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Spaces with an Abstract Convolution of Measures

ROBERT I. JEWETT

Department of Mathematics, Western Washington State College, Bellingham, Washington 98225

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1. INTRODUCTION

In the theory of locally compact groups there arise certain spaces which, though not groups, have some of the structure of groups. Often, the structure can be expressed in terms of an abstract convolution of measures on the space. The purpose of this paper is to study a class of spaces that have such convolutions. There is no reference to groups in the basic definitions but most of the examples here are related to groups in some way.

The Introduction is devoted to describing one example. The example is not simple, however, and involves a decomposition process that may not be familiar. This decomposition process has an analog in the theory of differential equations: Introduce a symmetry and thereby reduce the number of variables. While the rest of the paper does not make use of differential equations, it is convenient to consider here two problems: a differential equation problem that leads to a simpler differential equation and a problem in the theory of locally compact groups that leads to a space with an abstract convolution of measures.

It should be pointed out that there is no intention to find the most general approach to convolution. The theory arose out of a study of double coset spaces $G \,//\, H = \{HgH : g \in G\}$, where H is a compact subgroup of the locally compact group G; if H is not normal then $G \,//\, H$ does not inherit a multiplication from G, but the space of finite measures on $G \,//\, H$ does inherit a convolution from the measure algebra of G. We have merely abstracted the salient features of these convolution algebras.

We turn now to the two problems. Let $\mathbf{E}_2$ denote Euclidean 2-space.

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PROBLEM 1. Solve the Helmholtz equation in $\mathbf{E}_2$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u = 0.$$

PROBLEM 2. Determine the continuous unitary representations of the group of all orientation-preserving rigid motions of $\mathbf{E}_2$.

These two problems are closely related. One way to see that there is likely to be a connection is to note that the Helmholtz equation is invariant under each change of coordinates corresponding to a member of G, the group in Problem 2. That is, if u is a function on $\mathbf{E}_2$, g is an element of G, and $v = u \bigcirc g$, then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u = \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + v\right) \bigcirc g^{-1}.$$

The relationship between groups and differential equations, including the case just described, is studied in the book by Talman [15], based on the lectures of Eugene P. Wigner.

Let H be the rotation subgroup of G. That is, H consists of the members of G which leave the origin $\mathbf{O}$ fixed. There is a natural correspondence between the elements of $\mathbf{E}_2$ and the left cosets gH of H:

$$z \leftrightarrow \{g \in G : g(\mathbf{O}) = z\}.$$

Therefore, each function f on $\mathbf{E}_2$ determines a function F on G by the rule: $F(g) = f(g(\mathbf{O}))$; the function F is constant on each left coset of H.

There is a simplification which leads to partial solutions of the two problems. It involves the introduction of a symmetry based on the action of the compact group H. For each number $r \geqslant 0$ let

$$C_r = \{z \in E_2 : \| z \| = r\}.$$

These sets are circles (for $r > 0$) and form a decomposition of $\mathbf{E}_2$. Also, they are the orbits of H acting on $\mathbf{E}_2$.

For the first problem, we restrict attention to the functions on $\mathbf{E}_2$ which are rotationally invariant. These are the functions that are constant on the circles C_r . The Helmholtz equation can then be expressed more simply in polar coordinates: $z = (x, y)$, $r = \| z \|$, $u = f(r)$,

$$r^2 \frac{d^2u}{dr^2} + r \frac{du}{dr} + r^2 u = 0.$$

This is Bessel's equation of order 0, with solution $u = J_0(r)$.

For the second problem, we again restrict attention to the rotationally invariant functions on $\mathbf{E}_2$. The functions correspond to the functions on G which are constant on the double cosets HgH of H. This is based on the correspondence between the circles and the double cosets:

$$C_r \leftrightarrow \{g \in G: g(\mathbf{O}) \in C_r\}.$$

Thus, we have functions F on G such that $F(h_1 g h_2) = F(g)$ for all $g \in G$ and $h_1, h_2 \in H$. The functions of this type which are integrable with respect to Haar measure on G form a closed subalgebra A of the convolution algebra $L_1(G)$. Surprisingly enough, A is commutative, even though $L_1(G)$ is not commutative. Moreover, the representation theory of A leads to some, but not all, of the representations of G. In this way the theory of commutative Banach algebras can be used to get a partial solution of Problem 2.

By restricting the scope of the original problems we have produced two new problems: Solve Bessel's equation of order 0 on $\mathbf{R}^+ = [0, \infty)$; determine the representations of the subalgebra A of $L_1(G)$.

It would seem that the two new problems are not related. We have a differential equation (on a new domain) but no group of symmetries which leave it invariant. In simplifying Problem 2 we have not produced a new group, but only a new convolution algebra.

We have not used the fact that the points of $\mathbf{R}^+$ are in a natural one-to-one correspondence with the double cosets of H:

$$r \leftrightarrow \{g \in G: \| g(\mathbf{O}) \| = r\}.$$

Thus, each function in A corresponds to a function on $\mathbf{R}^+$, and these functions on $\mathbf{R}^+$ form an algebra A' with a (nonapparent) convolution for its multiplication. Convolutions are usually defined for functions and measures on groups, but $\mathbf{R}^+$ cannot be given the structure of a topological group. It is true that $\mathbf{R}^+$ is a semigroup under addition, but the convolution corresponding to this operation is not the correct one. This brings us to one of the purposes of this paper.

PROBLEM 3. Define and analyze the group-like structure of $\mathbf{R}^+$ which $\mathbf{R}^+$ inherits from G.

While Problems 1 and 2 are widely known, Problem 3 is not. This does not mean that it has not been raised before. In fact, there exists an elaborate theory designed to deal with just such problems. This theory, initiated by Delsarte [1] and developed mostly by Levitan [5–8] is based on the idea of a generalized translation operation. The idea comes from the translation of functions on groups:

$$(T_x f)(y) = f(x^{-1}y).$$

Brief descriptions of Levitan's theory are given by Naimark [13, pp. 427–430], Loomis [10, pp. 182–183], and Dunford and Schwartz [2, pp. 1622–1628].

The purpose of this paper is to develop another theory of group-like structures. Rather than start with the translation property we start with the convolution of measures. Since the convolution gives rise to translation operators, the objects studied here are special cases in Levitan's theory. But these objects are much more like groups than a typical space with a generalized translation operation; this approach seems to be more appropriate for structures that originate in the theory of locally compact groups.

Recently, Dunkl [17] has defined and studied hypergroups, which are spaces with a convolution of measures. Dunkl's theory is therefore much closer than Levitan's to the one developed here, except that the convolution of a hypergroup is assumed to be commutative. Dunkl also gives various examples for which the underlying space is compact.

In the remainder of the Introduction we merely indicate how the convolution of measures on $\mathbf{R}^+$ is constructed. As for the algebra A', the convolution of functions depends on a Haar measure on $\mathbf{R}^+$ which is uniquely determined by the convolution. The details are given in Section 9.

For any locally compact Hausdorff space X, let $M(X)$ denote the space of complex-valued regular Borel measures on X; for x in X, let p_x denote the unit point mass at x.

On the group G the convolution of unit point masses is quite simple: $p_g * p_h = p_{gh}$. The general convolution $\mu * \nu$ of a pair of measures in $M(G)$ is the (continuous) linearization of the operation on the unit point masses:

$$\begin{aligned}\int_G Fd(\mu * \nu) &= \int_G \int_G F(gh)\, \mu(dg)\, \nu(dh) \\ &= \int_G \int_G \left[\int_G Fd(p_g * p_h)\right] \mu(dg)\, \nu(dh).\end{aligned}$$

For $\mathbf{R}^+$ also it is enough to specify the basic convolutions $p_r * p_s$ for all r and s in $\mathbf{R}^+$. However, if r and s are positive then $p_r * p_s$ is not a point mass, but is a probability measure whose support is a compact interval. No binary operation on $\mathbf{R}^+$ is being used; we can convolve measures but not multiply points.

The convolution on $M(\mathbf{R}^+)$ is inherited from the convolution on $M(G)$. In this case there is a shortcut, which uses the convolution on $M(\mathbf{E}_2)$, based on the structure of $\mathbf{E}_2$ as an additive group: $p_z * p_w = p_{z+w}$ for $z, w \in \mathbf{E}_2$. The group G is not being ignored; the action of H on $\mathbf{E}_2$ will be used, and G is a semidirect product of H and $\mathbf{E}_2$.

There is a one-to-one correspondence between the measures on $\mathbf{R}^+$ and the rotationally invariant measures on $\mathbf{E}_2$. The rotationally invariant measures on $\mathbf{E}_2$ form a subalgebra of $M(\mathbf{E}_2)$ and $M(\mathbf{R}^+)$ is given the structure of this subalgebra.

Let r be an element of $\mathbf{R}^+$. The measure on $\mathbf{E}_2$ corresponding to p_r is by definition the unique rotationally invariant probability measure q_r whose support is equal to the circle C_r. It is apparent that this measure is a multiple of the length measure on C_r. Using appropriate topologies the mapping $p_r \mapsto q_r$ is extended to a (continuous) linear mapping of $M(\mathbf{R}^+)$ into $M(\mathbf{E}_2)$.

Let r and s be in R^+, with $0 < r < s$. Then q_r and q_s are probability measures on C_r and C_s, respectively. Therefore, $q_r * q_s$ is a probability measure supported by the set $C_r + C_s$. It is easily seen that $C_r + C_s$ is an annulus:

$$\begin{aligned}C_r + C_s &= \{z + w \colon \| z \| = r, \| w \| = s\} \\ &= \{z \in \mathbf{E}_2 \colon s - r \leqslant \| z \| \leqslant r + s\} \\ &= \bigcup \{C_t \colon s - r \leqslant t \leqslant r + s\}.\end{aligned}$$

Since $q_r * q_s$ is rotationally invariant, $q_r * q_s$ is a combination (using integrals) of the measures $\{q_t : s - r \leqslant t \leqslant r + s\}$. It turns out then that $p_r * p_s$, which corresponds to $q_r * q_s$, is a probability measure on $\mathbf{R}^+$ with support $[s - r, r + s]$.

With $*$ as its multiplication, $M(\mathbf{R}^+)$ is a commutative Banach algebra. The unit point mass at 0 is the unit of $M(\mathbf{R}^+)$. In a manner analogous to that used for locally compact abelian groups, continuous irreducible representations of $M(\mathbf{R}^+)$ are defined. Each such representation is one-dimensional and corresponds to a bounded continuous function on $\mathbf{R}^+$. These functions will be called multiplicative characters, as in the theory of LCA groups. This brings us back to Problems 1 and 2. One of the multiplicative characters on $\mathbf{R}^+$ is the Bessel function J_0.

2. Locally Compact Hausdorff Spaces

The results in this section are similar to known facts about functions and measures. We include most proofs, however.

2.1. *Notation*

Let X be a locally compact Hausdorff space. The notation below is used throughout the paper.

$\mathbf{R}$	The real numbers
$\mathbf{R}^+$	The nonnegative real numbers
$\mathbf{C}$	The complex numbers
$C(X)$	The continuous complex-valued functions on X
$C_b(X)$, $C_0(X)$, $C_c(X)$	The members of $C(X)$ which are: bounded, zero at infinity, with compact support
$C^+(X)$, $C_b^+(X)$, $C_0^+(X)$, $C_c^+(X)$	Those which are nonnegative.
Borel set	A member of the smallest σ-algebra which contains the open sets
cA	The closure of the set A
$B(X)$	The complex-valued Borel functions on X
$B^\infty(X)$	The Borel functions on X with values in $[0, \infty]$

Lower-semicontinuous	A function with values in $[0, \infty]$ such that $\{x : f(x) > c\}$ is open for all $c \geqslant 0$
i_A	The function equal to 1 on A and equal to 0 on the complement of A
$\|f\|_u$	$\sup \|f(x)\|$
$M(X)$	The regular complex-valued Borel measures on X
$M^+(X)$, $M_c(X)$, $M_c^+(X)$	Those which are: non-negative, with compact support, both
$M^\infty(X)$	The regular Borel measures on X with values in $[0, \infty]$
p_x	The unit point mass at x
spt μ	The support of the measure μ
spt f	The support of the function f
pos f	For nonnegative f, the set $\{x : f(x) > 0\}$
$\int f(x)\, \mu(dx)$	$\int f\, d\mu$
σ-finite function	With respect to a given measure, a function which is 0 off a σ-finite Borel set
$f\mu$	The measure, if it exists, such that $\int g\, d(f\mu) = \int gf\, d\mu$, for all g in $C_c(X)$

2.2. *The Cone Topology*

Let X be a locally compact Hausdorff space. The *cone topology* on $M^+(X)$ is the weakest topology such that, for each $f \in C_c^+(X)$, the mapping $\mu \mapsto \int_X f\, d\mu$ is continuous, and such that the mapping $\mu \mapsto \mu(X)$ is continuous. This is equal to the weak-* topology if and only if X is compact.

Throughout this paper, an unspecified topology on M^+ is the cone topology. References to the norm are explicit.

LEMMA 2.2A. *The set of measures in $M^+(X)$ with finite support is dense in $M^+(X)$.*

LEMMA 2.2B. *The mapping $x \mapsto p_x$ is a homeomorphism of X onto a closed subset of $M^+(X)$.*

LEMMA 2.2C. *Let D be a directed set. Let $\{\mu_\beta\}_{\beta \in D}$ be a net in $M^+(X)$ converging to μ. Let $\{f_\beta\}_{\beta \in D}$ be a net in $C_b{}^+(X)$ converging to f uniformly on compact subsets of X. Suppose that the numbers $\| f_\beta \|_u$ are bounded. Then*

$$\lim_\beta f_\beta \mu_\beta = f\mu.$$

Proof. The first two lemmas are obvious. For the third, let $N = \sup \| f_\beta \|_u$ and let $\epsilon > 0$. Let A be a compact subset of X such that $\mu(X - A) < \epsilon/2N$. Choose $g \in C_c{}^+(X)$ such that $g \leqslant 1$ on X and $g = 1$ on A. Then

$$\int f_\beta \, d\mu_\beta - \int f \, d\mu = \int gf \, d\mu_\beta - \int gf \, d\mu - \int g(f - f_\beta) \, d\mu_\beta$$
$$+ \int (1 - g) f_\beta \, d\mu_\beta - \int (1 - g) f \, d\mu.$$

Since g has compact support, it follows that

$$\limsup_\beta \left| \int f_\beta \, d\mu_\beta - \int f \, d\mu \right| \leqslant 2N \int (1 - g) \, d\mu < \epsilon.$$

The rest is straightforward.

THEOREM 2.2D. *Let L: $M(X) \to \mathbf{C}$ be a linear mapping. Then L is continuous on $M^+(X)$ if and only if there exists a bounded continuous function h on X such that*

$$L(\mu) = \int_X h \, d\mu$$

for each $\mu \in M(X)$.

Proof. If h is a bounded continuous function on X and L is defined as above, then L is continuous, by Lemma 2.2C.

Assume that L is continuous. Let h be defined by $h(x) = L(p_x)$. By Lemma 2.2B, h is continuous. Also, $L(\mu) = \int h \, d\mu$ for all measures with finite support. If h is bounded then, by Lemmas 2.2A and 2.2C,

$L(\mu) = \int h\, d\mu$ for all $\mu \in M^+(X)$, and hence for all $\mu \in M(X)$. Assume that h is unbounded. For each $n \geqslant 1$ there exists $x_n \in X$ such that $|h(x_n)| \geqslant n$. Let $\mu_n = (1/h(x_n))\, p_{x_n}$. Thus, $\mu_n \to 0$ and $L(\mu_n) = 1$ for all n. This is a contradiction, since L is continuous.

2.3. *Positive-Continuous Linear Mappings*

Let X and Y be locally compact Hausdorff spaces and let $\mu \mapsto \mu'$ be a linear mapping from $M(X)$ to $M(Y)$. This mapping will be called *positive-continuous* if:

(I) $\mu' \geqslant 0$ when $\mu \geqslant 0$;

(II) The restricted mapping, from $M^+(X)$ to $M^+(Y)$, is continuous.

We assume in this subsection that these conditions are satisfied. If g is a Borel function on Y then g' is defined on X by

$$g'(x) = \int_Y g\, d(p_x'),$$

whenever this integral exists.

LEMMA. *Let $\mu \in M(X)$ and $g \in C_b(Y)$.*

(2.3A) *The number $N = \sup \| p_x' \|$ is finite.*

(2.3B) $\| \mu' \| \leqslant N \| \mu \|$.

(2.3C) *g' is continuous and $\| g' \|_u \leqslant N \| g \|_u$.*

(2.3D) $\int_Y g\, d\mu' = \int_X g'\, d\mu$.

This lemma follows readily from Theorem 2.2D. In view of the lemma we may write

$$\mu' = \int_X p_x' \mu(dx),$$

regarding μ' as the integral of a measure-valued function on X. Sometimes it is possible to extend the mapping to an infinite nonnegative measure. Let $m \in M^\infty(X)$. Suppose that $\int g'\, dm$ is finite for all g in $C_c^+(Y)$. By the Riesz Representation Theorem there exists a unique measure m' in $M^\infty(Y)$ such that

$$\int_Y g\, dm' = \int_X g'\, dm$$

for all $g \in C_c^+(Y)$. In this case also we use the notation

$$m' = \int_X p_x' m(dx).$$

THEOREM. *Let m be a nonnegative measure on X and suppose that m' is defined.*

(2.3E) *If g is a lower-semicontinuous function on Y then g' is lower-semicontinuous on X and $\int g\, dm' = \int g'\, dm$.*

(2.3F) *If $g \in B^\infty(Y)$ then $g' \in B^\infty(X)$.*

(2.3G) *If $g \in B^\infty(Y)$ and g is σ-finite with respect to m' then g' is σ-finite with respect to m and $\int g\, dm' = \int g'\, dm$.*

Proof. Part E is apparent. We shall prove 2.3F and 2.3G together. This imposes no additional restriction on g since m could be a finite measure.

Our proof is essentially the proof by Karl Stromberg [15] of a closely related result for groups. See Hewitt and Ross [4, p. 727].

By the Monotone Convergence Theorem there is no loss of generality in assuming that $g = i_A$, where A is a Borel subset of Y and $m'(A)$ is finite. We must show that i_A' is a Borel function on X and that $\int i_A'\, dm = m'(A)$.

Let U be an open subset of Y such that $A \subset U$ and $m'(U)$ is finite. Let Σ be the collection of all Borel sets B such that $B \subset U$, i_B' is a Borel function, and $\int i_B'\, dm = m'(B)$. Now U is in Σ by (2.3E), and $i_B' \leqslant i_U' \leqslant N$ on X for all B in Σ, where N is given by (2.3A). Note that Σ contains each open subset of U.

Let Σ_0 be a subcollection of Σ which is maximal with respect to the properties of containing the open subsets of U and being closed under finite intersection. Let $E \in \Sigma_0$ and let

$$\Sigma_E = \{(B \cap E) \cup (C \cap F) : B, C \in \Sigma_0\},$$

where $F = U - E$. Thus Σ_E contains Σ_0 and is closed under finite intersection. Moreover, Σ_E is contained in Σ since

$$i_{(B \cap E) \cup (C \cap F)} = i_{B \cap E} + i_C - i_{C \cap E}\,.$$

Thus $\Sigma_E = \Sigma_0$ and $F \in \Sigma_0$. Hence Σ_0 is an algebra of subsets of U. Now let Σ_1 be the collection of all countable unions of members of Σ_0. Then $\Sigma_1 = \Sigma_0$, by the maximality of Σ_0 and the Monotone Convergence

Theorem. Thus Σ_0 is a σ-algebra. It follows that $\Sigma_0 = \Sigma$ and that $A \in \Sigma$.

The following lemma is apparent.

LEMMA 2.3H. *Let $x \mapsto \omega_x$ be a continuous mapping from X to $M^+(Y)$. Suppose that the numbers $\| \omega_x \|$ are bounded. Then the mapping $p_x \mapsto \omega_x$ has a unique extension to a positive-continuous linear mapping from $M(X)$ to $M(Y)$.*

2.4. *Positive-Continuous Bilinear Mappings*

Let X, Y and Z be locally compact Hausdorff spaces. Let $(\mu, \nu) \mapsto \mu \ast \nu$ be a bilinear mapping from $M(X) \times M(Y)$ to $M(Z)$. This mapping will be called *positive-continuous* if:

(I) $\mu \ast \nu \geqslant 0$ when $\mu \geqslant 0$ and $\nu \geqslant 0$;

(II) The restricted mapping, from $M^+(X) \times M^+(Y)$ to $M^+(Z)$, is continuous.

We assume in this subsection that these conditions are satisfied. The following statements are easily verified.

LEMMA 2.4A. *There exists a unique positive-continuous linear mapping $\pi \mapsto \pi'$ from $M(X \times Y)$ to $M(Z)$ such that $\mu \ast \nu = (\mu \times \nu)'$ for $\mu \in M(X)$ and $\nu \in M(Y)$.*

We refer to Hewitt and Ross for the properties of product measures on locally compact spaces. In view of Lemma 2.4A, the formulation

$$\mu \ast \nu = \int_Y \int_X (p_x \ast p_y)\, \mu(dx)\, \nu(dy)$$

may be used. If $\mu \in M^+(X)$ and $m \in M^\infty(Y)$ then $\mu \ast m$ is given by $\mu \ast m = (\mu \times m)'$, if this latter measure is defined. Note that, if $\mu \ast m$ is defined, then

$$\mu \ast m = \int_Y (\mu \ast p_y)\, m(dy).$$

LEMMA 2.4B. *Let $(x, y) \mapsto \omega_{x,y}$ be a continuous mapping from $X \times Y$ to $M^+(Z)$. Suppose that the numbers $\| \omega_{x,y} \|$ are bounded. Then $(p_x, p_y) \mapsto \omega_{x,y}$ has a unique extension to a positive-continuous bilinear mapping from $M(X) \times M(Y)$ to $M(Z)$.*

2.5. *The Space of Compact Subsets*

Let X be a nonvoid locally compact Hausdorff space. Let $\mathscr{C}(X)$ denote the collection of all nonvoid compact subsets of X. If A and B are subsets of X let $\mathscr{C}_A(B)$ be the collection of all C in $\mathscr{C}(X)$ such that $C \cap A$ is nonvoid and $C \subset B$. We give $\mathscr{C}(X)$ the topology generated by the subbasis of all $\mathscr{C}_U(V)$ for which U and V are open subsets of X. This topology is thoroughly examined by Michael [12]; note pp. 161–162. We review briefly.

(2.5A) *$\mathscr{C}(X)$ is a locally compact Hausdorff space.*

(2.5B) *If X is compact then $\mathscr{C}(X)$ is compact.*

(2.5C) *If Y is a subspace of X then $\mathscr{C}(Y)$ is a subspace of $\mathscr{C}(X)$, and if Y is closed then $\mathscr{C}(Y)$ is closed.*

(2.5D) *The mapping $x \mapsto \{x\}$ is a homeomorphism from X onto a closed subset of $\mathscr{C}(X)$.*

(2.5E) *The collection of nonvoid finite subsets of X is dense in $\mathscr{C}(X)$.*

(2.5F) *If Ω is a compact subset of $\mathscr{C}(X)$ then $B = \bigcup_{A\in\Omega} A$ is a compact subset of X.*

A proof of 2.5F is as follows. Let Σ be a collection of open subsets of X which covers B. Let Σ' be the collection of all unions of finite subcollections of Σ. Thus, if $A \in \Omega$ then $A \subset V$ for some $V \in \Sigma'$. Hence $\{\mathscr{C}(V) : V \in \Sigma'\}$ is an open cover of Ω. There exists a finite subcover $\{\mathscr{C}(V_i)\}$. But then the V_i cover B.

3. Semiconvos

A pair $(K, *)$ will be called a *semiconvo* if the following five conditions are satisfied:

(I) K is a nonvoid locally compact Hausdorff space.

(II) The symbol $*$ denotes a binary operation on $M(K)$, and with this operation $M(K)$ is a complex (associative) algebra.

(III) The bilinear mapping $(\mu, \nu) \mapsto \mu * \nu$ is positive-continuous.

(IV) If $x, y \in K$ then $p_x * p_y$ is a probability measure with compact support.

(V) The mapping $(x, y) \mapsto \operatorname{spt}(p_x * p_y)$ from $K \times K$ to $\mathscr{C}(K)$ is continuous.

In view of 2.4B, a semiconvo is determined by the measures $p_x * p_y$, and when considering examples we need only specify these measures. As for associativity, this can be checked by verifying that

$$p_x * (p_y * p_z) = (p_x * p_y) * p_z ,$$

which is the same as

$$\int_K (p_x * p_t)(p_y * p_z)(dt) = \int_K (p_t * p_z)(p_x * p_y)(dt).$$

The connection between semiconvos and semigroups is illustrated by the following two propositions. They need no proof.

PROPOSITION 1. *Let $(K, *)$ be a semiconvo. Suppose that for each pair of points x, y in K there exists a point $x \cdot y$ in K such that $p_x * p_y = p_{x \cdot y}$. Then $(K, \cdot)$ is a locally compact topological semigroup.*

PROPOSITION 2. *Let $(S, \cdot)$ be a locally compact semigroup. Let $*$ denote the standard convolution on $M(S)$, defined by: $\int f\, d(\mu * \nu) = \iint f(xy)\, \mu(dx)\, \nu(dy)$. Then $(S, *)$ is a semiconvo.*

In the remainder of Section 3 it is assumed that $(K, *)$ is a semiconvo.

3.1. *Translation of Functions*

If f is a Borel function on K and $x, y \in K$ then we define

$$f(x * y) = f_x(y) = f^y(x) = \int_K f\, d(p_x * p_y),$$

if this integral exists, though it need not be finite. Note that $f(x * y) = f(x \cdot y)$ if K is a semigroup. The following results are readily proved using (2.3) and (2.4). In the notation of those subsections, $f(x * y) = f'(x, y)$.

LEMMA. *Let f be a continuous function on K and let $x \in K$.*

(3.1A) *The mapping $(s, t) \mapsto f(s * t)$ is a continuous function on $K \times K$.*

(3.1B) *f_x and f^x are continuous functions on K.*

LEMMA. *Let $f \in B^\infty(K)$, let $\mu, \nu \in M^+(K)$, and let $x, y, z \in K$.*

(3.1C) *The mapping* $(s, t) \mapsto f(s * t)$ *is a Borel function on* $K \times K$.

(3.1D) f_x *and* f^x *are Borel functions on* K.

(3.1E) $\int_K f\, d(\mu * \nu) = \int_K \int_K f(s * t)\, \mu(ds)\, \nu(dt)$.

(3.1F) $\int_K f_x\, d\mu = \int_K f\, d(p_x * \mu)$.

(3.1G) $f_x(y * z) = f^z(x * y)$.

3.2. *Convolution of Sets*

If A and B are subsets of K then the set $A * B$ is defined by

$$A * B = \bigcup_{\substack{x \in A \\ y \in B}} \operatorname{spt}(p_x * p_y).$$

Note that $A * B = A \cdot B$ if K is a semigroup.

LEMMA. *Let* A, B *and* C *be subsets of* K.

(3.2A) $(cA) * (cB) \subset c(A * B)$.

(3.2B) *If* A *and* B *are compact then* $A * B$ *is compact.*

(3.2C) *Convolution is a continuous operation on* $\mathscr{C}(K)$.

(3.2D) *If* A *and* B *are compact and* U *is an open set containing* $A * B$ *then there exist open sets* V *and* W *such that* $A \subset V$, $B \subset W$, *and* $V * W \subset U$.

(3.2E) $(A * B) * C = A * (B * C)$.

Proof. Recall that cA is the closure of A. Also, 3.2B follows from 2.5F. The only thing needing proof in this subsection is 3.2C.

It is enough to consider subbasic open sets. Let V and W be open subsets of K and let

$$\Sigma = \{(A, B): A * B \in \mathscr{C}_V(W)\}.$$

We must show that Σ is open in $\mathscr{C}(K) \times \mathscr{C}(K)$. Let

$$P = \{x, y): \operatorname{spt}(p_x * p_y) \in \mathscr{C}_V(K)\}$$

$$Q = \{(x, y): \operatorname{spt}(p_x * p_y) \in \mathscr{C}(W)\}.$$

So P and Q are open subsets of $K \times K$. And Σ is the union of all $\mathscr{C}_R(S) \times \mathscr{C}_T(U)$ for which R, S, T, U are open in K and $R \times T \subset P$, $S \times U \subset Q$.

LEMMA. *Let* $\mu, \nu \in M^+(K)$.

(3.2F) $\operatorname{spt}(\mu * \nu) = c((\operatorname{spt}\mu) * (\operatorname{spt}\nu))$.

