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The Radon Transforms of a Combinatorial Geometry. II. Partition Lattices

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When $k < n/2$, the incidence matrix of rank-k versus rank-$(k+1)$ partitions in the partition lattice has maximum rank.

1. INCIDENCE MATRICES AND THE SPERNER PROPERTY

In his 1928 paper [29], Sperner proved that the maximum size of a collection of pairwise incomparable subsets of an n-element set equals

$$\max\left\{\binom{n}{k}\right\} = \binom{n}{\lceil n/2 \rceil},$$

where $\lceil x \rceil$, the *ceiling* of x, is the least integer m such that $m \geqslant x$; moreover, the collection consisting of all the subsets of size $\lceil n/2 \rceil$ is a collection attaining this maximum. Sperner's theorem founded two major areas of combinatorics, extremal set theory (see, for example, [1, 12]) and the study of the Sperner property in (partially) ordered sets. This paper belongs to the second area.

An *antichain* in an ordered set P is a subset of elements of P, no two of which are comparable. An ordered set P with a minimum $\hat{0}$ is said to be *ranked* if for every element x in P, every saturated chain from $\hat{0}$ to x has the same length. This common length is defined to be the *rank* of x. The *Whitney numbers* (of the second kind) $W(P; k)$ of P is the sequence defined by $W(P; k) = |P(k)|$, where $P(k)$ is the set of rank-k elements in P. A ranked ordered set is said to be *Sperner* if the maximum size of an antichain in P equals $\max\{W(P; k)\}$, the maximum Whitney number. In this terminology, Sperner's theorem says that the Boolean algebra of subsets of an n-element set ordered by containment is Sperner. Almost all the proofs (see [1, 2, 12, 13, 23, 24, 29, 31]) of Sperner's theorem are based on the following *regularity* property: the number of subsets covering (or

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covered by) a given k-element subset depends only on n and k. This regularity property is shared by subspaces and the proofs carry over to lattices of subspaces of a finite vector space. Motivated by these results, Rota [28] asked in 1967 whether partition lattices, and more generally, geometric lattices, are Sperner. The answer to both questions is negative. Dilworth and Greene [7] found examples of non-Sperner geometric lattices and Canfield [4] showed that for sufficiently large n, the partition lattice of rank n is not Sperner. (See also [11, 14, 16].)

The motivation behind this paper is the following conjecture. Let G be a finite geometric lattice. Let $\mathcal{M}(G; k, l)$ be the incidence matrix with columns indexed by $G(k)$ and rows indexed by $G(l)$ with the X, U-entry 1 if $X \geqslant U$ and 0 otherwise.

(1.1) *Conjecture.* Let G be a finite geometric lattice of rank n and $k < n/2$. Then the rank of the incidence matrix $\mathcal{M}(G; k, k+1)$ of rank-k elements versus rank-$(k+1)$ elements equals $W(G; k)$.

This conjecture implies two difficult conjectures. Let G be a geometric lattice. The *truncation* $\text{Trun}_m[G]$ *of* G *to rank* m is the lattice obtained by identifying all the elements in G of rank at least m; $\text{Trun}_m[G]$ is a geometric lattice of rank m.

(1.2) LEMMA. *Let G be a rank-n geometric lattice in which the incidence matrix $\mathcal{M}(G; k, k+1)$ has rank $W(G; k)$ for all k less than $n/2$. Then,*

$$W(G; 0) \leqslant W(G; 1) \leqslant W(G; 2) \leqslant \cdots \leqslant W(G; \lceil n/2 \rceil)$$

and the truncation $\text{Trun}_{\lceil n/2 \rceil + 1}(G)$ *is Sperner.*

Proof. The inequalities follow from that fact that the number of rows in a matrix is at least its rank. Now observe that when $k < n/2$, $\mathcal{M}(G; k, k+1)$ contains a nonsingular square submatrix of size $W(G; k)$. Because the determinant of this submatrix is nonzero, one of the terms in its expansion is nonzero. The permutation associated with that term gives an injection $g_k : G(k) \to G(k+1)$ such that for all $X \in G(k)$, $X \leqslant g_k(X)$. These injections yield a decomposition of $\text{Trun}_m[G]$ into $W(G; \lceil n/2 \rceil)$ chains. Hence, by Dilworth's chain decomposition theorem [6], the maximum size of an antichain in $\text{Trun}_{\lceil n/2 \rceil + 1}[G]$ is $W(G; \lceil n/2 \rceil)$. ∎

The inequality in Lemma 1.2 can be regarded as the "lower half" of the conjecture [17] that the Whitney numbers form a unimodal sequence.

These are three reasons for making Conjecture 1.1. First, the Dowling–Wilson theorem ([9]; also see Sections 2 and 3), viewed from the theory of combinatorial Radon transforms [20–22], suggests a way to prove the conjecture by first reconstructing upwards. The need to go up is why $n/2$

is a natural limit for a proof using Radon transforms. The second reason is that most of the proofs that lattices of subsets or subspaces are Sperner first prove that their truncation down to $\lceil n/2 \rceil$ is Sperner and then use the fact that these lattices are isomorphic to their order duals. [The "bracketing" proof (see [1, 12]) yielding an explicit chain decomposition for the lattice of subsets is an exception.] The last reason is that the conjecture holds for lattices of subsets and subspaces. This was proved by Kantor in [18] using elementary group theory; we give a more combinatorial proof in Section 3.

We remark that Conjecture 1.1 is false for arbitrary k. For example, Canfield's method in [4] shows that when n is sufficiently large, the matrix $\mathcal{M}(\Pi_n; n-K_n-1, n-K_n)$, where K_n is the integer at which the Stirling number $S(n,k)$ of the second kind is maximum, is singular. It is known [5, 15, 26] that $K_n \approx n/\log n$. It is also true that when $n \geqslant 10$, the matrix $\mathcal{M}(L(G_n); K, K+1)$, where $L(G_n)$ is the non-Sperner bond lattices constructed in [7], is singular for $K \approx 2n/3$.

Our main aim in this paper is to verify Conjecture 1.1 for partition lattices and hence, provide an affirmative answer to the lower half of Rota's question about the partition lattice.

