On the Lie Structure of the Skew Elements of a Simple Superalgebra with Superinvolution

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We investigate the Lie structure of the Lie superalgebra $K$ of skew elements of a unital simple associative superalgebra $A$ with superinvolution over a field of characteristic not 2. It is proved that if $A$ is more than 16-dimensional over its center $Z$, then any Lie ideal $U$ of $K$ satisfies either $U \subseteq Z$ or $U \supseteq [K, K]$; moreover, any Lie ideal $U$ of $[K, K]$ satisfies either $U \subseteq Z$ or $U = [K, K]$. In particular, $[K, K]/(Z \cap [K, K])$ is a simple Lie superalgebra.

INTRODUCTION

If we consider Kac's classification [8] of finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero, we will find in it two series of examples of classical simple Lie superalgebras (the orthosymplectic superalgebras $osp(m, n)$, $m, n \geq 1$, which comprise the families $B(m, n)$, $D(m, n)$ and $C(n)$, and the superalgebras $P(n)$, $n \geq 2$) that are superalgebras of skew-symmetric elements with respect to a superinvolution in a simple associative superalgebra.

It is natural to ask whether the simplicity of these Lie superalgebras of skew-symmetric elements is really a consequence of the simplicity of the
associative superalgebra with superinvolution. We will show in this paper that this is indeed the case.

Classically, this situation has been studied in the context of rings with involution by Herstein [5] and Baxter [1]. They have proved the following results, in which $R$ is a simple ring of characteristic different from 2 with an involution, $K$ is the Lie ring of skew-symmetric elements of $R$ under this involution, and $Z$ denotes the center of $R$.

**Theorem 1.** If $R$ is more than 4-dimensional over its center, then $K$, the subring of the simple ring $R$ generated by $K$, is $R$.

**Theorem 2.** If $R$ is a simple ring with involution of characteristic not 2 and if $R$ is more than 16-dimensional over $Z$, then any Lie ideal $U$ of $K$ must satisfy either $U \subseteq Z$ or $U \supseteq [K, K]$.

**Theorem 3.** If $R$ is as in the previous theorem and if $U$ is a Lie ideal of $[K, K]$, then either $U \subseteq Z$ or $U = [K, K]$. That is, $[K, K]/(Z \cap [K, K])$ is a simple Lie ring.

We generalize these results to the case of superalgebras with superinvolution. Our main results are Theorems 4.2, 5.7, and 6.4, which are just the superanalogues of Theorems 1–3.

Notice that the Lie structure of simple superalgebras without superinvolution was studied by F. Montaner [9] and S. Montgomery [11].

1. ASSOCIATIVE SUPERALGEBRAS

Let $F$ be a field characteristic different from 2. A superalgebra is a $Z_2$-graded algebra $A = A_0 \oplus A_1$, $A_\alpha A_\beta \subseteq A_{\alpha + \beta}$ ($\alpha, \beta \in Z_2$). If $a \in A_\alpha$, then $a$ is homogeneous of degree $\alpha$, and we write $\bar{a} = \alpha$. Elements from $A_0$ are called even and elements from $A_1$ are called odd. For the remainder of this paper, if $\bar{a}$ appears in some expression, it is tacitly assumed that $a$ is homogeneous and that the expression extends over the rest of the elements by linearity.

If $A = A_0 \oplus A_1$ is a superalgebra, then $A_0$ is a (nongraded) superalgebra of $A$. We will say that $A$ is trivial if $A_1 = 0$.

A superalgebra is said to be simple if it does not have nontrivial proper (graded) ideals and the multiplication is not trivial.

An associative superalgebra is just an associative $Z_2$-graded algebra, although this is not true for other varieties of algebras like Lie or Jordan, or even for proper subvarieties of associative algebras (see Section 2). Every associative superalgebra will be supposed to be nontrivial, and unital, in which case the unit is an even element (in particular, $A_0$ is also unital).
By the center $Z(A)$ of an associative superalgebra $A$ we mean a usual (not super!) center of $A$ as an algebra. Notice that $Z(A) \neq 0$, since $A$ is unital. It is clear that $Z(A)$ is a (graded) subalgebra of $A$; i.e., $Z(A) = Z(A)_0 \oplus Z(A)_1$. We will use the notation $Z = Z(A)_0$. It can be shown that if $A$ is simple, then $Z$ is an extension field of $F$ (see [10]). An associative superalgebra is called central if $Z = F$.

The following are basic examples of nontrivial unital central simple associative superalgebras (in examples 1 and 2 we use the notation of [7]).

**Example 1.1.** Let $V = V_0 \oplus V_1$ be a vector superspace. Then the associative algebra $\text{End}(V)$ is provided with the induced $Z_2$-grading $\text{End}(V) = \text{End}(V)_0 \oplus \text{End}(V)_1$, in which

$$\text{End}(V)_a = \{ \alpha \in \text{End}(V) | a(V_0) \subseteq V_{\beta+a} \}.$$

Suppose that $\dim V_0 = r \geq 1$ and $\dim V_1 = s \geq 1$. Translating this into the language of matrices, we obtain a superalgebra $M(r|s)$, whose underlying algebra is that of square matrices of order $r + s$ and whose $Z_2$-grading is determined in the following way:

$$M(r|s)_0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} | A \in M_r(F), D \in M_s(F) \right\},$$

$$M(r|s)_1 = \left\{ \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} | B \in M_r(F), C \in M_s(F) \right\}.$$

It can be proved that $Z(M(r|s))_1 = 0$.

We will use the notation $M(r) = M(r|r)$.

**Example 1.2.** Let $n \geq 1$ and let us consider the superalgebra $M(n)$ of Example 1.1. The following superalgebra $Q(n)$ is a (graded) subalgebra of $M(n)$:

$$Q(n) = \left\{ \begin{bmatrix} A & B \\ B & A \end{bmatrix} | A, B \in M_n(F) \right\}.$$

It can be proved that $Z(Q(n))_1 = Fu$, where

$$u = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

**Example 1.3.** A very simple example if $F \oplus Fu$, with $0 \neq u^2 \in F$. This superalgebra is isomorphic to $Q(1)$. Since it is commutative, $Z(F \oplus Fu)_1 = Fu$. 