(3.2G) *If* μ *and* ν *have compact support then* $\mu * \nu$ *has compact support and* $\operatorname{spt}(\mu * \nu) = (\operatorname{spt}\mu) * (\operatorname{spt}\nu)$.

3.3. *Subinvariant and Invariant Measures*

A measure m in $M^\infty(K)$ will be called *left-subinvariant* if $p_x * m$ is defined and $p_x * m \leqslant m$ for each x in K. A measure m in $M^\infty(K)$ will be called *left-invariant* if $p_x * m$ is defined and $p_x * m = m$ for each x in K. Thus, by definition, 3.3A and 3.3F are valid for all f in $C_c^+(K)$. Note that the natural analog of 3.3B for left-invariant measures ($=$ replacing $\leqslant$) is not valid. This is so because the measures $p_x * p_y$ are not point masses in general. For example, in 9.1D the function $f = \xi$ is not identically zero, but $f_b = 0$, since $f_b(x) = f(b * x) = f(b)f(x) = 0$ for all $x \in K$. This does not contradict 3.3F since $\int f\,dm = 0$.

LEMMA. *Let* m *be a left-subinvariant measure on* K. *Let* $f \in B^\infty(K)$, $x \in K$, *and* $\mu, \nu \in M^+(K)$.

(3.3A) $\int_K f_x\,dm \leqslant \int_K f\,dm$.

(3.3B) *If* $1 \leqslant p \leqslant \infty$ *then* $\|f_x\|_p \leqslant \|f\|_p$.

(3.3C) *If* A *is a compact subset of* K *then* $m(A) \leqslant m(\{x\} * A)$.

(3.3D) *The measure* $\mu * m$ *is defined and* $\mu * m \leqslant \mu(K)m$.

(3.3E) *If* ν *is absolutely continuous with respect to* m *then* $\mu * \nu$ *is absolutely continuous with respect to* m.

Proof A. This follows from 2.3G if f is integrable.

Proof B. Suppose that $p < \infty$. If $y \in K$ then $[f(x * y)]^p \leqslant f^p(x * y)$, by Hölder's Inequality. Hence $(f_x)^p \leqslant (f^p)_x$ on K. Thus $\|f_x\|_p \leqslant \|f\|_p$.

Suppose that $p = \infty$ and that $\|f\|_\infty < \infty$. Then $f = g + h$, where h is locally null and $g \leqslant \|f\|_\infty$ on K. It is enough to see that h_x is locally null. Let A be a compact subset of K and let $B = \{x\} * A$, which is also compact. Then $h_x = (i_B h)_x$ on A and

$$\int_A h_x\,dm \leqslant \int_K (i_B h)_x\,dm \leqslant \int_K i_B h\,dm = \int_B h\,dm = 0.$$

Proof C. Let $B = \{x\} * A$ and let $\epsilon > 0$. Choose $f \in C_c^+(K)$

such that $f = 1$ on B and $\int f\,dm < m(B) + \epsilon$. Then $f_x = 1$ on A and $m(A) \leqslant \int f_x\,dm \leqslant \int f\,dm < m(B) + \epsilon$.

Proof D. If $f \in C_c^+(K)$ then

$$\iint f(x * y)\,\mu(dx)\,m(dy) = \iint f_x(y)\,m(dy)\,\mu(dx)$$

$$\leqslant \iint f(y)\,m(dy)\,\mu(dx)$$

$$= \mu(K) \int f\,dm.$$

Proof E. There exist measures $\nu_n \in M^+(K)$ and positive numbers c_n such that each $\nu_n \leqslant c_n m$ and $\nu = \nu_1 + \nu_2 + \cdots$. Thus $\mu * \nu = \sum \mu * \nu_n$ and this sum converges in norm. Moreover, each $\mu * \nu_n$ is absolutely continuous with respect to m since $\mu * \nu_n \leqslant c_n \mu(K) m$.

THEOREM. *Let m be a left-invariant measure on K. Let $f \in B^\infty(K)$, $x \in K$, and $\mu \in M^+(K)$.*

(3.3F) *If f is σ-finite with respect to m then f_x is σ-finite with respect to m, and*

$$\int_K f_x\,dm = \int_K f\,dm.$$

(3.3G) $\mu * m = \mu(K)m$.

Proof. These are similar to 3.3A and 3.3D.

3.4. *Involutions*

Let X be a nonvoid locally compact Hausdorff space. A mapping $x \mapsto x^-$ will be called a *topological involution* of X if it is a homeomorphism of X and $(x^-)^- = x$ for all x in X. Let such a mapping be given. If f is a function on X, A is a subset of X, and μ is a Borel measure on X then f^-, A^- and μ^- are defined by

$$f^-(x) = f(x^-) \qquad A^- = \{x^- : x \in A\} \qquad \mu^-(B) = \mu(B^-).$$

Note that if $f \in B^\infty(X)$ and $\mu \in M^\infty(X)$ then

$$\int_X f^-\,d\mu = \int_X f\,d\mu^-.$$

A mapping $x \mapsto x'$ will be called an *involution* of the semiconvo K if it is a topological involution of K and

$$(\mu * \nu)^- = \nu^- * \mu^-$$

for all μ and ν in $M(K)$. Note that it is enough to check that $(p_x * p_y)^- = p_{y^-} * p_{x^-}$ for all $x, y \in K$. The following results are apparent.

LEMMA. *Let $x \mapsto x^-$ be an involution of K.*

(3.4A) *The mapping $\mu \mapsto \mu^-$, from $M(K)$ to $M(K)$, is linear and positive-continuous.*

(3.4B) *If $f \in B^\infty(K)$ and $x, y \in K$ then $f^-(x * y) = f(y^- * x^-)$.*

(3.4C) *If A and B are subsets of K then $(A * B)^- = B^- * A^-$.*

(3.4D) *If m is a left-invariant measure on K then m^- is right-invariant.*

4. CONVOS

A pair $(K, *)$ will be called a *convo* if the following three conditions are satisfied:

(I) $(K, *)$ is a semiconvo.

(II) There exists a (necessarily unique) element e of K such that $p_e * p_x = p_x = p_x * p_e$ for all x in K.

(III) There exists a (necessarily unique) involution $x \mapsto x^-$ of K such that (for $x, y \in K$) the element e is in the support of $p_x * p_y$ if and only if $x = y^-$.

The connection between convos and groups is illustrated by the following two propositions.

PROPOSITION 1. *Let $(K, *)$ be a convo. Suppose that, for each pair of points x, y in K, there exists a point $x \cdot y$ in K such that $p_x * p_y = p_{x \cdot y}$. Then $(K, \cdot)$ is a locally compact group.*

PROPOSITION 2. *Let $(G, \cdot)$ be a locally compact group. If $*$ denotes the standard convolution on G then $(G, *)$ is a convo.*

In the remainder of Section 4 it is assumed that $(K, *)$ is a convo.

The element e will be called the *identity* of K. For x in K, the element x^- will be called the *adjoint* of x.

4.1. *Convolution of Sets*

If $\{x_\beta\}_{\beta \in D}$ is a net in K then the expression $x_\beta \to \infty$ means that $x_\beta \in K\text{–}A$ eventually, for each compact subset A of K. If $\{A_\beta\}_{\beta \in D}$ is a net in $\mathscr{C}(K)$ then the expression $A_\beta \to \{\infty\}$ means that $A_\beta \subset K\text{–}A$ eventually, for each compact subset A of K. Note that $A_\beta \to \infty$ and $A_\beta \to \{\infty\}$ have different meanings.

LEMMA. *Let A, B and C be subsets of K.*

(4.1A) *$A^- * B$ contains e if and only if $A \cap B$ is nonvoid.*

(4.1B) *$(A * B) \cap C$ is nonvoid if and only if $B \cap (A^- * C)$ is nonvoid.*

(4.1C) *Let $\{A_\beta\}_{\beta \in D}$ and $\{B_\beta\}_{\beta \in D}$ be nets in $\mathscr{C}(K)$. If $A_\beta \to A$ and $B_\beta \to \{\infty\}$ then $A_\beta * B_\beta \to \{\infty\}$.*

(4.1D) *If B is open then $A * B$ is open, and $(cA) * B = A * B$.*

(4.1E) *If A is compact and B is closed then $A * B$ is closed.*

Proof. We shall prove only 4.1B and 4.1D. For 4.1B, assume that $(A * B) \cap C$ is nonvoid. By 4.1A, $C^- * (A * B)$ contains e. By 3.2E and 3.4C, $C^- * (A * B) = (A^- * C)^- * B$. The rest is clear.

For 4.1D, let $a \in A$. Then $x \in \{a\} * B$ if and only if $\{a^-\} * \{x\}$ is an element of $\mathscr{C}_B(K)$. Thus $\{a\} * B$ is an open subset of K. Hence $A * B$ is open. Now, let $x \in (cA) * B$. Then $(cA)^- * \{x\}$ meets B. Thus $A^- * \{x\}$ meets B, by (3.2A). Hence, $x \in A * B$.

4.2. *Convolution of Functions and Measures*

Let $\mu \in M^+(K)$ and $f \in B^\infty(K)$. Then the *convolutions* $\mu * f$ and $f * \mu$ are defined on K by

$$(\mu * f)(x) = \int_K f(y^- * x)\, \mu(dy)$$

$$(f * \mu)(x) = \int_K f(x * y^-)\, \mu(dy).$$

LEMMA. *Let $\mu \in M^+(K)$ and $f \in C_b{}^+(K)$.*

(4.2A) *$\mu * f$ is continuous.*

(4.2B) $\| \mu * f \|_u \leqslant \| \mu \| \cdot \| f \|_u$.

LEMMA. *Let $\mu \in M^+(K)$ and let f be lower-semicontinuous on K.*

(4.2C) $\mu * f$ *is lower-semicontinuous.*

(4.2D) $\operatorname{pos}(\mu * f) = (\operatorname{spt}\mu) * (\operatorname{pos} f)$.

LEMMA. *Let $\mu \in M^+(K)$ and $f \in C_0{}^+(K)$.*

(4.2E) $\mu * f \in C_0{}^+(K)$.

(4.2F) *If $\mu_\beta \to \mu$ in $M^+(K)$ then* $\lim \| \mu_\beta * f - \mu * f \|_u = 0$.

Proof. See 2.3C and 2.3E. For 4.2F, note that if μ and f have compact support then $\mu * f$ has compact support.

LEMMA. *Let $\mu, \nu \in M^+(K)$ and $f \in B^\infty(K)$.*

(4.2G) $\mu * f$ *is a Borel function.*

(4.2H) $\int_K (\mu^- * f)\, d\nu = \int_K f\, d(\mu * \nu)$.

(4.2I) $\mu * (\nu * f) = (\mu * \nu) * f$.

(4.2J) $\mu * (f * \nu) = (\mu * f) * \nu$.

(4.2K) $(\mu * f)^- = f^- * \mu^-$.

Proof. By 2.3F, $\mu * f$ is a Borel function. Also,

$$\int (\mu^- * f)\, d\nu = \int\int f(y^- * x)\, \mu^-(dy)\, \nu(dx)$$

$$= \int\int f(y * x)\, \mu(dy)\, \nu(dx) = \int f\, d(\mu * \nu).$$

Now, let $x \in K$ and let $\pi = p_x$.

For 4.2I,

$$[\mu * (\nu * f)](x) = \int [\mu * (\nu * f)]\, d\pi = \int (\nu * f)\, d(\mu^- * \pi)$$

$$= \int f\, d(\nu^- * \mu^- * \pi) = \int [(\mu * \nu) * f]\, d\pi.$$

For 4.2J,

$$[\mu * (f * \nu)](x) = \int [\mu * (f * \nu)]\, d\pi$$

$$= \int (f * \nu)\, d(\mu^- * \pi) = \int f\, d(\mu^- * \pi * \nu^-).$$

For 4.2K,

$$(\mu * f)^{-}(x) = (\mu * f)(x^{-}) = \int f(y^{-} * x^{-})\,\mu(dy)$$

$$= \int f^{-}(x * y)\,\mu(dy) = \int f^{-}(x * z^{-})\,\mu^{-}(dz).$$

LEMMA 4.2L. *Let $\mu \in M^{+}(K)$ and $m \in M^{\infty}(K)$. If μ has compact support then $\mu * m$ is defined, and* $\operatorname{spt}(\mu * m) = (\operatorname{spt} \mu) * (\operatorname{spt} m)$.

Proof. This is straightforward. Note 4.1E.

4.3. *Existence of a Subinvariant Measure*

The proof of the following theorem is adapted from Weil's proof, as given by Loomis [11], of the existence of an invariant measure on a locally compact group. However, the conclusion here is weaker. On a group, subinvariant implies invariant. On a convo, it does not. See Section 9.5.

LEMMA 4.3A. *Let f and k be in $C_c^{+}(K)$. Suppose that $k \neq 0$. Then there exists $\mu \in M_c^{+}(K)$ such that $f \leqslant \mu * k$.*

Proof. Choose $a \in K$ such that $k(a) > 0$. One readily sees that, if $x \in K$, then $(p_x * p_{a^-} * k)(x) > 0$. Thus, μ can be chosen to be a finite linear combination of measures of the form $p_x * p_{a^-}$.

LEMMA 4.3B. *Let $f \in C_c^{+}(K)$ and let $\epsilon > 0$. Then there exists an open neighborhood W of e with the following property: If $x, y \in K$ and $(p_{x^-} * p_y)(W) > 0$ then $|f(x) - f(y)| < \epsilon$.*

Proof. This follows from 4.1C.

THEOREM 4.3C. *There exists a measure m in $M^{\infty}(K)$ such that m is left-subinvariant and the support of m is equal to K.*

Proof. We make the following definition. If $f, k \in C_c^{+}(K)$ and $k \neq 0$ let

$$[f, k] = \inf\{\mu(K) : \mu \in M_c^{+}(K) \text{ and } f \leqslant \mu * k\}.$$

We now prove several lemmas.

LEMMA 4.3D. *Let $\mu \in M_c^+(K)$, $c \geqslant 0$, and $f, g, k \in C_c^+(K)$. Suppose that $k \neq 0$. Then*

$$\begin{aligned}
[\mu * f, k] &\leqslant \mu(K)[f, k], \\
[f + g, k] &\leqslant [f, k] + [g\ k], \\
[cf, k] &= c[f, k], \\
[f, k] &\leqslant [g, k] && \text{if } f \leqslant g, \\
[f, k] &\leqslant [f, g][g, k] && \text{if } g \neq 0, \\
[f, k] &> 0 && \text{if } f \neq 0.
\end{aligned}$$

Proof. For the first, if $f \leqslant \nu * k$ then $\mu * f \leqslant \mu * \nu * k$, and $[\mu * f, k] \leqslant (\mu * \nu)(K) = \mu(K)\,\nu(K)$. For the last, if $f \leqslant \mu * k$ then $\|f\|_u \leqslant \|\mu\| \cdot \|k\|_u = \mu(K)\,\|k\|_u$. The others are clear. The next lemma follows readily.

DEFINITION. Let F be a fixed nonzero element of $C_c^+(K)$. If $f, k \in C_c^+(K)$ and $k \neq 0$ then set

$$I_k f = \frac{[f, k]}{[F, k]}.$$

LEMMA 4.3E. *Let $\mu \in M_c^+(K)$, $c \geqslant 0$, and $f, g, k \in C_c^+(K)$. Suppose that $k \neq 0$. Then*

$$\begin{aligned}
I_k(\mu * f) &\leqslant \mu(K)\, I_k f, \\
I_k(f + g) &\leqslant I_k f + I_k g, \\
I_k(cf) &= c I_k f, \\
I_k f &\leqslant I_k g \qquad \text{if } f \leqslant g.
\end{aligned}$$

Moreover, if $f \neq 0$ then

$$\frac{1}{[F, f]} \leqslant I_k f \leqslant [f, F].$$

LEMMA 4.3F. *Let $f_1, f_2 \in C_c^+(K)$ and let $\epsilon > 0$. Then there exists an open neighborhood W of e with the following property: If $k \in C_c^+(K)$, $k \neq 0$, and $k = 0$ off W, then*

$$I_k f_1 + I_k f_2 < I_k(f_1 + f_2) + \epsilon.$$

Proof. Let $S = \operatorname{spt}(f_1 + f_2)$. Let V be an open set containing S such that cV is compact. Choose $g \in C_c^+(K)$ such that $g = 1$ on V. Let $a > 0$ and let $b = 3a + 2a^2$.

Let $h = f_1 + f_2 + ag$. Thus, $h \geqslant a$ on V. Define g_1 and g_2 by letting g_i equal f_i/h on V, and 0 elsewhere. Thus each g_i is continuous and is zero off S. Also, $g_1 + g_2 \leqslant 1$ on K.

By 4.3B, there exist open neighborhoods W_i of e such that $|g_i(x) - g_i(y)| < a$ when $(p_{x^-} * p_y)(W_i) > 0$. Let $W = W_1 \cap W_2$. Suppose that $k \in C_c^+(K)$, $k \neq 0$, and that $k = 0$ off W. Choose $\mu \in M_c^+(K)$ such that $h \leqslant \mu * k$ and such that $\mu(K) \leqslant (1 + a)[h, k]$.

If $x, y \in K$ and $k(x^- * y) > 0$ then $g_i(y) < a + g_i(x)$. Thus, if $y \in K$ then

$$
\begin{aligned}
f_i(y) = g_i(y)\, h(y) &\leqslant g_i(y)(\mu * k)(y) \\
&= \int g_i(y)\, k(x^- * y)\, \mu(dx) \\
&\leqslant \int [a + g_i(x)]\, k(x^- * y)\, \mu(dx) \\
&= \int k(x^- * y)[(a + g_i)\mu](dx) \\
&= ([(a + g_i)\mu] * k)(y).
\end{aligned}
$$

It follows that $[f_i, k] \leqslant \int (a + g_i)\, d\mu$. Combining we have

$$
\begin{aligned}
[f_1, k] + [f_2, k] &\leqslant \int (2a + g_1 + g_2)\, d\mu \\
&\leqslant (2a + 1)\, \mu(K) \\
&\leqslant (2a + 1)(a + 1)[h, k] \\
&= (1 + b)[h, k].
\end{aligned}
$$

After dividing by $[F, k]$ we have

$$
\begin{aligned}
I_k f_1 + I_k f_2 &\leqslant (1 + b)\, I_k(f_1 + f_2 + ag) \\
&\leqslant (1 + b)\, I_k(f_1 + f_2) + a(1 + b)\, I_k g \\
&< I_k(f_1 + f_2) + \epsilon,
\end{aligned}
$$

if a is sufficiently small.

Completion of Proof. Choose an appropriate net of functions $\{k_\beta\}_{\beta \in D}$ such that each $k_\beta \neq 0$, such that spt $k_\beta \to \{e\}$, and such that the functions I_{k_β} converge pointwise on the set $C_c^+(K)$. Let $J = \lim_\beta I_{k_\beta}$. Then J is nonnegative and semilinear. Moreover, if $f \neq 0$ then $Jf \geqslant 1/[F, f] > 0$. Also, if $\mu \in M_c^+(K)$ and $f \in C_c^+(K)$ then $J(\mu * f) \leqslant \mu(K)\, Jf$. By the Riesz Representation Theorem, there exists a unique m in $M^\infty(K)$ such that $Jf = \int f\, dm$ for all $f \in C_c^+(K)$. Also, the support of m is equal to K. Finally, if $f \in C_c^+(K)$ then

$$\int_K f\, d(p_x * m) = \int_K (p_{x^-} * f)\, dm \leqslant \int_K f\, dm.$$

5. Haar Measure

A nonzero left-invariant measure on a convo will be called a *left Haar measure.*

Conjecture. Every convo has a left Haar measure.

We shall see later that discrete convos, compact convos, and double coset convos have Haar measures.

In this section it is assumed that K is a convo and that m is a left Haar measure on K.

5.1. *The Adjoint Property*

In most computations it is the equation in 5.1D rather than the left-invariance of m that is used. On a group, of course, the equivalence of the two is obvious.

It follows from 5.1D that the mapping $f \mapsto f_{x^-}$ is the adjoint of the mapping $f \mapsto f_x$, both mappings being bounded linear operators on $L_2(m)$.

Lemma 5.1A. *The support of m is equal to K.*

Lemma. *Let $\{k_\beta\}_{\beta \in D}$ be a net in $C_c^+(K)$ such that each $\int k_\beta\, dm = 1$, and such that* spt $k_\beta \to \{e\}$.

(5.1B) *If $f \in C_c^+(K)$ then* $\lim_\beta \|(fm) * k_\beta^- - f\|_u = 0$.

(5.1C) *If $\mu \in M^+(K)$ then* $\lim_\beta (\mu * k_\beta) m = \mu$.

Proof. The first is clear. For 5.1B, let $\epsilon > 0$. By (4.3B), there exists

$\beta_0 \in D$ with the following property: If $x, y \in K$, $\beta \geqslant \beta_0$, and $k_\beta(x^- \ast y) > 0$, then $|f(x) - f(y)| < \epsilon$. Let $x \in K$ and let $k = k_\beta$, where $\beta \geqslant \beta_0$. Then

$$\begin{aligned}
|(fm \ast k^-)(x) - f(x)| &= \left| \int k^-(y^- \ast x) f(y)\, m(dy) - f(x) \int k(x^- \ast y)\, m(dy) \right| \\
&\leqslant \int k(x^- \ast y)\, |f(y) - f(x)|\, m(dy) \\
&\leqslant \epsilon \int k(x^- \ast y)\, m(dy) \\
&= \epsilon.
\end{aligned}$$

For 5.1C, let $g \in C_c^+(K)$. Using 4.2H, $\int g(\mu \ast k_\beta)\, dm = \int (\mu \ast k_\beta)\, d(gm) = \int k_\beta\, d(\mu^- \ast gm) = \int k_\beta^-\, d[(gm)^- \ast \mu] = \int (gm \ast k_\beta^-)\, d\mu$, and this last integral converges to $\int g\, d\mu$. Also,

$$\int (\mu \ast k_\beta)\, dm = \int\int k_\beta(y^- \ast x)\, \mu(dy)\, m(dx) = \int d\mu = \mu(K).$$

THEOREM 5.1D. *Let $f, g \in B^\infty(K)$ and let $x \in K$. If either f or g is σ-finite with respect to m, then*

$$\int_K f(x \ast y)\, g(y)\, m(dy) = \int_K f(y)\, g(x^- \ast y)\, m(dy).$$

Proof. By symmetry and the Monotone Convergence Theorem, we may assume that $\int g\, dm$ is finite. Let $\{k_\beta\}_{\beta \in D}$ be as in 5.1B. Let $h \in C_c^+(K)$. If $j \in C_c^+(K)$ then

$$\begin{aligned}
\int j\, d(h_x m) = \int h_x\, d(jm) &= \int (p_{x^-} \ast h)\, d(jm) \\
&= \int h\, d(p_x \ast jm) \\
&= \lim_\beta \int (hm \ast k_\beta^-)\, d(p_x \ast jm) \\
&= \lim_\beta \int (p_{x^-} \ast hm \ast k_\beta^-)\, d(jm) \\
&= \lim_\beta \int j\, d[(p_{x^-} \ast hm \ast k_\beta^-)m] \\
&= \int j\, d(p_{x^-} \ast hm).
\end{aligned}$$

Thus $h_x m = p_{x^-} * hm$. By 3.1E, we have

$$\int h\, d(p_x * gm) = \int h(x * y)\, g(y)\, m(dy) = \int h_x g\, dm = \int g\, d(p_{x^-} * hm)$$

$$= \int g_{x^-}\, d(hm) = \int h\, d(g_{x^-} m).$$

Hence,

$$p_x * gm = (g_{x^-})m.$$

Finally,

$$\int f(x * y)\, g(y)\, m(dy) = \int f\, d(p_x * gm) = \int f\, d(g_{x^-} m) = \int f g_{x^-}\, dm$$

$$= \int f(y)\, g(x^- * y)\, m(dy).$$

5.2. *The Uniqueness of Haar Measure*

The proof here is a copy of the proof by Loomis [11] of the uniqueness of Haar measure on locally compact groups.

THEOREM 5.2. *If n is a left-invariant measure on K then there exists a nonnegative real number c such that $n = cm$.*

Proof. Let n be a left-invariant measure on K. Let $\epsilon > 0$ and let $f, g \in C_c^+(K)$. Suppose that $f \neq 0$ and $g \neq 0$. Then

$$\lim_{y \to e} \| p_y * f - f * p_y \|_u = 0,$$

by 4.2F. Since f has compact support, we have

$$\lim_{y \to e} \int_K | f_y - f^y |\, dn = 0.$$

Thus there exists an open neighborhood U of e such that

$$\int_K | f_y - f^y |\, dn < (\epsilon/2) \int_K f\, dm,$$

$$\int_K | g_y - g^y |\, dn < (\epsilon/2) \int_K g\, dm,$$

for all $y \in U$. Choose $h \in C_c^+(K)$ such that $\int h\, dm > 0$, $h = h^-$, and $h = 0$ off U. Then

$$\begin{aligned}
\left(\int h\, dn\right)\left(\int f\, dm\right) &= \int f(y) \left(\int h_{y^-}\, dn\right) m(dy) \\
&= \int\int f(y)\, h(y^- * x)\, n(dx)\, m(dy) \\
&= \int\int f(y)\, h^-(x^- * y)\, m(dy)\, n(dx) \\
&= \int\int f(y)\, h(x^- * y)\, m(dy)\, n(dx) \\
&= \int\int f(x * y)\, h(y)\, m(dy)\, n(dx) \\
&= \int\int f(x * y)\, h(y)\, n(dx)\, m(dy) \\
&= \int h(y) \left(\int f^y\, dn\right) m(dy).
\end{aligned}$$

But $h = 0$ off U. Therefore,

$$\begin{aligned}
\left|\left(\int h\, dm\right)\left(\int f\, dn\right) - \left(\int h\, dn\right)\left(\int f\, dm\right)\right| &= \left|\int h(y) \left(\int (f_y - f^y)\, dn\right) m(dy)\right| \\
&\leqslant (\epsilon/2) \left(\int h\, dm\right)\left(\int f\, dm\right).
\end{aligned}$$

Dividing both sides of the inequality, we have

$$\left| \frac{\int f\, dn}{\int f\, dm} - \frac{\int h\, dn}{\int h\, dm} \right| \leqslant \frac{\epsilon}{2}.$$

The same argument applies to g, and thus

$$\left| \frac{\int f\, dn}{\int f\, dm} - \frac{\int g\, dn}{\int g\, dm} \right| \leqslant \epsilon.$$

This implies that $n = cm$ for some $c \geqslant 0$.

5.3. *The Modular Function*

The *modular function* Δ is defined on K by the identity

$$m * p_{x^-} = \Delta(x)m.$$

By 5.3A, this definition makes sense. The mapping $x \mapsto \Delta(x)$ is a homomorphism from the convo K to the multiplicative group of positive real numbers. Note that the constancy of Δ on the sets $\{x\} * \{y\}$ is essential. The multiplicative functions studied in Sections 6.3 and 7.3 are not usually homomorphisms from the convo to the multiplicative semigroup of complex numbers.

LEMMA 5.3A. *If $x \in K$ then there exists a unique positive real number c such that $m * p_x = cm$.*

THEOREM 5.3B. *The function Δ is continuous, and*

$$\Delta\Delta^- = 1 \qquad m = \Delta m^-.$$

THEOREM 5.3C. *Let $x, y \in K$. Then Δ is constant on $\{x\} * \{y\}$, and the value of Δ on this set is equal to*

$$\Delta(x * y) = \Delta(x)\,\Delta(y).$$

Proof A. The measure $m * p_x$ is defined, by (4.2L). It is clearly left-invariant.

Proof B. Let $f \in C_c^+(K)$, with $f \neq 0$. If $x \in K$ then

$$\Delta(x) \int f\,dm = \int f\,d(m * p_{x^-}) = \int (f * p_x)\,dm.$$

Thus Δ is continuous, by 4.2F. If $g \in C_c^+(K)$ then

$$\begin{aligned}
\int f\,dm \int g\,dm^- = \int f\,dm \int g^-\,dm &= \iint f(x)\,g^-(x^- * y)\,m(dy)\,m(dx)\\
&= \iint f(x)\,g(y^- * x)\,m(dx)\,m(dy)\\
&= \iint f(y * x)\,g(x)\,m(dx)\,m(dy)\\
&= \int g(x)\left(\int f^x\,dm\right) m(dx)\\
&= \left(\int f\,dm\right) \int g(x)\,\Delta(x^-)\,m(dx)\\
&= \int f\,dm \int g\,d(\Delta^- m).
\end{aligned}$$

Thus $m^- = \Delta^- m$. Therefore, $m = (m^-)^- = (\Delta^- m)^- = \Delta m^- = \Delta\Delta^- m$.

