - \rho B = P(M - \rho N)Q$$

for certain nonsingular constant matrices P, Q. Passing from $A - \rho B$ to $Q^{-1t}(A - \rho B)Q^{-1}$, we may assume $Q = E$. Since A and B are symmetric or skew, as are M and N, we obtain $MP^t = PM$, $NP^t = PN$. Thus $M(P^t)^d = P^d M$, $N(P^t)^d = P^d N$ for $d = 0, 1, 2, \ldots$, and hence

$$(M - \rho N)F(P)^t = F(P)(M - \rho N)$$

for any polynomial $F(\rho)$. Choose $F(\rho)$ so that $F(P) = P^{-1/2}$. This is always possible: see Section 1 of Chapter 5 of Gantmacher [8]. Set $R = F(P) = P^{-1/2}$. Then $(M - \rho N)R^t = R(M - \rho N)$ and thus

$$R(A - \rho B)R^t = RP(M - \rho N)R^t = RPR(M - \rho N) = M - \rho N.$$

This proves most of the following somewhat well-known result.

THEOREM 1. *Let A and B be complex symmetric or skew matrices. Then a simultaneous (complex) congruence of A and B exists reducing $A - \rho B$ to a direct sum of types as follows, for values of $\mathscr{E}$, e, α uniquely specified by the ordered pair of matrices A, B:*

(a) m_1, ∞_1, α_1 *when A and B are both symmetric;*
(b) $m_2, \infty_2, \infty_3, 0_1, 0_2, \alpha_2$ *when A is symmetric and B is skew;*
(c) $m_3, \infty_4, \infty_5, 0_3, 0_4, \alpha_3$ *when A is skew and B is symmetric;*
(d) m_4, ∞_6, α_4 *when A and B are both skew.*

The uniqueness assertion follows from the invariance of the minimal indices and elementary divisors of a polynomial matrix under strict equivalence.

6. THE REAL SYMMETRIC AND SKEW CASES

Similarities as well as differences in methods and results between the complex and real cases will become visible. The new features in the real cases arise from two sources: the action of the law of inertia, and the fact that the nonreal roots of a real polynomial occur in conjugate pairs. The law of inertia forces certain elementary divisors to have an attached plus or minus sign, called the *inertial signature*, and the conjugacy of nonreal roots induces canonical forms like those in the complex case but constructed from 2×2 real blocks. Throughout this section, A and B will be real symmetric or skew matrices.

Evidently the elementary divisors of $A-\rho B$ belonging to nonreal roots occur in complex conjugate pairs of equal degree. This is because the invariant factors of $A-\rho B$ are real polynomials, and the elementary divisors are obtained by splitting the invariant factors over the complex number field.

In the following, α will denote a real number (nonzero unless otherwise specified) and $\beta = a+ib$ will denote a nonreal number, with a,b real, b nonzero.

We form new real matrix blocks as follows. To avoid confusion with the blocks introduced in Section 5, we use symbols m', ∞', etc.

(a) *When A and B are both real and symmetric*:

(a i′) For a minimal index $\mathscr{E}$ of $A-\rho B$, a block $m_1' = m_1$.

(a ii′) For root ∞, belonging to an elementary divisor μ^e of $\mu A - B$, an $e\times e$ block

$$\infty_1' = \varepsilon\infty_1, \qquad \text{with} \quad \varepsilon = \pm 1.$$

(a iii′) For a finite real root α (possibly zero), belonging to an elementary divisor $(\alpha-\rho)^e$ of $A-\rho B$, an $e\times e$ block

$$\alpha_1' = \varepsilon\alpha_1, \qquad \text{with} \quad \varepsilon = \pm 1.$$

(a iv′) For a nonreal root $\beta = a+ib$, belonging to an elementary divisor pair $(\beta-\rho)^e, (\bar{\beta}-\rho)^e$ of $A-\rho B$, a $2e\times 2e$ block

$$\beta_1' = \begin{bmatrix} & & & & R \\ & & & R & S \\ & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \\ R & S & & & \end{bmatrix},$$

$$\text{where} \quad R = \begin{bmatrix} b & a-\rho \\ a-\rho & -b \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(b) *When A is real and symmetric, and B is real and skew*:

(bi′) For a minimal index $\mathscr{E}$ of $A-\rho B$, a block $m_2' = m_2$.

(bii′) For root ∞, belonging to an elementary divisor μ^e of $\mu A - B$ with e odd, an $e\times e$ block

$$\infty_2' = \varepsilon\infty_2, \qquad \text{with} \quad \varepsilon = \pm 1,$$

and belonging to an elementary divisor pair μ^e, μ^e of $\mu A - B$ with e even, a $2e\times 2e$ block

$$\infty_3' = \infty_3.$$

(biii′) For root zero, belonging to an elementary divisor ρ^e of $A-\rho B$ with e even, an $e\times e$ block

$$0_1' = \varepsilon 0_1, \qquad \text{with} \quad \varepsilon = \pm 1,$$

and belonging to an elementary divisor pair ρ^e, ρ^e of $A-\rho B$ with e odd, a $2e\times 2e$ block

$$0_2' = 0_2.$$

(biv′) for a finite real nonzero root α, belonging to an elementary divisor pair $(\alpha-\rho)^e, (\alpha+\rho)^e$ of $A-\rho B$, a $2e\times 2e$ block

$$\alpha_2' = \alpha_2.$$

(bv′) For a nonreal root $\beta = a + ib$, there are two types according as $a=0$ or $a\neq 0$. Belonging to an elementary divisor pair $(\beta-\rho)^e, (\bar\beta-\rho)^e$ of $A-\rho B$ with $\beta = bi$ purely imaginary, a $2e\times 2e$ block

$$\beta_2' = \varepsilon\begin{bmatrix} & & & & R \\ & & & R & S \\ & & \cdot & \cdot & \\ & \cdot & \cdot & & \\ R & S & & & \end{bmatrix},$$

where $\quad \varepsilon = \pm 1, \quad R = \begin{bmatrix} |b| & -\rho \\ \rho & |b| \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$

and belonging to an elementary divisor quadruple $(\beta-\rho)^e,(\bar{\beta}-\rho)^e,(\beta+\rho)^e,(\bar{\beta}+\rho)^e$ with β on neither coordinate axis, a $4e\times 4e$ block

$$\beta_3' = \begin{bmatrix} 0 & \beta_1'(\rho) \\ \beta_1'(-\rho) & 0 \end{bmatrix},$$

where $\beta_1'(\rho)$ is the matrix β_1' displayed above under type (a iv′).