Suppose that $\dim V_0 = r \geq 1$ and $\dim V_1 = s \geq 1$. Translating this into the language of matrices, we obtain a superalgebra $M(r|s)$, whose underlying algebra is that of square matrices of order $r + s$ and whose $Z_2$-grading is determined in the following way:

$$M(r|s)_0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} | A \in M_r(F), D \in M_s(F) \right\},$$

$$M(r|s)_1 = \left\{ \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} | B \in M_r(F), C \in M_s(F) \right\}.$$
Example 1.4. Now let $Q(\alpha, \beta)$ be a (generalized) quaternion algebra, with basis $\{1, u, v, w\}$, such that $u^2 = \alpha$, $v^2 = \beta$, and $uv = -vu$. The choice of $u$ and $v$ to be odd determines a $\mathbb{Z}_2$-grading in $Q(\alpha, \beta)$, in which $Q(\alpha, \beta)_0 = F(1, w)$ and $Q(\alpha, \beta)_1 = F(u, v)$. It is said that $Q(\alpha, \beta)$ is a quaternion superalgebra. Moreover, if $Q(\alpha, \beta)$ is split, then it is isomorphic to $M(1)$. It can be proved that $Z(Q(\alpha, \beta))_1 = 0$.

Example 1.5. Let $V$ be a vector space and $q$ be a quadratic form defined on $V$. The tensor algebra of $V$ is an associative superalgebra $\mathrm{T}(V) = \mathrm{T}(V)_0 \oplus \mathrm{T}(V)_1$, where $\mathrm{T}(V)_0 = F + V \otimes V + \cdots$ and $\mathrm{T}(V)_1 = V + V \otimes V \oplus V + \cdots$. The ideal $I(q) = (v \otimes v - q(v); v \in V)$ of $\mathrm{T}(V)$ is a graded ideal, since it is generated by homogeneous elements. Hence the Clifford algebra $C(V, q) = \mathrm{T}(V)/I(q)$ is in fact an associative superalgebra, which is called the Clifford superalgebra of the quadratic space $(V, q)$. If $q$ is nondegenerate, then $C(V, q)$ is a central simple superalgebra. Notice that Examples 1.3 and 1.4 are the particular cases of $C(V, q)$ with $\dim V = 1$ and $\dim V = 2$. It can be proved that $Z(C(V, q))_1 = 0$ if and only if $\dim V$ is even.

If $A$ is a simple associative superalgebra and $Z = Z(A)_0$, we will say that $A$ is $C(n)$ if there exist a $\mathbb{Z}$-vector space $V$ of dimension $n$ and a nondegenerate quadratic form $q$ defined on $V$ such that $A$ and $C(V, q)$ are isomorphic as superalgebras over $Z$. For example, a quaternion superalgebra is $C(2)$.

2. LIE AND JORDAN STRUCTURES IN ASSOCIATIVE SUPERALGEBRAS

A Grassmann superalgebra $G$ is defined to be the Clifford superalgebra $C(V, q)$, where $V$ is of denumerable dimension and $q$ is identically zero on $V$. In other words, $G = \mathrm{alg} < 1, e_1, e_2, \ldots > | e_i e_j + e_j e_i = 0, \forall i, j = 1, 2, \ldots >$ endowed with the natural $\mathbb{Z}_2$-grading $G = G_0 \oplus G_1$, where $e_i \cdots e_n \in G_0$ if $n$ is even and $e_i \cdots e_n \in G_1$ if $n$ is odd.

Given a superalgebra $A = A_0 \oplus A_1$, we consider the tensor product $G \otimes A$. Its subalgebra $G(A) = G_0 \otimes A_0 + G_1 \otimes A_1$ is called the Grassmann envelope of the superalgebra $A$.

Now let $V$ be a homogeneous variety of algebras, for example, that of associative, Lie, or Jordan algebras. We say that a superalgebra $A = A_0 \oplus A_1$ is a $V$-superalgebra if $G(A) \subseteq V$.

It can be proved that $G(A) \subseteq V$ if and only if $A$ satisfies a certain system of graded identities, depending on the variety. For example, the associativity of $G(A)$ is equivalent to that of $A$, which makes the foregoing
definition consistent with that in Section 1. On the other hand, a superalgebra is a Lie superalgebra if it satisfies the following graded identities (super-anticommutative and super-Jacobi):

\[ [a, b] = -(-1)^{\bar{x}\bar{y}}[b, a]; \]
\[ [a, [b, c]] = [[a, b], c] + (-1)^{\bar{a}\bar{b}}[b, [a, c]]. \]

where \( \bar{x} \) means the parity of \( x \): \( \bar{x} = i \) if \( x \in A_i \), \( i = 0, 1 \). A n analogous statement is true for a Jordan superalgebra (see [3], for example).

Let \( A \) be an associative superalgebra. We define two new multiplications on \( A \) by the following expressions:

\[ [a, b] = ab - (-1)^{\bar{x}\bar{y}}ba; \]
\[ a \circ b = ab + (-1)^{\bar{x}\bar{y}}ba. \]

We obtain in this way two new superalgebras \( A^- \) and \( A^+ \) with the same grading as \( A \). If \( A = A_0 \), then we have \([a, b] = ab - ba, a \circ b = ab + ba, \) and it is well known that in this case \( A^- \) is a Lie algebra and \( A^+ \) is a Jordan algebra. Straightforward calculations show that \( G(A^-) = G(A)^- \) and \( G(A^+) = G(A)^+ \), where \( G(A) \) is considered as an ordinary (even) algebra; hence \( A^- \) is a Lie superalgebra and \( A^+ \) is a Jordan superalgebra.

A (graded) subspace of \( A \) is said to be a Lie ideal of \( A \) if it is an ideal of \( A^- \). Notice that \([A, A], \) the subspace generated by all the elements of the form \([a, b], \) is a Lie ideal of \( A \). Hence, it also makes sense to speak about \( \) Lie ideals of \([A, A]. \)

### 3. SUPERINVOLUTIONS IN ASSOCIATIVE SUPERALGEBRAS

Let \( A \) be an associative superalgebra. A superinvolution on \( A \) is a \( \mathbb{Z}_2 \)-graded linear map \( * : A \rightarrow A \) such that, for all \( a, b \in A, (a^*)^* = a \) and \( (ab)^* = (-1)^{\bar{a}\bar{b}}b^*a^*. \)

If \( * \) is a superinvolution on \( A, \) then the restriction of \( * \) to \( A_0 \) is an involution on \( A_0, \) Moreover, it is easy to see that \( 1^* = 1. \)

The following are basic examples of superinvolutions [8].