Proof C. If $x, y \in K$ then

$$\Delta(x)\,\Delta(y)m = (m * p_{y^-}) * p_{x^-} = m * (p_x * p_y)^-.$$

Thus $\Delta(x)\,\Delta(y) = \Delta(x * y)$. It follows that

$$\int \Delta\, d(\mu * \nu) = \int \Delta\, d\mu \int \Delta\, d\nu$$

for all $\mu, \nu \in M_c(K)$.

Let $x, y \in K$. Set $\mu = p_x * p_y$, $S = \operatorname{spt}\mu$, and

$$\nu = \mu * \mu^- = p_x * p_y * p_{y^-} * p_{x^-}.$$

Thus $\nu^- = \nu$ and $\nu(K) = 1$. Moreover,

$$\int \Delta\, d\nu = \Delta(x)\,\Delta(y)\,\Delta(y^-)\,\Delta(x^-) = 1$$

$$\int \frac{1}{\Delta}\, d\nu = \int \frac{1}{\Delta}\, d\nu^- = \int \Delta^-\, d\nu^- \int \Delta\, d\nu = 1$$

$$\int \left(\Delta + \frac{1}{\Delta}\right) d\nu = 2.$$

Since $\nu(K) = 1$, it must be that $\Delta = 1$ on the support of ν. Thus, if $s, t \in S$ then $\Delta = 1$ on $\{s\} * \{t^-\}$, and so $1 = \Delta(s * t^-) = \Delta(s)/\Delta(t)$. It follows that Δ is constant on S, and its value there must be equal to $\Delta(x)\,\Delta(y)$.

5.4. *Convolution of Functions and Measures*

Hereinafter the expressions *σ-finite* and *almost everywhere* refer to the measure m. By 5.2 and 5.3B, their meanings do not actually depend on which Haar measure is used.

LEMMA. *Let $\{f_n\}$ be a nondecreasing sequence in $B^\infty(K)$. Suppose that $f_n \to f$.*

(5.4A) *If $x, y \in K$ then $f_n(x * y) \to f(x * y)$.*

(5.4B) *If $\mu \in M^+(K)$ then $\mu * f_n \to \mu * f$.*

Proof. By a straightupward use of the Monotone Convergence Theorem.

THEOREM. *Let $\mu \in M^+(K)$ and let $f, g \in B^\infty(K)$.*

(5.4C) *If f is σ-finite then $\mu * f$ is σ-finite.*

(5.4D) *If $\int f\,dm < \infty$ then $(\mu * f)m = \mu * (fm)$.*

(5.4E) *If $\int f\,dm < \infty$ then $\int_K (\mu * f)\,dm = \mu(K) \int_K f\,dm$.*

(5.4F) *If either f or g is σ-finite then*

$$\int_K (\mu * f)\,g\,dm = \int_K f(\mu^- * g)\,dm.$$

Proof. For 5.4D, we have by previous results that

$$\begin{aligned}
\int g(\mu * f)\,dm &= \int (\mu * f)\,d(gm) \\
&= \int f\,d(\mu^- * gm) \\
&= \iint f(x * y)(gm)(dy)\,\mu^-(dx) \\
&= \iint f(x^- * y)\,g(y)\,m(dy)\,\mu(dx) \\
&= \iint f(y)\,g(x * y)\,m(dy)\,\mu(dx) \\
&= \iint g(x * y)(fm)(dy)\,\mu(dx) \\
&= \int g\,d(\mu * fm).
\end{aligned}$$

The other parts follow readily.

LEMMA. *Let $\mu \in M^+(K)$, $f \in B^\infty(K)$, and $1 \leqslant p \leqslant \infty$.*

(5.4G) *If $\|f\|_p < \infty$ then $\|\mu * f\|_p \leqslant \|\mu\| \cdot \|f\|_p$.*

(5.4H) *If $p < \infty$, $\|f\|_p < \infty$, and $\{\mu_\beta\}_{\beta \in D}$ is a net in $M^+(K)$ converging to μ, then*

$$\lim_\beta \|\mu_\beta * f - \mu * f\|_p = 0.$$

Proof G. Recall the proof of 3.3B. Assume that $\mu(K) = 1$. First, suppose that $p < \infty$. Then

$$\begin{aligned}(\mu \divideontimes f)^p(x) &= \left(\int f(y^- \divideontimes x)\, \mu(dy)\right)^p \\ &\leqslant \int [f(y^- \divideontimes x)]^p\, \mu(dy) \\ &\leqslant \int f^p(y^- \divideontimes x)\, \mu(dy) \\ &= (\mu \divideontimes f^p)(x)\end{aligned}$$

for each $x \in K$. It follows that

$$\int (\mu \divideontimes f)^p\, dm \leqslant \int (\mu \divideontimes f^p)\, dm = \mu(K) \int f^p\, dm = \int f^p\, dm.$$

The case where $p = \infty$ is similar to 3.3B.

Proof H. Since $p < \infty$ and the numbers $\| \mu_\beta \| = \mu_\beta(K)$ converge to $\mu(K)$, it is enough to consider only $f \in C_c^+(K)$. By (4.2F),

$$\lim_\beta \| \mu_\beta \divideontimes f - \mu \divideontimes f \|_u = 0.$$

And by 5.4E, the $\int (\mu_\beta \divideontimes f)\, dm$ converge to $\int (\mu \divideontimes f)\, dm$. Thus $\mu_\beta \divideontimes f \to \mu \divideontimes f$ in both $L_1(m)$ and $L_\infty(m)$. This implies convergence in $L_p(m)$.

5.5. *Convolution of Functions*

Let f and g be in $B^\infty(K)$. If (and only if) at least one of these functions is σ-finite, the *convolution* $f \divideontimes g$ of f with g is defined on K by

$$(f \divideontimes g)(x) = \int_K f(x \divideontimes y)\, g(y^-)\, m(dy).$$

Note that the choice of m does introduce a scalar factor into the definition of $f \divideontimes g$. However, by Lemma 5.5A, below, the corresponding formula with the right Haar measure m^- yields the same function.

In the remainder of this subsection it is assumed that $p, q \in [1, \infty]$ and $1/p + 1/q = 1$.

LEMMA. *Let f and g be in $B^\infty(K)$.*

(5.5A) *If* $x \in K$ *and if either* f *or* g *is* σ*-finite, then*

$$(f * g)(x) = \int_K f_x g^- \, dm = \int_K f^- g^x \, dm^-.$$

(5.5B) *If either* f *or* g *is* σ*-finite, and if* $\{f_n\}$ *and* $\{g_n\}$ *are non-decreasing sequences in* $B^\infty(K)$ *converging to* f *and* g*, respectively, then* $f_n * g_n \to f * g$.

(5.5C) *If* $f_1 = f$ *almost everywhere,* $g_1 = g$ *locally almost everywhere, and* f *is* σ*-finite, then* $f_1 * g_1 = f * g$ *at all points of* K.

Proof. These results are apparent.

THEOREM. *Let* $g, h \in B^\infty(K)$*. Suppose that* $\| g \|_p < \infty$ *and* $\| h \|_q < \infty$.

(5.5D) $g * h^-$ *is continuous.*

(5.5E) $\| g * h^- \|_u \leqslant \| g \|_p \| h \|_q$.

Proof. These follow from 5.4G and 5.4H, since $(f * g^-)(x) = \int f_x g \, dm = \int f g_{x-} \, dm$.

LEMMA. *Let* $f, g \in B^\infty(K)$ *and suppose that either* f *or* g *is* σ*-finite.*

(5.5F) $f * g$ *is lower-semicontinuous.*

(5.5G) $(f * g)^- = g^- * f^-$.

(5.5H) *If both* f *and* g *are lower-semicontinuous, then*

$$\operatorname{pos}(f * g) = (\operatorname{pos} f) * (\operatorname{pos} g).$$

Proof. The first part follows from 5.5B and 5.5D. For the third, see 4.1B.

THEOREM. *Let* $\mu \in M^+(K)$ *and let* $f, g, h \in B^\infty(K)$.

(5.5I) *If* $\int f \, dm < \infty$ *then* $f * g = (fm) * g$.

(5.5J) *If either* f *or* g *is* σ*-finite then* $\mu * (f * g) = (\mu * f) * g$.

(5.5K) *If* $\int f \, dm < \infty$ *and* $\int g \, dm < \infty$ *then*

$$(f * g)m = (fm) * (gm).$$

(5.5L) *If* $\int f \, dm < \infty$ *and* $\int g \, dm < \infty$ *then* $\int_K (f * g) \, dm = \int_K f \, dm \int_K g \, dm$.

(5.5M) *If f and g are both σ-finite then $f * g$ is σ-finite.*

(5.5N) *If g is σ-finite, and if either f or h is σ-finite, then $f * (g * h) = (f * g) * h$.*

(5.5O) *In 5.5N, also $\int_K (f * g)\, h\, dm = \int_K f(h * g^-)\, dm$.*

Proof. For 5.5J, if $\int f\, dm < \infty$, then $\mu * (f * g) = \mu * (fm * g) = (\mu * fm) * g = (\mu * f)m * g = (\mu * f) * g$. And if $\int g\, dm^- < \infty$, then $\mu * (f * g) = \mu * (f * gm^-) = (\mu * f) * gm^- = (\mu * f) * g$.
For 5.5N, if $\int f\, dm < \infty$ and $\int g\, dm < \infty$, then $f * (g * h) = fm * (gm * h) = (fm * gm) * h = (f * g)m * h = (f * g) * h$.
For 5.5O, $\int (f * g)\, h\, dm = [(f * g) * h^-](e) = [f * (g * h^-)](e) = [f * (h * g^-)^-](e) = \int f(h * g^-)\, dm$.

THEOREM. *Let $f, g, h \in B^\infty(K)$.*

(5.5P) *If $1 < p < \infty$, $\| g \|_p < \infty$ and $\| h \|_q < \infty$ then $g * h^- \in C_0{}^+(K)$.*

(5.5Q) *If $\| f \|_1 < \infty$ and $\| g \|_p < \infty$ then $\| f * g \|_p \leqslant \| f \|_1 \| g \|_p$.*

Proof. For 5.5P, g and h can be approximated by functions with compact support.

5.6. *Absolutely Continuous Measures*

The set of measures in $M(K)$ which are absolutely continuous with respect to m (or, equivalently, to any other Haar measure, left or right) will be denoted by $M_a(K)$.

THEOREM. *Let $\nu \in M_a{}^+(K)$.*

(5.6A) *If $\mu \in M^+(K)$ then both $\mu * \nu$ and $\nu * \mu$ are in $M_a{}^+(K)$.*

(5.6B) *If $\{\mu_\beta\}_{\beta \in D}$ is a net in $M^+(K)$ converging to μ, then*

$$\lim_\beta \| \mu_\beta * \nu - \mu * \nu \| = 0.$$

Proof. The first part follows from 3.3E. For the second part, let $f \in C_c{}^+(K)$. Then, using 6.1D in advance,

$$\begin{aligned} &\| \mu_\beta * \nu - \mu * \nu \| \\ &\quad \leqslant \| \mu_\beta * \nu - \mu_\beta * fm \| + \| \mu_\beta * fm - \mu * fm \| + \| \mu * fm - \mu * \nu \| \\ &\quad \leqslant \| \mu_\beta \| \cdot \| \nu - fm \| + \| \mu_\beta * f - \mu * f \|_1 + \| \mu \| \cdot \| fm - \nu \|. \end{aligned}$$

Thus, by 5.4H,

$$\limsup_{\beta} \| \mu_\beta * \nu - \mu * \nu \| \leqslant 2\mu(K) \| \nu - fm \|.$$

But $\| \nu - fm \|$ can be made arbitrarily small.

LEMMA 5.6C. *Let* $\nu \in M^+(K)$. *Suppose that* $\lim_{x \to e} \| p_x * \nu - \nu \| = 0$. *Then* $\nu \in M_a^+(K)$.

Proof. Let $\epsilon > 0$. Let U be an open neighborhood of e such that $\| p_x * \nu - \nu \| < \epsilon$ for all $x \in U$. Choose $k \in C_c^+(K)$ such that $\int k \, dm^- = 1$ and such that spt k is contained in U. It is easily verified that $\| km^- * \nu - \nu \| \leqslant \epsilon$. But $km^- * \nu = (k * \nu) m^-$, which is in $M_a^+(K)$. Also, $M_a^+(K)$ is norm-closed. Hence, $\nu \in M_a^+(K)$.

6. CONVOLUTION ALGEBRAS

In preceding sections, the functions and measures considered were usually nonnegative, and the operations were semilinear or semibilinear. As a result the functions were defined everywhere on the convo. We shall assume here that the obvious extensions to complex-valued functions and measures have been made, insofar as the appropriate integrals exist, and state the results without proof in Sections 6.1 and 6.2.

In this section it is assumed that K is a convo. Recall that if $\mu \in M(K)$ and $f \in B(K)$ then there exist measures $\mu_k \in M^+(K)$ and functions $f_k \in B^+(K)$ such that each $\mu_k \leqslant |\mu|$, each $f_k \leqslant |f|$, and

$$\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4, \qquad f = f_1 - f_2 + if_3 - if_4.$$

6.1. *Complex-Valued Functions and Measures*

For $\mu \in M(K)$, the *adjoint* μ^* of μ is defined by $\mu^* = (\bar{\mu})^- = \overline{\mu^-}$.

LEMMA. *Let* $\mu, \nu \in M^+(K)$ *and* $f \in B(K)$.

(6.1A) *If* $x, y \in K$ *and* $|f|(x * y)$ *is finite, then* $f(x * y)$ *is defined, and* $|f(x * y)| \leqslant |f|(x * y)$.

(6.1B) *At the points of* K *where* $|\mu| * |f|$ *is finite,* $\mu * f$ *is defined, and* $|\mu * f| \leqslant |\mu| * |f|$.

(6.1C) $|\mu * \nu| \leqslant |\mu| * |\nu|$.

(6.1D) $\| \mu * \nu \| \leqslant \| \mu \| \cdot \| \nu \|$.

(6.1E) $(\mu * \nu)(K) = \mu(K)\,\nu(K)$.

(6.1F) *If f is bounded then*

$$\int_K f\,d(\mu * \nu) = \int_K \int_K f(x * y)\,\mu(dx)\,\nu(dy).$$

THEOREM 6.1G. *The space $M(K)$ is a Banach *-algebra with unit.*

6.2. *The L_p Spaces*

In this subsection it is assumed that m is a left Haar measure on K. For a function f in $B(K)$, the *adjoint* f^* of f is defined by $f^* = (\Delta \bar{f})^-$. Note that if $f \in L_1(m)$ then $f^* \in L_1(m)$ also.

For each μ in $M(K)$, a mapping $T_\mu : L_2(m) \to L_2(m)$ can be defined by $T_\mu f = \mu * f$. Each such T_μ is a bounded linear operator on $L_2(m)$. The mapping $\mu \mapsto T_\mu$ from $M(K)$ to the Banach *-algebra of bounded linear operators on $L_2(m)$ will be called the *left-regular representation* of K.

LEMMA 6.2A. *Let $f, g \in B(K)$. At those points where $|f| * |g|$ is finite, $f * g$ is defined, and $|f * g| \leqslant |f| * |g|$.*

THEOREM. *Let $\mu \in M(K)$ and $f \in L_1(m)$. Let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$, and let $g \in L_p(m)$ and $h \in L_q(m)$.*

(6.2B) $\mu * g \in L_p(m)$ *and* $\|\mu * g\|_p \leqslant \|\mu\| \cdot \|g\|_p$.

(6.2C) $f * g \in L_p(m)$ *and* $\|f * g\|_p \leqslant \|f\|_1 \|g\|_p$.

(6.2D) $\int_K (\mu * g)\,h\,dm = \int_K g(\mu^- * h)\,dm$.

(6.2E) $g * h^-$ *is continuous, and* $\|g * h^-\|_u \leqslant \|g\|_p \|h\|_q$.

(6.2F) *If p and q are finite then $g * h$ is in $C_0(K)$.*

THEOREM 6.2G. *The space $L_1(m)$ is a Banach *-algebra.*

THEOREM 6.2H. *The space $M_a(K)$ is a closed self-adjoint ideal in $M(K)$.*

THEOREM 6.2I. *The left-regular representation is a faithful norm-decreasing *-representation of $M(K)$.*

Proof. Let $\mu \in M(K)$ and suppose that $\mu \neq 0$. Then there exists $f \in C_c(K)$ such that $\int f\,d\mu \neq 0$. Let $h = f^-$. *Then* $(T_\mu h)(e) =$

$(\mu * h)(e) = \int f\,d\mu \neq 0$. And $\mu * h$ is continuous, by (4.2A). Thus $T_\mu \neq 0$.

LEMMA 6.2J. *Let $f \in L_2(m)$ and let $\{g_1, g_2, \ldots, g_n\}$ be an orthonormal set in $L_2(m)$. Then*

$$\left\| \sum_{k=1}^{n} |f * g_k^-|^2 \right\|_u \leqslant \|f\|_2^2 .$$

Proof. Note that the $\overline{g_k}$ also form an orthonormal set. Let $x \in K$. By Bessel's Inequality,

$$\sum_{k=1}^{n} |(f * g_k^-)(x)|^2 = \sum_{k=1}^{n} \left| \int f_x g_k \, dm \right|^2$$

$$= \sum_{k=1}^{n} |\langle f_x, \overline{g_k}\rangle|^2 \leqslant \|f_x\|_2^2 \leqslant \|f\|_2^2 .$$

6.3. *Multiplicative Functions*

A complex-valued function χ on K will be called a *multiplicative function* if χ is continuous and not identically zero, and has the property that

$$\chi(x * y) = \chi(x)\,\chi(y)$$

for all x and y in K. We denote the set of all such functions by $\mathfrak{X}(K)$, and give $\mathfrak{X}(K)$ the topology of uniform convergence on compact subsets of K. The set of all χ in $\mathfrak{X}(K)$ which are bounded will be denoted by $\mathfrak{X}_b(K)$. Note that the constant function 1 is in $\mathfrak{X}_b(K)$.

While a bounded multiplicative function is not in general a homomorphism from the convo to the semigroup of complex numbers under multiplication, it does give rise to an algebra-homomorphism. For χ in $\mathfrak{X}_b(K)$, let F_χ be defined on $M(K)$ by

$$F_\chi(\mu) = \int_K \bar{\chi}\, d\mu.$$

We assume in this subsection that there exists a left Haar measure m on K.

It turns out that if K is commutative then the space $\mathfrak{X}_b(K)$ is homeomorphic to the structure space of $M_a(K)$ with the Gel'fand topology.

Recall that a multiplicative linear functional on a complex algebra is a complex-valued homomorphism of the algebra which is not identically zero.

The first lemma below is apparent.

LEMMA. *Let* $\chi \in \mathfrak{X}_b(K)$.

(6.3A) *The function* F_χ *is a multiplicative linear functional on* $M(K)$.

(6.3B) F_χ *is not identically zero on* $M_a(K)$.

(6.3C) *The functions* χ^- *and* $\bar{\chi}$ *are in* $\mathfrak{X}_b(K)$.

(6.3D) $\chi(e) = 1 = \| \chi \|_u$.

LEMMA 6.3E. *Let* E *be a multiplicative linear functional on* $M_a(K)$. *Then there exists a unique multiplicative linear functional* F *on* $M(K)$ *such that* $F = E$ *on* $M_a(K)$.

THEOREM 6.3F. *Let* F *be a multiplicative linear functional on* $M(K)$ *which is not identically zero on* $M_a(K)$. *Then there exists a unique* $\chi \in \mathfrak{X}_b(K)$ *such that* $F = F_\chi$.

THEOREM 6.3G. *Let* $\{\chi_\beta\}_{\beta \in D}$ *be a net in* $\mathfrak{X}_b(K)$. *Then the following two statements are equivalent:*

(I) *The* χ_β *converge uniformly on compact subsets of* K *to a function* χ *on* K.

(II) *The restrictions of the* F_{χ_β} *to* $M_a(K)$ *converge pointwise to a function on* $M_a(K)$ *which is not identically zero.*

Suppose that these nets do so converge. Then $\chi \in \mathfrak{X}_b(K)$, *and the* F_{χ_β} *converge to* F_χ *on* $M(K)$.

THEOREM 6.3H. *The space* $\mathfrak{X}_b(K)$ *is a locally compact Hausdorff space. The two mappings* $\chi \mapsto \chi^-$ *and* $\chi \mapsto \bar{\chi}$ *are topological involutions of* $\mathfrak{X}_b(K)$.

Proof E. Recall that $M_a(K)$ is an ideal in $M(K)$. Thus F is given by

$$F(\mu) = \frac{E(\mu * \nu)}{E(\nu)},$$

where ν is an element of $M_a(K)$ such that $E(\nu) \neq 0$.

Proof F. Since $M(K)$ is a Banach algebra with unit, $\|F\| = 1$. Choose $\nu \in M_a(K)$ such that $F(\nu) \neq 0$. If $\mu_1, \mu_2 \in M(K)$ then

$$|F(\mu_1) - F(\mu_2)| \leqslant \frac{\|\mu_1 * \nu - \mu_2 * \nu\|}{|F(\nu)|}.$$

Thus, by 5.6B, F is positive-continuous. By 2.2D, there exists $f \in C_b(K)$ such that $F(\mu) = \int_K f\, d\mu$ for all $\mu \in M(K)$. Let $\chi = \bar{f}$.

Proof G. It is clear that (I) implies (II). Assume (II). Let the F_{χ_β} converge to E on $M_a(K)$. So E is a multiplicative linear functional on $M_a(K)$. By 6.3E and 6.3F, there exists $\psi \in \mathfrak{X}_b(K)$ such that $E = F_\psi$ on $M_a(K)$. Choose $\nu \in M_a(K)$ such that $E(\nu) \neq 0$.

Let $a \in K$. If $x \in K$, $\beta \in D$, $H_\beta = F_{\chi_\beta}$, and $H_\beta(\nu) \neq 0$, then

$$\begin{aligned}
&|\chi_\beta(x) - \psi(x)| \\
&\quad \leqslant |\chi_\beta(x) - \chi_\beta(a)| + |\chi_\beta(a) - \psi(a)| + |\psi(a) - \psi(x)| \\
&\quad \leqslant \frac{\|p_x * \nu - p_a * \nu\|}{|H_\beta(\nu)|} + \left| \frac{H_\beta(p_a * \nu)}{H_\beta(\nu)} - \frac{E(p_a * \nu)}{E(\nu)} \right| + |\psi(a) - \psi(x)|.
\end{aligned}$$

This shows, using 5.6B again, that

$$\lim_{\substack{\beta \\ x \to a}} |\chi_\beta(x) - \psi(x)| = 0.$$

And this implies that the χ_β converge to ψ uniformly on compact subsets of K. Thus $\psi \in \mathfrak{X}_b(K)$. It is clear that the F_{χ_β} converge to F_ψ pointwise on $M(K)$.

Proof H. This is a consequence of 6.3G and the Tihonov Product Theorem.

7. Some Special Convos

The convos that are most easily analysed are those that are either discrete, compact, or commutative. A similar statement can be made about locally compact groups, and for the same reasons.

7.1. *Discrete convos*

Unlike a topological group, a convo does not have a purely algebraic structure associated with it. In general, if a convo is given the discrete topology then the operation is no longer well defined. Another contrast is in that, while a topological group is homogeneous, there exists a convo that has isolated points but is not discrete. An example will be given elsewhere.

THEOREM 7.1A. *Let K be a discrete convo. Then there exists a left Haar measure on K. If m is the left Haar measure for which $m(\{e\}) = 1$, then*

$$m(\{x\}) = \frac{1}{(p_{x^-} * p_x)(\{e\})}$$

for each x in K.

THEOREM 7.1B. *Let K be a convo. Then the following three statements are equivalent:*

(I) *K is discrete.*

(II) *The identity e is an isolated point of K.*

(III) *There exists a Haar measure m on K such that $m(\{e\}) > 0$.*

Proof A. For $x, y, z \in K$ define $[x * y, z] = (p_x * p_y)(\{z\})$. Note that $[x * y, e] > 0$ if and only if $x = y^-$. For $x \in K$ let $[x] = 1/[x^- * x, e]$. If $x, y, z \in K$ then

$$(p_x * (p_y * p_z))(\{e\}) = ((p_x * p_y) * p_z)(\{e\}),$$

$$\sum_{t \in K} [x * t, e][y * z, t] = \sum_{t \in K} [x * y, t][t * z, e],$$

$$[x * x^-, e][y * z, x^-] = [x * y, z^-][z^- * z, e],$$

$$[z][y * z, x^-] = [x^-][x * y, z^-].$$

Let the measure m be defined by

$$m = \sum_{x \in K} [x]\, p_x .$$

If $x, y \in K$ then

$$\begin{aligned}(p_y * m)(\{x^-\}) &= \sum_{z \in K} [z](p_y * p_z)(\{x^-\}) \\ &= \sum_{z \in K} [z][y * z, x^-] \\ &= \sum_{z \in K} [x^-][x * y, z^-] \\ &= [x^-] = m(\{x^-\}).\end{aligned}$$

Proof B. We have just seen that (I) implies (III). Assume (III). Then $i_{\{e\}}$ is a nonzero element of $L_2(m)$. But $i_{\{e\}} * i_{\{e\}} = c\, i_{\{e\}}$ for some $c > 0$. Thus $i_{\{e\}}$ is continuous. Hence (II).

Assume (II). If $x \in K$ then $p_x * i_{\{e\}} = c\, i_{\{x\}}$ for some $c > 0$, and thus $i_{\{x\}}$ is continuous. Hence (I).

7.2. *Compact Convos*

The Theorem 7.2C is based on the theory of H^*-algebras. Two references for this theory are the books of Loomis [11] and Naimark [14]. The proof that the minimal closed ideals are finite-dimensional is modeled after a proof of Nachbin [13] that an irreducible unitary representation of a compact group is finite-dimensional. See Levitan [9, p. 22] for his version of 7.2C.

THEOREM 7.2A. *Let K be a compact convo. Then there exists a Haar measure on K. Moreover, K is unimodular.*

THEOREM 7.2B. *Let K be a convo. If there exists a finite Haar measure on K then K is compact.*

THEOREM 7.2C. *Let K be a compact convo, and let m be the Haar measure on K such that $m(K) = 1$. If $f, g \in L_2(m)$ then $f * g \in L_2(m)$ and*

$$\|f * g\|_2 \leqslant \|f\|_2 \|g\|_2 .$$

With convolution as the operation, $L_2(m)$ is a H^*-algebra, and is thus the (Hilbert space) direct sum of its minimal closed ideals. Each minimal closed ideal in $L_2(m)$ is finite-dimensional.

Proof A. By 4.3C, there exists a nonzero left-subinvariant measure m on K. It is clear that m is a Haar measure. And K is unimodular, since $\Delta(K)$ is a compact subgroup of $(0, \infty)$.

Proof B. Let $f = 1$ on K. Then $f \in L_2$ and $f * f = cf$ for some $c > 0$. But $f * f \in C_0(K)$. Thus K is compact.

Proof C. It is apparent from previous results that $L_2(m)$ is a H^*-algebra. Let J be a minimal closed ideal in $L_2(m)$. It is known that J is isomorphic to an algebra of complex matrices, where, for some $c \geqslant 1$, the norm of each matrix is c times the L_2-norm of the matrix. See Naimark [13, pp. 330–331]. Choose a column and consider the set of all matrices whose entries off that column are zero; let J_1 be the corresponding subset of J. In fact, J_1 is a minimal left ideal of $L_2(m)$. A simple computation with matrices shows that, for all $f, g \in J_1$,

$$c \| f * g^* \|_2 = \| f \|_2 \| g \|_2 .$$

Let $f \in J_1$ be such that $\| f \|_2 = 1$ and let $\{g_1, g_2, \ldots, g_n\}$ be an orthonormal set in J_1. The $\overline{g_k}$ form an orthonormal set in $L_2(m)$. By 6.2J,

$$\begin{aligned} \frac{n}{c^2} = \frac{1}{c^2} \sum_{k=1}^{n} \| f \|_2^2 \| g_k \|_2^2 &= \sum_{k=1}^{n} \| f * g_k^* \|_2^2 \\ &= \left\| \sum_{k=1}^{n} | f * g_k^* |^2 \right\|_1 \\ &\leqslant \left\| \sum_{k=1}^{n} | f * (\overline{g_k})^- |^2 \right\|_u \\ &\leqslant \| f \|_2^2 = 1. \end{aligned}$$

Therefore, $\dim J_1 \leqslant c^2$ and $\dim J \leqslant c^4$.

7.3. *Commutative Convos*

Let K be a convo. In this subsection it is assumed that K is *commutative*, which means that $p_x * p_y = p_y * p_x$ for all $x, y \in K$. It is easily seen that all convolutions of functions and measures commute whenever defined.