(c) *When A is real and skew, and B is real and symmetric*:

(c i′) For a minimal index $\mathscr{E}$ of $A-\rho B$, a block $m_3' = m_3$.

(c ii′) For root ∞, belonging to an elementary divisor μ^e of $\mu A - B$ with e even, an $e \times e$ block

$$\infty_4' = \varepsilon\infty_4, \qquad \text{with} \quad \varepsilon = \pm 1,$$

and belonging to an elementary divisor pair μ^e, μ^e of $\mu A - B$ with e odd, a $2e\times 2e$ block

$$\infty_5' = \infty_5.$$

(c iii′) For root zero, belonging to an elementary divisor ρ^e of $A-\rho B$ with e odd, an $e\times e$ block

$$0_3' = \varepsilon 0_3, \qquad \text{with} \quad \varepsilon = \pm 1,$$

and belonging to an elementary divisor pair ρ^e, ρ^e of $A-\rho B$ with e even, a $2e\times 2e$ block

$$0_4' = 0_4.$$

(c iv′) For a finite real nonzero root α, belonging to an elementary divisor pair $(\alpha-\rho)^e,(\alpha+\rho)^e$ of $A-\rho B$, a $2e\times 2e$ block

$$\alpha_3' = \alpha_3.$$

(c v′) For a nonreal root $\beta = a+ib$, there are two cases according as $a=0$, $a\neq 0$: belonging to an elementary divisor pair $(\beta-\rho)^e$,

$(\bar{\beta}-\rho)^e$ of $A-\rho B$ with $\beta = bi$ purely imaginary, a $2e\times 2e$ block

$$\beta_4' = \varepsilon\begin{bmatrix} & & & R \\ & & R & S \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ R & S & & \end{bmatrix},$$

$$\text{where}\quad \varepsilon = \pm 1,\quad R = \begin{bmatrix} -\rho & |b| \\ -|b| & -\rho \end{bmatrix},\quad S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and belonging to an elementary divisor quadruple $(\beta-\rho)^e, (\bar{\beta}-\rho)^e, (\beta+\rho)^e, (\bar{\beta}+\rho)^e$ with β on neither coordinate axis, a $4e\times 4e$ block

$$\beta_5' = \begin{bmatrix} 0 & \beta_1'(\rho) \\ -\beta_1'(-\rho) & 0 \end{bmatrix},$$

where $\beta_1'(\rho)$ is the matrix β_1' displayed above under type (a iv′).

(d) *When A and B are both real and skew*:

(d i′) For a minimal index $\mathscr{E}$ of $A-\rho B$, a block $m_4' = m_4$.

(d ii′) For root ∞, belonging to an elementary divisor pair μ^e, μ^e of $\mu A - B$, a $2e\times 2e$ block

$$\infty_6' = \infty_6.$$

(d iii′) For a finite real root α (possibly zero), belonging to an elementary divisor pair $(\alpha-\rho)^e, (\alpha-\rho)^e$ of $A-\rho B$, a $2e\times 2e$ block

$$\alpha_4' = \alpha_4.$$

(d iv′) For a nonreal root $\beta = a+ib$, belonging to an elementary divisor quadruple $(\beta-\rho)^e, (\beta-\rho)^e, (\bar{\beta}-\rho)^e, (\bar{\beta}-\rho)$, a $4e\times 4e$ block

$$\beta_6' = \begin{bmatrix} 0 & \beta_1'(\rho) \\ -\beta_1'(\rho) & 0 \end{bmatrix},$$

where $\beta_1'(\rho)$ is the matrix β_1' displayed above under type (a iv′).

The numerical factors $\varepsilon = \pm 1$ in types $\infty_1', \alpha_1', \infty_2', 0_1', \beta_2', \infty_4', 0_3', \beta_4'$ are the inertial signatures. They may be different for different blocks belonging to the same root.

To see that $\beta_1', \beta_2', \beta_3', \beta_4', \beta_5', \beta_6'$ have the claimed elementary divisors, argue as follows: Let

$$D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \Delta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and let U be a unitary matrix for which $UDU^* = \operatorname{diag}(i, -i)$. In the cases of $\beta_1', \beta_3', \beta_5', \beta_6'$ set $R_1 = U\Delta$, $S_1 = U^*$; in the case of β_2' set $R_1 = UD$, $S_1 = U^*\Delta$; and in the case of β_4' set $R_1 = \Delta U$, $S_1 = U^*$. Now put $R = \operatorname{diag}(R_1, R_1, \ldots, R_1)$, $S = \operatorname{diag}(S_1, S_1, \ldots, S_1)$. Then, in all cases, $R\beta_i'S$ has a form from which the elementary divisors may be read off after a row and a column rearrangement.

Our objective is to prove the following theorem.

THEOREM 2. *Let A and B be real symmetric or skew matrices. Then a simultaneous (real) congruence of A and B exists reducing $A - \rho B$ to a direct sum of types as follows, for values of $\mathscr{E}$, e, α, β, ε uniquely specified by the ordered pair of matrices A, B:*

(a) *Types $m_1', \infty_1', \alpha_1', \beta_1'$ when both A and B are symmetric.*
(b) *Types $m_2', \infty_2', \infty_3', 0_1', 0_2', \alpha_2', \beta_2', \beta_3'$ when A is symmetric and B is skew.*
(c) *Types $m_3', \infty_4', \infty_5', 0_3', 0_4', \alpha_3', \beta_4', \beta_5'$ when A is skew and B symmetric.*
(d) *Types $m_4', \infty_6', \alpha_4', \beta_6'$ when A and B are both skew.*

Proof. Existence: We imitate, as far as possible, the proof of Theorem 1. We use the fact, essentially proved in Chapter 12 of [8], that pencils $A - \rho B$ and $A_1 - \rho B_1$, where A, B, A_1, B_1 have elements in an infinite base field, are strictly equivalent by nonsingular matrices P, Q with elements in the base field,

$$A - \rho B = P(A_1 - \rho B_1)Q,$$

if and only if the two pencils have the same minimal indices and the same elementary divisors. (The elementary divisors may be taken over an extension field.)