#### Example 3.1.
Let \( V = V_0 \oplus V_1 \) be a vector superspace with \( \dim V_0 = r \geq 1 \) and \( \dim V_1 = s \geq 1. \) Let \( (, ) \) be a nondegenerate super-symmetric bilinear form on \( V; \) i.e., the restriction of \( (, ) \) to \( V_0 \) is symmetric, the restriction of \( (, ) \) to \( V_1 \) is skew-symmetric (in particular, \( s = 2r \) is even),
and $V_0$ and $V_1$ are orthogonal. The adjoint endomorphism $a^*$ of $a \in \text{End}(V)$ is defined by

$$(a(v), w) = (-1)^{ar}(v, a^*(w)).$$

Straightforward calculations show that the map $a \mapsto a^*$ defines a superinvolution on $\text{End}(V)$, which we call orthosymplectic superinvolution and will denote by $osp$.

Let us translate this into the language of matrices. Let $H \in M_r(F)$ (respectively, $K \in M_t(F)$) be the matrix associated with the restriction of $(\cdot, \cdot)$ to $V_0$ (respectively, to $V_1$). Notice that $H$ is a symmetric matrix, $K$ is a skew-symmetric matrix, and both $H$ and $K$ are invertible. It is easily checked that the orthosymplectic superinvolution on $M(r,s)$ is given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{op} = \begin{bmatrix} H & 0 \\ 0 & K \end{bmatrix}^{-1} \begin{bmatrix} A & -B \\ C & D \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & K \end{bmatrix},$$

where $t$ denotes the usual matrix transposition.

**Example 3.2.** Let us consider now the superalgebra $M(r)$. We will call the following superinvolution defined on $M(r)$ a transposition superinvolution, and we will denote it by $trp$:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{trp} = \begin{bmatrix} D^t & -B^t \\ C^t & A^t \end{bmatrix}.$$

Let $A$ be an associative superalgebra with superinvolution $\ast$. An element $a \in A$ is called symmetric if $a^\ast = a$ and skew-symmetric if $a^\ast = -a$. Both the set $H = H(A, \ast) = \{a \in A | a^\ast = a\}$ of symmetric elements and the set $K = K(A, \ast) = \{a \in A | a^\ast = -a\}$ of skew elements are graded subspaces of $A$. Since the characteristic of $F$ is different from 2, it is obvious that $H \cap K = 0$; moreover, given $a \in A$, $a = (a + a^\ast)/2 + (a - a^\ast)/2$, with $(a + a^\ast)/2 \in H$ and $(a - a^\ast)/2 \in K$; hence $A = H \oplus K$.

The following containments are straightforward to check, and they will be used throughout without explicit mention: $[K, K] \subseteq K$, $[K, H] \subseteq H$, $[H, H] \subseteq K$, $H \circ H \subseteq H$, $H \circ K \subseteq K$, and $K \circ K \subseteq H$. In particular, $K$ is a subalgebra of $A^\ast$; hence it is a Lie superalgebra. And $H$ is a subalgebra of $A^\ast$; hence it is a Jordan superalgebra.

The interplay between the superinvolution and the center will be shown to be decisive in our work. First, notice that $Z(A)^\ast \subseteq Z(A)$; hence $\ast$ induces an involution on $Z$. If this involution is the identity map, then $\ast$ is said to be of the *first kind*, and it is said to be of the *second kind* in the contrary case. In other words, $\ast$ is of the first kind if $Z \subseteq H$, and of the
second kind if $Z \not\subseteq H$. We will use the notation $Z_H = Z \cap H$, $Z_K = Z \cap K$. Notice that $Z_H = Z$ if and only if $*$ is of the first kind.

Let $A$ be a simple associative superalgebra with superinvolution $*$. Then one has the following lemmas, whose proof can be found in [2].

**Lemma 3.1.** If the superinvolution $*$ is of the second kind, then $Z$ is a subfield of $Z$ with $[Z : Z_H] = 2$. Furthermore, if $0 \neq q \in Z_K$, then $H = qK$, $K = qH$, $Z_H = qZ_K$, and $Z_K = qZ_H$.

Notice that $*$ is always a $Z_H$-superinvolution and that it is a $Z$-superinvolution if and only if $*$ is of the first kind.

**Lemma 3.2.** If $Z(A)_1 \neq 0$, then the superinvolution $*$ is of the second kind.

This lemma is used essentially in the proof of the following.

**Theorem 3.1.** The following simple associative superalgebras do not admit superinvolutions:

1. $Q(n)$ for all $N \geq 1$.
2. $F \oplus Fu$ with $0 \neq u^2 \in F$.
3. $C(V, q)$ if $V$ is of odd dimension and $q$ is nondegenerate.
4. A division quaternion superalgebra $Q(\alpha, \beta)$.

**Proof.** Suppose first that $A$ is one of the superalgebras of items (i), (ii), or (iii) in the list above and that $*$ is a superinvolution on $A$. It was noticed in Section 1 that $A$ is central, i.e., $Z = F$; hence $*$ is a $Z$-superinvolution and consequently it is of the first kind. On the other hand, it was also noticed in Section 1 that $Z(A)_1 \neq 0$. Therefore, by Lemma 3.2, $*$ is of the second kind, a contradiction.

Suppose now that $*$ is a superinvolution on $Q(\alpha, \beta)$, and let $\{1, u, v, uv\}$ be a basis of $Q(\alpha, \beta)$, with $u^2 = \alpha$, $v^2 = \beta$, and $uv = -uv$. Let $a, b, c, d \in F$ such that $u^* = au + bv$ and $v^* = cu + dv$. Imposing the necessary conditions for $*$ to be a superinvolution and making calculations, we obtain $a = d = 0$; hence $*$ is determined by $1^* = 1$, $(uv)^* = -uw$, $u^* = bv$, and $v^* = cu$, with $bc = 1$. Furthermore, it must satisfy the relation $\beta b = -\alpha c$. Now let us consider the nonzero element $u + bv$. Its norm is $n(u + bv) = -\alpha - \beta b^2 = 0$, which proves that $Q(\alpha, \beta)$ must be split.