We also assume here that K has a Haar measure m. It is apparent that K is unimodular.

Let $\hat{K}$ be the set of all χ in $\mathfrak{X}_b(K)$ such that

$$\chi(x^-) = \overline{\chi(x)}$$

for $x \in K$. Note that $\hat{K}$ is nonvoid, since it contains the constant function 1, and that if $\chi \in \hat{K}$ then $\chi^- = \bar{\chi}$ is in $\hat{K}$ also.

For $\mu \in M(K)$, the *Fourier–Stieltjes transform* $\hat{\mu}$ of μ is defined on $\hat{K}$ by

$$\hat{\mu}(\chi) = \int_K \bar{\chi}\, d\mu.$$

For $f \in L_1(m)$, the *Fourier transform* $\hat{f}$ of f with respect to m is defined on $\hat{K}$ by

$$\hat{f}(\chi) = \int_K f\bar{\chi}\, dm.$$

The set S is the set of all $\chi \in \hat{K}$ such that $|\hat{\mu}(\chi)| \leqslant \| T_\mu \|$ for all $\mu \in M(K)$. For any function k defined on $\hat{K}$, let

$$\| k \|_S = \sup_{\chi \in S} | k(\chi) |.$$

We credit 7.3I to Levitan [6], but shall include a proof of this basic result. The measure π is called the *Plancherel measure* on $\hat{K}$ associated with m.

THEOREM.

(7.3A) *The space $\hat{K}$ is a nonvoid (locally compact) closed subspace of $\mathfrak{X}_b(K)$.*

(7.3B) *If $\chi \in \hat{K}$ and $|\hat{\nu}(\chi)| \leqslant \| T_\nu \|$ for all $\nu \in M_a(K)$, then $\chi \in S$.*

(7.3C) *S is a closed nonvoid subset of $\hat{K}$.*

(7.3D) *If $\mu \in M(K)$ then $\| T_\mu \| = \| \hat{\mu} \|_S$.*

(7.3E) *The mapping $\mu \mapsto \hat{\mu}$ is a norm-decreasing *-algebra isomorphism from $M(K)$ into $C_b(\hat{K})$.*

(7.3F) *If $\nu \in M_a(K)$ then $\hat{\nu} \in C_0(\hat{K})$.*

(7.3G) *If $f \in L_1(m)$ then $\hat{f} = (fm)^\wedge$ and $\hat{f} \in C_0(\hat{K})$.*

(7.3H) *The set $\{\hat{f} : f \in C_c(K)\}$ is a dense self-adjoint subalgebra of $C_0(\hat{K})$.*

THEOREM 7.3I (Levitan). *There exists a unique nonnegative measure π on $\hat{K}$ such that*

$$\int_K |f|^2 \, dm = \int_{\hat{K}} |\hat{f}|^2 \, d\pi$$

for all f in $L_1(m) \cap L_2(m)$. The support of π is equal to S. The set $\{\hat{f} : f \in C_c(K)\}$ is dense in $L_2(\pi)$.

Proof A. Apparent.

Proof B. This is straightforward, since $M_a{}^+(K)$ is dense in $M^+(K)$.

Proof C. Let $\nu_0 \in M_a(K)$, with $\nu_0 \neq 0$. Then $T_{\nu_0} \neq 0$. The algebra $\{T_\nu : \nu \in M_a(K)\}$ is self-adjoint and commutative. Thus there exists a multiplicative linear functional H on this algebra such that $| H(T_{\nu_0})| = \| T_{\nu_0} \|$. Also, H must be a *-homomorphism. But the mapping $\nu \mapsto H(T_\nu)$

is a multiplicative linear functional on $M_a(K)$. By 6.3E and 6.3F, there exists $\psi \in \mathfrak{X}_b(K)$ such that

$$H(T_\nu) = F_\psi(\nu) = \hat{\nu}(\psi)$$

for all $\nu \in M_a(K)$. Also, $\psi \in \hat{K}$, since H preserves adjoints. It follows that $\psi \in S$, since $\| H \| = 1$. It is clear that S is closed, since the transforms $\hat{\mu}$ are continuous.

Proof D. In the notation of the previous proof, $\| T_{\nu_0} \| = | \nu_0\hat{}(\psi)|$. Thus, if $\nu \in M_a(K)$ then $| \hat{\nu} |$ attains its supremum on S, and the supremum is $\| T_\nu \|$. For $\mu \in M(K)$, one can approximate by members of $M_a(K)$. In this case, though, the supremum may not be attained.

Proof E. This is clear, since $T_\mu \neq 0$ if $\mu \neq 0$.

Proofs F, G. This follows from (6.3G), since $\mathfrak{X}_b(K)$ is just the structure space of $M_a(K)$.

Proof H. This follows from the Stone–Weierstrass Theorem.

Proof I. Let H be the uniform closure of the set $\{\hat{\mu} : \mu \in M(K)\}$ in $C_b(\hat{K})$. Thus H is a closed self-adjoint subalgebra of $C_b(\hat{K})$ containing $C_0(\hat{K})$. There exists a unique *-homomorphism $k \mapsto V_k$ from H onto the closure (with respect to norm) of the algebra $\{T_\mu : \mu \in M(K)\}$ with the property that

$$V_{\hat{\mu}} = T_\mu \qquad \text{and} \qquad \| V_k \| = \| k \|_S$$

for all $\mu \in M(K)$ and $k \in H$.

The remainder of the proof is divided into several lemmas.

LEMMA 7.3J. *Let $k \in C_c(\hat{K})$. Then there exists a unique function $k' \in C_0(K)$ such that $\| k' \|_2 < \infty$ and such that $V_k g = k' * g$ for all $g \in L_2(m)$.*

Proof. There exists $f \in C_c(K)$ such that $| \hat{f} | > 0$ on spt k, by 7.3H. There exists $j \in C_c(\hat{K})$ such that $k = j \cdot \hat{f} \cdot \bar{\hat{f}}$. Thus $V_k = V_j V_{\hat{f}} V_{\bar{\hat{f}}} = V_j T_{fm} T_{\bar{f}m}$. Let $k' = (V_j f) * \bar{f}$. The rest is straightforward.

LEMMA 7.3K. *The mapping $k \mapsto k'$ from $C_c(\hat{K})$ to $C_0(K)$ is linear, and $k' = 0$ if and only if $\| k \|_S = 0$.*

LEMMA 7.3L. *If $j, k \in C_c(\hat{K})$ and $\mu \in M(K)$ then*

$$(\bar{k})' = (k')^* \qquad (jk)' = j' * k' \qquad (\hat{\mu}k)' = \mu * k'.$$

Proof. Let $g \in L_2(m)$. Then

$$\bar{k}' * g = V_{\bar{k}}g = (V_k)^*g = (k')^* * g,$$
$$(jk)' * g = V_{jk}g = V_jV_kg = j' * k' * g = (j' * k') * g,$$
$$(\hat{\mu}k)' * g = V_{\hat{\mu}k}g = V_{\hat{\mu}}V_kg = \mu * k' * g.$$

LEMMA 7.3M. *Let $k \in C_c(\hat{K})$. Suppose that $k \geqslant 0$. Then $k'(e) \geqslant 0$. Also, $k'(e) > 0$ if and only if $\| k \|_S > 0$.*

Proof. There exists $j \in C_c(\hat{K})$ such that $j \geqslant 0$ and $k = j^2$. Thus $k' = j' * j'^*$. Therefore, $k(e) = \int | j' |^2 \, dm$. The rest is clear.

LEMMA 7.3N. *There exists a unique measure $\pi \in M^\infty(\hat{K})$ such that $\int k \, d\pi = k'(e)$ for all $k \in C_c(\hat{K})$. The support of π is equal to S.*

Proof. This follows from the Riesz Representation Theorem.

LEMMA 7.3O. *If $k \in C_c(\hat{K})$ and $f \in L_1(m) \cap L_2(m)$ then*

$$\int_{\hat{K}} | k |^2 \, d\pi = \int_K | k' |^2 \, dm, \qquad \int_{\hat{K}} | \hat{f} |^2 \, d\pi = \int_K | f |^2 \, dm.$$

Moreover, the set $\{k' : k \in C_c(\hat{K})\}$ is dense in $L_2(m)$, and the set $\{\hat{f} : f \in C_c(K)\}$ is dense in $L_2(\pi)$.

Proof. If $k \in C_c(\hat{K})$ then $\int | k |^2 \, d\pi = (k\bar{k})'(e) = (k' * k'^*)(e) = \int | k' |^2 \, dm$.

Suppose that $h \in L_2(m)$ and that $\int k'h \, dm = 0$ for all $k \in C_c(\hat{K})$. If $x \in K$ and $k \in C_c(\hat{K})$ then $j = (p_{x^-})^\wedge k$ is in $C_c(\hat{K})$, and

$$0 = \int_K j'h \, dm = \int_K (p_{x^-} * k')h \, dm = (k' * h^-)(x).$$

Thus $V_k(h^-) = 0$ for all $k \in C_c(\hat{K})$. It follows that $f * h = 0$ for all $f \in L_1(m)$. Hence $h = 0$.

The rest is apparent, since the mapping $k \mapsto k'$ extends to an isometry of $L_2(\pi)$ onto $L_2(m)$.

LEMMA 7.3P. *Let $\pi_0 \in M^\infty(\hat{K})$ and suppose that $\int |\hat{f}|^2 \, d\pi_0 = \int |f|^2 \, dm$ for all $f \in C_c(K)$. Then $\pi_0 = \pi$.*

Proof. Let $k \in C_c(\hat{K})$. Let F be the set of all $f \in C_c(K)$ such that $\hat{f} \geqslant 0$ on $\hat{K}$. Then $\hat{F} = \{\hat{f} : f \in F\}$ is dense in $C_0{}^+(\hat{K})$, by 7.3H. Choose $k_0 \in \hat{F}$ such that $k_0 > k$ on spt k. It is possible to define a sequence $\{k_n\}$ in $\hat{F}$ such that

$$\begin{aligned} k_n &\leqslant 1 && \text{on } \hat{K} \\ k < k_0 k_1 \cdots k_n &= h_n && \text{on spt } k \\ h_n &< k + \frac{1}{n} && \text{on } \hat{K} \end{aligned}$$

for $n \geqslant 1$. Note that each $h_n \in \hat{F}$ also. Thus $\int h_n \, d\pi_0 = \int h_n \, d\pi$ for $n \geqslant 1$. Hence, $\int k \, d\pi_0 = \int k \, d\pi$, by the Lebesgue Dominated Convergence Theorem.

8. CONVOS FROM GROUPS

The main result here is the Theorem 8.2B, which asserts that: If G is a locally compact group and H is a compact subgroup of G, then the double coset space $G//H = \{HxH : x \in G\}$ is a convo in a natural way.

8.1. *Actions*

A *continuous action* of a topological group H on a topological space X is a continuous mapping $(x, s) \mapsto x^s$ from $X \times H$ to X such that

$$x^1 = x \qquad \text{and} \qquad (x^s)^t = x^{(st)}$$

for $x \in X$ and $s, t \in H$. If $x \in X$ then the orbit of x under H is denoted by $x^H = \{x^s : s \in H\}$. The set of orbits is denoted by $X^H = \{x^H : x \in X\}$.

Let G be a group. A mapping $A: G \to G$ is called *affine* if there exists $c \in G$ and an automorphism B of G such that $A(x) = cB(x)$ for all $x \in G$. Note that, if $a, b \in G$ and B is an automorphism of G, then aBb is affine, since $aBb = (ab)(b^{-1}Bb)$ and $b^{-1}Bb$ is also an automorphism.

Let G and H be topological groups. A *continuous affine action* of H on G is a continuous action $(x, s) \mapsto x^s$ for which each mapping $x \mapsto x^s$ is affine.

THEOREM 8.1A. *Let $(x, s) \mapsto x^s$ be a continuous action of a compact* *group H on a nonvoid locally compact Hausdorff space X. Then X^H is a decomposition of X into compact subsets, and X^H is a closed subset of $\mathscr{C}(X)$. The quotient topology on X^H and the relative topology on X^H are equal. With this topology, X^H is a locally compact Hausdorff space, and the natural projection, $x \mapsto x^H$, is a continuous open mapping from X onto X^H.*

Proof. Let π be the natural projection. It is clear that X^H is a closed subset of $\mathscr{C}(X)$. If U and V are open subsets of X then

$$W = \pi^{-1}(\mathscr{C}_U(V)) = \{x \in X : x^H \subset V, x^H \cap U \text{ nonvoid}\}$$

is an open subset of X. Thus the relative topology is contained in the quotient topology. Now let Q be a subset of X^H open in the quotient topology. Then $U = \pi^{-1}(Q)$ is open in X. But $Q = \mathscr{C}(U^H) \cap X^H$. Thus the topologies are equal. Also, if W is an open subset of X then $\pi(W) = \mathscr{C}(W^H) \cap X^H$ is an open subset of X^H.

THEOREM 8.1B. *Let G be a locally compact group and let H be a compact group, and suppose that $(x, s) \mapsto x^s$ is a continuous affine action of H on G. Let λ be a left Haar measure on G and let σ be the normalized Haar measure on H. Let G^H have the quotient topology. For $x, y \in G$ define*

$$(p_{x^H}) * (p_{y^H}) = \int_H \int_H p_{(x^s y^t)^H} \sigma(ds)\, \sigma(dt).$$

Then this operation is well defined, and, with it, G^H is a semiconvo. Moreover, the measure

$$m = \int_G p_{x^H} \lambda(dx)$$

is a left-invariant measure on G^H.

Proof. It is apparent that the operation is well defined. For each $x \in G$ let $\omega_x \in M_c^+(G)$ be the probability measure given by: $\int_G f\, d\omega_x = \int_H f(x^s)\, \sigma(ds)$. The mapping $x^H \mapsto \omega_x$ from G^H to $M^+(G)$ is well defined and continuous. Thus 2.3H applies, and we have an extended mapping, $\mu \mapsto \mu'$, from $M(G^H)$ to $M(G)$. One readily sees that the measures μ' are precisely the measures on G which are invariant under the action of H.

For each $s \in H$ let $a_s = (1^s)^{-1}$, where 1 is the identity of G. It is easy to verify that $(xy)^s = x^s a_s y^s$ for $s \in H$ and $x, y \in G$. Let the measure $\pi \in M_c^+(G)$ be defined by: $\int_G f\, d\pi = \int_H f(a_s)\, \sigma(ds)$. The computation

$(x^s y^t)^u = x^{su} a_u y^{tu}$ shows that the measure $\omega_x * \pi * \omega_y$ is invariant under H, and

$$\omega_x * \pi * \omega_y = (p_{xH} * p_{yH})'.$$

Thus the convolution, for $\mu, \nu \in M(G^H)$, is given by

$$(\mu * \nu)' = \mu' * \pi * \nu'.$$

Thus G^H is a semiconvo. It is apparent, since H is compact and since the action is affine, that each mapping $x \mapsto x^s$ of G leaves λ invariant. Thus the measure m is left-invariant.

8.2. *Cosets*

In this subsection, G is a locally compact group, H is a compact subgroup of G, λ is a left Haar measure on G, and σ is the normalized Haar measure on H. Let the set of left cosets

$$G/H = \{xH : x \in G\}$$

and the set of double cosets

$$G//H = \{HxH : x \in G\}$$

have the quotient topologies.

THEOREM 8.2A. *The space G/H, with the operation*

$$p_{xH} * p_{yH} = \int_H p_{xtyH}\sigma(dt),$$

is a semiconvo. A left-invariant measure on G/H is given by

$$m = \int_G p_{xH}\lambda(dx).$$

THEOREM 8.2B. *The space $G//H$, with the operation*

$$p_{HxH} * p_{HyH} = \int_H p_{HxtyH}\sigma(dt),$$

is a convo. The identity element is $e = H = H1H$. *If* $x \in G$ *then* $(HxH)^- = Hx^{-1}H$. *A left Haar measure on* $G//H$ *is given by*

$$m = \int_G p_{HxH}\lambda(dx).$$

Proof. The mapping $(x, s) \mapsto xs$ is a continuous affine action of H on G, with orbit space G/H. The mapping

$$(x, (s, t)) \leftrightarrow s^{-1}xt$$

is a continuous affine action of $H \times H$ on G, with orbit space $G//H$. The theorems follow readily from 8.1B.

8.3. *Groups of Automorphisms*

Let $(x, s) \mapsto x^s$ be a continuous action of a compact group H on a locally compact group G. Suppose that each mapping $x \mapsto x^s$ is an automorphism of G. Let σ be the normalized Haar measure on H. Let G^H have the quotient topology.

THEOREM 8.3A. *The space* G^H, *with the operation*

$$(p_{x^H}) * (p_{y^H}) = \int_H p_{(x^s y)^H}\sigma(ds) = \int_H p_{(xy^t)^H}\sigma(dt),$$

is a convo, and has a Haar measure. The identity is $1^H = \{1\}$. *If* $x \in G$ *then* $(x^H)^- = (x^{-1})^H$.

THEOREM 8.3B. *Let* G *and* H *be as above. Let* G' *be the product space* $G \times H$ *and let* $H' = \{1\} \times H$. *Define a binary operation on* G' *by*

$$(x, s) \cdot (y, t) = (x^t y, st).$$

Then G' *is a locally compact group,* H' *is a compact subgroup of* G', *and the mapping*

$$H'(x, s)H' \mapsto x^H$$

is an isomorphism from the convo $G'//H'$ *onto the convo* G^H.

Proof A. Since the action is affine, 8.1B applies. The operation given here is correct, since

$$x^s y^t = (x^{st^{-1}}y)^t = (xy^{ts^{-1}})^s$$

for $x, y \in G$ and $s, t \in H$. Also, $(x^{-1})^s = (x^s)^{-1}$ for $x \in G$ and $s \in H$.

Proof B. Here, G' is just a semidirect product of the groups G and H. If $(x, s) \in G'$ then

$$H'(x, s)H' = x^H \times H.$$

Note also that $(x, 1) \cdot (1, s) \cdot (y, 1) = (x^s y, s)$.

8.4. *Compact Groups*

Let G be a compact group, with normalized Haar measure σ. A function ψ on G will be called a *normalized character* if $\psi = (1/n)T$, where T is the trace function of an irreducible unitary representation of G on an n-dimensional Hilbert space. For $x \in G$ let

$$x^G = \{t^{-1}xt \colon t \in G\},$$

the conjugacy class of x. Let $K = \{x^G : x \in G\}$ have the quotient topology.

THEOREM 8.4A. *The space K, with the operation*

$$(p_{x^G}) * (p_{y^G}) = \int_G P_{(t^{-1}xty)^G}\sigma(dt),$$

is a compact commutative convo. The identity is $\{1\}$. If $x \in G$ then $(x^G)^- = (x^{-1})^G$. A function ψ on G is a normalized character if and only if there exists a multiplicative character $\chi \in \hat{K}$ such that $\psi(x) = \chi(x^G)$ for all $x \in G$.

THEOREM 8.4B. *Let K be as above. Let χ_1 and χ_2 be in $\hat{K}$. Then there exist positive numbers a_k and elements ψ_k of $\hat{K}$ such that*

$$\chi_1\chi_2 = \sum_{k=1}^n a_k\psi_k \qquad 1 = \sum_{k=1}^n a_k .$$

This representation is unique. Define

$$p_{\chi_1} * p_{\chi_2} = \sum_{k=1}^n a_k p_{\psi_k}.$$

With this operation, $\hat{K}$ is a commutative discrete convo. The identity of $\hat{K}$ is the constant function 1. *The Plancherel measure π associated with a Haar measure on K is a Haar measure on $\hat{K}$.*

Proof A. The mapping $(x, s) \mapsto s^{-1}xs$ is a continuous action of G on G, and 8.3A applies. Recall that a function ψ on G is a normalized character if and only if ψ is continuous and

$$\psi(x)\,\psi(y) = \int_G \psi(t^{-1}xty)\,\sigma(dt)$$

for all $x, y \in G$. Moreover, a character of G is constant on the conjugacy classes of G.

Proof B. Let m be the normalized Haar measure on K. The space $\hat{K}$ is discrete, since the elements of $\hat{K}$ are orthogonal functions in $L_2(m)$. That an operation on $\hat{K}$ can be defined as stated above follows from known facts about the characters of a compact group. It is clear that the resulting convolution in $M(\hat{K})$ is associative and commutative. Since the mapping $x \mapsto x^-$ is an involution of K, the mapping $\chi \mapsto \chi^-$ is an involution of the semiconvo $\hat{K}$. Let χ_1 and χ_2 be in $\hat{K}$. Suppose that

$$\chi_1\overline{\chi_2} = c + \sum_{k=1}^{n} c_k\psi_k\,,$$

where the ψ_k are nonconstant elements of $\hat{K}$. Then $\int \chi_1\overline{\chi_2}\,dm = c$. Now $c > 0$ if and only if $\chi_1 = \chi_2$. Thus $\hat{K}$ is a convo.

Let π be the Plancherel measure on $\hat{K}$ associated with m. Let $\chi \in \hat{K}$. Then $\int |\chi|^2\,dm = \int |\hat{\chi}|^2\,d\pi$. But $\hat{\chi}(\psi) = 0$ if $\psi \neq \chi$, and $\hat{\chi}(\chi) = \int |\chi|^2\,dm$. Thus

$$\pi = \sum_{\chi\in\hat{K}} \left(\frac{1}{\int |\chi|^2\,dm}\right) p_\chi\,.$$

Let $\pi' = \sum (1/a_\chi)\,p_\chi$ be the Haar measure on $\hat{K}$ specified in (7.1A). It was noted above that $a_\chi = \int |\chi|^2\,dm$. Thus $\pi' = \pi$.

9. Examples

The question of whether or not $\hat{K}$ is a convo, where K is a given commutative convo, will be considered elsewhere. For compact K, the idea is illustrated by 8.4B. It turns out that: In 9.3, $\hat{K}$ is a convo isomorphic to K; in 9.5, $\hat{K}$ is not a convo.

9.1. *Some Small Convos*

An element x of a convo K will be called *self-adjoint* if $x^- = x$. If each element of K is self-adjoint, then K will be called a *Hermitian convo.* Hermitian convos are not rare. The process used in the Introduction to construct an operation on $\mathbf{R}^+$ from that on $\mathbf{R}$ can be applied to any locally compact Abelian group. The process used in 8.4B to construct an operation on $\hat{K}$ does not work in general, as is shown by the example 9.1C.

THEOREM 9.1A. *If K is a Hermitian convo then K is commutative.*

Proof. If $x, y \in K$ then

$$p_x * p_y = (p_x * p_y)^- = p_{y^-} * p_{x^-} = p_y * p_x\,.$$

EXAMPLE 9.1B. Let $K = \{e, a\}$ be a discrete space with two elements. Let β be a real number such that $0 < \beta \leqslant 1$. An operation, depending on β, is defined on K as follows:

$$\begin{aligned} p_e * p_e &= p_e \qquad & p_e * p_a &= p_a \\ p_a * p_e &= p_a \qquad & p_a * p_a &= \beta p_e + (1 - \beta)\, p_a\,. \end{aligned}$$

Then K is a Hermitian convo. The identity is e. Note that K is a group if and only if $\beta = 1$. A Haar measure on K is

$$m = p_e + (1/\beta)\, p_a\,.$$

Let $\hat{K} = \{1, \chi\}$. Then $\chi(e) = 1$ and $\chi(a) = -\beta$. The Plancherel measure on $\hat{K}$ associated with m is

$$\pi = \frac{\beta}{1 + \beta} p_1 + \frac{1}{1 + \beta} p_\chi\,.$$

Also, $\hat{K}$ is a convo and $\hat{K}$ is isomorphic to K, since

$$\chi^2 = \beta \cdot 1 + (1 - \beta)\chi.$$

EXAMPLE 9.1C. Let $K = \{e, a, b\}$ be a discrete space with three elements. An operation, with e the identity, is defined on K as follows:

$$p_a * p_a = \frac{1}{4}p_e + \frac{1}{20}p_a + \frac{7}{10}p_b,$$

$$p_a * p_b = p_b * p_a = \frac{7}{20}p_a + \frac{3}{10}p_b,$$

$$p_b * p_b = \frac{1}{4}p_e + \frac{3}{10}p_a + \frac{9}{20}p_b.$$

Then K is a Hermitian convo. A Haar measure on K is

$$m = p_e + 4p_a + 4p_b.$$

And $\hat{K} = \{1, \chi, \psi\}$, where the values at e, a, b are

$$1 : 1, 1, 1$$

$$\chi : 1, -\frac{3}{4}, \frac{1}{2}$$

$$\psi : 1, \frac{1}{10}, -\frac{7}{20}.$$

The Plancherel measure on $\hat{K}$ associated with m is

$$\pi = \frac{1}{9}p_1 + \frac{4}{17}p_\chi + \frac{100}{153}p_\psi.$$

But $\hat{K}$ is not a convo, since

$$\chi^2 = \frac{17}{36} \cdot 1 - \frac{3}{68}\chi + \frac{175}{306}\psi.$$

EXAMPLE 9.1D. Let S_4 denote the group of all permutations of the set $\{1, 2, 3, 4\}$. The elements of S_4 will be written in cyclic notation, and multiplication is computed from left to right. For example, $(12)(13) = (123)$. Let A_4 be the subgroup of even permutations. We apply here the results of 8.4 to the group $G = A_4$. The elements of $K = \{e, a, b, c\}$ are given below in columns:

e	a	b	c
1	(12)(34)	(123)	(124)
	(13)(24)	(134)	(132)
	(14)(23)	(142)	(143)
		(243)	(234)

Then K is a commutative convo with identity e. Moreover,

$$p_a * p_a = \frac{1}{3}p_e + \frac{2}{3}p_a \qquad p_a * p_b = p_b \qquad p_a * p_c = p_c$$

$$p_b * p_b = p_c \qquad p_b * p_c = \frac{1}{4}p_e + \frac{3}{4}p_a$$

$$p_c * p_c = p_b .$$

Note that $a^- = a$ and $b^- = c$. The normalized Haar measure on K is

$$m = \frac{1}{12}p_e + \frac{1}{4}p_a + \frac{1}{3}p_b + \frac{1}{3}p_c .$$

Let $\alpha = e^{2\pi i/3}$ and $\beta = e^{4\pi i/3}$. Then $\hat{K} = \{1, \chi, \psi, \xi\}$, where the values at e, a, b, c are

$$1 : 1, 1, 1, 1$$

$$\chi : 1, 1, \alpha, \beta$$

$$\psi : 1, 1, \beta, \alpha$$

$$\xi : 1, -\frac{1}{3}, 0, 0.$$

The Plancherel measure on $\hat{K}$ associated with m is

$$\pi = p_1 + p_\chi + p_\psi + 9p_\xi .$$

And $\hat{K}$ is a convo. We have

$$\xi^2 = \frac{1}{9} \cdot 1 + \frac{1}{9}\chi + \frac{1}{9}\psi + \frac{2}{3}\xi.$$

The other products are obvious. For example, $\chi^2 = \psi$ and $\chi\xi = \xi$. For more information about the group A_4 see Hewitt and Ross [4, p. 48].