(1.3) THEOREM. *Let $k < n/2$. Then the incidence matrix $\mathcal{M}(\Pi_{n+1}; k, k+1)$ has rank $W(\Pi_{n+1}; k)$. In particular, the truncation* $\mathrm{Trun}_{\lceil n/2 \rceil + 1}[\Pi_{n+1}]$ *is Sperner.*

In Sections 2 and 4, we recall those portions of the theory of combinatorical Radon transform and Möbius functions needed in the proof. The proof itself occupies Section 5.

2. RADON TRANSFORMS AND A MÖBIUS FUNCTION IDENTITY

Incidence matrices are matrices associated with (combinatorial) Radon transforms. Let G be a geometric lattice and let $f: G \to \mathbb{Q}$ be a function defined from the flats in G to the rational numbers $\mathbb{Q}$. The function f is said to be *supported* on a subset H of flats if $f(X)$ equals zero unless X is in H. The Radon transform T is the linear transformation defined from the rational vector space of functions defined from G to $\mathbb{Q}$ given by

$$Tf(X) = \sum_{Y:\, Y \leqslant X} f(Y).$$

A function f supported on H is said to be *reconstructible* from its Radon transform Tf restricted to the subset K if f is uniquely determined by the function $Tf|_K: K \to \mathbb{Q}$, or, equivalently, if the linear transformation

$f \mapsto Tf|_K$ is injective. The matrix of the Radon transform $f|_H \mapsto Tf|_K$ relative to the standard basis of delta functions $\delta_X(Y) = 1$ if $Y = X$ and 0 otherwise is the incidence matrix of H versus K. Hence, the rank of the incidence matrix equals $|H|$ if and only if every function f supported on H is reconstructible from its restricted Radon transform $Tf|_K$.

A fundamental fact used in reconstructing Radon transforms is the following Möbius function identity due to Doubilet [8] and Dowling and Wilson [9]. Algebraic proofs are given in [8, 9]; a simple-minded proof is given in [21, 22].

(2.1) LEMMA. *Let $f\colon G \to \mathbb{Q}$ be a function defined on the finite lattice G. Then*

$$\sum_{Y:\, X \leqslant Y \leqslant Z} \mu(Y, Z)\, Tf(Y) = \sum_{U:\, U \vee X = Z} f(U).$$

Our first use of (2.1) is to prove a result due to Dowling and Wilson ([9]; see, in particular, the remark on p. 510) which is needed in Section 3.

(2.2) THEOREM (Dowling and Wilson). *Let G be a geometric lattice of rank n and $k < n/2$. A function $f\colon G \to \mathbb{Q}$ supported on $G(1) \cup G(2) \cup \cdots \cup G(k)$ can be reconstructed from its Radon transform Tf restricted to $G(n-k) \cup G(n-k+1) \cup \cdots \cup G(n-1)$.*

Proof. We use implicitly the fact (due to Rota [27]) that the Möbius function $\mu(X, \hat{1})$ is nonzero in a geometric lattice.

By the submodular inequality, if $\operatorname{rank}(U) \leqslant k$ and $\operatorname{rank}(X) < n-k$, then $\operatorname{rank}(U) + \operatorname{rank}(X) < n$ and $U \vee Z \neq \hat{1}$. Hence, by Lemma 2.1, if $\operatorname{rank}(X) < n-k$, then

$$Tf(X) = -\frac{1}{\mu(X, \hat{1})} \left[\sum_{Y:\, X < Y \leqslant \hat{1}} \mu(Y, \hat{1})\, Tf(Y) \right]. \tag{2.1}$$

(2.3) LEMMA.

$$Tf(X) = \sum_{Y:\, n-k \leqslant \operatorname{rank}(Y) \leqslant n} \left(\frac{\mu(Y, \hat{1})\, \mu^{\dagger}(X, Y)}{\mu(X, \hat{1})} \right) Tf(Y),$$

where $\mu^{\dagger}(X, Y)$ is the Möbius function evaluated from the minimum X to the maximum Y in the truncation $\operatorname{Trun}_{n-k-\operatorname{rank}(X)+1}[[X, Y]]$ *of the interval* $[X, Y]$.

Proof. Using Eq. (2.1) and induction down the lattice, we obtain

$$Tf(X) = \sum_{Y:\, n-k \leqslant \operatorname{rank}(Y) \leqslant n} \gamma(X, Y)\, Tf(Y),$$

where

$$\gamma(X, Y) = \sum_{(A_i)} (-1)^m \prod_{i=0}^{m-1} \mu(A_{i+1}, \hat{1})/\mu(A_i, \hat{1}),$$

the sum being over all strictly increasing chains $X = A_0 < A_1 < \cdots < A_m = Y$ such that $\text{rank}(A_{m-1}) < n-k$. However, because the numerator and denominator of adjacent terms are equal, the product inside the sum telescopes and equals $\mu(Y, \hat{1})/\mu(X, \hat{1})$. Hence,

$$\gamma(X, Y) = \frac{\mu(Y, \hat{1})}{\mu(X, \hat{1})} \left(\sum_{(A_i)} (-1)^m \right).$$

But, by Philip Hall's theorem (see [27, p. 346]),

$$\sum_{(A_i)} (-1)^m = \mu^{\dagger}(X, Y). \quad \blacksquare$$

Let M be a variable standing for the unknown Radon transform $Tf(\hat{1})$. By Lemma 2.3 and the fact that f is supported on $G(1) \cup G(2) \cup \cdots \cup G(k)$,

$$0 = f(\hat{0}) = Tf(\hat{0}) = \alpha + \beta M,$$

where α and β are known rational numbers and $\beta = \mu^{\dagger}(\hat{0}, \hat{1})/\mu(\hat{0}, \hat{1})$ is nonzero. Thus, we can solve for M and use Lemma 2.3 to reconstruct the Radon transform Tf on all of G. Once we know Tf, f can be reconstructed using Möbius inversion. $\blacksquare$

3. Locally Projective Lattices

Before embarking in the proof of Theorem 1.3, we prove Conjecture 1.1 for geometric lattices with a strong regularity property.

(3.1) Theorem. *Let $0 < k \leqslant m \leqslant n-k$ and let G be a geometric lattice of rank n satisfying the following regularity property: There exist positive integers c_r, $m+1 \leqslant r \leqslant m+k-1$, such that for every rank-k flat L and rank-r flat X, the number of rank-m flats U such that $L < U < X$ equals the constant c_r.*

Then $W(G, k) \leqslant W(G, m)$ and the rank of the incidence matrix $\mathcal{M}(G; k, m)$ equals $W(G, k)$.