Notice that this theorem does not exclude the possibility of the existence of a superinvolution of the second kind in a simple (not central) associative superalgebra that is of one of the above-mentioned types as a superalgebra over $Z$.

If a quaternion superalgebra $Q(\alpha, \beta)$ is split, then it is isomorphic to $M(1)$. The following theorem classifies all superinvolutions in this superalgebra.
**Theorem 3.2.** The only two superinvolutions on $M(1)$ are $trp$ and $(trp)p$, where $p$ is the automorphism of $M(1)$ given by $p(a_0 + a_1) = a_0 - a_1$ (the parity automorphism).

**Proof.** Let $*$ be a superinvolution on $M(1)$, and let us consider the standard basis $\{e_{11}, e_{22}, e_{12}, e_{21}\}$ of $M(1)$. It is easy to see that $0, 1, e_{11},$ and $e_{22}$ are the only even idempotents of $M(1)$, and that if $e$ is one of them, then $e^*$ is also idempotent. Since $0^* = 0$ and $1^* = 1$, there are exactly two possibilities: (i) $e_{11}^* = e_{11}, e_{22}^* = e_{22}$; (ii) $e_{11}^* = e_{22}, e_{22}^* = e_{11}$. Let us write in any case, $e_{12}^* = xe_{12} + ye_{21}, e_{21}^* = ze_{12} + te_{21}$. Imposing the necessary conditions for $*$ to be a superinvolution and making calculations, we arrive at the following: in case (i), an inconsistent system of equations, and in case (ii), $x = \pm 1, y = z = 0, t = \mp 1$. Hence there are exactly two possibilities for $*$:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ c & a \end{pmatrix};
\]

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & b \\ -c & a \end{pmatrix}.
\]

In the first case, $* = trp$ is the transposition superinvolution, and in the second case, $* = (trp)p$ is the composition of $trp$ with the parity automorphism. \(\square\)

For the remainder of this paper we assume that $A$ is a simple associative superalgebra with superinvolution $*$, $K$ is the Lie superalgebra of skew elements of $A$, $H$ is the Jordan superalgebra of symmetric elements of $A$, and $Z$ is $Z(A)_e$, the even part of the center of the algebra $A$.

### 4. Generation of $A$ by $K$

Let $\overline{K}$ be the subsuperalgebra of $A$ generated by $K$. We prove in this section that, with some exceptions, $\overline{K} = A$.

**Lemma 4.1.** $K^2$ is a Lie ideal of $A$.

**Proof.** Let $k, l \in K$ and $a \in A$. Since $la - (la)^* = la - (-1)^{|a|}l^*a = la + (-1)^{|a|}l^*a \in K$ and $ka - (ka)^* = ka - (-1)^{|a|}a^*k = ka + (-1)^{|a|}a^*k \in K$, we get that $[kl, a] = kla - (-1)^{|a|}a^*kla = k(la + (-1)^{|a|}a^*l) - (-1)^{|a|}a^*k(la + (-1)^{|a|}a^*l) \in K^2$; hence $[K^2, A] \subseteq K^2$. \(\square\)

**Theorem 4.1.** If $U$ is a subalgebra and a Lie ideal of $A$, then either $U \subseteq Z$ or $U = A$, except if $A$ is $C(2)$. 


Proof. Let us first support that \([U, U] \neq 0\). Then for some homogeneous \(u, v \in U, [u, v] \neq 0\). For any \(a \in A, [u, va]\) is in \(U\); that is, \([u, v]a + (-1)^{pr}v[u, a] \) is in \(U\). The second member of this is in \(U\), since both \(v\) and \([u, a]\) are in \(U\) (since \(U\) is both a Lie ideal and a subalgebra). The net result of all this is that \([u, v]A \subseteq U\). But then for \(a, b \in A, [[u, v], b] = [u, v][ab - (-1)^{pr}x \sigma y b[u, v]a \in U, leading to \(A[u, v]A \subseteq U\). Since \(A\) is simple and \(A[u, v]A\) is a nontrivial ideal of \(A\), we arrive in this case at \(U = A\).

Suppose now that \([U, U] = 0\). Then, by Theorem 3.1 of [9], \(U \subseteq Z\), except if \(A = C(2)\). 

Theorem 4.2. The subsuperalgebra \(\overline{K}\) of \(A\) generated by \(K\) coincides with \(A\), except if \(A = C(2)\).

Proof. Let \(\overline{K^2}\) be the subsuperalgebra of \(A\) generated by \(K^2\). Since \(K^2\) is a Lie ideal of \(A\) by Lemma 4.1, \(\overline{K^2}\) is both a subalgebra and a Lie ideal of \(A\). We will suppose in the sequel that \(A\) is not \(C(2)\). Then, by Theorem 4.1, either \(\overline{K^2} \subseteq Z\) or \(\overline{K^2} = A\). In the second case it is obvious that \(\overline{K} = A\), and we are finished.

So, suppose that \(\overline{K^2} \subseteq Z\). Equivalently, \((K^2)_{0} \subseteq Z\) and \((K^2)_{1} = 0\). Let us first consider the possibility \(K_{0} = 0\), i.e., \(A_{0} = H_{0}\). If \(a, b \in A_{0}\), then \(ab = (ab)^{0} = b^{a}a^{0} = ba\); hence \(A_{0}\) is commutative. Since \(A\) is not \(C(2)\), \(A\) is commutative by Lemma 2.5 of [9]; in fact, \(A = Z \oplus Z\) with \(0 \neq u^{2} \in Z\), and \(\ast\) is a \(Z\)-superinvolution. This contradicts Theorem 3.1. Hence \(K_{0} \neq 0\), and we choose a nonzero element \(a \in K_{0}\).