9.2. *A Finite Noncommutative Convo*

Here we apply the results of (8.3) to the group $G = A_4$, acted on by the group $H = \{1, (12)\}$ by inner automorphisms in S_4. Let the resulting convo be $K = \{e, a, b, c, d, u, v\}$. It is easily seen that S_4 can play the role of G' and that H corresponds to H', where G' and H' are defined in (8.3B). The elements of K are given below in columns:

e	a	b	c	d	u	v
1	(12)(34)	(13)(24)	(123)	(124)	(134)	(143)
		(14)(23)	(132)	(142)	(234)	(243)
(12)	(34)	(1324)	(13)	(14)	(1234)	(1243)
		(1423)	(23)	(24)	(1342)	(1423)

Note that the top half of each column is an orbit under H, and that the entire column is a double coset of H. Note also that $u^{-} = v$, and that the other five elements of K are self-adjoint. The convo K is not commutative, since $p_u * p_v \neq p_v * p_u$. In the adjoining convolution table, we have put x in place of p_x. The normalized Haar measure on K is

$$m = \frac{1}{12}p_e + \frac{1}{12}p_a + \frac{1}{6}p_b + \frac{1}{6}p_c + \frac{1}{6}p_d + \frac{1}{6}p_u + \frac{1}{6}p_v .$$

There are four minimal ideals in $L_2(m)$. Three are one-dimensional, giving rise to $\mathfrak{X}(K) = \{1, \chi, \psi\}$. The values of χ and ψ are, in order,

$$\chi : 1, 1, 1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}$$

$$\psi : 1, -1, 0, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} .$$

The other ideal has dimension 4 and corresponds to a two-dimensional irreducible representation. This gives rise to the normalized character ξ, with values

$$\xi : 1, 0, -\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4} .$$

Even though ψ is a central function on K, the function $f = \psi^2$ is not central, since $f(a \ast u) \neq f(u \ast a)$.

	e	a	b	c	d	u	v
e	e	a	b	c	d	u	v
a	a	e	b	u	v	c	d
b	b	b	$\frac{1}{2}e + \frac{1}{2}a$	$\frac{1}{2}d + \frac{1}{2}v$	$\frac{1}{2}c + \frac{1}{2}u$	$\frac{1}{2}d + \frac{1}{2}v$	$\frac{1}{2}c + \frac{1}{2}u$
c	c	v	$\frac{1}{2}d + \frac{1}{2}u$	$\frac{1}{2}e + \frac{1}{2}c$	$\frac{1}{2}b + \frac{1}{2}u$	$\frac{1}{2}b + \frac{1}{2}d$	$\frac{1}{2}a + \frac{1}{2}v$
d	d	u	$\frac{1}{2}c + \frac{1}{2}v$	$\frac{1}{2}b + \frac{1}{2}v$	$\frac{1}{2}e + \frac{1}{2}d$	$\frac{1}{2}a + \frac{1}{2}u$	$\frac{1}{2}b + \frac{1}{2}c$
u	u	d	$\frac{1}{2}c + \frac{1}{2}v$	$\frac{1}{2}a + \frac{1}{2}u$	$\frac{1}{2}b + \frac{1}{2}c$	$\frac{1}{2}b + \frac{1}{2}v$	$\frac{1}{2}e + \frac{1}{2}d$
v	v	c	$\frac{1}{2}d + \frac{1}{2}u$	$\frac{1}{2}b + \frac{1}{2}d$	$\frac{1}{2}a + \frac{1}{2}v$	$\frac{1}{2}e + \frac{1}{2}c$	$\frac{1}{2}b + \frac{1}{2}u$

9.3. *Rotations of the Plane*

Let G be the group $\mathbf{R} \times \mathbf{R}$, and let H be the circle group. There is a natural continuous action of H on G, each mapping being a rotation. These rotations are automorphisms, as in 8.3. The orbits are concentric circles, with center (0, 0). If $x, y \geqslant 0$ then

$$\|(x, 0) + (y \cos t, y \sin t)\| = (x^2 + y^2 + 2xy \cos t)^{1/2}.$$

We use $\mathbf{R}^+$ as a model, the positive number x representing the circle of radius x. The convolution operation is specified by the identity

$$f(x \ast y) = \frac{1}{2\pi} \int_0^{2\pi} f((x^2 + y^2 + 2xy \cos t)^{1/2})\, dt.$$

This can be rewritten as

$$p_x \ast p_y = \int_{|x-y|}^{x+y} \left(\frac{2s/\pi}{[(x+y+s)(x+y-s)(x-y+s)(-x+y+s)]^{1/2}} \right) p_s\, ds,$$

for positive x and y. The identity element is 0. The Haar measure inherited from $G = \mathbf{R} \times \mathbf{R}$ is given by

$$m(dx) = 2\pi x\, dx,$$

where dx denotes Lebesgue measure on $\mathbf{R}^+$. Also, this convo is Hermitian.

Let J_0 be the Bessel function of order zero. Let J be defined on $\mathbf{R}^+$ by $J(x) = J_0(2x)$. Thus,

$$J(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2}\, x^{2n}.$$

So J is a bounded continuous function on $\mathbf{R}^+$. If $x, y \in \mathbf{R}^+$ then

$$J(x * y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2} \cdot \frac{1}{2\pi} \int_0^{2\pi} (x^2 + y^2 + 2xy \cos t)^n \, dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2} \sum_{u+v+w=n} \binom{n}{u \;\; v \;\; w} x^{2u} y^{2v} (2xy)^w \frac{1}{2\pi} \int_0^{2\pi} \cos^w t \, dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2} \sum_{u+v+2k=n} \binom{n}{u \;\; v \;\; 2k} x^{2u} y^{2v} (2xy)^{2k} \binom{2k}{k} 4^{-k}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2} \sum_{u+v+2k=n} x^{2(u+k)} y^{2(v+k)} \frac{n!}{u!\, v!\, k!\, k!}$$

$$= \sum_{n=0}^{\infty} \sum_{u+v+2k=n} \frac{(-1)^{u+v+2k}}{(u+v+2k)!(u+k)!(v+k)!} \times x^{2(u+k)} y^{2(v+k)} \binom{u+k}{k} \binom{v+k}{k}$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k \leqslant i, k \leqslant j} \frac{(-1)^{i+j}}{(i+j)!\, i!\, j!} x^{2i} y^{2j} \binom{i}{k} \binom{j}{k}$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j}}{(i+j)!\, i!\, j!} x^{2i} y^{2j} \binom{i+j}{i}$$

$$= J(x)\, J(y).$$

Thus J is an element of $\hat{K}$, where $K = \mathbf{R}^+$.

For each $c \geqslant 0$ define χ_c on K by $\chi_c(x) = J(cx)$. Thus $\chi_0 = 1$, and each χ_c is bounded and continuous. Moreover, if $x, y \in K$ then

$$\chi_c(x * y) = \frac{1}{2\pi} \int_0^{2\pi} \chi_c(\sqrt{x^2 + y^2 + 2xy \cos t}) \, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} J(c \sqrt{x^2 + y^2 + 2xy \cos t}) \, dt$$

$$= J(cx * cy) = J(cx)\, J(cy) = \chi_c(x)\, \chi_c(y).$$

Thus each χ_c is in $\hat{K}$ also.

9.4. *Rotations of the Sphere*

Let Σ be the unit sphere,

$$\Sigma = \{(x, y, z): x^2 + y^2 + z^2 = 1\},$$

in $\mathbf{R}^3$. Let G be the group of all orientation-preserving isometries of Σ. Thus G is isomorphic to the special orthogonal group $SO(3)$. Let $NP = (0, 0, 1)$ and let $SP = (0, 0, -1)$. Finally, let

$$H = \{g \in G: g(NP) = NP\}.$$

Thus H is a closed subgroup of G, and H is isomorphic to the circle group. We now examine three structures associated with G and H.

EXAMPLE 9.4A. We apply the results of 8.4 to G. Two elements of G are conjugate if they rotate Σ through the same angle, which we represent by a number in the compact interval $[0, \pi]$. A convolution operation is thereby defined on $K_1 = [0, \pi]$, and K_1 is a compact commutative convo. It is clear that K_1 is Hermitian, that 0 is the identity element, and the support of a convolution $p_x * p_y$ is either an interval or a singleton. In fact, $p_\pi * p_\pi$ has support equal to K_1.

EXAMPLE 9.4B. Consider the semiconvo G/H. For each $x \in \Sigma$ the set

$$C_x = \{g \in G: g(NP) = x\}$$

is a left coset of H. Thus it may be assumed that $K_2 = \Sigma$ is a semiconvo, and that the mapping $x \mapsto c_x$ is an isomorphism from K_2 onto G/H.

Let $x, y \in \Sigma$ and let $d_y = \| y - NP \|$, the distance measured in R^3. Then

$$\operatorname{spt}(p_x * p_y) = S_{x,y} = \{z \in \Sigma: \| z - x \| = d_y\}.$$

If $y \neq NP$, SP then $S_{x,y}$ is a circle, and $p_x * p_y$ is a multiple of the length measure. In the other two cases,

$$p_x * p_{NP} = p_x \qquad \text{and} \qquad p_x * p_{SP} = p_{-x}.$$

Note that NP is not an identity, but just a right identity.

EXAMPLE 9.4C. Now we consider $G//H$. For each number x in the interval $[-1, 1]$ let

$$D_x = \{g \in G: g(NP) = (s, t, x) \text{ for some } s, t\}.$$

Each set D_x is a double coset of H. We thereby have a convo $K_3 = [-1, 1]$. The identity element is 1. The convo K_3 is Hermitian, since the only topological involution of $[-1, 1]$ which leaves 1 fixed is the identity mapping. Thus K_3 is commutative. But K_3 is not isomorphic to K_1, since $p_{-1} * p_{-1} = p_1$.

9.5. *An Example of Naimark*

The convo studied here is essentially the same as a structure given by Naimark [14, p. 274]. However, due to the difference in notation, a certain amount of verification is required.

Let $x, y \in \mathbf{R}^+$. If b is a nonzero complex number, then

$$\int_{|x-y|}^{x+y} \frac{\sin bt}{b}\, dt = 2\frac{\sin bx}{b} \cdot \frac{\sin by}{b}.$$

When $b = i$ this becomes

$$\int_{|x-y|}^{x+y} \sinh t\, dt = 2(\sinh x)(\sinh y).$$

Now we define the convo. Let $K = \mathbf{R}^+$. Let 0 be the identity element. If $x, y \in K$, $x > 0$, $y > 0$, let

$$p_x * p_y = \frac{1}{2(\sinh x)(\sinh y)} \int_{|x-y|}^{x+y} (\sinh t)\, p_t\, dt.$$

It is clear that these measures are probability measures, that the operation is commutative, and that the mappings $(x, y) \mapsto p_x * p_y$ and $(x, y) \mapsto \operatorname{spt}(p_x * p_y)$ are continuous. Associativity will be verified later.

For $a \in \mathbf{C}$ let χ_a be defined on K by

$$\chi_a(x) = \Big[\sum_{n=0}^{\infty} a^n \frac{(-1)^n}{(2n+1)!} x^{2n}\Big] \Big/ \Big[\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n}\Big].$$

Note that, for each $a \in \mathbf{C}$, the function χ_a is continuous, and $\chi_a(0) = 1$. Also, $\chi_{-1} = 1$, and $\chi_0(x) = x/\sinh x$ for $x > 0$.

Let $x, y \in K$ and $a, b \in \mathbf{C}$, with $a = b^2$. If x and a are non-zero then

$$\chi_a(x) = \sin bx/b \sinh x.$$

If, in addition, $y \neq 0$ then

$$\begin{aligned}\chi_a(x \ast y) &= \frac{1}{2(\sinh x)(\sinh y)} \int_{|x-y|}^{x+y} \frac{\sin bt}{b}\, dt \\ &= \chi_a(x)\, \chi_a(y).\end{aligned}$$

Now let $z \in K$, with $z > 0$. It is apparent from the definition of the convolution that there exist finite nonnegative measures μ and ν on K such that

$$\begin{aligned}p_x \ast (p_y \ast p_z) &= (\sinh)\mu \\ (p_x \ast p_y) \ast p_z &= (\sinh)\nu.\end{aligned}$$

It follows from previous computations that

$$\begin{aligned}\int_K \frac{\sin bt}{b}\, \mu(dt) &= \int_K \chi_a \sinh d\mu = \chi_a(x)\, \chi_a(y)\, \chi_a(z) \\ \int_K \frac{\sin bt}{b}\, \nu(dt) &= \int_K \chi_a \sinh d\nu = \chi_a(x)\, \chi_a(y)\, \chi_a(z).\end{aligned}$$

Thus $\mu = \nu$, by the faithfulness of the Fourier transform on the real line.

It has just been proved that K is a semiconvo. It is apparent that K is actually a Hermitian convo and that each χ_a is an element of $\mathfrak{X}(K)$. Naimark proves that the χ_a are the only multiplicative functions on K and that χ_a is bounded if and only $|\operatorname{Im} b| \leqslant 1$, where $a = b^2$. Thus,

$$\begin{aligned}\mathfrak{X}_b(K) &= \{\chi_{c+id}\colon c \geqslant d^2/4 - 1\} \\ \hat{K} &= \{\chi_c\colon -1 \leqslant c < \infty\}.\end{aligned}$$

Recall that for a commutative group G it must be that $\hat{G} = \mathfrak{X}_b(G)$.

A Haar measure m on K is given by

$$m(dx) = (\sinh x)^2\, dx.$$

To verify this, let $f \in C_c^+(K)$ and $x > 0$. Then

$$\int_K f(x * y)\, m(dy) = \int_0^\infty \frac{1}{2(\sinh x)(\sinh y)} \left[\int_{|x-y|}^{x+y} f(t) \sinh t\, dt\right] m(dy)$$

$$= \frac{1}{2 \sinh x} \int_0^\infty \int_{|x-y|}^{x+y} f(t)(\sinh t)(\sinh y)\, dt\, dy$$

$$= \frac{1}{2 \sinh x} \int_0^\infty \int_{|x-t|}^{x+t} f(t)(\sinh t)(\sinh y)\, dy\, dt$$

$$= \frac{1}{2 \sinh x} \int_0^\infty f(t)\, 2(\sinh x)(\sinh t)^2\, dt$$

$$= \int_K f(t)\, m(dt).$$

Note that χ_0 is positive on K. Let $m_0 = \chi_0 m$. Then $p_x * m_0 = \chi_0(x)\, m_0$ for all $x \in K$. Thus m_0 is a subinvariant measure on K, but m_0 is not a Haar measure.

Finally, we shall show that the Plancherel measure on $\hat{K}$ associated with m is given by

$$\int_{\hat{K}} h\, d\pi = \frac{1}{\pi} \int_0^\infty h(\chi_t)\, \sqrt{t}\, dt.$$

The topology on $\hat{K}$ is the obvious one. The formula for π given above shows that

$$\operatorname{spt} \pi = \{\chi_t \colon 0 \leqslant t < \infty\} \neq \hat{K}.$$

For each number $c > 1$ let f_c be defined on $(0, \infty)$ by

$$f_c(x) = \frac{e^{-cx}}{\sinh x}.$$

These functions, while not defined at 0, are in $L_1(m) \cap L_2(m)$. Let $c, d > 1$. Then

$$\int_K f_c f_d\, dm = \int_0^\infty e^{-(c+d)x}\, dx = \frac{1}{c+d}.$$

If $s > 0$ and $t = s^2$ then

$$f_c\hat{}(\chi_t) = \int_0^\infty e^{-cx} \frac{\sin sx}{s}\, dx = \frac{1}{c^2 + s^2} = \frac{1}{c^2 + t}.$$

And if $c \neq d$ then

$$\begin{aligned}\int_{\hat{K}} f_c\hat{\ } f_d\hat{\ }\, d\pi &= \frac{1}{\pi}\int_0^\infty \frac{1}{c^2+t}\,\frac{1}{d^2+t}\,\sqrt{t}\,dt\\ &= \frac{1}{\pi}\int_0^\infty \frac{1}{c^2+s^2}\,\frac{1}{d^2+s^2}\,2s^2\,ds\\ &= \frac{2}{\pi(c^2-d^2)}\int_0^\infty\left(\frac{c^2}{c^2+s^2}-\frac{d^2}{d^2+s^2}\right)ds\\ &= \frac{2}{\pi(c^2-d^2)}\left(\frac{\pi c}{2}-\frac{\pi d}{2}\right)\\ &= \frac{1}{c+d} = \int_K f_c f_d\, dm.\end{aligned}$$

By continuity, this is true even if $c = d$.

It only remains to show that the linear span of the functions f_c is dense in $L_2(m)$. Let $f \in L_2(m)$ and let $g = f \sinh$. Then g is square-integrable with respect to Lebesgue measure. By the properties of the Laplace transform, there exists a number $c > 1$ such that

$$\int_0^\infty g(x)\, e^{-cx}\, dx \neq 0.$$

Thus $\int_K f f_c\, dm \neq 0$. This completes the proof.

10. Subconvos

The notions of subgroup and coset extend naturally to the larger class of convos. One could also generalize the notions of normal subgroup and homomorphism, but this will not be done here. For one thing, there are certain difficulties in the case of noncompact subconvos. The main reason, though, is that the morphisms which are most useful in the theory of convos are not usually homomorphisms, even when the domain is a group.

In this section, K is a convo.

10.1. *Subconvos*

A subset H of K will be called a *subconvo* of K if the following three conditions are satisfied:

(i) H is a closed nonvoid set,

(ii) $H^- = H$,

(iii) $H * H \subset H$.

It follows readily from (II) and (III) that

(iv) $e \in H$.

If μ, $\nu \in M(H)$, then these measures may be regarded as members of $M(K)$, and then $\mu * \nu$ may be regarded as a member of $M(H)$. This defines a convolution on H. Recall that the topology on $\mathscr{C}(H)$ is equal to its relative topology as a subset of $\mathscr{C}(K)$, by 2.5C. Also, a function f is in $C_c^+(H)$ if and only if $f = g \mid H$, where $g \in C_c^+(K)$. Thus, with the operation defined above, H is a convo and has the same identity and adjoint mapping as K.

The condition that a subconvo be closed cannot easily be relaxed, as is shown by 10.1A.

LEMMA 10.1A. *Let H be a nonvoid subset of K, with $H * H \subset H$ and $H^- = H$. Suppose that H is locally compact in the relative topology. Then H is a (closed) subconvo of K.*

Proof. It is clear that e is in H. Let $x \in cH$, the closure of H, and let $\{x_\beta\}_{\beta \in D}$ be a net in H converging to x. Let U be an open subset of K such that $e \in U$ and such that $A = c(U \cap H)$ is a compact subset of H. There exists $\beta_0 \in D$, such that $\{x_\alpha^-\} * \{x_\beta\}$ meets U for all α, $\beta \geqslant \beta_0$. Thus, $\{x_\alpha^-\} * \{x_\beta\}$ meets A for all α, $\beta \geqslant \beta_0$. Let $B = \{x_{\beta_0}\} * A$. This is a compact subset of H. Also, $x_\beta \in B$ for $\beta \geqslant \beta_0$, by 4.1B. Hence, $x \in B \subset H$.

LEMMA 10.1B. *Let A be a subset of K. Then there exists a smallest subconvo H of K which contains A.*

Proof. The intersection of all subconvos of K which contain H is a subconvo of K.

LEMMA 10.1C. *Let A be a σ-compact subset of K. Then there exists a subconvo of K which contains A and is both open and σ-compact.*

Proof. Let $\{U_n\}$ be an increasing sequence of open subsets of K whose union contains A and which has the property that cU_n is compact and $e \in U_n = U_n^-$ for each $n \geqslant 1$. Then

$$U_n^n = (cU_n)^{n-1} * U_n \supset (cU_n)^{n-1} \supset (U_{n-1})^{n-1},$$

by 4.1D. Thus the union H of the sets U_n^n is an open σ-compact subconvo of K containing A, by 10.1A.

10.2. *Special Subconvos*

The following results are well known for groups. It is not true that the convolution of connected subsets of a convo is a connected set; consider any finite convo that is not a group.

LEMMA 10.2A. *Let H be a subconvo of K. If the interior of H is nonvoid then H is open.*

Proof. Let U be the nonvoid interior of H. Then $e \in U * U^-$. By 4.1D, the set $H = (U * U^-) * H = U * (U^- * H)$ is open in K.

LEMMA 10.2B. *Let A and B be connected subsets of K. Suppose that there exist $a \in A$, $b \in B$, and a connected set C such that*

$$\{a\} * \{b\} \subset C \subset A * B.$$

Then $A * B$ is connected.

Proof. Suppose that $A * B \subset V \cup W$, where V and W are open subsets of K, and where $(A * B) \cap V$ and $(A * B) \cap W$ are disjoint. Then C is contained in one of these sets, say $(A * B) \cap V$. The two sets

$$P = \{(x, y) \colon \{x\} * \{y\} \in \mathscr{C}(V)\},$$

$$Q = \{(x, y) \colon \{x\} * \{y\} \in \mathscr{C}_W(K)\}$$

are open in $K \times K$. Moreover, $P \cup Q$ contains $A \times B$, and $(A \times B) \cap P$ and $(A \times B) \cap Q$ are disjoint. Thus $A \times B \subset P$, since $(a, b) \in P$ and $A \times B$ is connected. Hence, $A * B \subset V$.

LEMMA 10.2C. *Let H be the component of e in K. Then H is a subconvo of K.*

Proof. Since H^- is connected, $H^- = H$. And $H * H$ is connected, since $\{e\} * \{e\} \subset \{e\} \subset H * H$. Thus $H * H \subset H$.

LEMMA 10.2D. *Let A be a compact subset of K and let $a \in A$. Suppose that $A * A \subset A$. Then there exists a probability measure $\mu \in M^+(K)$ such that* spt $\mu \subset A$, $p_a * \mu = \mu$, *and* $\mu * \mu = \mu$.

Proof. For each $n \geqslant 1$ let

$$\mu_n = \frac{1}{n}[p_a + (p_a * p_a) + \cdots + (p_a)^n].$$

Thus,

$$\| p_a * \mu_n - \mu_n \| = \left\| \frac{1}{n}[p_a - (p_a)^n] \right\| \leqslant \frac{2}{n}.$$

Since A is compact and contains the supports of all these measures, there exists a limit-point μ of the sequence $\{\mu_n\}$ in $M^+(K)$. It is clear that $\mu(K) = 1$ and $p_a * \mu = \mu$. But this implies that $\mu_n * \mu = \mu$ for all $n \geqslant 1$. Thus $\mu * \mu = \mu$.

THEOREM. *Suppose that there exists a Haar measure on K.*

(10.2E) *If $\mu \in M^+(K)$, $\mu \neq 0$, and $\mu * \mu = \mu$, then $\mu^- = \mu$, the set $H =$* spt μ *is a compact subconvo of K, and μ is the normalized Haar measure on H.*

(10.2F) *If H is a compact nonvoid subset of K and $H * H \subset H$, then $H^- = H$ and H is a subconvo of K.*

Proof E. Let m be a left Haar measure on K. It is clear that $\mu(K) = 1$. Thus $0 < \| T_\mu \| \leqslant 1$ and $T_\mu^2 = T_{\mu * \mu} = T_\mu$. It follows that T_μ is an orthogonal projection on $L_2(m)$. This implies that $T_\mu^* = T_\mu$. Hence, $\mu^- = \mu^* = \mu$ and $H^- = H$.

We have shown that H is a subconvo of K. The following computations take place on H.

Let $f \in C_c^+(H)$. Let $g = \mu * f$, which is in $C_0^+(H)$. Thus there exists $b \in H$ such that $g(b) = \| g \|_u$. Since $\mu * g = g$, we have that

$$g(b) = \int_H g(y^- * b)\, \mu(dy) = \int_H g(x * b)\, \mu(dx).$$

It follows that $g(b) = g(x * b)$ for all $x \in H$. This implies that g is constant on H. Therefore, H is compact. Moreover, if $x \in H$ then

$$\int_H f(y)\, \mu(dy) = g(e) = g(x) = \int_H f(y * x)\, \mu(dy).$$

This implies that μ is a right Haar measure on H.

Proof F. Let $a \in H$. It follows readily from the two previous results that $a^- \in H$ also.

10.3. *Cosets*

In this subsection, H is a subconvo of K. For $x \in K$, let $xH = \{x\} * H$. These sets will be called *left cosets* of H. The collection $K/H = \{xH : x \in K\}$ will be given the quotient topology.

LEMMA 10.3A. *Let x, $y \in K$. Then xH and tH are either equal or disjoint.*

Proof. Suppose that $z \in xH \cap yH$. Then $\{x^-\} * \{z\}$ meets H, by 4.1B. Thus $x \in zH^- = zH$. It follows that $xH = zH$. By symmetry, $yH = zH$.

THEOREM 10.3B. *The space K/H is a locally compact Hausdorff space. The natural projection, $x \mapsto xH$, is an open continuous mapping from K onto K/H.*

Proof. Let π be the natural projection. If U is an open subset of K then $\pi^{-1}(\pi(U)) = U * H$ is also open, and thus $\pi(U)$ is open. So π is an open continuous mapping, and this implies that K/H is locally compact.

Let xH and yH be distinct elements of K/H. Then $\{x^-\} * \{y\}$ is disjoint from H. By 3.2D, there exist open neighborhoods U and V of x and y such that $(U^- * V) \cap H = (U^- * V) \cap (H * H^-)$ is void. Using 4.1B again we see that $U * H$ and $V * H$ are disjoint. This implies that $\pi(U)$ and $\pi(V)$ are disjoint open neighborhoods of xH and yH.

LEMMA 10.3C. *The space K is the union of a collection of disjoint open and closed σ-compact subsets.*

Proof. This is a consequence of 10.1C.

10.4. *Subgroups*

If x, y, $z \in K$ then the formula $x * y = z$ will be used to say that $p_x * p_y = p_z$. Let the set G be defined by

$$G = \{x \in K: x * x^- = x^- * x = e\}.$$

We shall call G the *maximum subgroup* of K.

LEMMA 10.4A. *The domain of the mapping* $(x, y) \mapsto x \ast y$ *is a closed subset of* $K \times K$, *and the mapping is continuous on this set.*

LEMMA 10.4B. *Let* $x \in G$. *Then* $x \ast y$ *is defined for all* $y \in K$, *and the mapping* $y \mapsto x \ast y$ *is a homeomorphism from* K *onto* K.

THEOREM 10.4C. *The set* G *is a closed subconvo of* K. *If* $x, y \in G$ *then* $x \ast y$ *is defined and in* G. *With the operation* $(x, y) \mapsto x \ast y$, G *is a locally compact group.*

LEMMA 10.4D. *Let* $x \in G$. *Then the mapping* $y \mapsto x^- \ast y \ast x$ *is an automorphism of* K.

Proof A. This is apparent.

Proof B. To simplify the notation, let ab stand for $\{a\} \ast \{b\}$. Let $y \in K$ and choose $s \in xy$. Then $x^-s \subset x^-xy = ey = y$. Thus $x^-s = y$. Also, $xy = xx^-s = s$.

Thus the mapping $y \mapsto xy$ is well-defined and continuous. The inverse is $y \mapsto x^-y$.

Proof. C. Let $x, y \in G$. Then $x \ast y = z$ is defined, and $zz^- = xyy^-x^- = xx^- = e$. The rest is clear.

Proof D. If $y, z \in K$ then $(x^- \ast y \ast x)^- = x^- \ast y^- \ast x$ and

$$p_{x^- \ast y \ast x} \ast p_{x^- \ast z \ast x} = p_{x^-} \ast (p_y \ast p_z) \ast p_x .$$

10.5. *Products and Joins*

Let J be a convo.

Product. The set $J \times K$ can be made into a convo in the following way. The topology is the product topology. If (s, t) and (x, y) are in $J \times K$ then

$$p_{(s,t)} \ast p_{(x,y)} = (p_s \ast p_x) \times (p_t \ast p_y).$$

The details are not difficult.

Join. Suppose that J is compact and that K is discrete. Suppose also that $J \cap K = \{e\}$, where e is the identity of both convos. Let $J \vee K = J \cup K$ have that unique topology for which J and K are closed subspaces of $J \vee K$. Let σ be the normalized Haar measure on J. The operation $\bullet$ is defined as follows:

(i) If $x, y \in J$ then $p_x \bullet p_y = p_x \ast p_y$.
(ii) If $x, y \in K$ and $x \neq y^-$ then $p_x \bullet p_y = p_x \ast p_y$.
(iii) If $x \in J$ and $e \neq y \in K$ then $p_x \bullet p_y = p_y = p_y \bullet p_x$.
(iv) If $x \in K$ and $x \neq e$ and $p_{x^-} \ast p_x = \sum_{t\in K} c_t p_t$ then

$$p_{x^-} \bullet p_x = c_e \sigma + \sum_{\substack{t\in K \\ t\neq e}} c_t p_t .$$

We omit further detail.

Note that J is a compact subconvo of $J \vee K$. But K is not a subconvo unless either J or K is equal to $\{e\}$. Hence, if J and K are both finite but nontrivial, then $J \vee K$ and $K \vee J$ are not equal as convos.

11. Representations of Convos

In this section the theory of representations of convos on Hilbert spaces is developed. The results and their proofs are practically identical with those for groups. Certain proofs are therefore omitted.

Though the fact plays no role here, there exist unbounded positive-definite (continuous) functions on certain convos. The unbounded real-valued multiplicative functions on the convo of Subsection 9.5 are examples.

In this section, K is a convo.

11.1. *Positive-definite Functions*

A complex-valued function f on K will be called *positive-definite* if f is continuous and

$$0 \leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} f(x_i \ast x_j^-)$$

for each choice of complex numbers a_i and points x_i in K. A positive-definite function need not be bounded.

LEMMA. *Let f be a bounded positive-definite function on K.*

(11.1A) *$\bar{f}$ is positive-definite.*

(11.1B) *If $\mu \in M(K)$ then $\int_K f\, d(\mu * \mu^*) \geqslant 0$.*

(11.1C) *If $\mu \in M(K)$ then $\mu * f * \mu^*$ is positive-definite.*

(11.1D) *If $x \in K$ then $f(x^-) = \overline{f(x)}$.*

(11.1E) $f(e) = \| f \|_u$.

Proof A. This follows directly from the definition.