Proof. It suffices to show that a function $f: G \to \mathbb{Q}$ supported on $G(k)$ can be reconstructed from its Radon transform $Tf: G(m) \to \mathbb{Q}$ restricted to the rank-m flats.

We begin by reconstructing the values of the Radon transform $Tf(X)$ for a rank-r flat X, where $m+1 \leqslant r \leqslant m+k-1$. This is done using the equation

$$\sum_{U:\, \text{rank}(U)=m \text{ and } U<X} Tf(U) = \sum_{U:\, \text{rank}(U)=m \text{ and } U<X} \left[\sum_{L:\, \text{rank}(L)=k \text{ and } L<U} f(L) \right]$$
$$= \sum_{L,\, U:\, \text{rank}(L)=k,\ \text{rank}(U)=m \text{ and } L<U<X} f(L).$$

Each function value $f(L)$ contributes as many times to this sum as there are flats U such that $L<U<X$. Hence, this sum equals

$$\sum_{L:\, L<X} c_r f(L) = c_r Tf(X).$$

Since c_r is nonzero, $Tf(X)$ is reconstructed. We can now reconstruct f using Lemma 2.2. ∎

Besides Boolean algebras of subsets, natural examples of lattices satisfying the regularity property in Theorem 3.1 are *locally projective* lattices introduced by Kantor [19]. These are lattices in which every proper upper interval $[U, \hat{1}]$ is isomorphic to the lattice of flats of a projective geometry. In particular, projective and affine geometries have locally projective lattices of flats. Affine geometries are obtained by deleting a hyperplane from projective geometries; in fact, if one deletes $q-1$ hyperplanes $H_1, H_2, ..., H_{q-1}$ with intersection $\bigcap H_i$ a flat of codimension 2 from the projective geometry $PG(n, q)$ of dimension n over the finite field $GF(q)$, then the remaining points, consisting of the disjoint union of two affine geometries of dimension $n-1$, has a locally projective lattice of flats.

(3.2) COROLLARY. *Let G be a locally projective lattice and k and l be integers satisfying $0 \leqslant k \leqslant l \leqslant n-k$. Then $\mathcal{M}(G; k, l)$ has rank equal to $W(G, k)$.*

Since the lattice $L(n, q)$ of flats of $PG(n-1, q)$ is isomorphic to its order dual, Corollary 3.2 yields another proof of a result of Kantor [18]. The q-binomial coefficient $\binom{n}{m}_q$ is the number of flats of dimension m in $PG(n-1, q)$.

(3.3) COROLLARY. *The rank of the incidence matrix between the flats of dimension k and the flats of dimension l in $PG(n-1, q)$ equals $\min\{\binom{n}{k}_q, \binom{n}{l}_q\}$.*

4. Partition Lattices

We prove Theorem 1.3 using the fact that partition lattices have regularity properties similar to the property in Theorem 3.1. We first recall several elementary facts about the partition lattice, due to Birkhoff [3] and Ore [25].

A *partition* π of the set S is a collection of nonempty pairwise-disjoint subsets $B_1, B_2, ..., B_p$ called *parts* such that the union $B_1 \cup B_2 \cup \cdots \cup B_p$ equals S. A part containing exactly one element is said to be *trivial*. We write

$$\pi = B_1 \oplus B_2 \oplus \cdots \oplus B_p.$$

If $\pi = \bigoplus B_i$ is a partition of S, $\sigma = \bigoplus C_j$ is a partition of T, and S and T are disjoint, the partition $\pi \oplus \sigma$ is the partition of $S \cup T$ with parts B_i and C_j. Partitions can be ordered by *reverse refinement*: $\pi \leqslant \sigma$ if every part of π is contained in a part of σ. Under this order, the partitions of a finite set S form a geometric lattice $\Pi(S)$. Its maximum $\hat{1}$ is the partition with exactly one part and its minimum $\hat{0}$ is the partition with $|S|$ trivial parts.

We denote the lattice of partitions of $\{1, 2, ..., n\}$ by Π_n. The rank of a partition π in Π_n is given by

$$\text{rank}(\pi) = n - \text{number of parts in } \pi.$$

In particular, the maximum partition $\{1, 2, ..., n\}$ has rank $n-1$. This shifting of the rank leads to a notational morass. We avoid most of this by working with the rank-n lattice Π_{n+1} of partitions of the set $\{1, 2, ..., n+1\}$. To avoid shifting subscripts, we denote the rank-n partition lattice by Q_n.

The partition lattice Q_n satisfies two regularity properties. The first is that it is *upper homogeneous*, that is,

$$[\pi, \hat{1}] \cong Q_{n - \text{rank}(\pi)}$$

for every partition π. The second property concerns all intervals. Let $\pi \leqslant \sigma$. If $\sigma = C_1 \oplus C_2 \oplus \cdots \oplus C_p$, then $\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_p$, where π_i is a partition of C_i. The interval $[\pi, \sigma]$ is determined up to isomorphism by the ranks of the partitions π_i in the following way:

$$\begin{aligned}[\pi, \sigma] &\cong [\pi_1, C_1] \times [\pi_2, C_2] \times \cdots \times [\pi_p, C_p] \\ &\cong Q_{\text{rank}(C_1) - \text{rank}(\pi_1)} \times Q_{\text{rank}(C_2) - \text{rank}(\pi_2)} \times \cdots \times Q_{\text{rank}(C_p) - \text{rank}(\pi_p)}.\end{aligned}$$

Note that $\text{rank}(C_i) = |C_i| - 1$. Suppose that $[\hat{0}, \sigma] \cong Q_{t_1} \times Q_{t_2} \times \cdots \times Q_{t_m}$, where $t_i \geqslant 1$ for all i. The *type* of σ is the multiset $t_1, t_2, ..., t_m$, written in the following way: $t_1 \oplus t_2 \oplus \cdots \oplus t_m$. The *height* $\mathrm{ht}(\sigma)$ is the maximum integer $\max\{t_i\}$ in its type. The *extent* $\mathrm{ex}(\sigma)$ of σ is defined by

$$\mathrm{ex}(\pi) = t_1 + t_2 + \cdots + t_m + m.$$

The extent is the size of the set obtained by taking the union of the nontrivial parts of σ.