Let us now suppose that \(aK_{0} = 0\). If \(h \in H_{0}\), then \(hah \in K_{0}\); hence \(ahah = 0\). Let \(x = h + k \in A_{0}\), with \(h \in H_{0}\) and \(k \in K_{0}\). Then \((ax)^{2} = a(h + k)(h + k) = 0\). Since \(aA_{0} \neq 0\), \(A_{0}\) contains a nonzero nilpotent ideal by Lemma 1.1 of [5], which is a contradiction, since \(A_{0}\) is semiprime by Lemma 1.1 of [9]. Therefore \(aK_{0} \neq 0\), and we can choose an element \(b \in K_{0}\) such that \(ab \neq 0\). Since \(0 \neq ab \in (K^2)_{0} \subseteq Z\), \(a\) must be invertible in \(A\).

Let \(c \in K_{1}\). Then \(ac \in (K^2)_{1} = 0\); hence \(c = 0\), since \(a\) is invertible. Thus \(K_{1} = 0\) and \(A_{1} = H_{1}.\) If there exists an element \(u \in Z(A)_{1}\) satisfying \(u^{2} = 1\), then \(1 = (wu)^{0} = -u^{0}w^{0} = -u^{2} = -1\), which is a contradiction. Hence \(A\) is a simple algebra by Lemma 3 of [10].

Now let \(z \in Z(A)_{1}\) and \(x \in A_{1}\). Since \(A_{1} = H_{1}\), \((z^{2}x)^{0} = x^{0}(zz)^{0} = -x^{0}z^{0}z^{0} = -xz^{2} = -z^{2}x\); hence \(z^{2}x \in K_{1} = 0.\) This proves \(z^{2}A_{1} = 0\). By Lemma 1 of [10], \(A_{0} = A_{0}^{2}\); thus \(AZ = z^{2}A_{0} = 0.\) But \(A\) is semiprime by Lemma 1.1 of [9]; therefore \(z = 0\). This proves \(Z(A)_{1} = 0.\) We deduce from this that since \(Z(A) = Z\), \(A\) is central simple as an algebra over \(Z\).
Let us again consider the element \( a \in K_0 \), which is invertible in \( A \).

0 \( \neq a^2 = \xi \in Z \) for \( (K^2)_0 \subseteq Z \); hence \( a^{-1} = \xi^{-1}a \). Since \( aK_0 \subseteq (K^2)_0 \subseteq Z, K_0 \subseteq Za^{-1} = Za \).

If \( h \in H_0 \) commutes with \( a \), then \( ha \) must be in \( K_0 \); hence \( ha = \alpha a \) for some \( \alpha \in Z \), leading us to \( h = \alpha \in Z \). In other words, the only even symmetrics commuting with \( a \) are those in the center. Given \( h \in H_0, h \circ a \in K_0 \); hence \( ha + ah = \mu a \) with \( \mu \in Z_h \) (notice that \( K_0 = Z_h a \) by the above). Thus \((h - \mu/2)a + a(h - \mu/2) = 0 \). But then \((h - \mu/2)^2\) is an even symmetric element that commutes with \( a \); therefore \((h - \mu/2)^2\) must be in \( Z \).

Let \( x = h + K \in A_0 \) with \( h \in H_0, k = \lambda a \in K_0, \lambda \in Z \). As before, \( ha + ah = \mu a \) with \( \mu \in Z_h \), and \((h - \mu/2)^2 \in Z \). Moreover, \((h - \mu/2)a + a(h - \mu/2) = 0 \); hence \((x - \mu/2)^2 = (h - \mu/2 + \lambda a)^2 = (h - \mu/2)^2 + \lambda((h - \mu/2)a + a(h - \mu/2)) + \lambda^2a^2 = (h - \mu/2)^2 + \lambda^2a^2 \). But \((h - \mu/2)^2, \lambda^2a^2 \in Z \); hence \((x - \mu/2)^2 \in Z \). Therefore, \( A_0 \) is a quadratic algebra over \( Z \).

It is not difficult to prove that \( A_0 \) is in fact a composition algebra over \( Z \). In particular, \( A_0 \) is finite-dimensional over \( Z \), and either \( A_0 \) is commutative or \( A_0 \) is central simple over \( Z \). Since \( A_0^1 = A_0 \), we get that \( A \) is also finite-dimensional over \( Z \).

If \( \hat{Z} \) is an algebraic closure of \( Z \), then \( \hat{A} = \hat{Z} \otimes Z \). \( A \) is a finite-dimensional simple superalgebra over \( Z \), which is also simple as an algebra (Theorem 2 of [10]). By Theorem 2.6 of [7], we get that there exist \( r, s \geq 1 \) such that \( A \) is isomorphic to \( M(r|s) \) over \( Z \).

Suppose that \( A_0 \) is central simple over \( Z \) (in particular, \( Z(A_0) = Z \)). Then \( A_0 = \hat{Z} \otimes Z \). \( A_0 \) is a simple algebra over \( Z \). On the other hand, \( A_0 \) is isomorphic to \( M_r(\hat{Z}) \oplus M_s(\hat{Z}) \), which is not a simple algebra \((r, s \geq 1) \). This contradiction proves that \( A_0 \) is commutative.

Since \( Z(A_0) = 0 \) and \( A \) is nontrivial, \( A \) is not commutative. Since \( A_0 \) is commutative, \( A \) must be \( C(2) \) by Lemma 2.5 of [9], a contradiction. \( \blacksquare \)

With respect to the exceptional case of the theorem, let us suppose that \( A \) is \( C(2) \) and that the superinvolution \(*\) is of the first kind; that is, \(*\) is a \( Z\)-superinvolution. By Theorem 3.1, \( A \), being a quaternion superalgebra over \( Z \), must be split and hence isomorphic to \( M(1) \) as a superalgebra over \( Z \). By Theorem 3.2, we may suppose that the superinvolution \(*\) is \( trp \); i.e., it is given by

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}^{trp} = \begin{bmatrix}
d & -b \\
c & a
\end{bmatrix}.
\]

Straightforward calculations show that

\[
K = K(M(1), trp) = \left\{ \begin{bmatrix}
a & b \\
0 & -a
\end{bmatrix} \right| a, b \in Z \right\}.
\]
Therefore $K \neq A$, since it is evident that every matrix from $K$ has 0 in its $(2,1)$-entry.

The following results give some relationships between $H$ and $K$.