Proof B. The statement is true if μ has finite support. One can use 2.2A.

Proof C. If $\mu, \nu \in M(K)$ and $\pi = \mu^- * \nu$, then

$$\int_K (\mu * f * \mu^*)\, d(\nu * \nu^*) = \int_K f\, d(\pi * \pi^*) \geqslant 0.$$

Proofs D, E. Recall the following fact: If A, B, C, D are complex numbers, and $0 \leqslant A + Bz + C\bar{z} + Dz\bar{z}$ for all complex numbers z, then $A \geqslant 0$, $B = \bar{C}$, $D \geqslant 0$, $| B |^2 \leqslant AD$, and $2 \, | B | \leqslant A + D$.

Let $x \in K$. If $z \in \mathbf{C}$ then

$$0 \leqslant f(e) + zf(x) + \bar{z}f(x^-) + z\bar{z}f(x * x^-).$$

Thus, $f(e) \geqslant 0, f(x^-) = \overline{f(x)}, f(x * x^-) \geqslant 0$, $| f(x)|^2 \leqslant f(e)f(x * x^-)$, and $2 \, | f(x)| \leqslant f(e) + f(x * x^-)$. The last inequality implies that $2 \, | f(x)| \leqslant f(e) + \| f \|_u$. Thus, $2 \, \| f \|_u \leqslant f(e) + \| f \|_u$.

11.2. *The Pseudo-Inner Product*

In this subsection, f is a bounded positive-definite function on K. If $\mu, \nu \in M(K)$ let

$$[\mu, \nu]_f = \int_K \bar{f}\, d(\nu^* * \mu).$$

If $\mu \in M(K)$ let

$$\| \mu \|_f = ([\mu, \mu]_f)^{1/2}.$$

LEMMA. *Let $\mu, \nu, \pi \in M(K)$.*

(11.2A) *The form $[\ ,\ ]_f$ is a pseudo-inner product on $M(K)$.*

(11.2B) $[\pi * \mu, \nu]_f = [\mu, \pi^* * \nu]_f = \int_K (\mu * \bar{f} * \nu^*)\, d\pi^-$.

(11.2C) $\| \mu \|_f \leqslant (f(e))^{1/2} \, \| \mu \|$.

(11.2D) $\| \pi * \mu \|_f \leqslant \| \pi \| \cdot \| \mu \|_f$.

Proof. This is straightforward.

LEMMA. *Let* $H = \{x \in K : f(x) = f(e)\}$.

(11.2E) *If* $x \in H$ *then* $\| p_x - p_e \|_f = 0$.

(11.2F) H *is a subconvo of* K.

Proof. If $x, y \in H$ then

$$\begin{aligned}[p_x - p_e, p_y - p_e]_f &= \overline{f(y^- * x)} - \overline{f(x)} - \overline{f(y^-)} + \overline{f(e)} \\ &= f(x^- * y) - \overline{f(x)} - f(y) + f(e) \\ &= f(x^- * y) - f(e).\end{aligned}$$

Thus, if $x \in H$ then $0 \leqslant \| p_x - p_e \|_f^2 = f(x^- * x) - \| f \|_u \leqslant 0$, and so $\| p_x - p_e \|_f = 0$. Using this, if $x, y \in H$ then $0 = f(x^- * y) - f(e)$, which implies that $\{x^-\} * \{y\} \subset H$. Thus, $e \in H$ and $H^- * H \subset H$.

11.3. *Representations*

Let $\mathscr{H}$ be a Hilbert space, possible with dimension zero. Let $B(\mathscr{H})$ be the Banach *-algebra of all bounded linear operators on $\mathscr{H}$, and let I be the identity operator.

We shall say that U is a *representation* of K on $\mathscr{H}$ if the following four conditions are satisfied:

(i) The mapping $\mu \mapsto U_\mu$ is a *-homomorphism from $M(K)$ into $B(\mathscr{H})$.

(ii) If $\mu \in M(K)$ then $\| U_\mu \| \leqslant \| \mu \|$.

(iii) $U_{p_e} = I$.

(iv) If $a, b \in \mathscr{H}$ then the mapping $\mu \mapsto \langle U_\mu a, b \rangle$ is continuous on $M^+(K)$ with respect to the cone topology.

Let U be as above. We shall write U_x for U_{p_x} for $x \in K$. By condition (iv), if $a, b \in \mathscr{H}$ then the mapping $x \mapsto \langle U_x a, b \rangle$ is bounded and continuous, and

$$\langle U_\mu a, b \rangle = \int_K \langle U_x a, b \rangle \, \mu(dx)$$

for all $\mu \in M(K)$.

The only notion of equivalence between representations that will be used is that of unitary equivalence.

LEMMA. *Let* U *be a representation of* K *on the Hilbert space* $\mathscr{H}$. *Let* $a \in \mathscr{H}$.

(11.3A) *If f is defined on K by $f(x) = \langle U_x a, a\rangle$ then f is positive-definite.*

(11.3B) *If $\{\mu_\beta\}_{\beta \in D}$ is a net in $M^+(K)$ converging to μ then*

$$\lim_\beta \| U_{\mu_\beta} a - U_\mu a \| = 0.$$

(11.3C) *If $\mu \in M(K)$ then $U_\mu a$ is in the closed linear span of the set $\{U_x a : x \in K\}$.*

Proof. The first part is obvious. For the second, using the same function f, we have that

$$\| U_{\mu_\beta} a - U_\mu a \|^2 = \int_K f \, d(\mu_\beta^- * \mu_\beta - \mu_\beta^- * \mu - \mu^- * \mu_\beta + \mu^- * \mu).$$

Recall that convolution is continuous on $M^+(K)$. The rest is clear.

11.4. *Irreducible Positive-Definite Functions*

In this subsection, f is a bounded positive-definite function.

We shall say that f *generates* a function g on K if there exists a sequence $\{\mu_n\}$ in $M(K)$ such that

$$\lim_{m,n\to\infty} \| \mu_m * f * \mu_n{}^* - g \|_u = 0.$$

Of course, if f generates g then g is bounded and positive-definite.

We shall say that f is *irreducible* if $f \neq 0$ and whenever $f = g + h$, where g and h are bounded positive-definite functions, then $g = cf$ and $h = (1 - c)f$ for some number c in $[0, 1]$.

LEMMA 11.4A. *Let $\{\mu_n\}$ be a sequence in $M(K)$. Then the following three conditions are equivalent:*

(i) *$\{\mu_n\}$ is a Cauchy sequence with respect to the pseudo-norm $\| \ \|_f$.*

(ii) *The $\lim_{m,n\to\infty} (\mu_m * f * \mu_n{}^*)(e)$ exists.*

(iii) *There exists a function g on K such that*

$$\lim_{m,n\to\infty} \| \mu_m * f * \mu_n{}^* - g \|_u = 0.$$

Proof. It is clear that (iii) implies (ii). And (ii) implies (i), since $[\mu_m, \mu_n]_f = (\mu_m * f * \mu_n{}^*)(e)$, by 11.2B. It is also true, by 11.2B, that

$$(\mu_m * f * \mu_n{}^*)(x^-) = [p_x * \mu_m, \mu_n]_f.$$

So (i) implies (iii), since

$$|[p_x * \mu_i, \mu_j]_f - [p_x * \mu_m, \mu_n]_f| \leqslant \|\mu_i - \mu_m\|_f \|\mu_j\|_f + \|\mu_m\|_f \|\mu_j - \mu_n\|_f .$$

THEOREM.

(11.4B) *There exists a Hilbert space $\mathscr{H}$ and a cyclic representation U of K on $\mathscr{H}$, with cyclic vector a, such that $\overline{f(x)} = \langle U_x a, a\rangle$ for all $x \in K$.*

(11.4C) *If $\mu, \nu \in M(K)$ then $\langle U_\mu a, U_\nu a\rangle = [\mu, \nu]_f$.*

(11.4D) *This representation is unique up to unitary equivalence.*

(11.4E) *A function g is generated by f if and only if there exists $b \in \mathscr{H}$ such that $\overline{g(x)} = \langle U_x b, b\rangle$ for all $x \in K$.*

(11.4F) *If $f = g + h$, where g and h are bounded positive-definite functions, then f generates g and h.*

(11.4G) *U is irreducible if and only if f is irreducible.*

Proof. The first three parts are standard results.

For 11.4E, let $\{\mu_n\}$ be a sequence in $M(K)$, let $b_n = U_{\mu_n} a$, and let $g_{m,n} = \mu_m * f * u_n{}^*$. Then

$$\overline{g_{m,n}(x)} = \langle U_x b_m , b_n\rangle.$$

The result follows from the previous lemma, since a is a cyclic vector.

For 11.4F, recall that the mapping $\mu \mapsto \int_K f\, d\mu$ is a positive functional on $M(K)$. This part follows from 11.4E. See Hewitt and Ross [3, p. 325].

The same considerations apply to 11.4G.

11.5. *Absolutely Continuous Measures*

In this subsection it is assumed that there exists a left Haar measure m on K. The algebra $M_a(K)$ of all measures which are absolutely continuous with respect to m is discussed in subsections (5.6) and (6.2).

THEOREM 11.5A. *Let $\mathscr{H}$ be a Hilbert space and let $\nu \mapsto V_\nu$ be a bounded $*$-homomorphism from the Banach $*$-algebra $M_a(K)$ into $B(\mathscr{H})$. Suppose that, if $a \in \mathscr{H}$ and $V_\nu a = 0$ for all $\nu \in M_a(K)$, then $a = 0$. Then there exists a unique representation U of K on $\mathscr{H}$ such that $U_\nu = V_\nu$ for each $\nu \in M_a(K)$.*

Proof. Let S be the linear span of the set $\{V_\nu a : \nu \in M_a(K),\ a \in \mathscr{H}\}$.

Then S is dense in $\mathscr{H}$. Let $\mu \in M^+(K)$ and let $\{\nu_\beta\}_{\beta \in D}$ be a net in $M_a^+(K)$ converging to μ. Let $\nu_k \in M_a(K)$ and $a_k \in \mathscr{H}$ for $1 \leqslant k \leqslant n$. Then

$$\left\| V_{\nu_\beta}\left(\sum_{k=1}^{n} V_{\nu_k} a_k\right) - \sum_{k=1}^{n} V_{\mu * \nu_k} a_k \right\| \leqslant \sum_{k=1}^{n} \| \nu_\beta * \nu_k - \mu * \nu_k \| \cdot \| a_k \|.$$

It follows that the operators V_{ν_β} converge pointwise on S. Each $\| V_{\nu_\beta} \| \leqslant \| \nu_\beta \| = \nu_\beta(K)$, and $\nu_\beta(K) \to \mu(K) = \| \mu \|$. Since S is dense, there exists $U_\mu \in B(\mathscr{H})$ such that $V_{\nu_\beta} \to U_\mu$ pointwise on $\mathscr{H}$ and such that $\| U_\mu \| \leqslant \| \mu \|$. The rest is clear.

THEOREM 11.5B *Let h be a bounded Borel function on K. Suppose that $\int_K h\, d(\nu * \nu^*) \geqslant 0$ for all $\nu \in M_a(K)$. Then there exists a (continuous) bounded positive-definite function f on K such that $h = f$ locally almost everywhere.*

Proof. The mapping $\nu \mapsto \int h\, d\nu$ is a bounded positive linear functional on $M_a(K)$. Thus there exists a cyclic representation $\nu \mapsto V_\nu$ of $M_a(K)$ which satisfies the conditions of the previous theorem. That is, there exists $a \in \mathscr{H}$ such that $\langle V_\nu a, a\rangle = \int h\, d\nu$ for all $\nu \in M_a(K)$. Let U be as described in the theorem, and let f be defined on K by $f(x) = \langle U_x a, a\rangle$. Then $\int f\, d\nu = \int h\, d\nu$ for all $\nu \in M_a(K)$.

LEMMA 11.5C. *Let $f \in L_2(m)$. Then $f * \bar{f}^-$ is a bounded positive-definite function on K.*

Proof. If $\mu \in M(K)$ then $\int_K (f * \bar{f}^-)\, d(\mu * \mu^*) = \| \mu^- * f \|_2^2$.

THEOREM. *Let P be the set of all bounded positive-definite functions f on K such that $0 \leqslant f(e) \leqslant 1$. Let P have the topology determined by the mappings $f \mapsto \int f\, d\nu$, for all $\nu \in M_a(K)$. If $\mu \in M(K)$ let $N(\mu) = \sup_{f \in P} \| \mu \|_f$.*

(11.5D) *P is a compact space, and P is closed under convex combination.*

(11.5E) *A nonzero $f \in P$ is an extreme point of P if and only if f is irreducible and $f(e) = 1$.*

(11.5F) *If $\mu \in M(K)$ and U is a representation of K then $\| U_\mu \| \leqslant N(\mu)$.*

(11.5G) *If $\mu \in M(K)$, $f \in P$, and U is the representation determined by f as in* 11.4B, *then $\| \mu \|_f \leqslant \| U_\mu \|$.*

(11.5H) *If* $\nu \in M_a(K)$ *then there exists an irreducible* $f \in P$ *such that* $\| \nu \|_f = N(\nu)$ *and* $f(e) = 1$.

Proof. The first two parts are standard results.

For 11.5F, let U be a representation of K on $\mathscr{H}$ and let $a \in \mathscr{H}$ be such that $\| a \| \leqslant 1$. Define f on K by $\overline{f(x)} = \langle U_x a, a \rangle$. Then $f \in P$ and

$$\| U_\mu a \|^2 = \langle U_{\mu^* * \mu} a, a \rangle = \int_K f \, d(\mu^* * \mu) = \| \mu \|_f^2 .$$

Thus $\| \mu \|_f \leqslant \| U_\mu \| \leqslant N(\mu)$. This proves 11.5G also.

For 11.5H, let $\nu \in M_a(K)$. It may be assumed that $\nu \neq 0$. The mapping $g \mapsto \int g \, d(\nu^* * \nu) = \| \nu \|_g^2$ is linear and continuous. Therefore, the supremum $N(\nu)^2$ is achieved at an extreme point h of P. Let $f = \bar{h}$.

12. Commutative Convos

The main results here are the Inversion Theorem and Bochner's Theorem. We also consider the question of when the dual of a commutative convo is a convo and prove a (rather weak) Duality Theorem.

In this section, K is a commutative convo, m is a Haar measure on K, and π is the Plancherel measure on $\hat{K}$ associated with m. The notation is as in Section 7.3.

12.1. *The Inverse Fourier Transform*

Note that the mapping $(\chi, x) \to \chi(x)$ is continuous on $\hat{K} \times K$. If $a \in M(\hat{K})$ and $k \in L_1(\pi)$ then $\check{a} = a^\vee$ and $k^\vee$ are defined on K by

$$a^\vee(x) = \int_{\hat{K}} \chi(x) \, a(d\chi),$$

$$k^\vee(x) = \int_{\hat{K}} \chi(x) \, k(\chi) \, \pi(d\chi).$$

Lemma. *Let* $f, g \in L_1(m) \cap L_2(m)$ *and let* $h = f * g$.

(12.1A) $h \in C_0(K) \cap L_1(m)$.

(12.1B) $\hat{h} \in C_0(\hat{K}) \cap L_1(\pi)$.

(12.1C) *If* $\mu \in M(K)$ *then* $\int_K h \, d\mu^- = \int_{\hat{K}} \hat{h} \hat{\mu} \, d\pi$.

Proof. Recall that $\hat{h} = \hat{f}\hat{g}$ and that $\hat{f}$ and $\hat{g}$ are both in $C_0(\hat{K}) \cap L_2(\pi)$. If $\mu \in M(K)$ then

$$\int_K h\, d\mu^- = \int_K (f * gm)\, d\mu^- = \int_K (\mu * f)\,\overline{g^*}\, dm = \int_{\hat{K}} \hat{\mu}\hat{f}\hat{g}\, d\pi.$$

LEMMA. *Let $\mu \in M(K)$ and $a \in M(\hat{K})$.*

(12.1D) *$\check{a}$ is continuous and $\| \check{a} \|_u \leqslant \| a \|$.*

(12.1E) $(\mu^-)^\wedge = (\hat{\mu})^-$ $\quad (\bar{\mu})^\wedge = (\hat{\mu})^*$ $\quad (\mu^*)^\wedge = \bar{\hat{\mu}}$.

(12.1F) $(a^-)^\vee = (\check{a})^-$ $\quad (\bar{a})^\vee = (\check{a})^*$ $\quad (a^*)^\vee = \bar{\check{a}}$.

(12.1G) $\int_K \check{a}\, d\mu^- = \int_{\hat{K}} \hat{\mu}\, da$.

(12.1H) $\mu * \check{a} = (\hat{\mu}a)^\vee$.

Proof. This is straightforward.

THEOREM 12.1I. *Let $k \in L_1(\pi) \cap L_2(\pi)$. Then*

$$\int_K | k^\vee |^2\, dm = \int_{\hat{K}} | k |^2\, d\pi.$$

Proof. Let $f \in L_1(m) \cap L_2(m)$. By (12.1G),

$$\int_K k^\vee \bar{f}\, dm = \int_K k^\vee d(f^* m)^- = \int_K (f^* m)^\wedge k\, d\pi = \int_K k\bar{\hat{f}}\, d\pi.$$

In view of 7.3I, $\| k^\vee \|_2 = \| k \|_2$.

12.2 *The Inversion Formula*

Note that the functions f which satisfy the conditions of (12.2C) form a dense subspace of $L_1(m)$.

THEOREM 12.2A. *Let $a \in M(\hat{K})$. If $\check{a} = 0$ then $a = 0$.*

LEMMA 12.2B. *Let $\mu \in M(K)$ and $a \in M(\hat{K})$. Then $\mu = \check{a}m$ if and only if $a = \hat{\mu}\pi$.*

THEOREM 12.2C. *Let $f \in C(K)$. Suppose that f is integrable and that $\hat{f}$ is also integrable. Then $f = (\hat{f})^\vee$. That is, if $x \in K$ then*

$$f(x) = \int_{\hat{K}} \chi(x) \hat{f}(\chi)\, \pi(d\chi).$$

Proof A. Suppose that $a \neq 0$. Since the set $\{\hat{\mu} : u \in M_a(K)\}$ is dense in $C_0(\hat{K})$, there exists $\mu \in M_a(K)$ such that $0 \neq \int_{\hat{K}} \hat{\mu}\, da = \int_K \check{a}\, d\mu^-$.

Proof B. Let h be as in the previous subsection. Then, by 12.1C and 12.1G,

$$\int_K h\, d\mu^- = \int_{\hat{K}} \hat{h}\, d(\hat{\mu}\pi),$$

$$\int_K h\, d(\check{a}m)^- = \int_{\hat{K}} \hat{h}\, da.$$

Note that the h are dense in $C_0(K)$ and that the $\hat{h}$ are dense in $C_0(\hat{K})$.

Proof C. This follows from the previous result, with $\mu = fm$ and $a = \hat{f}\pi = \hat{\mu}\pi$.

12.3. *Bochner's Theorem*

THEOREM 12.3A. *Let $a \in M^+(\hat{K})$. Then $\check{a}$ is a bounded positive-definite function on K.*

THEOREM 12.3B. *Let f be a bounded positive-definite function on K. Then there exists a unique $a \in M^+(\hat{K})$ such that $f = \check{a}$.*

Proof A. Let $\mu \in M(K)$. Then

$$\int_K \check{a}\, d(\mu * \mu^*) = \int_{\hat{K}} (\mu * \mu^*)^\wedge\, da^- = \int_{\hat{K}} |\, \hat{\mu}\, |^2\, da^-.$$

Proof B. Assume that $0 < f(e) \leqslant 1$. Let $\nu \in M_a(K)$. By 11.5H, there exists an irreducible bounded positive-definite function χ on K such that $\chi(e) = 1$ and $\| \nu \|_f \leqslant \| \nu \|_\chi$. Since K is commutative, $\chi \in \hat{K}$. Thus $|\int f\, d\nu^-| = |\int \bar{f}\, d\nu| = |[\nu, p_e]_f| \leqslant \| \nu \|_f \| p_e \|_f \leqslant \| \nu \|_f \leqslant \| \nu \|_\chi = |\int \bar{\chi}\, d\nu| = |\hat{\nu}(\chi)| \leqslant \| \hat{\nu} \|_u$.

The mapping $\hat{\nu} \mapsto \int f\, d\nu^-$ is therefore bounded and linear on a dense subspace of $C_0(\hat{K})$. Thus there exists $a \in M(\hat{K})$ such that $\int_K f\, d\nu^- = \int_{\hat{K}} \hat{\nu}\, da$ for all $\nu \in M_a(K)$. By (12.1G), $f = \check{a}$. One can see from the previous proof that $a \geqslant 0$.

12.4 *The Dual Convo*

In general, the product of characters is not positive-definite. See the Example 9.1C. However, if $\chi, \psi \in \hat{K}$ and $\chi\psi$ is positive-definite then, by Bochner's Theorem, there exists a measure $a \in M^+(\hat{K})$ such that $\check{a} = \chi\psi$.

It is clear that a is a probability measure, and we might set $p_\chi * p_\psi = a$.

The statement $\hat{K}$ is a convo means that the formula

$$(a * b)^\vee = \check{a}\check{b} \qquad \text{for} \quad a, b \in M(\hat{K})$$

determines a convolution on $\hat{K}$, that with this convolution $\hat{K}$ is a convo, and that the adjoint of χ is χ^- for each $\chi \in \hat{K}$. Note that the identity of $\hat{K}$ must be the constant function 1.

THEOREM. *Suppose that $\hat{K}$ is a convo. For each $x \in K$ let $\tilde{x}$ be defined on $\hat{K}$ by $\tilde{x}(\chi) = \overline{\chi(x)}$. Let $\tilde{K} = \{\tilde{x} : x \in K\}$ and let $L = \hat{K}$.*

(12.4A) *The measure π is a Haar measure on L, and* $\operatorname{spt} \pi = L$.

(12.4B) *The mapping $x \mapsto \tilde{x}$ is a homeomorphism from K onto the closed subset $\tilde{K}$ of $\hat{L}$.*

(12.4C) *The measure $\tilde{m} = \int_K p_{\tilde{x}} m(dx)$ is the Plancherel measure on $\hat{L}$ associated with π.*

(12.4D) *If $\hat{L}$ is a convo then it is isomorphic to K.*

Proof A. Let $k \in L_1(\pi)$ and $\chi \in L$. Suppose that $f = k^\vee \in L_1(m)$. If $\psi \in L$ then $k(\chi^- * \psi) = \int k \, d(p_{\chi^-} * p_\psi) = \int \chi^- \psi f^- \, dm = \int \chi \bar{\psi} f \, dm$, by (12.1G). Thus $(p_\chi * k)^\vee = \chi k^\vee$.

Let $j \in C_c^+(L)$. As in the proof of 7.3P, j can be approximated simultaneously in $L_1(\pi)$ and $L_2(\pi)$ by functions such as k above. Thus, $(p_\chi * j)^\vee = \chi j^\vee$ for $\chi \in L$. Recall the definition of π in 7.3N and note that $j' = j^\vee$. Thus $\int (p_\chi * j) \, d\pi = (p_\chi * j)^\vee(e) = (\chi j^\vee)(e) = j^\vee(e) = \int j \, d\pi$.

Proof B. It is obvious that $\tilde{K} \subset \hat{L}$ and that the mapping is both continuous and one-to-one. Let $\{x_\beta\}_{\beta \in D}$ be a net in K such that $x_\beta \to \infty$. All we must show is that it is not possible for $\tilde{x}_\beta \to \phi$, where $\phi \in \hat{L}$. If $\tilde{x}_\beta \to \phi$ then there exists $k \in C_c(L)$ such that $\hat{k}(\phi) \neq 0$, and this implies that the $\hat{k}(\tilde{x}_\beta)$ do not converge to 0. But it follows from previous results that $k^\vee \in C_0(K)$. This is a contradiction, since each $\hat{k}(\tilde{x}_\beta) = k^\vee(x_\beta)$.

Proofs C, D. These are apparent.

13. ORBITAL MORPHISMS

Most of the examples of convos considered previously have been (or have been isomorphic to) decompositions of locally compact groups. Here we shall study the mappings, $\phi : G \to K$, associated with these decompositions; that is: G is a locally compact group, K is a convo, and ϕ is a continuous mapping from G onto K which (somehow) takes the operation on G to the convolution on K.

Actually, we consider mappings $\phi : J \to K$, where J is a convo also. The mappings will be called orbital morphisms. The concept of an orbital morphism is not really a generalization of the concept of a group homomorphism. The observations below should illustrate the distinction.

The following proposition is well known. If G is a group, S is a set with a binary operation, and $\phi : G \to S$ is an operation-preserving mapping from G onto S, then S is a group. This can be restated in terms of decompositions. If $\mathscr{A}$ is a decomposition of a group G, and each product AB of a pair of members of $\mathscr{A}$ is a subset of a member of $\mathscr{A}$, then $\mathscr{A} = G/H$ for some normal subgroup H of G.

Recall the convo $K = \mathbf{R}^+$ of Section 9.3. Let $\phi : \mathbf{R} \times \mathbf{R} \to K$ be given by $\phi(x, y) = (x^2 + y^2)^{1/2}$. This is the sort of mapping to be studied here. The sets $C_r = \phi^{-1}(r)$ form a decomposition of the group $G = \mathbf{R} \times \mathbf{R}$. Rather than use the fact that these circles are the orbits associated with the action of a compact group on G, we wish to define the convolution on K solely in terms of the structure of G and the decomposition $\mathscr{A} = \{C_r : r \in \mathbf{R}^+\}$. This can be done as follows. Let λ be Lebesgue measure on G. This is a Haar measure. The idea is to decompose λ with respect to $\mathscr{A}$. That is, to put a probability measure q_r on each set C_r in such a way that q_r depends continuously on r, and λ can be expressed (as an integral) in terms of the q_r. There is exactly one way to do this. For $r > 0$, q_r must be a multiple of the length measure on C_r. The mapping $r \mapsto q_r$ is called the *recomposition* of ϕ consistent with λ. To convolute two point masses p_r and p_s on K, we carry the measure $q_r * q_s$ on G to the corresponding measure $\phi_*(q_r * q_s)$ on K. This mapping, $\phi_* : M(G) \to M(K)$, depends only on ϕ, and has nothing to do with the fact that G is a group.

The previous paragraph was meant to show how the mapping $\phi : G \to K$ and the structure of G impose a convolution on K. Let G_d denote the group G with the discrete topology. The identity mapping $i : G_d \to G$ is a continuous homomorphism. But it is clear that the composition $\phi \circ i : G_k \to K$ does not in any way respect the convolutions on G_d and K, since each nontrivial convolution on K is a continuous measure. This illustrates one contrast between homomorphisms and orbital morphisms. For another contrast, consider the closed subgroup $H = \mathbf{R} \times \{0\}$ of G. Even though $\phi(H) = K$, the convolution on K

imposed by H and $\phi \mid H$ is not the given one. Rather, it is specified by the rule: $p_r * p_s = \frac{1}{2} p_{|r-s|} + \frac{1}{2} p_{r+s}$.

In the first two subsections here topological and measure-theoretic questions are considered. It is assumed that X, Y and Z are nonvoid locally compact Hausdorff spaces.

The definition and certain properties of orbital morphisms are given in Subsection 13.3. In the examples studied in Section 8, Haar measure played an indirect role; that is, it was not used to define the convolution. Here, however, Haar measure is used in the definition of orbital morphism.

There are two main classes of orbital morphisms: the unary morphisms and the double coset morphisms. Section 14 is devoted to double coset morphisms and includes the basic factorization theorem, 14.3B.

Suppose that J is a convo with a Haar measure and that $\mathscr{A}$ is a decomposition of J into compact subsets. The statement that *$\mathscr{A}$ is a convo* means that $\mathscr{A}$ can be given the structure of a convo in such a way that the natural projection, $\pi : J \to \mathscr{A}$, is an orbital morphism. This structure is unique, as will be seen. Referring to the examples above, we may say that $\{C_r : r \in \mathbf{R}^+\}$ and $\{\{x, -x\} : x \in \mathbf{R}^+\}$ are convos.

The main results in this section are Theorems 13.5A and 13.7B. The first theorem gives sufficient conditions for a decomposition $\mathscr{A}$ of a convo J to be a convo. These conditions are satisfied by the two decompositions of the previous paragraph. The second theorem gives sufficient conditions, if $\mathscr{A}$ is a commutative convo, for a character of $\mathscr{A}$ to determine a representation of J.