We denote the Whitney numbers of Q_n by $W(n, k)$. These numbers are related to the Stirling numbers by the relation

$$W(n, k) = S(n+1, n+1-k).$$

In particular, $W(n, 1) = \binom{n+1}{2}$. The Möbius function of Q_n was computed by Rota in [27]. For upper intervals, it is given by the formula

$$\mu(\pi, \hat{1}) = (-1)^{n-\mathrm{rank}(\pi)} (n - \mathrm{rank}(\pi))!.$$

(4.1) LEMMA. $\sum_{j=0}^{r} (-1)^{r-j+1} (r-j+1)!\, W(r, j) = (-1)^{r+1}$.

Proof. The identity follows from Stirling's formula [30]

$$x^{r+1} = \sum_{j=1}^{r+1} S(r+1, j)\, x(x-1)(x-2)\cdots(x-j+1)$$

evaluated at $x = -1$. ∎

We use the following notation. If $\mathbf{s} = (s_1, s_2, ..., s_p)$ is a p-tuple and $\mathbf{t} = (t_1, t_2, ..., t_r)$ is an r-tuple, then $(\mathbf{s}, \mathbf{t})$ is the $(p+r)$-tuple $(s_1, s_2, ..., s_p, t_1, t_2, ..., t_r)$ obtained by concatenating $\mathbf{s}$ and $\mathbf{t}$.

5. RECONSTRUCTION

In this section, we prove Theorem 1.3 by showing that when $k < n/2$, a function $f\colon Q_n \to \mathbb{Q}$ supported on the rank-k partitions can be reconstructed from its Radon transform Tf restricted to the rank-$(k+1)$ partitions.

We first observe that since $k < n/2$, $n \geqslant 2k+1$. Moreover, every rank-k partition π is less than a partition consisting of a single nontrivial part of size $2k+2$. Relabelling if necessary, we may assume that this partition is $\{1, 2, ..., 2k+2\} \oplus \{2k+3\} \oplus \cdots \oplus \{n+1\}$. By deleting the trivial parts $\{2k+3\}, ..., \{n+1\}$, π may be regarded as a partition of $\{1, 2, ..., 2k+2\}$ in the lattice Q_{2k+1}. We show that $f(\pi)$ can reconstructed from the values of the Radon transforms $Tf(\sigma)$, where σ ranges over all the rank-$(k+1)$ partitions in Q_{2k+1}.

Let π be the partition $B_1 \oplus B_2 \oplus \cdots \oplus B_p$ and $\mathbf{r} = (r_1, r_2, ..., r_p)$ be the p-tuple of integers such that $r_i = |B_i| - 1$. Let $\mathbf{s} = (s_1, s_2, ..., s_p)$ be a p-tuple of integers such that $0 \leqslant s_i \leqslant r_i$ and $s_1 + s_2 + \cdots + s_p = k$. Define $X_\pi(\mathbf{s})$ as

$$X_\pi(\mathbf{s}) = \sum_{\alpha : \alpha = \bigoplus \alpha_i \leqslant \pi, \, \mathrm{rank}(\alpha_i) = s_i} f(\alpha),$$

where the sum ranges over all rank-k partitions $\alpha = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_p$ such that α_i is a rank-s_i partition of B_i. Let $\mathscr{S}_{\mathbf{r}, k}$ be the set of p-tuples $(s_1, s_2, ..., s_p)$ such that $0 \leqslant s_i \leqslant r_i$ and $s_1 + s_2 + \cdots + s_p = k$.

The idea behind the reconstruction is to find and solve systems of linear equations in the unknowns $X_\pi(\mathbf{s})$. When the unknowns $X_\pi(\mathbf{s})$ are found, $Tf(\pi)$ can be obtained using the following obvious lemma.

(5.1) LEMMA. $Tf(\pi) = \sum_{\mathbf{s} \in \mathscr{S}_{\mathbf{r}, k}} X_\pi(\mathbf{s})$.

Most of our equations for the unknowns $X_\pi(\mathbf{s})$ are a consequence of the following counting lemma.

(5.2) LEMMA. *Let* $\mathbf{t} = (t_1, t_2, ..., t_p)$ *be a p-tuple of integers such that* $0 \leqslant t_i \leqslant r_i$. *Then*

$$\sum_{\mathbf{s} : \mathbf{s} \leqslant \mathbf{t}} C(\mathbf{t}, \mathbf{s}) X_\pi(\mathbf{s}) = \sum_{\sigma : \sigma = \bigoplus \sigma_i \leqslant \pi, \, \mathrm{rank}(\sigma_i) = t_i} Tf(\sigma), \qquad (E_{\mathbf{t}})$$

where the left-hand sum ranges over all p-tuples $\mathbf{s}$ *such that* $s_i \leqslant t_i$ *the right-hand sum ranges over all rank-*$(t_1 + t_2 + \cdots + t_p)$ *partitions* $\sigma = \sigma_1 \oplus \sigma_2 \oplus \cdots \oplus \sigma_p$, *where* σ_i *is a rank-*t_i *partition of* B_i, *and*

$$C(\mathbf{t}, \mathbf{s}) = \prod_{i=1}^{p} W(r_i - s_i, t_i - s_i).$$

Proof. Consider a rank-k partition $\alpha = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_p$, where α_i is a rank-s_i partition of B_i. The interval $[\alpha, \pi]$ is isomorphic to the product $Q_{r_1 - s_1} \times Q_{r_2 - s_2} \times \cdots \times Q_{r_p - s_p}$. Because there are $W(r_i - s_i, t_i - s_i)$ partitions σ_i of rank t_i in the interval $[\alpha_i, B_i]$, the function value $f(\alpha)$ occurs in exactly $\prod_{i=1}^{p} W(r_i - s_i, t_i - s_i)$ Radon transforms in the right-hand sum. Since the number $\prod_{i=1}^{p} W(r_i - s_i, t_i - s_i)$ depends only on the type of α, we can group together function values $f(\alpha)$ of rank-k partitions α of the same type to obtain the left-hand sum. ∎

The *zeroth step* in the reconstruction is to reconstruct the Radon transform for partitions in Q_{2k+1} having height greater than k.