**Theorem 4.3.** $H = K \circ K$ and $A = K + K \circ K$, except if $A$ is $C(2)$.  

**Proof.** If $a, b \in K$, then $[a, b] = ab - (-1)^{\sigma \tau} ba \in K$, $a \circ b = ab + (-1)^{\sigma \tau} ba \in K \circ K$, and hence $2ab = [a, b] + a \circ b \in K + K \circ K$. Therefore $K^2 \subseteq K + K \circ K$.

Let $a, b, c \in K$. On one hand, $(a \circ b) \circ c \in (K \circ K) \circ K \subseteq H \circ K \subseteq K$; on the other, $[a \circ b, c] = [a \circ b, c] = a \circ [b, c] + (-1)^{\sigma \tau} [a, c] \circ b \in K \circ K$. Adding up as before gives $2(a \circ b)c \in K + K \circ K$. Therefore $(K \circ K)K \subseteq K + K \circ K$.

We deduce from the preceding containments that $K^3 = K^2 K \subseteq K^2 + (K \circ K)K \subseteq K + K \circ K$. Continuing, we get by induction that $K^n \subseteq K + K \circ K$ for all $n$.

Now taking Theorem 4.2 into account, we obtain that $A = K = \Sigma K^n \subseteq K + K \circ K$. Therefore $A = K + K \circ K$, and it is now obvious that $H = K \circ K$.

**Corollary 4.1.** If $A$ is not $C(2)$ and $\dim_{\mathbb{Z}} K = n$, then $\dim Z A \leq n + n^2$.

**Proof.** By the theorem, $A = K + K \circ K$. Now if $(k_1, \ldots, k_n)$ is a basis of $K$ over $Z_H$, then $(k_i, k_j | i, j = 1, \ldots, n)$ generates $K \circ K$ over $Z_H$. Hence, $A$ is at most of dimension $n + n^2$ over $Z$.

5. **Lie Structure of $K$**

We start with superinvolutions of the second kind, which are easier to deal with.

**Theorem 5.1.** If the superinvolution $\ast$ is of the second kind and $U$ is a Lie ideal of $K$, then either $U \subseteq Z$ or $U \supseteq [K, K]$, except if $A$ is $C(2)$.

**Proof.** Let $0 \neq q \in Z_k$ and consider the $Z$-subspace $ZU$ generated by $U$. By Lemma 3.1, $H = qK$; hence $[ZU, A] = [ZU, K + qK] = [ZU, K] + [ZU, qK] \subseteq ZU$. Therefore, $ZU$ is a Lie ideal of $A$.

Suppose first that $A_q$ is not commutative; in particular, $A$ is not $C(2)$. By Theorem 3.1 of [9], either $ZU \subseteq Z$ or $ZU \supseteq [A, A]$. In the first case it is obvious that $U \subseteq Z$.

Let us now consider the second case, namely $ZU \supseteq [A, A]$. We have $Z_H U + qZ_H U = ZU \supseteq [A, A] = [K, K] + q[K, K]$; hence $Z_H U \supseteq [K, K]$ for $Z_H U$ is the skew part of $ZU$. 


Suppose now that $A_0$ is not commutative, but $A$ is not $C(2)$; then $A$ is commutative by Lemma 2.5 of [9]. We distinguish two cases (Lemma 3 of [10]):

(i) $A$ is a simple algebra. Then $A = Z(A)$ is a field. If we take $0 \neq q \in Z_K$, $0 \neq u \in K_1 \subseteq A_1 = Z(A)_1$, then it is easily checked that $(1, q, u, qu)$ is a $Z_H$-basis of $A$ with $1^* = 1$, $q^* = -q$, $u^* = -u$, $(qu)^* = qu$. In this case, $(q, u)$ is a $Z_H$-basis of $K$. Since $u^2$ is skew, $[u, u] = 2u^2 = q$ with $0 \neq z \in Z_H$; hence $[K, K] = Z_H q = Z_K$. Let $0 \neq \lambda q + \mu u \in U$ with $\lambda, \mu \in Z_H$. If $\mu = 0$, it is clear that $[K, K] \subseteq U$. If $\mu \neq 0$, then $[\lambda q + \mu u, u] = [\mu, u] = \mu z q \in U$; hence $[K, K] \subseteq U$ also. In fact, there are only three possibilities for $U$, namely: $U = 0$, $U = [K, K] = Z_K$, or $U = K$.

(ii) $A_0$ is simple and $A_1 = A_0 u$ with $u \in Z(A)_1$ and $u^2 = 1$. Since $u^* \in A_1$, there exists $q \in A_0$ such that $u^* = qu$. Hence $q^* = -q$; i.e., $0 \neq q \in K_0 = Z_K$. Now it is easy to check that $(1, q, u, qu)$ is a $Z_H$-basis of $A$ with $1^* = 1$, $q^* = -q$, $u^* = qu$, $(qu)^* = u$. In this case, $(q, u - qu)$ is a $Z_H$-basis of $K$. Taking into account that $q^2 = -1$ and $u^2 = 1$, we get $[u - qu, u - qu] = [u, u] - 2q[u, u] + q^2[u, u] = -4q$; hence $[K, K] = Z_H q = Z_K$. Let $0 \neq \lambda q + \mu (u - qu) \in U$ with $\lambda, \mu \in Z_H$. If $\mu = 0$, it is clear that $[K, K] \subseteq U$. If $\mu \neq 0$, then $[\lambda q + \mu (u - qu), u - qu] = -4\mu q \in U$; hence $[K, K] \subseteq U$ also. In fact, in this case, too, there are only three possibilities for $U$, namely: $U = 0$, $U = [K, K] = Z_K$, or $U = K$.

We will need in the next section the following facts about the structure of $[K, K]$ obtained during the proof of this theorem.

**Corollary 5.1.** Under the conditions of Theorem 5.1, if $A_0$ is not commutative, then $[K, K] = [[K, K], [K, K]]$, and if $A_0$ is commutative, then $[K, K] = Z_K$.

Now we can focus our attention on the case of superinvolutions of the first kind. While many of the results to follow hold irrespective of the nature of the superinvolution, we henceforth assume in this section that the superinvolution $\circ$ is of the first kind.