Let A and B be symmetric or skew real matrices. Changing again slightly the use of the symbols M and N, let $M - \rho N$ be a direct sum of blocks of the types $m_1, \ldots, \beta_6'$ as described above, deleting however the factors $\varepsilon = \pm 1$ multiplying certain of these blocks. Choose this direct sum such that $A - \rho B$ and $M - \rho N$ have the same minimal indices and elementary divisors. This is possible because the types listed cover all possible configurations for the minimal indices and elementary divisors of a pencil $A - \rho B$ when A and B are real and symmetric or skew. Then M (or N) will be symmetric or skew according as A (or B, respectively) is symmetric or skew. Thus we have

$$A - \rho B = P(M - \rho N)Q$$

for certain real nonsingular matrices P, Q. Passing to $Q^{-1t}(A - \rho B)Q^{-1}$, we may assume that $Q = E$. Because A and B are symmetric or skew, as are M and N, we get

$$PM = MP^t, \qquad PN = NP^t,$$

so that $P(M - \rho N) = (M - \rho N)P^t$. We now argue by induction on the matrix dimensions, considering three cases, only the last of which involves the induction hypothesis. Dimension 1×1 is covered by the first case. The three cases are:

(i) P has just one distinct real eigenvalue and no nonreal eigenvalues;
(ii) P has just one distinct pair of conjugate nonreal eigenvalues and no real eigenvalues;
(iii) all other possibilities.

Case (i): First suppose that the single distinct eigenvalue of P is positive. Then $P^{-1} = F(P)^2$ for some real polynomial $F(\rho)$. Indeed, let $(\rho - \gamma)^n$ with $\gamma > 0$ be the characteristic polynomial of P^{-1}. By the lemma in Section 9, $\rho \equiv F_n(\rho)^2 \pmod{(\rho - \gamma)^n}$ for some real polynomial $F_n(\rho)$. Then $P^{-1} = F_n(P^{-1})^2$, and since P^{-1} is a polynomial in P, we get $P^{-1} = F(P)^2$. The proof is now identical with the proof in the complex case. In fact, with $R = F(P) = P^{-1/2}$,

$$R(A - \rho B)R^t = RP(M - \rho N)R^t = RPR(M - \rho N) = M - \rho N.$$

In this case we have obtained the desired form with each inertial signature $\varepsilon = +1$.

Now suppose that the single distinct eigenvalue of P is negative. Then $-P^{-1}=F(-P)^2$ for a real polynomial $F(\rho)$. Taking $R=F(-P)=(-P)^{-1/2}$, we get

$$R(A-\rho B)R^t = RP(M-\rho N)R^t = RPR(M-\rho N) = -(M-\rho N).$$

This produces a factor -1 multiplying each block in $M-\rho N$. We have to make a further congruence to remove this factor from the blocks of types $m_1', m_2', m_3', m_4', \infty_3', \infty_5', \infty_6', 0_2', 0_4', \alpha_2', \alpha_3', \alpha_4', \beta_1', \beta_3', \beta_5', \beta_6'$. All of these blocks, except β_1', have the form

$$\begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix},$$

and on each of these blocks we remove the factor -1 by the congruence

$$\begin{bmatrix} E & 0 \\ 0 & -E \end{bmatrix}\begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix}\begin{bmatrix} E & 0 \\ 0 & -E \end{bmatrix}^t = -\begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix}.$$

As for block β_1', let

$$D=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

as before, and set $R=\operatorname{diag}(D, D, \ldots, D)$. Then $R\beta_1'R^t=-\beta_1'$. We have now obtained the desired form with each inertial signature $\varepsilon=-1$.

Case (ii): In this case there again exists, as in the first part of case (i) (see Section 9), a polynomial $F(\rho)$ with real coefficients such that $P^{-1}=F(P)^2$. Set $R=F(P)$. Then the proof is completed as in the first part of case (i). Each inertial signature $\varepsilon=+1$.

Case (iii): Let S be a real nonsingular matrix, and use it to effect a similarity of P,

$$SPS^{-1}=\operatorname{diag}(P_1, \ldots, P_r),$$

such that each block P_i has either just one distinct real eigenvalue or just one distinct pair of conjugate nonreal eigenvalues, and blocks P_i, P_j with $i \neq j$ have no common eigenvalue. This similarity always exists: for example, use the real version of the Jordan canonical form; a block P_i may comprise several Jordan blocks. The possibility $r = 1$ was covered in cases (i) and (ii). Thus $r > 1$. We obtain

$$S(A - \rho B)S^t = (SPS^{-1})\{S(M - \rho N)S^t\}.$$

Let $\tilde{P} = SPS^{-1}$, and set $S(M - \rho N)S^t = \tilde{M} - \rho\tilde{N}$. Then $\tilde{M}, \tilde{N}$ are symmetric or skew according as A, B are symmetric or skew, but not necessarily in block diagonal form. Since $SAS^t = \tilde{P}\tilde{M}$, $SBS^t = \tilde{P}\tilde{N}$, evidently $\tilde{P}\tilde{M}$ and $\tilde{P}\tilde{N}$ are also symmetric or skew. Partition

$$\tilde{M} - \rho\tilde{N} = \left[M_{ij} - \rho N_{ij}\right]_{1 \leqslant i,j \leqslant r}.$$