(5.3) LEMMA. *Let* $\pi = B_1 \oplus B_2 \oplus \cdots \oplus B_p$, *where* $|B_p| - 1 > k$, *and* $r_i = |B_i| - 1$. *Let* $\hat{\mathscr{S}}_{\mathbf{r},k}$ *be the set of p-tuples in* $\mathscr{S}_{\mathbf{r},k+1}$ *which are in the image of the injection* $\mathscr{S}_{\mathbf{r},k} \to \mathscr{S}_{\mathbf{r},k+1}$ *defined by*

$$\mathbf{s} \mapsto \hat{\mathbf{s}}, \qquad (s_1, s_2, ..., s_{p-1}, s_p) \mapsto (s_1, s_2, ..., s_{p-1}, s_p + 1).$$

Then, the system of equations $E_{\mathbf{t}}$, *where* $\mathbf{t}$ *ranges over* $\hat{\mathscr{S}}_{\mathbf{r},k}$, *is a system of* $|\hat{\mathscr{S}}_{\mathbf{r},k}|$ *linearly independent equations in the same number* $|\mathscr{S}_{\mathbf{r},k}|$ *of unknowns* $X_\pi(\mathbf{s})$.

Proof. Let $C = [C(\mathbf{t}, \mathbf{s})]$ be the matrix of coefficients of the system $E_{\mathbf{t}}$. Index the columns $\mathbf{s}^1, \mathbf{s}^2, ...$ of C so that the pth component is nondecreasing, that is, $s_p^i \leqslant s_p^j$ whenever $i < j$. Next, index the rows of C using the injection $\mathbf{s} \to \hat{\mathbf{s}}$. Suppose that $i > j$. Because

$$s_1^i + s_2^i + \cdots + s_{p-1}^i \leqslant s_1^j + s_2^j + \cdots + s_{p-1}^j$$

and $\mathbf{s}^i \neq \mathbf{s}^j$, $\hat{s}_m^i = s_m^i < s_m^j$ for some m. When this is the case, $W(r_m - s_m^j, \hat{s}_m^i - s_m^j) = 0$ and hence, $C(\hat{\mathbf{s}}^i, \mathbf{s}^j) = 0$. In addition, the diagonal entry $C(\hat{\mathbf{s}}^i, \mathbf{s}^i)$ is $W(r_p - s_p^i, 1)$, which equals the nonzero binomial coefficient

$$\binom{r_p - s_p^i + 1}{2}.$$

We conclude that C is upper triangular with nonzero diagonal entries. ∎

EXAMPLE. For $k = 3$ and $\pi = \{1, 2, 3\} \oplus \{4, 5, 6, 7, 8\}$, the matrix C has row indices $2 \oplus 2$, $1 \oplus 3$, $0 \oplus 4$ and column indices $2 \oplus 1$, $1 \oplus 2$, $0 \oplus 3$ and equals

$$\begin{pmatrix} 6 & 1 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

By Lemma 5.3, we can solve for the unknowns $X_\pi(\mathbf{s})$ in the system $E_{\mathbf{t}}$ and reconstruct $Tf(\pi)$ using Lemma 5.1.

With the reconstruction of function values of partitions with height greater than k, we have completed the zeroth step of the reconstruction. The remaining steps consist of reconstructing inductively function values of partitions with height r, where $r = k, k-1, ..., 2, 1$. The crucial partitions are those of the type $r \oplus 1 \oplus \cdots \oplus 1 \oplus r$. For these partitions, the system $E_{\mathbf{t}}$ has more unknowns than there are equations. Fortunately, there is an extra equation arising from the Möbius function identity in Lemma 2.1.

We use $[m]$ to denote, depending on the context, the multiset with m 1's or the m-tuple with m 1's; that is,

$$[m] = \overbrace{1 \oplus \cdots \oplus 1}^{k \text{ times}} \qquad \text{or} \qquad [m] = \overbrace{(1, ..., 1)}^{k \text{ times}}.$$

(5.4) ALTERNATING SUM LEMMA. *Let l and r be integers such that $l \leqslant k-r$, $k \geqslant r$, and $2r \geqslant k+1$. Let π be the partition*

$$\{1, 2, ..., r+1\} \oplus \{r+2, r+3\} \oplus \{r+4, r+5\} \oplus \cdots \oplus \{r+2l, r+2l+1\}$$
$$\oplus \{r+2l+2, ..., 2r+2l+2\} \oplus \{2r+2l+3\}$$
$$\oplus \{2r+2l+4\} \oplus \cdots \oplus \{2k+2\}$$

having type $r \oplus [l] \oplus r$. Then

$$\sum_{m=0}^{r} (-1)^{r-m-1} X_\pi(k-l-m, [l], m) = A + B, \qquad (\mathrm{Alt}(r, l))$$

where

(1) *A is linear combination of Radon transforms $Tf(\omega)$, where ω has height at least $r+1$, and*

(2) *B is a linear combination of values $X_\pi(\mathbf{s})$, where $\mathbf{s} = (m', ..., 0, ..., m'')$ is a $(l+2)$-tuple in which at least one of the middle l components is zero, or, equivalently, $m' + m'' > k - l$.*

Proof. Let σ be the partition

$$\{1, 2, ..., r+1\} \oplus \{r+2, r+3\} \oplus \{r+4, r+5\} \oplus \cdots$$
$$\oplus \{r+2l, r+2l+1\} \oplus \{r+2l+2\} \oplus \{r+2l+3\}$$
$$\oplus \cdots \oplus \{2k+2\}$$

of type $r \oplus [l]$ and ω be the partition

$$\{1, 2, ..., r+1\} \cup \{r+2l+2, ..., 2r+2l+2\} \oplus \{r+2, r+3\} \oplus \{r+4, r+5\}$$
$$\oplus \cdots \oplus \{r+2l, r+2l+1\} \oplus \{2r+2l+3\}$$
$$\oplus \{2r+2l+4\} \oplus \cdots \oplus \{2k+2\},$$

of type $2r \oplus [l]$. The interval $[\sigma, \omega]$ is isomorphic to Q_r. Applying Lemma 2.1, we obtain the equation

$$\sum_{\tau : \sigma \leqslant \tau \leqslant \omega} \mu(\tau, \omega)\, Tf(\tau) = \sum_{\xi : \xi \vee \sigma = \omega} f(\xi).$$

We simplify this equation in three steps.