**Lemma 5.1.** Let $U$ be a Lie ideal of $K$, and let $u, v \in U$, $h \in H$. Then $[u \circ v, h] \in U$. 


Proof. The following graded identity is straightforward:

\[ [x \circ y, z] = [x, y \circ z] + (-1)^{xy}[y, x \circ z]. \]

By this identity, \([u \circ v, h] = [u, v \circ h] + (-1)^{uv}[v, u \circ h]\). Now \(u \circ h, v \circ h \in K \circ H \subseteq K\); hence \([u, v \circ h], [v, u \circ h] \in [U, K] \subseteq U\). □

**Lemma 5.2.** Let \(U\) be a Lie subsuperalgebra of \(K\). Then the subsuperalgebra \(\bar{U}\) generated by \(U\) in \(A\) is given by \(\bar{U} = \sum_{i=0}^{n} (U, \circ)^i\), where \((U, \circ)^1 = U\), and for \(i > 1\), \((U, \circ)^i = (U, \circ) \circ U\). In particular, \(\bar{U} \cap K = \sum_{i=0}^{n} (U, \circ)^{2i-1}\).

Proof. Denote the sum in the lemma by \(\bar{U}\). It suffices to show that \(\bar{U} \subseteq \bar{U}\). It is obvious that \(\bar{U} \circ U \subseteq \bar{U}\). Furthermore, we have \([U, U] \subseteq U\); so assume that \([U, \circ)^i U, U] \subseteq U\) for some \(i \geq 1\). Then \([U, \circ)^i U, U] = \sum_{i=0}^{n} (U, \circ)^{2i-1}\), which proves that \([U, U] \subseteq \bar{U}\). Since \(2ab = [a, b] + a \circ b\) for homogeneous elements \(a, b\), this proves that \(\bar{U} \subseteq U\). Finally, the inclusions \(K \circ H \subseteq K, K \circ H \subseteq K\) imply \(\sum_{i=0}^{n} (U, \circ)^{2i-1} \subseteq K\).

**Lemma 5.3.** Let \(U\) be a Lie ideal of \(K\) such that \(\bar{U} = A\). Then \(U \supseteq [K, K]\).

Proof. By Lemma 5.2, \(K = \sum_{i=0}^{n} (U, \circ)^{2i-1}\). Therefore, it suffices to prove that \([U, (U, \circ)^{2i-1}, K] \subseteq U\) for any \(i \geq 1\). Since \([U, K] \subseteq U\), we may make the induction assumption that \([U, (U, \circ)^{2i-1}, K] \subseteq U\) for some \(i \geq 1\). Then,

\[
(U, \circ)^{2i+1} = (U, \circ)^{2i} \circ U, K
\]

\[
\subseteq [(U, \circ)^2i, U \circ K] + [U, (U, \circ)^2i \circ K]
\]

\[
\subseteq [(U, \circ)^{2i-1} \circ U, H] + [U, H \circ K]
\]

\[
\subseteq [(U, \circ)^{2i-1} \circ U, H] + [U, (U, \circ)^{2i-1} \circ H] + [U, K]
\]

\[
\subseteq [(U, \circ)^{2i-1}, K] + [U, K \circ H] + U \subseteq U + [U, K] = U,
\]

and the lemma is proved. □

**Lemma 5.4.** Let \(M\) and \(S\) be subsuperspaces of \(A\) such that \([M, S] \subseteq S\) and \(M = A\); that is, the set \(M\) generates \(A\) as a superalgebra. Then

(i) If \(S \neq \{0\}\), then the set \(W = \{w \in A | wS = 0\} = 0\).

(ii) If there exists \(0 \neq a \in A\) such that \(Aa \subseteq \bar{S}\), then \(\bar{S} = A\).
Proof. It is obvious that $AW \subseteq W$. Furthermore, let $w \in W$, $s \in S$, $m \in M$. Then we have
\[
0 = w[m, s] = wms - (-1)^mwxsm = wms,
\]
which proves that $WM \subseteq W$ and $W\overline{M} \subseteq W$. Therefore, $WA \subseteq W$, and $W$ is an ideal of $A$. Since $A$ is simple and unital, it follows that $W = 0$, and (i) is proved.

To prove (ii), consider the set $L = \{ a \in A | aA \subseteq \overline{S} \}$. Evidently, $L$ is a left ideal of $A$. We claim that $LA \subseteq L$ as well. Observe first that the inclusion $[M, S] \subseteq S$ implies $[M, \overline{S}] \subseteq \overline{S}$. Furthermore, let $m \in M$, $a \in L$, $x \in A$; then $xa \in \overline{S}$. Hence $[m, xa] = mxa - (-1)^{m+\sigma}xam \in [M, \overline{S}] \subseteq \overline{S}$. Since $a \in L$, we have $mxa = (mx)a \in \overline{S}$, which gives $xam \in S$ and $am \in L$. Since $\overline{M} = A$, this proves that $LA \subseteq L$ and $L$ is an ideal of $A$. By the condition of the lemma $L \neq 0$; hence $L = A$ and $A = A^2 = AL \subseteq \overline{S}$.

**Lemma 5.5.** Suppose that $K = A$, and let $U$ be a Lie ideal of $K$. Then either $\overline{U} = A$ or $[u \circ v, w] = 0$ for all $u, v, w \in U$.

**Proof.** Let $u, v \in U$, $h \in H$, $k \in K$. By Lemma 5.1, $[u \circ v, h] \in U$. On the other hand,
\[
[u \circ v, k] = u \circ [v, k] + (-1)^t[u, k] \circ v \in \overline{U},
\]
since $[u, k], [v, k] \in [K, U] \subseteq U$. Since $A = H \oplus K$, this proves that $[u \circ v, A] \subseteq \overline{U}$.

Now let $w \in U$, $a \in A$. By the above, $[u \circ v, aw] \in \overline{U}$. But
\[
[u \circ v, aw] = [u \circ v, a]w + (-1)^{t+t}a[u \circ v, w],
\]
and clearly $[u \circ v, aw] \in \overline{U}$, since $[u \circ v, a] \in \overline{U}$ and $w \in U$. Consequently, $a[u \circ v, w] \in \overline{U}$; that is, $A[u \circ v, w] \subseteq \overline{U}$.