From $\tilde{P}\tilde{M} = \tilde{M}\tilde{P}^t$, $\tilde{P}\tilde{N} = \tilde{N}\tilde{P}^t$, we get

$$P_i M_{ij} = M_{ij} P_j^t, \qquad P_i N_{ij} = N_{ij} P_j^t.$$

Hence $P_i^k M_{ij} = M_{ij}(P_j^t)^k$ for $k = 0, 1, \dots,$ and therefore $F(P_i)M_{ij} = M_{ij}F(P_j)^t$ for any polynomial $F(\rho)$. As P_i and P_j have no common eigenvalues for $i \neq j$, we may choose $F(\rho)$ such that $F(P_i) = 0$, $F(P_j)$ is nonsingular. But then $M_{ij} = 0$. Similarly $N_{ij} = 0$ if $i \neq j$. That is, the congruence transformation of $A - \rho B$ by S splits $A - \rho B$:

$$S(A - \rho B)S^t = \operatorname{diag}(A_{11} - \rho B_{11}, \dots, A_{rr} - \rho B_{rr}), \qquad r > 1.$$

We may apply the induction hypothesis to each of the direct summands $A_{11} - \rho B_{11}, \dots, A_{rr} - \rho B_{rr}$, and by suitable congruence transformations on each obtain diagonal blocks of the desired types. This completes the existence part of the proof of Theorem 2.

7. UNIQUENESS

We still have to prove the uniqueness of the decomposition of $A - \rho B$ into a direct sum of blocks of the various types described in Section 6. The number, size, and roots of the blocks actually present are determined by the minimal indices and elementary divisors of $A - \rho B$ or $\mu A - B$. Therefore only the uniqueness of the inertial signatures needs to be established. They occur only in certain cases:

(i) when A and B are both symmetric, for elementary divisors belonging to infinite or finite real roots;

(ii) when A is symmetric and B skew, for odd degree elementary divisors for root ∞, even degree elementary divisors belonging to root 0, and elementary divisors belonging to a pair of conjugate purely imaginary roots;

(iii) when A is skew and B symmetric, for even degree elementary divisors belonging to root ∞, odd degree elementary divisors belonging to root 0, and elementary divisors belonging to a pair of conjugate purely imaginary roots;

(iv) when A and B are both skew, inertial signatures do not occur, so this case needs no further study.

Let $M-\rho N$ be a direct sum of blocks of the various types, as in Section 6, with [see (2)] the blocks belonging to minimal indices placed first on the block diagonal, then the blocks belonging to the infinite and finite roots, with blocks belonging to the same root placed consecutively in order of increasing size, and among the blocks of fixed size for a given root, those with positive inertial signatures ε placed ahead of those with negative inertial signatures. The last constraint is understood to be automatically satisfied for blocks without inertial signatures. Let $\tilde{M}-\rho\tilde{N}$ be a like direct sum of blocks, differing from $M-\rho N$ only in that the inertial signatures are possibly different, say $\tilde{\varepsilon}$'s in place of ε's, but otherwise consisting of the same blocks in the same positions. To prove the uniqueness of the inertial signatures, we assume that $M-\rho N$ and $\tilde{M}-\rho\tilde{N}$ are congruent, and wish to prove that the ε's and $\tilde{\varepsilon}$'s are the same.

We have

$$T(M-\rho N)T^{t}=\tilde{M}-\rho\tilde{N}$$

for some nonsingular matrix T with real elements. Hence

$$T(M-\rho N)=(\tilde{M}-\rho\tilde{N})S, \tag{8}$$

where $S=T^{-1t}$. Our first objective is to "cancel away" the minimal indices; for this we use the method in Section 4, slightly modified. Suppose that

$$M-\rho N=\operatorname{diag}\big(\mathscr{M}_{\mathscr{E}_1}(\rho),\dots,\mathscr{M}_{\mathscr{E}_s}(\rho),\mathbb{B}_{s+1}(\rho),\dots,\mathbb{B}_{s+k}(\rho)\big),$$

$$\tilde{M}-\rho\tilde{N}=\operatorname{diag}\big(\mathscr{M}_{\mathscr{E}_1}(\rho),\dots,\mathscr{M}_{\mathscr{E}_s}(\rho),\tilde{\mathbb{B}}_{s+1}(\rho),\dots,\tilde{\mathbb{B}}_{s+k}(\rho)\big),$$

where $\mathscr{M}_{\mathscr{E}}(\rho)$ denotes a block belonging to a minimal index $\mathscr{E}$, and

$\mathbb{B}_{s+1}(\rho),\ldots,\mathbb{B}_{s+k}(\rho)$ are each single blocks of the types described in Section 6, belonging to infinite or finite roots, with $\tilde{\mathbb{B}}_i(\rho)=\pm\mathbb{B}_i(\rho)$. Note that a block $\mathbb{B}_i(\rho)$ belongs to an infinite or finite real root α, or to a pair $\beta,\bar{\beta}$ of nonreal roots, or to a quadruple $\beta,-\beta,\bar{\beta},-\bar{\beta}$ of nonreal roots. Also note that a block associated with a pair of nonreal roots never has a root in common with a block associated with a quadruple of nonreal roots. Thus two blocks $\mathbb{B}_i(\rho),\mathbb{B}_j(\rho)$ either have no common root or have coincident roots.

Conforming to the direct sum structure of $M-\rho N$, partition

$$T=\left[T_{ij}\right]_{1\leqslant i,j\leqslant s+k},\qquad S=\left[S_{ij}\right]_{1\leqslant i,j\leqslant s+k};$$

then partition the lower left T_{ij} and S_{ij} to conform to the structure of $\mathscr{M}_{\mathscr{E}_j}(\rho)$ as

$$T_{ij}=\left[U_{ij},V_{ij}\right],\quad S_{ij}=\left[Y_{ij},Z_{ij}\right],\qquad s<i\leqslant s+k,\quad 1\leqslant j\leqslant s,$$

where V_{ij} and Z_{ij} have $\mathscr{E}_j$ columns. From (8) we obtain $T_{ij}\mathscr{M}_{\mathscr{E}_j}(\rho)=\tilde{\mathbb{B}}_i(\rho)S_{ij}$ for $j\leqslant s<i$, and hence

$$V_{ij}\mathscr{L}_{\mathscr{E}_j}(\rho)^{\pm t}=\tilde{\mathbb{B}}_i(\rho)Y_{ij},\qquad j\leqslant s<i.\tag{9}$$