(1) If $\tau \notin [\sigma, \pi]$, then at least one of the trivial parts $\{j\}$, $r+2l+2 \leqslant j \leqslant 2r+2l+2$, of σ is contained in the part containing $\{1, 2, ..., r+1\}$ in τ. Hence, τ has a part of size greater than $r+2$, $\mathrm{ht}(\tau) \geqslant r+1$, and we can take the term involving $Tf(\tau)$ to the right-hand side.

(2) If $\xi \vee \sigma = \omega$, then ξ has a part containing all the elements $r+2l+2$, $r+2l+3, ..., 2r+2l+2$ and at least one element from the part

$\{1, 2, \ldots, r+1\}$. Hence, ξ has a part of size greater than $r+2$ and height at least $r+1$. Moreover, since ξ is a rank-k partition, $f(\xi) = Tf(\xi)$.

(3) The sum remaining on the right hand side equals

$$\sum_{\tau : \tau \in [\sigma, \pi]} \mu(\tau, \omega)\, Tf(\tau).$$

If τ is a partition in $[\sigma, \pi]$, $r \leqslant \operatorname{rank}(\tau) \leqslant 2r$ and the Möbius function $\mu(\tau, w)$ equals

$$(-1)^{2r - \operatorname{rank}(\tau)}\,(2r - \operatorname{rank}(\tau))!,$$

an integer depending only on $\operatorname{rank}(\tau)$. Hence, we can group together all the partitions of rank $r+j$ in $[\sigma, \pi]$ to obtain

$$\sum_{j=0}^{r} (-1)^{r-j+1}\,(r-j+1)! \left[\sum_{\tau : \tau \in [\sigma, \pi] \text{ and } \operatorname{rank}(\tau) = j} Tf(\tau) \right]. \qquad (5.1)$$

The inner sum can be expressed in terms of $X_\pi(\mathbf{s})$ by counting the number of rank-$(r+j)$ partitions containing a given rank-k partition as in the proof of Lemma 5.2. Doing so, we obtain

$$\sum_{\tau : \tau \in [\sigma, \pi] \text{ and } \operatorname{rank}(\tau) = j} Tf(\tau) = \sum_{m=0}^{k} W(r-m, j-m)\, X_\pi(k-l-m, [l], m)$$
$$+ \text{ terms involving } X_\pi(m', \ldots, 0, \ldots, m''),$$
$$\text{where} \quad m' + m'' > k - l.$$

We substitute this into Eq. (5.1), move the terms involving $X_\pi(m', \ldots, 0, \ldots, m'')$ to the right, and interchange summations to obtain

$$\sum_{m=0}^{k} \left[\sum_{j=0}^{r} (-1)^{r-j+1}\,(r-j+1)!\; W(r-m, j-m) \right] X_\pi(k-l-m, [l], m).$$

Since $W(r-m, j-m) = 0$ when $j < m$, the inner sum is really from $j = m$ to $j = r$. Changing index of summation from j to $j - m$ and using (4.1), we conclude that the inner sum equals

$$\sum_{j=0}^{r-m} (-1)^{r-m-j+1}\,(r-m-j+1)!\; W(r-m, j) = (-1)^{r-m+1}.$$

This completes the proof of Lemma 5.4. ▌

Let π be the partition of type $r \oplus [l] \oplus r$ defined in Lemma 5.4 and let $\mathbf{t} = (t_1, t_2, \ldots, t_{l+2})$ be an $(l+2)$-tuple such that $0 \leqslant t_1 \leqslant r$, $t_i = 1$ for $t = 2, 3, \ldots, l+1$, and $0 \leqslant t_{l+2} \leqslant r$. The *expurgated* equation $E_{\mathbf{t}}^*$ is the equation obtained from $E_{\mathbf{t}}$ by moving all the terms involving $X_\pi(m', \ldots, 0, \ldots, m'')$, where $m' + m'' > k - l$, to the right-hand side. The reason for moving such terms to the right is that the values $X_\pi(m', ..., 0, ..., m'')$ equal values computed earlier in the reconstruction. This is made precise in the next lemma.

(5.5) LEMMA. *Let* π *be as defined in the previous lemma. Suppose* $\mathbf{s} = (m', ..., 0, ..., m'')$ *is a* $(l+2)$*-tuple with the* $(j+1)$*st component zero. Then* $X_\pi(\mathbf{s}) = X_{\tilde{\pi}}(\tilde{\mathbf{s}})$, *where* $\tilde{\pi}$ *is the partition of type* $r \oplus [l-1] \oplus r$ *obtained from* π *by splitting the part* $\{r+2j, r+2j+1\}$ *into two trivial parts* $\{r+2j\}$ *and* $\{r+2j+1\}$ *and* $\tilde{\mathbf{s}} = (m', ..., ..., m'')$, *the* $(l+1)$*-tuple obtained from* $\mathbf{s}$ *by deleting the* $(j+1)$*st component.*

Proof. Observe that $\text{rank}(\alpha_{j+1}) = 0$ in the rank-k partition $\alpha = \bigoplus \alpha_i$ if and only if the elements $r+2j$ and $r+2j+1$ are in two trivial parts. ∎

The equation $(\text{Alt}(r, l))$ arises from an identity true in all lattices while the expurgated equations E_t^* arise from rather specific facts about the partition lattice. Thus, we expect them to form a linearly independent system of equations. This is in fact the case.

(5.6) LEMMA. *Let* r *and* l *be integers such that* $k \geq r$ *and* $2r \geq k-l+1$. *Then the* $2r-k+l$ *row vectors of the coefficient matrix* D *of the system of expurgated equations* E_t^*, *for* $\mathbf{t}$ *ranging over* $r \oplus [l] \oplus k-l-r+1$, $r-1 \oplus [l] \oplus k-l-r+2, ..., k-l-r+1 \oplus [l] \oplus r$, *and the vector*

$$\mathbf{a} = (-1, 1, -1, ,1, ..., 1, -1)$$

are linearly independent.