13.1. *Continuous Decompositions*

Let $\mathscr{A}$ be a decomposition of X into compact subsets, and let $\pi : X \to \mathscr{A}$ be the natural projection. Thus $\mathscr{A} \subset \mathscr{C}(X)$ and π is a mapping from X into $\mathscr{C}(X)$. We shall say that $\mathscr{A}$ is a *continuous decomposition* and that π is a *continuous decomposition projection* if π is continuous with respect to the topologies on X and $\mathscr{C}(X)$.

LEMMA. *Let $\mathscr{A}$ and π be as above, with π continuous.*

(13.1A) *The quotient topology on $\mathscr{A}$ and the relative topology on $\mathscr{A}$ are equal.*

(13.1B) *π is an open mapping from X onto $\mathscr{A}$.*

(13.1C) *If Σ is a compact subset of $\mathscr{A}$ then $\pi^{-1}(\Sigma)$ is a compact subset of X.*

(13.1D) *If* $\{x_\beta\}_{\beta \in D}$ *is a net in* X *such that* $x_\beta \to \infty$ *then* $\pi(x_\beta) \to \{\infty\}$.

(13.1E) $\mathscr{A}$ *is a closed subset of* $\mathscr{C}(X)$.

(13.1F) $\mathscr{A}$ *is a locally compact Hausdorff space.*

(13.1G) π *is a closed mapping.*

(13.1H) *If* g *is a function on* $\mathscr{A}$ *with values in some space, and if* $g \bigcirc \pi$ *is continuous, then* g *is continuous.*

Proof. For 13.1A, let Σ be a subset of $\mathscr{A}$ and let $S = \pi^{-1}(\Sigma)$. If Σ is open in the quotient topology then (by definition) S is open in X, and this implies that $\Sigma = \mathscr{A} \cap \mathscr{C}(X)$ is relatively open. If Σ is relatively open then $\pi^{-1}(\Sigma)$ is open (by the continuity of π) and this implies that Σ is open in the quotient topology.

For 13.1B, let U be an open subset of X. Then $\pi(U) = \mathscr{A} \cap \mathscr{C}_U(X)$.

For 13.1C, let Σ be a compact nonvoid subset of $\mathscr{A}$. Then $\pi^{-1}(\Sigma) = \bigcup \Sigma$ is compact, by 2.5F.

For 13.1D, suppose that $x_\beta \to \infty$ but that the $\pi(x_\beta)$ do not converge to $\{\infty\}$. Then there exists a compact subset C of X and a subnet $\{y_\alpha\}_{\alpha \in E}$ such that each $\pi(y_\alpha)$ meets C. Thus $y_\alpha \to \infty$ and each $\pi(y_\alpha)$ is contained in $\pi^{-1}(\pi(C))$. This contradicts 13.1C.

The other parts are apparent.

13.2. *Orbital Mappings*

An open continuous mapping ϕ from X onto Y will be called *orbital* if it satisfies the four equivalent conditions of the following lemma. The compact sets $\phi^{-1}(y)$ will be called the *ϕ-orbits.*

Consider the following example. The projection $(x, y) \mapsto x$ from the solid square $[0, 1] \times [0, 1]$ onto the interval $[0, 1]$ is orbital. But the restriction of this mapping to the boundary is not orbital.

Lemma 13.2D is going to be used in the following way. Suppose that we have two continuous decompositions, $\mathscr{A}$ and $\mathscr{B}$, of X into compact subsets. Suppose that $\mathscr{A}$ is finer than $\mathscr{B}$, which means that each member of $\mathscr{A}$ is contained in a member of $\mathscr{B}$. Then $\mathscr{B}$ induces a continuous decomposition of $\mathscr{A}$ into compact subsets.

Let $\phi : X \to Y$ be continuous. There is a natural positive-continuous linear mapping $\phi_* : M(X) \to M(Y)$ associated with ϕ. It can be defined by the formula $\phi_*(\mu) = \int_X p_{\phi(x)}\mu(dx)$. Another way is by the formula

$$\int_Y g \, d(\phi_*(\mu)) = \int_X (g \bigcirc \phi) \, d\mu,$$

for all $g \in C_c(Y)$. Note that $\| \phi_*(\mu)\| \leqslant \| \mu \|$ for all $\mu \in M(X)$. In view of 13.2E, it is correct to write $\phi_*(\mu) = \mu \bigcirc \phi^{-1}$.

Let ϕ be an orbital mapping from X onto Y, and let $\ell \in M^\infty(X)$ have support equal to X. A recomposition of ϕ consistent with ℓ is a continuous mapping $y \mapsto q_y$ from Y to $M^+(X)$ such that each q_y is a probability measure on X with support equal to $\phi^{-1}(y)$, and such that

$$\ell = \int_X q_{\phi(x)} \ell(dx).$$

LEMMA 13.2A. *Let ϕ: $X \to Y$ be an open continuous mapping from X onto Y. Then the following four conditions are equivalent*:

(i) *The set $\mathscr{A} = \{\phi^{-1}(y) : y \in Y\}$ is a continuous decomposition of X into compact subsets, and the mapping $y \mapsto \phi^{-1}(y)$ is a homeomorphism from Y onto $\mathscr{A}$.*

(ii) *If C is a compact subset of Y then $\phi^{-1}(C)$ is a compact subset of X.*

(iii) *If $\{x_\beta\}_{\beta \in D}$ is a net in X such that $x_\beta \to \infty$ then $\phi(x_\beta) \to \infty$.*

(iv) *ϕ is a closed mapping, and $\phi^{-1}(y)$ is compact for each $y \in Y$.*

Proof. By 13.1C, (i) implies (ii). It is apparent that (ii) implies (iii) and (iii) implies (iv).

Assume (iv). Let π: $X \to \mathscr{A}$ be the natural projection. Then $\pi(x) = \phi^{-1}(\phi(x))$ for $x \in X$. Let ψ: $Y \to \mathscr{A}$ be given by $\psi(y) = \phi^{-1}(y)$. Thus $\pi = \psi \bigcirc \phi$. To see that ψ is continuous, let U and V be open subsets of X. Then $\Sigma = \mathscr{A} \cap \mathscr{C}_U(V)$ is a subbasic open subset of $\mathscr{A}$. Note that $\psi^{-1}(\Sigma) = \phi(U) - \phi(X - V)$. This is an open subset of Y, since ϕ is an open and closed mapping. Thus ψ is continuous. This implies that $\pi = \psi \bigcirc \phi$ is continuous. And ψ^{-1} is continuous, by 13.1H, since $\psi^{-1} \bigcirc \pi = \phi$ is continuous.

LEMMA. *Let ϕ be an orbital mapping from X onto Y.*

(13.2B) *The mapping $A \mapsto \phi(A)$ from $\mathscr{C}(X)$ to $\mathscr{C}(Y)$ is continuous.*

(13.2C) *The mapping $B \mapsto \phi^{-1}(B)$ from $\mathscr{C}(Y)$ to $\mathscr{C}(X)$ is continuous.*

Proof. For the first, let U and V be open subsets of Y. Then

$$\{A \in \mathscr{C}(X) : \phi(A) \in \mathscr{C}_U(V)\} = \mathscr{C}_S(T),$$

where $S = \phi^{-1}(U)$ and $T = \phi^{-1}(V)$.

For the second, let S and T be open subsets of X. Then

$$\{B \in \mathscr{C}(Y): \phi^{-1}(B) \in \mathscr{C}_S(T)\} = \mathscr{C}_U(V),$$

where $U = \phi(S)$ and $V = Y - \phi(X - T)$.

LEMMA 13.2D. *Let the mappings* $\phi_1 : X \to Y$, $\phi_2 : Y \to Z$, *and* $\phi_3 : X \to Z$ *be surjective. Suppose that* $\phi_3 = \phi_2 \bigcirc \phi_1$. *If any two of these mappings are orbital then so is the third.*

Proof. One need only use the fact that a mapping ϕ is orbital if and only if both ϕ and ϕ^{-1} preserve openness and compactness.

LEMMA. *Let ϕ be an orbital mapping from X onto Y, and let $\ell \in M^\infty(X)$ have support equal to X. Suppose that the mapping $y \mapsto q_y$ is a recomposition of ϕ consistent with ℓ.*

(13.2E) *The measure* $m = \phi_*(\ell) = \int_X p_{\phi(x)}\ell(dx)$ *is defined.*

(13.2F) $\ell = \int_Y q_y m(dy)$.

(13.2G) *A function g is in $B^\infty(Y)$ if and only if $g \bigcirc \phi \in B^\infty(X)$.*

(13.2H) *If $g \in B^\infty(Y)$ then* $\int_Y g\, dm = \int_X (g \bigcirc \phi)\, d\ell$.

(13.2I) *The mapping $y \mapsto q_y$ is the unique recomposition of ϕ consistent with ℓ.*

Proof. The measure m is defined, since ϕ^{-1} preserves compactness. The equation $\ell = \int q_y m(dy)$ is just a restatement of the definition of recomposition.

For 13.2G, use 2.3F. Note that there are two positive-continuous linear mappings, determined by $p_x \mapsto p_{\phi(x)}$ and $p_y \mapsto q_y$.

For 13.2H, use 2.3G. The condition of σ-compactness is not needed here, since ϕ is an open mapping.

For 13.2I, let $y \in Y$. Let V be an open subset of Y containing y and having compact closure. Let

$$\mu_V = \frac{1}{m(V)} \int_V q_y m(dy).$$

This is a probability measure on X. If V is a small neighborhood of y then μ_V is close to q_y in the cone topology. That is, $\mu_V \to q_y$ as $cV \to \{y\}$. But the μ_V do not depend on the recomposition, since, with $U = \phi^{-1}(V)$,

$$\mu_V = \frac{1}{\ell(U)} i_U \ell.$$

13.3. *Orbital Morphisms*

Let J and K be convos, and let ℓ be a left Haar measure on J. An *orbital morphism* from J onto K is a mapping ϕ which satisfies the following four conditions:

(i) ϕ is an orbital mapping from J onto K.

(ii) There exists a (necessarily unique) recomposition $y \mapsto q_y$ of ϕ consistent with ℓ.

(iii) If $y \in K$ then $q_{y^-} = (q_y)^-$.

(iv) If $y, z \in K$ then $p_y * p_z = \phi_*(q_y * q_z)$. Recall that the recomposition gives rise to a positive-continuous linear mapping from $M(K)$ to $M(J)$. This mapping will be denoted by ϕ^*. That is, $\phi^*(p_y) = q_y$ and

$$\phi^*(\nu) = \int_K q_z \nu(dz)$$

for $y \in K$ and $\nu \in M(K)$. It is clear that the q_y do not depend on the choice of the left Haar measure ℓ. By 13.3F, below, a right Haar measure could be used also.

THEOREM. *Let ϕ be an orbital morphism from J onto K. Let ℓ, the q_y, and ϕ^* be as above. Set $m = \phi_*(\ell)$.*

(13.3A) *m is a left Haar measure on K.*

(13.3B) *$\phi_* \circ \phi^*$ is the identity mapping on $M(K)$.*

(13.3C) *If $\nu \in M(K)$ then $\phi^*(\nu^-) = \phi^*(\nu)^-$ and $\|\phi^*(\nu)\| = \|\nu\|$.*

(13.3D) *If $x \in J$ and $y \in K$ then $\phi(x^-) = \phi(x)^-$ and $\phi^{-1}(y^-) = \phi^{-1}(y)^-$.*

(13.3E) *If $\mu \in M(J)$ then $\phi_*(\mu^-) = \phi_*(\mu)^-$.*

(13.3F) *The mapping $y \mapsto q_y$ is a recomposition of ϕ consistent with ℓ^-.*

(13.3G) *If Δ is the modular function of K then $\Delta \circ \phi$ is the modular function of J.*

(13.3H) *If $\mu, \nu \in M(K)$ then $\mu * \nu = \phi_*(\phi^*(\mu) * \phi^*(\nu))$.*

THEOREM 13.3I. *Let J, K and L be convos and let the mappings $\phi_1 : J \to K$, $\phi_2 : K \to L$, and $\phi_3 : J \to L$ be surjective. Suppose that $\phi_3 = \phi_2 \circ \phi_1$. If any two of these mappings are orbital morphisms then so is the third.*

Proof. For 13.3A, let $y \in K$. Then

$$p_y * m = \int_Y (p_y * p_z)\, m(dz) = \int_Y \phi_*(q_y * q_z)\, m(dz)$$
$$= \phi_*(q_y * \ell) = \phi_*(\ell) = m.$$

For 13.3B, $\phi_* \bigcirc \phi^*$ is positive-continuous on $M(K)$, and

$$(\phi_* \bigcirc \phi^*)(p_y) = \phi_*(q_y) = p_y$$

for all $y \in Y$.

For 13.3C, let $\nu \in M(K)$. By condition (iii), $\phi^*(\nu^-) = \phi^*(\nu)^-$. By 13.3B, ϕ^* must be an isometry, since $\| \phi_* \| \leqslant 1$ and $\| \phi^* \| \leqslant 1$.

For 13.3D and 13.3E, note that $\phi^{-1}(y^-) = \operatorname{spt} q_{y^-} = \operatorname{spt}(q_y{}^-) = (\operatorname{spt} q_y)^- = \phi^{-1}(y)^-$.

For 13.3F, we have that $\ell^- = \int (q_{\phi(x)})^- \ell(dx) = \int q_{\phi(x)^-} \ell(dx) = \int q_{\phi(x^-)}\, \ell(dx) = \int q_{\phi(t)}\, \ell^-(dt)$.

For 13.3G, recall that $m = \Delta m^-$. Thus $\ell = \phi^*(m) = \phi^*(\Delta m^-) = (\Delta \bigcirc \phi)\, \phi^*(m^-) = (\Delta \bigcirc \phi)\, \ell^-$.

And 13.3H follows directly from condition (iv).

For 13.3I, one can use 13.2D and 13.3H.

13.4. *Consistent Measures*

In the next subsection a certain class of orbital morphisms will be constructed. The idea is illustrated by the following result.

LEMMA 13.4A. *Let A be an algebra, L a linear space, and h: $A \to L$ a linear mapping. Let B be the set of all $x \in A$ such that $h(xy) = 0 = h(yx)$ whenever $h(y) = 0$. Suppose that $h(B) = L$. Then B is a subalgebra of A, and there exists a unique multiplication on L such that L is an algebra with this multiplication, and such that $h \mid B$ is an algebra homomorphism. Moreover, $h(xy) = h(x)\, h(y)$ if either x or y is in B.*

Proof. Let $I = h^{-1}(0)$. It is apparent that B is a subalgebra of A and that $I \cap B$ is an ideal of B. Thus the multiplication on L exists. Now let $x \in A$ and $y \in B$. There exists $x' \in B$ such that $h(x') = h(x)$. Thus $h(xy) = h((x - x')y) + h(x'y) = h(x'y) = h(x')\, h(y) = h(x)\, h(y)$.

The context of interest here is when $A = M(J)$, $L = M(Y)$, and $h = \phi_*$, where J is a convo, Y is a locally compact space, and ϕ is an orbital mapping from J onto Y. Under these circumstances, a measure $\mu \in M(J)$ will be said to be *left ϕ-consistent* if $\phi_*(\mu * \nu) = 0$ whenever

$\phi_*(\nu) = 0$, and μ will be said to be *ϕ-consistent* if $\phi_*(\mu * \nu) = 0 = \phi_*(\nu * \mu)$ whenever $\phi_*(\nu) = 0$.

LEMMA 13.4B. *Let ϕ be an orbital mapping from J onto Y. Let $\mu \in M(J)$. Then the following three conditions are equivalent*:

(i) *μ is left ϕ-consistent.*

(ii) *If $s, t \in J$ and $\phi(s) = \phi(t)$ then $\phi_*(\mu * p_s) = \phi_*(\mu * p_t)$.*

(iii) *If $f \in C_c(J)$ and f is constant on each ϕ-orbit then $\mu^- * f$ is constant on each ϕ-orbit.*

Proof. It is apparent that (i) implies (ii).

Assume (ii). Let $f = g \circ \phi$, where $g \in C_c(Y)$. If $\phi(s) = \phi(t)$ then $(\mu^- * f)(s) = \int f \, d(\mu * p_s) = \int g \, d(\phi_*(\mu * p_s)) = (\mu^- * f)(t)$. Thus f is constant on each ϕ-orbit. Hence (iii).

Assume (iii). Suppose that $\phi_*(\nu) = 0$. If $g \in C_c(Y)$ and $f = g \circ \phi$ then $\int g \, d(\phi_*(\mu * \nu)) = \int f \, d(\mu * \nu) = \int (\mu^- * f) \, d\nu = \int (h \circ \phi) \, d\nu = \int h \, d(\phi_*(\nu)) = 0$, for some $h \in C_c(Y)$. Hence (i).

13.5. *Unary Morphisms*

A *unary morphism* is an orbital morphism ϕ such that $\phi^{-1}(e) = \{e\}$. The decomposition projection determined by the action of a compact group of automorphisms of a locally compact group is a unary morphism. Not every unary morphism satisfies the hypotheses of the following theorem. See (15.1C).

THEOREM 13.5A. *Let J be a convo with left Haar measure ℓ, and let Y be a locally compact Hausdorff space. Let ϕ be an orbital mapping from J onto Y, and set $\tilde{e} = \phi(e)$. Suppose that the following three conditions are satisfied*:

(i) $\phi^{-1}(\tilde{e}) = \{e\}$.

(ii) *If A is a ϕ-orbit then so is A^-.*

(iii) *For each $y \in Y$ there exists a probability measure q_y on J such that* spt $q_y \subset \phi^{-1}(y)$ *and such that q_y is ϕ-consistent.*

Then there exists a unique convolution $*$ on Y such that $(Y, *)$ is a convo and ϕ is a unary morphism.

LEMMA 13.5B. *Let μ be a left ϕ-consistent measure on J and suppose that $\phi_*(\mu) = 0$. Then $\mu = 0$.*

Proof. Let $f \in C_c(J)$ and suppose that f is constant on the ϕ-orbits. If $x \in J$ and $y = \phi(x)$ then

$$(\mu^- * f)(x) = \int_J (\mu^- * f)\, dq_y = \int_J (f * q_y^-)\, d\mu = 0.$$

Thus $\mu^- * f = 0$. Hence $\mu^- * (f\ell) = 0$. But p_e can be approximated in $M^+(J)$ by measures of the form $f\ell$, since $\phi^{-1}(\tilde{e}) = \{e\}$. Therefore $\mu = 0$.

LEMMA 13.5C. *There exists a topological involution $y \mapsto y^-$ of Y such that $\phi(x^-) = \phi(x)^-$ for all $x \in J$.*

Proof. In view of condition (ii), the involution can be defined. By 13.2D, it is continuous.

LEMMA 13.5D. *Let $y \in Y$. Then q_y is the unique left ϕ-consistent probability measure on J whose support is contained in $\phi^{-1}(y)$. Moreover, $q_{y^-} = (q_y)^-$ and* spt $q_y = \phi^{-1}(y)$.

Proof. Suppose that μ is a left ϕ-consistent probability measure supported by $\phi^{-1}(y)$. Then $\phi_*(\mu - q_y) = p_y - p_y = 0$. Thus $\mu - q_y = 0$, by the first lemma.

It is easy to see that $(q_y)^-$ is also ϕ-consistent. Since it is supported by $\phi^{-1}(y^-)$ it must equal q_{y^-}.

Now let $x \in \phi^{-1}(y)$. The following three conditions on x are equivalent: (i) $x \in \text{spt } q_y$, (ii) $e \in \text{spt } (p_{x^-} * q_y)$, (iii) $\tilde{e} \in \text{spt } \phi_*(p_{x^-} * q_y)$. Since q_y is left ϕ-consistent, condition (iii) is either satisfied by all $x \in \phi^{-1}(y)$ or by none. In view of condition (i), $\phi^{-1} = \text{spt } q_y$.

Proof of Theorem. If $\{y_\beta\}_{\beta \in D}$ is a net in Y converging to y, then the sets spt q_{y_β} converge to $\phi^{-1}(y)$, and each limit point of $\{q_{y_\beta}\}_{\beta \in D}$ in $M^+(J)$ is a left ϕ-consistent probability measure, which must then be equal to q_y. Thus the mapping $y \mapsto q_y$ from Y to $M^+(J)$ is continuous, and there exists a unique positive-continuous linear mapping $\phi^*: M(Y) \to M(J)$ such that $\phi^*(p_y) = q_y$ for each $y \in Y$. This is by 2.3H.

It is not hard to see that $\phi^*(M(Y))$ is just the set of left ϕ-consistent measures on J. In fact, it is just the set of ϕ-consistent measures, also. It follows from the definition that these measures form an algebra. Since $\phi_* \bigcirc \phi^*$ is the identity mapping on $M(Y)$, ϕ^* is an isometry. Also, ϕ^* preserves adjoints.

The restriction of ϕ_* to the range of ϕ^* is a linear isomorphism, and thus carries the convolution on $\phi^*(M(Y))$ to an operation on $M(Y)$.

It is straightforward to verify that Y is a convo with this operation. In particular, if y, $z \in Y$ then

$$p_y * p_z = \phi_*(q_y * q_z).$$

The adjoint mapping is specified in 13.5C and the identity of Y is $\tilde{e}$. Continuity of supports can by proved using 13.2B and 13.2C.

All that remains to be proved is that $\ell = \phi^*(m)$, where m is defined by $m = \phi_*(\ell)$. Let $\lambda = \phi^*(m)$, and note that $\int_J f\, d\lambda = \int_J f\, d\ell$ if f is constant on each ϕ-orbit.

If $x \in J$, $y = \phi(x)$, $f \in C_c(J)$, and f is constant on each ϕ-orbit, then

$$\begin{aligned}
\int_J (p_x * f)\, d\lambda &= \int_Y \int_J (p_x * f)\, dq_z m(dz) \\
&= \int_Y \int_J (f * q_{z^-})\, dp_{x^-} m(dz) \\
&= \int_Y \int_J (f * q_{z^-})\, dq_{y^-} m(dz) \\
&= \int_Y \int_J (q_y * f)\, dq_z m(dz) \\
&= \int_J (q_y * f)\, d\lambda \\
&= \int_J (q_y * f)\, d\ell \\
&= \int_J f\, d\ell \\
&= \int_J (p_x * f)\, d\ell.
\end{aligned}$$

Thus, if $\mu \in M_c(J)$, $f \in C_c(J)$, and f is constant on each ϕ-orbit, then $\int (\mu * f)\, d\lambda = \int (\mu * f)\, d\ell$. But $\phi^{-1}(\tilde{e}) = \{e\}$, and each function in $C_c(J)$ can be approximated by these $\mu * f$. The precise statement is in 5.1B. Therefore, $\lambda = \ell$, and the proof is complete.

13.6. *Consistent Orbital Morphisms*

In this subsection, J and K are convos and ϕ is an orbital morphism from J onto K. Let ℓ be a left Haar measure on J and let $m = \phi^*(\ell)$ be the corresponding left Haar measure on K. For 13.6D the measures must be related in this way.

We shall say that ϕ is *consistent* if the mapping $\phi^*: M(K) \to M(J)$ is an algebra homomorphism. By 13.6C, if ϕ is consistent then ϕ^* is a Banach *-algebra isomorphism.

LEMMA 13.6A. *Suppose that each measure $q_y = \phi^*(p_y)$ is left ϕ-consistent. Then ϕ is consistent.*

THEOREM. *Suppose that ϕ is consistent. Let $\nu \in M(K)$ and $g, h \in C_c(K)$.*

(13.6B) *$\phi^*(\nu)$ is a ϕ-consistent measure on J.*

(13.6C) $(\nu * g) \bigcirc \phi = \phi^*(\nu) * (g \bigcirc \phi)$.

(13.6D) $(g * h) \bigcirc \phi = (g \bigcirc \phi) * (h \bigcirc \phi)$.

Proof A. Let $y \in K$ and $g \in C_c(K)$. It is enough to show that

$$\phi^*(p_y * gm) = \phi^*(p_y) * \phi^*(gm),$$

since ϕ^* is continuous on $M^+(K)$. We have:

$$\phi^*(p_y * gm) = \phi^*((p_y * g)m) = [(p_y * g) \bigcirc \phi]\ell = f_1\ell,$$

$$\phi^*(p_y) * \phi^*(gm) = q_y * (g \bigcirc \phi)\ell = [q_y * (g \bigcirc \phi)]\ell = f_2\ell.$$

Since $q_y = (q_{y^-})^-$ is left ϕ-consistent, f_2 is constant on each ϕ-orbit. Moreover, $\phi_*(f_1\ell) = p_y * gm = \phi_*(f_2\ell)$. Thus $f_1 = f_2$.

Proofs B, C, D. Note that 13.6C implies that $\phi^*(\nu)^-$ is left ϕ-consistent. By symmetry, this implies 13.6B. And 13.6D implies 13.6C, since ν can be suitably approximated by measures of the form gm. Finally, 13.6D merely says that $\phi^*(gm * hm) = \phi^*(gm) * \phi^*(hm)$, which is true by assumption.

13.7. *Positive-Definite Functions*

In this subsection ϕ is an orbital morphism from J onto K.

LEMMA 13.7A. *Let $g \in C(K)$ and let $f = g \bigcirc \phi$. If f is a positive-definite function on J then g is a positive-definite function on K.*

THEOREM 13.7B. *Suppose that ϕ is a consistent orbital morphism from J onto K and that K is a commutative convo. Let χ be a (bounded self-*

adjoint multiplicative) character of K. If χ is an element of the support of the Plancherel measure on $\hat{K}$ then $\chi \bigcirc \phi$ is a positive-definite function on J.

Proof A. Let $\nu \in M_c(K)$ and let $\mu = \phi^*(\nu)$. By 13.3H,

$$\begin{aligned}\int_K g\, d(\nu * \nu^*) &= \int_K g\, d[\phi_*(\phi^*(\nu) * \phi^*(\nu^*))]\\ &= \int_J f\, d[\phi^*(\nu) * \phi^*(\nu)^*]\\ &= \int_J f\, d(\mu * \mu^*).\end{aligned}$$

Proof B. Let m be a Haar measure on K and suppose that χ is in the support of the Plancherel measure on $\hat{K}$. Using the inverse Fourier transform, we can construct a net of (continuous) functions of the form $g * g^*$ which converge to χ uniformly on compact sets, where each g is in $L_2(m)$. Since $C_c(K)$ is dense in $L_2(m)$, we may assume that each $g \in C_c(K)$. By 13.6D, $\chi \bigcirc \phi$ is the limit (uniformly on compact sets) of a net of functions of the form $f * f^*$, where each $f \in C_c(J)$. But such functions are positive-definite, by 11.5C. Note that J is unimodular, by 13.3G.

14. Double Coset Convos

We have already seen that the collection of double cosets of a compact subgroup of a group is a convo in a natural way. The corresponding statement for convos is valid. That is, if H is a compact subconvo of the convo K, then the sets $H * \{x\} * H = HxH$ form a decomposition of K, and $K // H = \{HxH : x \in K\}$ has a natural convolution. The existence of a Haar measure on K is not needed for this.

One question (especially interesting in the case where K is a group) that is considered here is as follows: What can be said about the representations of K, given the representations of $K // H$? An answer is given in 14.4D. This answer is expressed in terms of positive-definite functions. There are two reasons for this. First, the notation is simpler. Second, in the examples of particular interest, $K // H$ is commutative (though K is not) and it is natural to work with characters, rather than one-dimensional representations, of $K // H$. One can see from 13.7B that there are many characters of $K // H$ which determine irreducible representations of K if $K // H$ is commutative.

To work directly with the representations, one need only note that $M(K)$ contains a subalgebra $M(K \| H)$ isomorphic to $M(K \,//\, H)$. Each representation of K on a Hilbert space $\mathscr{H}$ gives rise, in a canonical way, to a representation of $K \,//\, H$ on a (possibly trivial) subspace $\mathscr{H}'$ of $\mathscr{H}$. A representation of $K \,//\, H$ can sometimes be raised to a representation of K on a larger Hilbert space; if it exists the raised representation of K is essentially unique. We omit the detailed statements and proofs.

In this section, K is a convo, H is a compact subconvo of K, and σ is the normalized Haar measure on H.

14.1. *Double Cosets*

For $x \in K$, let $HxH = H * \{x\} * H$. These sets will be called *double cosets* of H. Clearly, each double coset is compact. In view of 14.1A, the collection $K \,//\, H = \{HxH : x \in K\}$ is a decomposition of K into compact subsets. The natural projection, $x \mapsto HxH$, will be denoted by π. In view of 14.1C, the quotient topology and the relative topology on $K \,//\, H$ are equal. We give $K \,//\, H$ this common topology.

THEOREM. *Let $x, y \in K$.*

(14.1A) *Either HxH and HyH are equal or they are disjoint.*

(14.1B) $\sigma * p_x * \sigma = \sigma * p_y * \sigma$ *if and only if* $HxH = HyH$.