Write $\mathbb{B}_i(\rho)=M_i-\rho N_i$. Then (9) yields

$$\pm\left[V_{ij},0\right]=N_iY_{ij},\qquad \pm\left[0,V_{ij}\right]=M_iY_{ij}.$$

When $\mathbb{B}_i(\rho)$ belongs to finite roots, N_i is nonsingular and therefore

$$\left[0,V_{ij}\right]=\pm M_iN_i^{-1}\left[V_{ij},0\right].$$

Recursive comparison of the columns in this equation, beginning with the last, yields $V_{ij}=0$. If, however, $\mathbb{B}_i(\rho)$ belongs to ∞, then M_i is nonsingular, and we get

$$\left[V_{ij},0\right]=\pm N_iM_i^{-1}\left[0,V_{ij}\right],$$

yielding again $V_{ij}=0$. Thus

$$T_{ij}=\left[U_{ij},0\right],\qquad j\leqslant s<i.\tag{10}$$

Using (10), we calculate the (i, j) block in $T(M-\rho N)T^t$, for $s < i, j \leqslant s+k$, to be

$$\sum_{p=1}^{s}\left[U_{ip}, 0\right]\begin{bmatrix} 0 & \mathscr{L}_{\mathscr{E}_p}(\rho) \\ \mathscr{L}_{\mathscr{E}_p}(\rho)^{\pm t} & 0 \end{bmatrix}\begin{bmatrix} U_{jp}^t \\ 0 \end{bmatrix} + \sum_{p=s+1}^{k} T_{ip}\mathbb{B}_p(\rho)T_{jp}^t$$

$$= \sum_{p=s+1}^{k} T_{ip}\mathbb{B}_p(\rho)T_{jp}^t.$$

That is, if $\mathbf{T} = [T_{ij}]_{s<i,j\leqslant s+k}$ is the lower right block in T, then

$$\mathbf{T}\operatorname{diag}\left(\mathbb{B}_{s+1}(\rho),\ldots,\mathbb{B}_{s+k}(\rho)\right)\mathbf{T}^t = \operatorname{diag}\left(\tilde{\mathbb{B}}_{s+1}(\rho),\ldots,\tilde{\mathbb{B}}_{s+k}(\rho)\right). \quad (11)$$

Since the right hand side of (11) has determinant not the zero polynomial, it follows that the lower right sections of $M-\rho N$ and $\tilde{M}-\rho\tilde{N}$ are congruent. That is, the blocks $\mathscr{M}_{\mathscr{E}}(\rho)$ belonging to minimal indices have been canceled. We may therefore assume from the outset that these blocks are absent, that is, $s=0$.

From (8) we now get

$$T_{ij}\mathbb{B}_j(\rho) = \tilde{\mathbb{B}}_i(\rho)S_{ij}.$$

Writing, as above, $\tilde{\mathbb{B}}_i(\rho) = M_i - \rho N_i$, $\mathbb{B}_j(\rho) = \pm(M_j - \rho N_j)$, we obtain

$$T_{ij}N_j = \pm N_i S_{ij}, \qquad T_{ij}M_j = \pm M_i S_{ij}.$$

If $\mathbb{B}_i(\rho)$, $\mathbb{B}_j(\rho)$ both belong to finite roots, then N_i, N_j are both nonsingular, and hence

$$T_{ij}\left(M_j N_j^{-1}\right) = \left(M_i N_i^{-1}\right)T_{ij}.$$

Note that the roots of a block $\mathbb{B}_i(\rho) = M_i - \rho N_i$ with finite roots are the eigenvalues of $M_i N_i^{-1}$, since $\det\mathbb{B}_i(\rho) = \det N_i \det(M_i N_i^{-1} - \rho E)$. By a familiar argument,

$$T_{ij}F\left(M_j N_j^{-1}\right) = F\left(M_i N_i^{-1}\right)T_{ij},$$

for any polynomial $F(\rho)$. Unless $\mathbb{B}_i(\rho)$, $\mathbb{B}_j(\rho)$ belong to the same roots,

$M_iN_i^{-1}$ and $M_jN_j^{-1}$ will not have a common eigenvalue, whence $T_{ij}=0$ follows if $F(\rho)$ is suitably chosen. If, however, $\mathbb{B}_i(\rho)$ belongs to ∞ and $\mathbb{B}_j(\rho)$ to finite roots, then M_i and N_j are nonsingular, and we get

$$T_{ij}=\left(N_iM_i^{-1}\right)T_{ij}\left(M_jN_j^{-1}\right).$$

Iterating this equation yields $T_{ij}=0$, since $N_iM_i^{-1}$ is nilpotent when $\mathbb{B}_i(\rho)$ belongs to root ∞.

Thus T splits: in our uniqueness proof we may assume that a single type of block is present. Those types without inertial signatures may henceforth be ignored.

Now let both A and B be symmetric. We give two arguments. First let the root α be real and finite. With $\mathbb{B}_i(\rho)=M_i-\rho N_i$, $M_iN_i^{-1}=\alpha E+H^t$, so that $T_{ij}H^t=H^tT_{ij}$. Thus T_{ij}, because it commutes with the nonderogatory matrix H^t, is essentially a polynomial in H^t,

$$T_{ij}=\begin{bmatrix}\cdot & & & \\ \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \end{bmatrix}\quad\text{or}\quad T_{ij}=\begin{bmatrix} & & \\ \cdot & & \\ \cdot & \cdot & \\ \cdot & \cdot & \cdot\end{bmatrix},\tag{12}$$

and is constant along each diagonal parallel to the main diagonal and zero on each such diagonal not meeting both the left hand and bottom edges of T_{ij}. Let the inertial signatures belonging to the blocks $\mathbb{B}_1(\rho),\dots,\mathbb{B}_k(\rho)$ be $\varepsilon_1,\dots,\varepsilon_k$, respectively, and for the blocks $\tilde{\mathbb{B}}_1(\rho),\dots,\tilde{\mathbb{B}}_k(\rho)$ be $\tilde{\varepsilon}_1,\dots,\tilde{\varepsilon}_k$, respectively. Assume that the blocks $\mathbb{B}_i(\rho)$ of a given dimension are $\mathbb{B}_{u+1}(\rho),\dots,\mathbb{B}_v(\rho)$, with $\mathbb{B}_u(\rho)$ if present having strictly fewer rows than $\mathbb{B}_{u+1}(\rho)$, and $\mathbb{B}_{v+1}(\rho)$ if present strictly more rows than $\mathbb{B}_v(\rho)$.