Proof. By Lemma 5.3 and the fact that $W(n, 1) = \binom{n+1}{2}$, the coefficient matrix D is the following $(2r-k+l) \times (2r-k+l+1)$ rectangular matrix with the nonzero entries on a "diagonal" of width 2:

$$\begin{bmatrix} \binom{2r-k+l+1}{2} & 1 & 0 & 0 \cdots 0 & 0 & 0 \\ 0 & \binom{2r-k+l}{2} & 3 & 0 \cdots 0 & 0 & 0 \\ 0 & 0 & \binom{2r-k+l-1}{2} & 6 \cdots 0 & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 \cdots 3 & \binom{2r-k+l}{2} & 0 \\ 0 & 0 & 0 & 0 \cdots 0 & 1 & \binom{2r-k+l+1}{2} \end{bmatrix}$$

We need to show that the vector $\mathbf{a}$ is not in the row space of D. To do this, we observe that the vector

$$\mathbf{u}=\left(1,\ -\frac{\binom{2r-k+l+1}{2}}{\binom{2}{2}},\ \frac{\binom{2r-k+l+1}{2}\binom{2r-k+l}{2}}{\binom{2}{2}\binom{3}{2}},\ \ldots,\ \pm\frac{\binom{2r-k+l+1}{2}\binom{2r-k+l}{2}\binom{2r-k+l-1}{2}\cdots}{\binom{2}{2}\binom{3}{2}\binom{4}{2}\cdots},\ \ldots,\ \frac{\binom{2r-k+l+1}{2}\binom{2r-k+l}{2}}{\binom{2}{2}\binom{3}{2}},\ -\frac{\binom{2r-k+l+1}{2}}{\binom{2}{2}},\ 1\right)$$

is orthogonal (under the usual dot product) to all the rows of D. [This can be established by an easy computation, but the underlying reason why is that the ith coordinate of $\mathbf{u}$ is the quotient

$$(-1)^{i-1}\det D_{i-1}/\det D_{i-1}^{\rightarrow},$$

where D_j is the $j\times j$ matrix obtained by taking the first j rows and columns of D and $D_j^{\rightarrow}$ is the $j\times j$ matrix obtained by taking the second to $(j+1)$st rows and the first j columns of D.] Hence, the codimension-1 row space of D is the kernel of the linear functional $\mathbf{x}\to\mathbf{x}\cdot\mathbf{u}$. However, $\mathbf{a}\cdot\mathbf{u}$ is a sum of strictly negative terms. We conclude that $\mathbf{a}$ and the rows of D form a linearly independent set. ∎

The next two lemmas are used to reconstruct the Radon transforms of those partitions not having type $r\oplus[l]\oplus r$.

(5.7) Lemma. *Let $r<k$. Suppose that for all rank-k partitions α with height at least r, the function values $f(\alpha)$ have already been constructed. Let π be a partition such that* $\operatorname{rank}(\pi)\geqslant k$ *and* $\operatorname{ht}(\pi)=r$. *Then $Tf(\pi)$ can be reconstructed.*

Proof. Let $\pi=B_1\oplus B_2\oplus\cdots\oplus B_{p-1}\oplus B_p$, $r_i=|B_i|-1$, $r_i\leqslant r$, and $r_p=r$. As in the first step of the reconstruction, consider the set $\mathscr{S}_{\mathbf{r},k}$ of p-tuples (s_i) such that $s_i\leqslant r_i$ and $\sum s_i=k$. Let $\mathscr{T}$ be the set of p-tuples in $\mathscr{S}_{\mathbf{r},k+1}$ of the form $\hat{\mathbf{s}}$, where the pth component s_p of $\mathbf{s}$ is strictly less than r.

Consider the system of equations $E_{\mathbf{t}}$, where $\mathbf{t}$ ranges over $\mathscr{T}$. This is a rectangular system of $|\mathscr{T}|$ equations in $|\mathscr{S}_{r,k}|$ unknowns. Indexed as earlier, the matrix C of coefficients of this system is upper triangular in the sense that its i, j-entry is zero whenever $i > j$. Moreover, the diagonal entries are nonzero binomial coefficients. The columns of C not in the principal square submatrix are indexed by p-tuples $(\mathbf{s}', r)$, where $\mathbf{s}'$ is a $(p-1)$-tuple with $s_i' \leqslant r_i$ and $s_1 + s_2 + \cdots + s_{p-1} = k - r$.

By hypothesis, the values $X_\pi(\mathbf{s}', r)$ are sums of already reconstructed function values and can be computed. Substituting the actual values of $X_\pi(\mathbf{s}', r)$ into $E_{\mathbf{t}}$ results in an upper triangular system of $|\mathscr{S}_{r,k}|$ equation in the same number of unknowns with nonzero diagonal entries. We can now reconstruct $Tf(\pi)$ by solving this system for the remaining values of $X_\pi(\mathbf{s})$ and using Lemma 5.1. ∎

EXAMPLE. Let $k = 3$, $r = 2$, and the type of π be $1 \oplus 2 \oplus 2$. The matrix C has row indices

$$1 \oplus 2 \oplus 0, \quad 1 \oplus 1 \oplus 1, \quad 0 \oplus 2 \oplus 1, \quad 1 \oplus 0 \oplus 2, \quad 0 \oplus 1 \oplus 2$$

and columns indices

$$1 \oplus 2 \oplus 1, \quad 1 \oplus 1 \oplus 2, \quad 0 \oplus 2 \oplus 2$$

and equals

$$\begin{pmatrix} 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

The values $f(\alpha)$ where α has $1 \oplus 0 \oplus 2$, $0 \oplus 1 \oplus 2$ are assumed to have been reconstructed. Hence, $X_\pi(1 \oplus 0 \oplus 2)$ and $X_\pi(0 \oplus 1 \oplus 2)$ are known and the last two columns of C and be eliminated, leaving us with a square upper triangular matrix.

(5.8) LEMMA. *Let $r < k$. Suppose that for all rank-k partitions α with height greater than r, the function values $f(\alpha)$ has already been reconstructed. Let α be a rank-k partition such that* $\mathrm{ex}(\alpha) \leqslant 2k - r + 1$. *Then $f(\alpha)$ can be reconstructed.*

Proof. Relabelling if necessary, we may assume that α is a partition all of whose nontrivial parts lie in $\{1, 2, ..., 2k - r\}$. Let σ be the rank-$(r+1)$ partition with the single nontrivial part $\{2k - r + 1, 2k - r + 3, ..., 2k + 2\}$ and $\pi = \alpha \vee \sigma$, the partition whose nontrivial parts are the nontrivial parts of α and σ.