(14.1C) *The mapping π is a continuous decomposition projection.*

(14.1D) *The mapping* $HtH \mapsto \sigma * p_t * \sigma$ *is a homeomorphism from $K \,//\, H$ onto a closed subset of $M^+(K)$.*

Proof. For 14.1A, let $z \in HxH$. Then $HzH \subset HxH$. Using 4.1B, we have that $z \in HxH$, H^-z meets xH, and H^-zH^- contains x. Since $H^- = H$, it must be that $HzH = HxH$. The result follows from this.

For 14.1B, if $HxH \neq HyH$ then the measures are unequal because they have disjoint supports. Suppose that $HxH = HyH$. Let $f \in C_c^+(K)$ and let $h = \sigma * f * \sigma$. Since $h \in C_c^+(K)$ also, there exists $z \in HxH$ such that $h(z) = \sup \{h(t) : t \in HxH\}$. If $s, t \in H$ then $p_{s^-} * h * p_{t^-} = h$ and $h(s * z * t) = (p_{s^-} * h * p_{t^-})(z) = h(z)$, which implies that h is constant on the set $\{s\} * \{z\} * \{t\}$. It follows that h is constant on HzH, which contains x and y. Thus

$$\int f \, d(\sigma * p_x * \sigma) = h(x) = h(y) = \int f \, d(\sigma * p_y * \sigma).$$

This implies that the measures are equal.

Parts 14.1C and 14.1D are apparent, since convolution is continuous on $\mathscr{C}(K)$ and $M^+(K)$.

14.2. *The Operation*

In view of 14.1B and 14.1D, there exists a (unique) positive-continuous linear mapping $\pi^*: M(K\,//\,H) \to M(K)$ such that $\pi^*(p_{HxH}) = \sigma * p_x * \sigma$ for each $x \in K$. The convolution on $K\,//\,H$ is defined by:

$$\mu * \nu = \pi_*(\pi^*(\mu) * \pi^*(\nu))$$

for $\mu, \nu \in M(K\,//\,H)$. Let

$$M(K \,\|\, H) = \{\mu \in M(K): \sigma * \mu * \sigma = \mu\}.$$

THEOREM. *Let $M(K\,//\,H)$ have the operation specified above.*

(14.2A) *$K\,//\,H$ is a convo.*

(14.2B) *The identity of $K\,//\,H$ is H.*

(14.2C) *If $x \in K$ then $(HxH)^- = Hx^-H$.*

(14.2D) *$M(K \,\|\, H)$ is a closed self-adjoint subalgebra of $M(K)$.*

(14.2E) *π^* is an isomorphism from the Banach *-algebra $M(K\,//\,H)$ onto $M(K \,\|\, H)$.*

(14.2F) *If $x, y \in K$ then*

$$p_{HxH} * p_{HyH} = \pi_*(p_x * \sigma * p_y) = \int_H p_{HtH}(p_x * \sigma * p_y)(dt).$$

(14.2G) *If $x, y \in K$, $g \in B^\infty(K\,//\,H)$, and $f = g \circ \pi$ then*

$$g(HxH * HyH) = \int_H f(x * t * y)\, \sigma(dt).$$

(14.2H) *If there exists a left Haar measure ℓ on K then π is an orbital morphism from K onto $K\,//\,H$, the mapping $HxH \mapsto \sigma * p_x * \sigma$ is the recomposition of π consistent with ℓ, and*

$$m = \int_K p_{HxH}\ell(dx)$$

is a left Haar measure on $K\,//\,H$.

Proof. Everything follows readily from 14.2D and the fact that π^*

is a norm-preserving adjoint-preserving positive-continuous linear mapping from $M(K \,//\, H)$ onto $M(K \,\|\, H)$. The operation on $M(K \,//\, H)$ is, by definition, the operation inherited from $M(K \,\|\, H)$ through π^* and its inverse $\pi_* \mid M(K \,\|\, H)$.

14.3. *Factorization Theorems*

The first result here is an analog of the First Isomorphism Theorem for groups. For convos with Haar measure it is a special case of 13.3I.

The second result has no analog in the theory of groups. The existence and uniqueness of Haar measure is crucial in this case, since all we prove is that the decomposition $\{IxI : x \in J\}$ of J is finer than the decomposition $\{\phi^{-1}(y) : y \in K\}$.

THEOREM 14.3A. *Let J be a compact subconvo of K and suppose that $H \subset J$. Then $J \,//\, H$ is a compact subconvo of $K \,//\, H$, and*

$$K//J \cong K//H \,//\, J//H.$$

THEOREM 14.3B. *Let J be a convo and let ϕ be an orbital morphism from J onto K. Let $I = \phi^{-1}(e)$. Then I is a compact subconvo of J, and the diagram*

$$\begin{tikzcd} J \arrow[r, "\pi"] \arrow[dr, "\phi"'] & J//I \arrow[d, "\psi"] \\ & K \end{tikzcd}$$

commutes, where π is the natural projection and ψ is a unary morphism.

Proof A. We omit the details. The isomorphism is given by:

$$JxJ \to (J//H)\, HxH (J//H).$$

Proof B. Let $y \mapsto q_y$ be the recomposition of ϕ. Then $q_e^- = q_{e^-} = q_e$ and $p_e = p_e \ast p_e = \phi_*(q_e \ast q_e)$. Since $I = \operatorname{spt} q_e$, we have that $I^- = I$ and $I \ast I \subset I$. Thus I is a compact subconvo of J. Let $y \in K$. Then $\phi_*(q_e \ast q_y) = p_e \ast p_y = p_y$. Thus $I \ast \phi^{-1}(y) \subset \phi^{-1}(y)$. Similarly, $\phi^{-1}(y) \ast I \subset \phi^{-1}(y)$. This implies that $\phi^{-1}(y)$ is a union of double cosets of I. The rest follows from 13.3I.

14.4. *Positive-Definite Functions*

These results do not generalize to arbitrary orbital morphisms. Referring to the example in Subsection 9.3, the function $f(x, y) = J_0((x^2 + y^2)^{1/2})$ is positive-definite on $\mathbf{R} \times \mathbf{R}$, but is not irreducible. In fact, f is a combination of functions of the form $(x, y) \to e^{i(ax+by)}$, none of which is constant on the circles.

THEOREM. *Let f be a bounded positive-definite function on K and suppose that f is constant on H.*

(14.4A) *f is constant on each double coset of H.*

(14.4B) *$f = g \bigcirc \pi$, where g is a bounded positive-definite function on $K \,//\, H$.*

(14.4C) *If $f = f_1 + f_2$, where f_1 and f_2 are bounded positive-definite functions on K, then f_1 and f_2 are constant on each double coset of H.*

(14.4D) *f is an irreducible positive-definite function on K if and only if g is an irreducible positive-definite function on $K \,//\, H$.*

Proof A. By 11.2E, $\| p_x - p_e \|_f = 0$ for all $x \in H$. Thus $\| \sigma - p_e \|_f = 0$. If $x \in K$ then

$$\begin{aligned}\int_K \bar{f}\, d(\sigma * p_x * \sigma) &= \int_K \bar{f}\, d(\sigma^* * p_x * \sigma)\\ &= (p_x * \sigma, \sigma)_f\\ &= [p_x * p_e, p_e]_f\\ &= \overline{f(x)}.\end{aligned}$$

By 14.1B, f is constant on each double coset.

Proof B. This follows from 13.7A.

Proof C. It is enough to show that f_1 and f_2 are constant on H. If $x \in H$ then

$$0 = f(e) - f(x) = [f_1(e) - f_1(x)] + [f_2(e) - f_2(x)].$$

The two expressions on the right are nonnegative. Hence, they are zero.

Proof D. If f is reducible then so is g, by the preceding results. Suppose now that $g = g_1 + g_2$, where g_1 and g_2 are bounded positive-definite functions on $K \,//\, H$. It is enough to show that each $f_k = g_k \circ \pi$ is positive-definite. By 11.4F, g generates g_1 and g_2. That is, there exist two sequences $\{\nu_n{}^1\}$ and $\{\nu_n{}^2\}$ in $M(K \,//\, H)$ such that

$$\lim_{m,n\to\infty} \| \nu_m{}^k * g * (\nu_n{}^k)^* - g_k \|_u = 0$$

for $k = 1, 2$. Let $\mu_n{}^k = \pi^*(\nu_n{}^k)$. By 13.6C,

$$\lim_{m,n\to\infty} \| \mu_m{}^k * f * (\mu_n{}^k)^* - f_k \|_u = 0$$

for $k = 1, 2$. By 11.1C, f_1 and f_2 are positive-definite.

15. Examples

Except in the first subsection, the object of study here is the group $SL(2, \mathbf{C})$ of all two-by-two complex matrices with determinant 1. The convos, mappings and functions constructed should illustrate our results on orbital morphisms. Recall that a convo K is Hermitian if $x^- = x$ for each x in K.

15.1. *Several Convos*

Example 15.1A. Let $\mathbf{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ be the additive group of order 6. The decomposition $\{0\}$, $\{3\}$, $\{1, 5\}$, $\{2, 4\}$ of $\mathbf{Z}_6$ is a convo, determined by the group of automorphisms $x \mapsto x$, $x \mapsto -x$.

Example 15.1B. The decomposition $\{0\}$, $\{3\}$, $\{1, 4\}$, $\{2, 5\}$ of $\mathbf{Z}_6$ is a convo, and the natural projection is consistent.

Example 15.1C. The decomposition $\{0\}$, $\{3\}$, $\{1, 2\}$, $\{4, 5\}$ of $\mathbf{Z}_6$ is a convo, but the natural projection is not consistent.

Example 15.1D. For each positive integer n let b_n be a number such that $0 < b_n \leqslant 1$. Let $c_0 = 1$ and define numbers c_n inductively by the rule:

$$c_n = (1/b_n)(c_0 + c_1 + \cdots + c_{n-1}).$$

For each $n \geqslant 1$ the two-point set $K_n = \{0, n\}$ can be made into a convo by letting 0 be the identity and setting

$$p_n * p_n = b_n p_0 + (1 - b_n)\, p_n .$$

We consider here the infinite discrete join $(K, \bullet)$, where

$$K = K_1 \vee K_2 \vee \cdots = \{0, 1, 2, \ldots\}.$$

The operation on K is defined by

$$p_m \bullet p_n = p_n \bullet p_m = p_n$$

$$p_n \bullet p_n = \frac{c_0}{c_n} p_0 + \frac{c_1}{c_n} p_1 + \cdots + \frac{c_{n-1}}{c_n} p_{n-1} + (1 - b_n) p_n$$

for $0 \leqslant m < n$. Thus K is a Hermitian convo with identity 0. A Haar measure m on K is given by

$$m = \sum_{k=0}^{\infty} c_k p_k .$$

An interesting fact about this convo is that each set $H_n = \{0, 1, 2, \ldots, n\}$ is a subconvo of K.

EXAMPLE 15.1E. Let G_1 be the circle group and let G_2 be a two-element group, with $G_1 \cap G_2 = \{1\}$. Then $G_1 \vee G_2$ is a nondiscrete convo with an isolated point.

15.2. *The Group* $SL(2, \mathbf{C})$

In the remaining subsections several related convos will be constructed. They depend on the structure of the group $G = SL(2, \mathbf{C})$ of all two-by-two complex matrices with determinant equal to 1, and the subgroup $H = SU(2)$ of unitary matrices in G. The idea is illustrated by the diagram:

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & G^H \arrow[d] \arrow[dr] & \\ & G//H \arrow[r] & G^H/I. \end{tikzcd}$$

The expression G^H refers to the convo of H-conjugacy classes of G.

We write G^H/I because G^H is commutative. Of course, it is more convenient to use models for these convos. The notation is:

$$\begin{tikzcd} G \arrow[r, "\text{uny}"] \arrow[dr, "\text{dbc}"'] & J \arrow[d] \arrow[dr, "\text{dbc}"] & \\ & L \arrow[r, "\text{iso}"'] & K \end{tikzcd}$$

The convo K is the one defined and studied in Section 9.5. Here, uny $=$ unary morphism, dbc $=$ double coset morphism, and iso $=$ isomorphism. The mappings studied explicitly are as follows, where the convos are replaced by spaces of which they are subsets:

$$\begin{tikzcd} G \arrow[r, "(\tau,\phi)"] \arrow[dr, "\phi"'] & \mathbf{C} \times \mathbf{R} \arrow[d, "\text{proj}"] & \\ & \mathbf{R} & \mathbf{R} \arrow[l, "\cosh"] \end{tikzcd}$$

The two functions τ and ϕ are defined on G by

$$\tau\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \tfrac{1}{2}(a+d)$$

$$\phi\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \tfrac{1}{2}(|a|^2 + |b|^2 + |c|^2 + |d|^2).$$

In his thesis Andrew Bao-Hwa Wang [16] constructs the spherical functions on $SL(2, \mathbf{C})$ associated with the various irreducible representations of $SU(2)$. The main object of study in this thesis is the convolution algebra $I_c(G)$ of all functions f on $G = SL(2, C)$ which are infinitely differentiable, have compact support, and have the property that $f(t^{-1}xt) = f(x)$ for all $x \in G$ and $t \in H$. It is easily seen that $I_c(G)$ is isomorphic to a dense subalgebra of $L_1(J)$. Some of Wang's results can thereby be expressed in terms of J and $\hat{J}$, but this will not be done here.

15.3. *Double Cosets in $SU(2)$*

The group $H = SU(2)$ is compact. The elements of H are of the form

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

where $a, b \in \mathbf{C}$ and $|a|^2 + |b|^2 = 1$. We define:

$$r_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$d_{it} = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}$$

$$D = \{d_{it} : t \in \mathbf{R}\}$$

$$[a : b] = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

$$h([a : b]) = |a|^2 - |b|^2.$$

The purpose here is to study the convo $H \,//\, D$, which is isomorphic to E since the diagram below commutes:

$$\begin{tikzcd} H \arrow[r, "\pi"] \arrow[dr, "h"'] & H//D \arrow[d] \\ & E \end{tikzcd}$$

PROPOSITION.

(15.3A) *Each double coset of D in H is equal to $Dr_\theta D$ for a unique $\theta \in [0, \pi/2]$.*

(15.3B) *The function h is an orbital mapping from H onto $E =$* $[-1,1]$. *Each h-orbit is a double coset of D. If $\theta \in R$ then*

$$h^{-1}(\cos 2\theta) = Dr_\theta D.$$

(15.3C) *E is a Hermitian convo with identity* 1.

(15.3D) *The convolution on E is specified by the formulas*:

$$f(\cos s \ast \cos t) = \frac{1}{\pi}\int_0^\pi f(\cos s \cos t - \sin s \sin t \cos u)\, du,$$

$$f(x \ast y) = \frac{1}{\pi}\int_{A-B}^{A+B} \frac{1}{(1+2xyz - x^2 - y^2 - z^2)^{1/2}} f(z)\, dz,$$

where $A = xy$ *and* $B = (1-x^2)^{1/2}(1-y^2)^{1/2}$.

(15.3E) *Lebesgue measure is a Haar measure on E.*

(15.3F) *The normalized Haar measure σ on H is given by the formula*:

$$\int_H f\, d\sigma = \frac{1}{4\pi^2}\int_{t=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{s=0}^{2\pi} f(d_{is} r_\theta d_{it}) \sin 2\theta\, ds\, d\theta\, dt.$$

Proof. Note that $h(r_\theta) = \cos 2\theta$ and

$$d_{is}[a : b]\, d_{it} = [e^{i(s+t)}a : e^{i(s-t)}b].$$

The first four parts now follow readily. For the convolution, the definition says that

$$f(h(r_s) \ast h(r_t)) = \frac{1}{2\pi}\int_0^{2\pi} f(h(r_s d_{iu} r_t))\, du.$$

This gives the first formula, though with $2s$, $2t$, $2u$ instead of s, t, u. The second formula can be deduced from the first by an obvious change of variable.

For Haar measure on E, it must be shown that

$$\int_{-1}^{1} f(x \ast y)\, dy = \int_{-1}^{1} f(y)\, dy.$$

This can be checked directly, using the second formula in 15.3D and the fact that (for $C < D$)

$$1 = \frac{1}{\pi}\int_C^D \frac{dy}{((D-y)(y-C))^{1/2}}.$$

For 15.3F, we know that σ is of this form, though the correctness of the expression $\sin 2\theta\, d\theta$ is not obvious. In view of 15.3A and 15.3B we need only verify that

$$\int_0^{\pi/2} f(\cos 2\theta) \sin 2\theta \, d\theta = \tfrac{1}{2} \int_{-1}^{1} f(x)\, dx.$$

15.4. *The Conjugacy Classes of $SU(2)$*

It was proved in Subsection 8.4 that the conjugacy classes of a compact group form a convo. Here, we use $F = [-1, 1]$ as a model for the convo of conjugacy classes of H. The dual $\hat{F}$ is also a convo. Note that each character χ_n of F is a polynomial of degree n. For example: $\chi_0(x) = 1$, $\chi_1(x) = x$, $\chi_2(x) = \frac{4}{3}x^2 - \frac{1}{3}$.

PROPOSITION.

(15.4A) *The function $\tau \mid H$ is an orbital mapping from H onto $F = [-1, 1]$. Each orbit is a conjugacy class of H.*

(15.4B) *F is a Hermitian convo with identity* 1.

(15.4C) *The convolution on F is specified by the formulas*:

$$f(\cos s * \cos t) = \tfrac{1}{2} \int_0^{\pi} f(\cos s \cos t - \sin s \sin t \cos u) \sin u \, du,$$

$$f(\cos s * \cos t) = \frac{1}{2 \sin s \sin t} \int_{s-t}^{s+t} f(\cos u) \sin u \, du,$$

$$f(x * y) = \frac{1}{2B} \int_{A-B}^{A+B} f(z)\, dz,$$

where $A = xy$ and $B = (1 - x^2)^{1/2}(1 - y^2)^{1/2}$.

(15.4D) *The normalized Haar measure ρ on F is given by the formulas*:

$$\int_F f\, d_\rho = \frac{2}{\pi} \int_{-1}^{1} f(y)(1 - y^2)^{1/2}\, dy,$$

$$= \frac{2}{\pi} \int_0^{\pi} f(\cos \theta) \sin^2 \theta \, d\theta.$$

(15.4E) *The members of the dual $\hat{F} = \{\chi_0, \chi_1, \ldots\}$ of F are given by the formula*:

$$\chi_{n-1}(\cos \theta) = \frac{\sin n\theta}{n \sin \theta}.$$

Proof. It is well known and easily shown that two matrices in H are conjugate if and only if they have the same trace. Since $\tau(d_{it}) = \cos t$ and $\tau(H) = [-1, 1]$, each conjugacy class of H contains d_{it} for some $t \in R$. Note that

$$\begin{aligned}\tau((d_{iu}r_\theta d_{iv})^{-1}d_{is}(d_{iu}r_\theta d_{iv})\, d_{it}) &= \tau(d_{-iv}r_{-\theta}d_{is}r_\theta d_{iv}d_{it}) \\ &= \tau(r_{-\theta}d_{is}r_\theta d_{iv}d_{it}d_{-iv}) \\ &= \tau(r_{-\theta}d_{is}r_\theta d_{it}) \\ &= \cos s \cos t - \sin s \sin t \cos 2\theta.\end{aligned}$$

The first formula in 15.4C now follows from 15.3F, with $u = 2\theta$. The others are easily deduced from this.

For 15.4D, one can use the Haar measure σ on H, since $\rho = \tau_*(\sigma)$. But it is easier to use the formula for $f(x \ast y)$ to verify directly that

$$\int_{-1}^{1} f(x \ast y)(1 - y^2)^{1/2}\, dy = \int_{-1}^{1} f(z)(1 - z^2)^{1/2}\, dz.$$

Hewitt and Ross [4, p. 134] compute the characters of $H = SU(2)$. The relationship between the characters of H and F is noted in 8.4A. One can check directly using the second formula in 15.4C. Since each χ_n is a polynomial of degree n, the linear span of the χ_n is just the set of polynomials on F. This set is dense in $C(F)$, which implies that there are no characters other than the χ_n.

15.5. *The Double Cosets of $SU(2)$ in $SL(2, \mathbf{C})$*

Here, we take $L = [1, \infty)$ as a model for $G \,//\, H$. Recall that K is the convo of Section 9.5. In view of 15.5E, which is obvious from 15.5D, the dual of L can be constructed using $\hat{K}$. The characters of L then determine the zonal spherical functions on G relative to H. By 14.4D, those which are positive-definite on G are irreducible. Of course, formulas for these functions are well known.

For $z \in \mathbf{C}$, let

$$d_z = \begin{bmatrix} e^z & 0 \\ 0 & e^{-z} \end{bmatrix}.$$

PROPOSITION.

(15.5A) *Each double coset of H in G contains d_t for a unique $t \geqslant 0$.*

(15.5B) *The function* ϕ *is an orbital mapping from* G *onto* $L = [1, \infty)$. *Each* ϕ*-orbit is a double coset of* H. *If* $t \in R$ *then* $\phi^{-1}(\cosh 2t) = Hd_tH$.

(15.5C) L *is a Hermitian convo with identity* 1.

(15.5D) *The convolution on* L *is specified by the formulas*:

$$f(\cosh s \ast \cosh t) = \tfrac{1}{2}\int_0^\pi f(\cosh s \cosh t - \sinh s \sinh t \cos u) \sin u \, du,$$

$$f(\cosh s \ast \cosh t) = \frac{1}{2 \sinh s \sinh t}\int_{s-t}^{s+t} f(\cosh u) \sinh u \, du,$$

$$f(x \ast y) = \frac{1}{2B}\int_{A-B}^{A+B} f(z)\, dz,$$

where $A = xy$ *and* $B = (x^2 - 1)^{1/2}(y^2 - 1)^{1/2}$.

(15.5E) *The mapping* $x \mapsto \cosh x$ *is an isomorphism from* K *onto* L.

Proof. Here we regard the elements of G as linear operators on the two-dimensional Hilbert space $\mathbf{C} \times \mathbf{C}$ by the rule:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(s, t) = (as + bt, cs + dt).$$

Let $g \in G$ and set $e^t = \| g \|$. Since $\det g = 1$, it must be that $t \geqslant 0$. There exists $x \in \mathbf{C} \times \mathbf{C}$ such that $\| x \| = 1$ and $\| g(x) \| = e^t$. There exist $h_1, h_2 \in H$ such that $h_1(1, 0) = x$ and $h_2(g(x)) = (e^t, 0)$. Let $g' = h_2 \bigcirc g \bigcirc h_1$. Thus $g'(1, 0) = (e^t, 0)$. Since $\| g' \| = \| g \| = e^t$, it is necessary that $g' = d_t$.

One can prove by direct computation that ϕ is constant on double cosets. The rest of the proof is similar to that of the previous subsection.

15.6. *The* $SU(2)$*-Conjugacy Classes of* $SL(2, \mathbf{C})$

The group H acts on G by inner automorphisms: $(g, h) \mapsto h^{-1}gh$. Each orbit $\{h^{-1}gh : h \in H\}$ will be called a *H-conjugacy class* of G. The space of orbits is a convo, which was denoted by G^H in Subsection 8.3. As a model for G^H the set

$$J = \{(z, r) \in \mathbf{C} \times \mathbf{R} : | z - 1 | + | z + 1 | \leqslant (2(r + 1))^{1/2}\}$$

will be used. This set is something like a solid cone, but has a line-segment $I = [-1, 1] \times \{1\}$ at its base instead of a point. In fact, I is the image of H under the mapping (τ, ϕ) and is isomorphic to the convo F of 15.4. The projection $(z, r) \mapsto r$ from J onto L is a double coset morphism with kernel I. Thus each horizontal cross-section of J is a coset of I. These cosets are solid planar ellipses.

PROPOSITION.

(15.6A) *Each H-conjugacy class of G contains a matrix of the form $r_\theta d_{is} d_t$, where $\theta \in [0, \pi/2]$, $s \in R$, and $t \in [0, \infty)$.*

(15.6B) *The mapping (τ, ϕ) is an orbital mapping from G onto J. Each (τ, ϕ)-orbit is a H-conjugacy class of G. If θ, s, $t \in R$ then*

$$\tau(r_\theta d_{is} d_t) = [\cos s \cosh t + i \sin s \sinh t] \cos \theta,$$

$$\phi(r_\theta d_{is} d_t) = \cosh 2t.$$

(15.6C) *J is a Hermitian convo with identity* (1, 1).

Proof. Let $g \in G$ and let g^H denote the H-conjugacy class of G containing g. By 15.5B, there exists $t \in [0, \infty)$ such that $g = h_1 d_t h_2$ for some h_1, $h_2 \in H$. Let $h = h_2^{-1} h_1$. Then $g^H = (hd_t)^H$. By 15.3A, there exist $\theta \in [0, \pi/2]$ and u, $v \in \mathbf{R}$ such that $h = d_{iu} r_\theta d_{iv}$. Let $s = u + v$. Then $g^H = (r_\theta d_{is} d_t)^H$.

By 15.5B, ϕ is constant on g^H. Certainly τ is also. Thus g^H is contained in a (τ, ϕ)-orbit. We can see from the second of the two formulas in 15.6B, which are easily checked, that t is uniquely determined by g. If $t = 0$ then g^H is a (τ, ϕ)-orbit, by 15.4A. Assume that $t > 0$. Then θ is uniquely determined by g. If $\cos \theta \neq 0$ then e^{is} is determined by g, and this would imply that g^H is a (τ, ϕ)-orbit. Assume that $\cos \theta = 0$. That is, $\theta = \pi/2$. The equation

$$d_{-iu}(r_{\pi/2} d_t)\, d_{iu} = r_{\pi/2} d_{2iu} d_t$$

shows that g^H is a (τ, ϕ)-orbit.

The convo is Hermitian, since $\tau(g^{-1}) = (\tau(g)$ and $\phi(g^{-1}) = \phi(g)$.

Now we verify that the range of (τ, ϕ) is equal to J. Let $g = r_\theta\, d_{is}\, d_t$ and set $x + iy = \tau(g)$, $r = \phi(g)$. Suppose that $t \neq 0$. Then x and y can be any real numbers such that

$$\frac{x^2}{\cosh^2 t} + \frac{y^2}{\sinh^2 t} \leqslant 1,$$

by 15.6B. Thus $z = x + iy$ can be any complex number such that

$$| z - 1 | + | z + 1 | \leqslant 2 \cosh t = (2(r + 1))^{1/2}.$$

References

1. J. Delsarte, Sur une extension de la formula de Taylor, *J. Math. Pures Appl.* **17** (1938), 213–230.
2. N. Dunford and J. T. Schwartz, "Linear Operators," Part II, Interscience, New York, 1963.
3. E. Hewitt and K. A. Ross, "Abstract Harmonic Analysis," Vol. I, Springer-Verlag, Berlin, 1963.
4. E. Hewitt and K. A. Ross, "Abstract Harmonic Analysis," Vol. II, Springer-Verlag, Berlin, 1970.
5. B. M. Levitan, Plancherel's Theorem for a generalized translation operation, *Dokl. Akad. Nauk SSSR* **47** (1945), 318–321 (Russian).
6. B. M. Levitan, Normed rings generated by the generalized operation of translation, *Dokl. Akad. Nauk SSSR* **47** (1945), 3–6 (Russian).
7. B. M. Levitan, The application of generalized translation operators to second-order linear differential equations, *Uspehi Mat. Nauk* **4**, No. 1 (29), (1949), 3–112 (Russian); Amer. Math. Soc. Translation No. 59 (1951).
8. B. M. Levitan, "Generalized Translation Operators and Some of Their Applications," Israel Program for Scientific Translations, 1964.
9. B. Levitan and A. Povzner, Sturm-Liouville differential equations on the semi-axis and the Plancherel theorem, *Dokl. Akad. Nauk SSSR* **52** (1946), 483–486 (Russian).
10. L. H. Loomis, "An Introduction to Abstract Harmonic Analysis," Van Nostrand, Princeton, 1963.
11. E. Michael, Topologies on spaces of subsets, *Trans. Amer. Math. Soc.* **71** (1951), 152–182.
12. L. Nachbin, On the finite dimensionality of every irreducible representation of a compact group, *Proc. Amer. Math. Soc.* **12** (1961), 11–12.
13. M. A. Naimark, "Normed Rings," P. Noordhoff N. V., Groningen, 1964.
14. K. Stromberg, Doctoral Thesis, University of Washington, 1958.
15. J. D. Talman, "Special Functions," Benjamin, New York, 1968.
16. A. B-H. Wang, On spherical functions and Paley-Wiener theorem on $SL(2, C)$, Doctoral Thesis, University of Washington, 1972.
17. C. F. Dunkl, The measure algebra of a locally compact hypergroup, *Trans. Amer. Math. Soc.* **179** (1973), 331–348.