From (11) we obtain

$$\sum_{p=1}^{k}T_{ip}N_pT_{jp}^t=\begin{cases}0 & \text{if}\quad i\neq j,\\ \tilde{N}_i & \text{if}\quad i=j.\end{cases}$$

Let i and j lie in the range $u<i,j\leqslant v$. Rewrite the left hand sum as

$$\sum_{p=1}^{u}T_{ip}N_pT_{jp}^t+\sum_{p=u+1}^{v}T_{ip}N_pT_{jp}^t+\sum_{p=v+1}^{k}T_{ip}N_pT_{jp}^t.\tag{13}$$

Each term in the first and third parts of this sum has zero for its extreme lower left element, whereas the extreme lower left element for the term

$T_{ip}N_pT_{jp}^t$ in the second part is $t_{ip}\varepsilon_p t_{jp}$, where t_{ip} denotes the constant element along the main diagonal of the square block T_{ip}. Therefore

$$\sum_{p=u+1}^{v} t_{ip}\varepsilon_p t_{jp} = \begin{cases} 0 & \text{if} \quad i \neq j, \\ \tilde{\varepsilon}_i & \text{if} \quad i = j. \end{cases}$$

Letting $\mathbf{T} = [t_{ij}]_{u<i,j\leqslant v}$, we thus have

$$\mathbf{T}\operatorname{diag}(\varepsilon_{u+1},\dots,\varepsilon_v)\mathbf{T}^t = \operatorname{diag}(\tilde{\varepsilon}_{u+1},\dots,\tilde{\varepsilon}_v).$$

Thus $\operatorname{diag}(\varepsilon_{u+1},\dots,\varepsilon_v)$ and $\operatorname{diag}(\tilde{\varepsilon}_{u+1},\dots,\tilde{\varepsilon}_v)$ are congruent, with $\mathbf{T}$ nonsingular because each $\tilde{\varepsilon}_p$ is nonzero. By the law of inertia, the number of positive terms among $\varepsilon_{u+1},\dots,\varepsilon_v$ is the same as among $\tilde{\varepsilon}_{u+1},\dots,\tilde{\varepsilon}_v$. Since each ε_p and each $\tilde{\varepsilon}_p$ is also ± 1, and since we agreed that positive inertial signatures precede negative ones for each fixed block size, it follows that $\varepsilon_{u+1},\dots,\varepsilon_v$ and $\tilde{\varepsilon}_{u+1},\dots,\tilde{\varepsilon}_v$ coincide term by term.

The argument if $\alpha = \infty$ is similar, now using

$$\left(N_iM_i^{-1}\right)T_{ij} = T_{ij}\left(N_jM_j^{-1}\right)$$

and $N_iM_i^{-1} = H^t$ to deduce the structure (12) for the blocks T_{ij}. We omit the similar details, which use (13) with M_p in place of N_p.

An alternative proof is as follows. Let α be finite. From $T(M-\rho N)T^t = \tilde{M} - \rho\tilde{N}$, we deduce

$$T\{(M-\alpha N)N^{-1}\}T^{-1} = (\tilde{M}-\alpha\tilde{N})\tilde{N}^{-1}.$$

Raising to the rth power and multiplying by $TNT^t = \tilde{N}$, we obtain

$$T\{(M-\alpha N)N^{-1}\}^r NT^t = \{(\tilde{M}-\alpha\tilde{N})\tilde{N}^{-1}\}^r\tilde{N}, \qquad r = 0,1,2,\dots. \tag{14}$$

Now $(M-\alpha N)N^{-1}$ is a nilpotent, in fact a direct sum of nilpotents $(M_i - \alpha N_i)N_i^{-1}$. If this latter block is $e_i \times e_i$, then for $r = e_i - 1$,

$$\{(M_i - \alpha N_i)N_i^{-1}\}^r N_i = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon_i \end{bmatrix} \tag{15}$$

is entirely zero except for a single element, ε_i, in the extreme lower right.

For $r = e_i$ we instead get zero. For any r, the matrix on the left side of (15) is symmetric.

Now take $r = e_k - 1$. If $e_u < e_{u+1} = \cdots = e_k$, then the left side of (14) has inertia given by $\varepsilon_{u+1}, \dots, \varepsilon_k$ and the right side by $\tilde{\varepsilon}_{u+1}, \dots, \tilde{\varepsilon}_k$. Thus $\varepsilon_{u+1} = \tilde{\varepsilon}_{u+1}, \dots, \varepsilon_k = \tilde{\varepsilon}_k$. Now take $r = e_u - 1$. If $e_{v-1} < e_v = \cdots = e_u < e_{u+1}$, the left hand side has inertia $\varepsilon_{v+1}, \dots, \varepsilon_u \oplus$(terms involving $\varepsilon_{u+1}, \dots, \varepsilon_k$), whereas the right hand side has inertia $\tilde{\varepsilon}_{v+1}, \dots, \tilde{\varepsilon}_u \oplus$(the same terms). Hence $\varepsilon_{v+1} = \tilde{\varepsilon}_{v+1}, \dots, \varepsilon = \tilde{\varepsilon}_u$. Continuing in this way, we establish the equality of the ε_i and $\tilde{\varepsilon}_i$. If $\alpha = \infty$ the argument is similar, using $T(NM^{-1})T^{-1} = \tilde{N}\tilde{M}^{-1}$.

Now let A be symmetric, B skew. Inertial signatures appear when $\alpha = 0$, $\alpha = \infty$, or β is pure imaginary. We shall adapt the above two methods: the second for the two α cases, the first for the β case.