Consider the interval $[\sigma, \pi]$. All the partitions τ in this interval have a part, namely the part containing $\{2k-r+1, 2k-r+2, ..., 2k+2\}$, having size at least $r+2$. Hence, $\mathrm{ht}(\tau) > r$ and by Lemma 5.7, the Radon transforms $Tf(\tau)$ can be reconstructed. By Lemma 2.1 and the fact that α is the unique rank-k partition whose join with σ is π,

$$f(\alpha) = \sum_{\tau : \sigma \leqslant \tau \leqslant \pi} \mu(\tau, \pi)\, Tf(\tau).$$

We can now reconstruct $f(\alpha)$. ∎

The next lemma shows that Lemma 5.8 can be applied in many cases.

(5.9) LEMMA. *Except for the partitions of type $r \oplus [k-r]$, all the partitions of rank-k and height r have extent at most $2k-r$.*

Proof. The maximum extent of a rank-k height-r partition is $r+1+2(k-r) = 2k-r+1$. A partition attains this maximum extent if and only if it has type $r \oplus [k-r]$. ∎

With the technical lemmas in place, we can reconstruct the values of $f(\alpha)$ inductively. The function values of height-k partitions are reconstructed first; next, the function values of partitions of height $k-1$ are reconstructed, and so on. The reconstruction is completed at the kth step when the function values of height-1 partitions are reconstructed.

At the $(i+1)$st *step* of the reconstruction, the Radon transforms or function values of partitions of height greater than $k-i$ have already been reconstructed. Let $r = k-i$. We first reconstruct $f(\alpha)$ for α having height r and extent less than $2k-r$. By Lemma 5.8, the only remaining values to be reconstructed are $f(\alpha)$, where α has type $r \oplus [k-r]$. We go through the reconstruction of $f(\alpha)$, where

$$\begin{aligned} \alpha = {} & \{1, 2, ..., r+1\} \oplus \{r+2, r+3\} \oplus \{r+4, r+5\} \\ & \oplus \cdots \oplus \{r+2i, r+2i+1\} \oplus \{r+2i+2\} \\ & \oplus \{r+2i+2\} \oplus \cdots \oplus \{2k+2\}. \end{aligned}$$

We first reconstruct $f(\pi_0)$ for the partition

$$\begin{aligned} \pi_0 = {} & \{1, 2, ..., r+1\} \oplus \{r+2\} \oplus \{r+3\} \oplus \{r+4\} \oplus \cdots \oplus \{r+2i+1\} \\ & \oplus \{r+2i+2, ..., 2r+2i+2\} \oplus \{2r+2i+3\} \\ & \oplus \{2r+2i+4\} \oplus \cdots \oplus \{2k+2\} \end{aligned}$$

by solving the expurgated equations E_r^* and the equation $(\mathrm{Alt}(r, 0))$. Next, we reconstruct $f(\pi_1)$ for all partitions π_1 of type $r \oplus 1 \oplus r$, obtained from π_0 by merging into one part one of the pairs $\{r+j\}$, $\{r+j+1\}$, where $j=2, 4, 6, ..., 2i$. Proceeding in this way, we reconstruct, after $k-r+1$ iteration, $f(\pi_{k-r})$, where π_{k-r} is the partition

$$\begin{aligned}\pi_{k-r} = \{1, 2, ..., r+1\} &\oplus \{r+2, r+3\} \oplus \{r+4, r+5\} \\ &\oplus \cdots \oplus \{r+2i+1, r+2i+2\} \\ &\oplus \{r+2i+2, ..., 2r+2i+2\} \oplus \{2r+2i+3\} \\ &\oplus \{2r+2i+4\} \oplus \cdots \oplus \{2k+2\}\end{aligned}$$

of type $r \oplus [k-r] \oplus r$ by solving E_r^* and $(\mathrm{Alt}(r, k-r))$. When we solve these equations, we also obtain the quantity $X_\pi(r, [k-r], 0)$, which equals $f(\alpha)$. We remark that when i is large, partitions π_l of type $k-i \oplus [l] \oplus k-i$ may have rank less than k. When this is the case, $Tf(\pi_l)$ and $X_{\pi_l}(\mathrm{s})$ are empty sums and equal zero. We finish the $(i+1)$st step by reconstructing $Tf(\pi)$ for all partitions of height $k-i$ using Lemma 5.7.

At the kth and last step, all the Radon transforms of partitions of rank greater than k have been reconstructed. (This is because in the partition

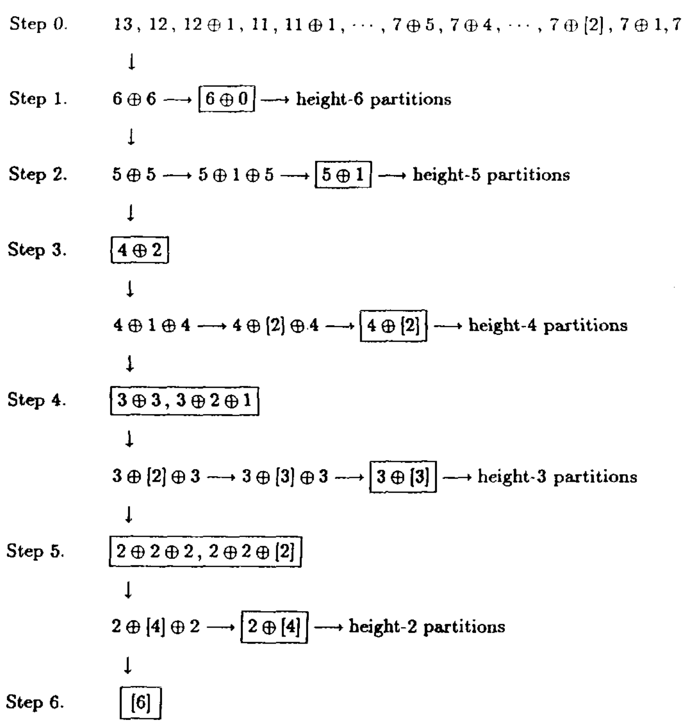

FIG. 1. Reconstruction scheme for $k=6$.

lattice Q_{2k+1}, partitions of rank greater than $k+1$ have height at least 2. Hence, their Radon transforms have been reconstructed in an earlier step.) Thus, we can reconstruct $f(\alpha)$ where α has type $[k]$ by using Lemma 2.1.

This completes our proof of Theorem 1.3. The proof is designed so that it extends in a straightforward way to Dowling's group-labelled partition lattices [10] when appropriate q's are inserted. A flow-chart of the reconstruction process when $k=6$ is given schematically in Fig. 1.

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