First let $\alpha = 0$. We have

$$T\{(MN^{-1})^r N\}T^t = (\tilde{M}\tilde{N}^{-1})^r \tilde{N}, \qquad r = 0, 1, 2, \dots.$$

Using $M^t = M$ and $N^t = -N$, we get $\{(MN^{-1})^r N\}^t = (MN^{-1}M \cdots MN^{-1}M)^t = (-1)^{r-1}\{(MN^{-1})^r N\}$. Thus the matrix $(MN^{-1})^r N$ is symmetric whenever r is odd. For an $e_i \times e_i$ block $M_i - \rho N_i$ of type $0_1'$, with $r = e_i - 1$, we obtain [analogous to (15)],

$$\left(M_i N_i^{-1}\right)^r N_i = \begin{bmatrix} 0 & 0 \\ 0 & (-1)^{\frac{1}{2}e_i - 1}\varepsilon_i \end{bmatrix}.$$

For $r = e_i$ we get instead zero. If $M_i - \rho N_i$ is of type $0_2'$, then the matrices $(M_i N_i^{-1})^r N_i$ (for odd r) will have the same signatures as the equal matrices $(\tilde{M}_i \tilde{N}_i^{-1})^r \tilde{N}_i$. The argument of the preceding paragraph may now be repeated; take first $r = e_k - 1$, then $r = e_u - 1$, etc., where $e_1 \leqslant \cdots \leqslant e_k$ are the sizes of the various blocks of type $0_1'$.

For root ∞ the argument is similar, the roles of M and N being interchanged.

Now, suppose that $M - \rho N$ has only blocks of type β_2' belonging to roots $\beta = \pm bi$. Set

$$D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then M and N may be regarded as matrices with entries of the form

$\gamma E + \delta D$, where γ, δ are real scalars. Note that $D^2 = -E$. We again have

$$T(MN^{-1})T^{-1} = \tilde{M}\tilde{N}^{-1}.$$

Since the blocks in $M - \rho N$ are the same as the blocks in $\tilde{M} - \rho\tilde{N}$ except for the signatures, we have $\tilde{M}\tilde{N}^{-1} = MN^{-1}$, and thus

$$T(MN^{-1}) = (MN^{-1})T.$$

Write $M - \rho N = \operatorname{diag}(M_1 - \rho N_1, \dots, M_k - \rho N_k)$ with $M_i - \rho N_i$ a block of type β_2' of size $2e_i \times 2e_i$. Each $M_i N_i^{-1}$ has the form

$$\begin{bmatrix} |b|D & & & & \\ D & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & D & |b|D \end{bmatrix}. \tag{16}$$

Partitioning $T = [T_{ij}]_{1 \leqslant i,j \leqslant k}$, we have

$$T_{ij}\left(M_j N_j^{-1}\right) = (M_i N_i)^{-1} T_{ij}. \tag{17}$$

From (16) and (17) it follows that T_{ij} is composed of 2×2 blocks of the form $xE + yD$, with x and y real scalars. This is a recursive computation on the 2×2 entries of T_{ij} that begins with the upper right entry, then evaluates each 2×2 block in terms of blocks nearer the top right corner. The key step is that if X is a 2×2 matrix such that $XD - DX$ is a polynomial in D, then in fact X is also a polynomial in D and $XD - DX = 0$. Thus M, N, T may now be viewed as matrices with elements from the algebra of real polynomials in D, and this is the view taken in the rest of the proof. Moreover, (16) and (17) imply that the matrix T_{ij} with entries 2×2 blocks of type $xE + yD$ has the structure shown in (12), constant along each block diagonal parallel to the main block diagonal, and zero along each block diagonal not meeting the left or lower edges of T_{ij}.

The (i, j) block in $\tilde{N} = TNT^t$ is

$$\sum_{p=1}^{k} T_{ip} N_p T_{jp}^t.$$

Take $u < i, j \leqslant v$. Again we have the split into three parts, as shown in (13). We calculate the extreme lower left element (which as a polynomial in D is a 2×2 block) in each term in the three parts: terms from the first and third part have zero in the lower left 2×2 position, and from the terms in the second part we get

$$\sum_{p=u+1}^{v} t_{ip}(\varepsilon_p D)t_{jp}^t = \sum_{p=u+1}^{v} \{(-v_{ip}\varepsilon_p u_{jp} + u_{ip}\varepsilon_p v_{jp})E$$
$$+(u_{ip}\varepsilon_p u_{jp} + v_{ip}\varepsilon_p v_{jp})D\}, \qquad (18)$$

where $t_{ip} = u_{ip}E + v_{ip}D$ denotes the 2×2 block along the block diagonal of the (square) matrix T_{ip}, with u_{ip} and v_{ip} real scalars. The expression (18) equals $\tilde{\varepsilon}_i D$ if $i = j$ and 0 if $i \neq j$. Let

$$U = [u_{ij}]_{u<i,j\leqslant v}, \qquad V = [v_{ij}]_{u<i,j\leqslant v}.$$

Let also $\mathbb{E} = \operatorname{diag}(\varepsilon_{u+1},\dots,\varepsilon_v)$, $\tilde{\mathbb{E}} = \operatorname{diag}(\tilde{\varepsilon}_{u+1},\dots,\tilde{\varepsilon}_v)$. From (18) we get

$$-V\mathbb{E}U^t + U\mathbb{E}V^t = 0, \qquad U\mathbb{E}U^t + V\mathbb{E}V^t = \tilde{\mathbb{E}},$$

and therefore

$$\begin{bmatrix} U & V \\ -V & U \end{bmatrix}\begin{bmatrix} \mathbb{E} & 0 \\ 0 & \mathbb{E} \end{bmatrix}\begin{bmatrix} U^t & -V^t \\ V^t & U^t \end{bmatrix} = \begin{bmatrix} \tilde{\mathbb{E}} & 0 \\ 0 & \tilde{\mathbb{E}} \end{bmatrix}.$$

Since the right hand side is nonsingular, evidently $\operatorname{diag}(\mathbb{E},\mathbb{E})$ and $\operatorname{diag}(\tilde{\mathbb{E}},\tilde{\mathbb{E}})$ are congruent; thus $\mathbb{E}$ and $\tilde{\mathbb{E}}$ have the same numbers of positive terms. This forces $\varepsilon_i = \tilde{\varepsilon}_i$ for $u < i \leqslant v$. This completes the proof of the uniqueness of the inertial signatures associated with blocks of type β_2'.

The corresponding discussion when A is skew and B symmetric is parallel—almost exactly the same formulas apply—and is omitted. [In (18) the left hand sum has E in place of D, and the right side has D and E interchanged.] ∎

8. THE HERMITIAN CASE

If instead of two real symmetric matrices we consider two Hermitian matrices, and replace congruence by conjunctivity, results analogous to Theorem 2(a) may be obtained, with the exception that the block β_1' is replaced with

$$\beta_1' = \begin{bmatrix} 0 & (\beta-\rho)\Delta+\Lambda \\ (\bar{\beta}-\rho)\Delta+\Lambda & 0 \end{bmatrix},$$

a $2e\times 2e$ block belonging to a conjugate pair $(\beta-\rho)^e,(\bar{\beta}-\rho)^e$ of elementary divisors of $A-\rho B$, with β not real. Since the elementary divisors of $(A-\rho B)^*$ are the conjugates of those of $A-\rho B=(A-\rho B)^*$, it indeed is true that the elementary divisors belonging to nonreal roots must occur in conjugate pairs. The proof of the modified version of Theorem 2(a) is not significantly different from the proof given above. See Section 2 of [120] for a complete discussion.

For an analysis of some other cases, with A complex symmetric or complex skew and B Hermitian, see a paper by Ermolaev [83] and another by Li Santi and Thompson [100].

9. A LEMMA

The following lemma (see [92]) was used in the proof of Theorem 2. Let a real polynomial $p(\rho)$ have one of the forms

$$p(\rho)=\rho-\gamma \qquad \text{with} \quad \gamma>0,$$

$$p(\rho)=(\rho-\gamma)(\rho-\gamma) \qquad \text{with } \gamma \text{ not real}.$$

Then:

Lemma. *If m is a positive integer, a polynomial $F_m(\rho)$ exists with real coefficients such that*

$$\rho\equiv F_m(\rho)^2 \pmod{p(\rho)^m}.$$

Proof. First let $m=1$. If $p(\rho)=\rho-\gamma$, take $F_1(\rho)=\gamma^{1/2}$. Otherwise take $F_1(\rho)=\{|\gamma|+\rho\}\{2(|\gamma|+\operatorname{Re}\gamma)\}^{-1/2}$. Continue by induction on m. If $F_m(\rho)$ has already been found, then $\rho=F_m(\rho)^2+t(\rho)p(\rho)^m$ for some real polynomial $t(\rho)$. Since $F_m(\rho)$ is relatively prime to $p(\rho)$, there is a real polynomial $g(\rho)$ such that

$$g(\rho)F_m(\rho) \equiv \tfrac{1}{2}t(\rho) \pmod{p(\rho)}.$$

Now set

$$F_{m+1}(\rho) = F_m(\rho) + g(\rho)p(\rho)^m. \qquad \blacksquare$$

10. REMARK ON THE SYMMETRIC/SKEW CASES

The results when A is symmetric and B skew should be dual to those when A is skew and B symmetric, under an interchange of A and B. And this is largely true, but not completely so. The reason is that one of the matrices A, B has a preferred role relative to the other, and an interchange of the two without transferring the preferred role in some cases changes the appearance of the results.

11. SUBPENCILS

The relation of the invariants of a pencil to those of a principal subpencil is rather intricate, sometimes involving interlacing for real roots, and sometimes not. A very detailed study for Hermitian pencils is in [120], and a later, somewhat simpler study in [98]. Similar results should hold for real symmetric or skew pencils.

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Much of the abundant literature on linear pencils is cited below, very loosely classified. A more detailed classification seems hopeless. The equally abundant literature on nonlinear pencils is largely omitted; for this the Gohberg-Lancaster-Rodman books [9, 87, 209] may be consulted. MR = *Mathematical Reviews*; RZ = *Referativnyĭ Zhurnal*.

Some of the literature from the 1930–1940 era overlooks the fact that pencils involving Hermitian matrices have more structure than those involving complex symmetric matrices, the extra structure arising from the law of inertia. This inertial structure seems first to have been noticed by Bromwich [29] and Muth [107] about 1905. A good historical summary up to about 1935 is in the book by Turnbull and Aitken [22, p. 142].

The modern literature includes the application of the Krull-Schmidt theorem of algebra to deduce pencil structure ([5, 6] and elsewhere), and an interaction between pencil theory and the Hasse-Minkowski principle of number theory. The Krull-Schmidt theorem speaks about the decomposition of a module into indecomposable submodules, so it is a natural device to employ. The Hasse-Minkowski principle, briefly, asserts that algebraic (usually quadratic or bilinear) equations are solvable over a field precisely when they are solvable over all the closures of the field induced by all the valuations (metrics) on the field. It has a central position in the number theory of quadratic forms. Surprisingly, it turns out to be false [211] for the simultaneous vanishing of two forms, but true for the simultaneous congruence of a pair of forms to another pair [136].

Another modern development is the application of pencil theory, including minimal indices, to control theory [19], where the minimal indices are called control indices. Minimal indices also found an unexpected role in the proof by Zaballa [224, 225] of the Sa-Thompson theorem classifying the relation between the similarity invariants of a matrix and those of a principal submatrix.

Since about 1975, the numerical analysis of pencils has been extensively studied, a process that still continues, particularly in the hands of Kublanovskaya [14, 178–194], and more recently, van Dooren [201, 202], Demmel [169–171], Kågström and Ruhe (see the book [177]), and others. A natural by-product has been an interest in pencil perturbation theory; see for example Elsner et al. [145, 146], Stewart [151–154], Sun [155–160], and others. The compilation of this part of the literature appearing below should be reasonably useful.

While the study of matrix pairs can be, and often is (see Wall [220] and Malcev [105]) viewed with the beauty and elegance of the modern abstract style, the very concrete classical approach continues to flourish, partly because of its contact with numerical computation.

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