Now applying Lemma 5.4(ii) to $M = K$, $S = U$, $a = [u \circ v, w]$, we get that either $\overline{U} = A$ or $[u \circ v, w] = 0$ for all $u, v, w \in U$.

This lemma, together with Lemma 5.3, enables us to establish the key fact contained in the next theorem.

**Theorem 5.2.** Suppose that $A$ is not $C(2)$ and let $U$ be a Lie ideal of $K$. Then either $U \supseteq [K, K]$ or $[u \circ v, w] = 0$ for all $u, v, w \in U$.

**Proof.** Since $A$ is not $C(2)$, $K = A$ by Theorem 4.2. Assume that $[u \circ v, w] \neq 0$ for some $u, v, w \in U$; then $\overline{U} = A$ by Lemma 5.5, and $U \supseteq [K, K]$ by Lemma 5.3.
To obtain further consequences of the second case of Theorem 5.2, we will need the following.

**Lemma 5.6.** Let $h$ be a symmetric element satisfying $[h, [h, A]] = 0$. Then, if $h$ is even, then $h \in Z$, and if $h$ is odd, then $h^2 = 0$.

**Proof.** If $h$ is even, then $h \in Z$ by Lemma 1.19 of [6]. Now suppose that $h$ is odd. For any $a \in A$ we have

$$0 = [h, [h, a]] = [h, ha - (-1)^{ah}h]$$

$$= h(ha - (-1)^{ah}) + (-1)^{ah}(ha - (-1)^{ah})h = h^2a - ah^2;$$

hence $h^2 \in Z$. Moreover, $2h^2 = [h, h] \in K$; thus $h^2 \in Z \cap K = 0$ ($\ast$ is of the first kind).

**Theorem 5.3.** Let $U$ be a Lie ideal of $K$ such that $[u, v, w] = 0$ for all $u, v, w \in U$. Then

(i) $u \circ v \in Z$ for all $u, v \in U_0$,

(ii) $u \circ v = 0$ for all $u, v \in U_1$,

(iii) $(u \circ v)^2 = 0$ for all $u \in U_0, v \in U_1$.

Moreover, assertion (ii) is true for any Lie subsuperalgebra $U$ of $K$ such that $u \circ v \in Z$ for all $u, v \in U_i, i = 0, 1$.

**Proof.** Let $u, v \in U$, $h \in H$, $k \in K$. By Lemma 5.1, $[u \circ v, h] \in U$; hence $[u \circ v, [u \circ v, h]] = 0$. On the other hand, $[u \circ v, k] = u \circ [v, k] + (-1)^{\gamma}(u \circ v)(k) \circ v \in U \circ U$, hence $[U, [u \circ v, k]] = 0$. Similarly, $[U \circ U, U \circ U] = 0$, and so $[u \circ v, [u \circ v, k]] = 0$. This proves that $[u \circ v, [u \circ v, a]] = 0$ for all $a \in A$. Now assertions (i) and (iii) follow immediately from Lemma 5.6.

It remains to show that if $U$ is a Lie subsuperalgebra of $K$ such that $U_i \circ U_j \subseteq Z$ for $i = 0, 1$, then $u \circ v = 0$ for any $u, v \in U_1$. Suppose, on the contrary, that $0 \neq u \circ v \in Z$ and notice first that

$$2u^2 = [u, u], \quad 2v^2 = [v, v], \quad [u, v] \in U_0;$$

therefore

$$2u^4 = u^2 \circ u^2, \quad 2v^4 = v^2 \circ v^2, \quad u^2 \circ v^2, \quad [u, v] \circ u^2, \quad [u, v] \circ v^2 \in Z$$

by the assumption. We also have that

$$[u^2, v] = u \circ (u \circ v) = 2\gamma u.$$ 

But then

$$4\gamma u^3 = (2\gamma u) \circ u^2 = [u^2, v] \circ u^2 = [u^4, v] = 0;$$
hence $u^3 = 0$ and, analogously, $v^3 = 0$. Now,

$$0 = [u, u^2 \circ v^2] = u^2 \circ [u, v^2] = u^2 \circ (v \circ (u \circ v)) = 2 \gamma (u^2 \circ v);$$

therefore $u^2 \circ v = 0$, and in the same way, $v^2 \circ u = 0$. On the other hand,

$$0 = [u, u^2 \circ v^2] = u^2 \circ [u, v^2] = u^2 \circ [[u, v], v],$$

$$= [[u, v] \circ u^2, v^2] - [u, v] \circ [u^2, v],$$

$$= - [u, v] \circ [u^2, v] = -2 \gamma [u, v] \circ u;$$

hence $[u, v] \circ u = 0$ and, interchanging the role of $u$ and $v$, $[u, v] \circ v = 0$. We deduce from this that $wu = vwu = 0$, and thus

$$0 = wu^2 = vu^2 + (u \circ v)u^2 = \gamma u^2,$$

which implies that $u^2 = 0$, and, consequently, $2 \gamma u = [u^2, v] = 0$. Hence $u = 9$, a contradiction. This finally proves (ii).

At this point, our study of Lie ideals will be divided into two cases. We will consider first the case in which $u^2 = 0$ for all $u \in U_0$, and then the opposite case. Motivated by future applications to the Lie structure of $[K, K]$, we will sometimes make slightly more general assumptions on $U$.

**Lemma 5.7.** Let $U$ be a Lie subsuperalgebra of $K$, such that $[U, [U, K]] \subseteq U$ and $U$ satisfies the conclusions (i)–(iii) of Theorem 5.3. If $u^2 = 0$ for all $u \in U_0$, then $U^4 = 0$.

**Proof.** Linearizing $u^2 = 0$, we get $uv + vu = 0$ for all $u, v \in U_0$. Thus $wu = -uw = 0$ for any $w, v \in U_0$. Given $u, v \in U_0$, $k \in K_0$, then $2uku = (uk - ku)u - u(uk - ku) = [[u, k], u] \in U$ (for $u^2 = 0$); hence $uku \in U_0$. But then $vuku = 0$. Since $uv = -wu$, we arrive at $wK_0u = 0$. Now $(uv)^* = v^*u^* = vu = -wu$; that is, $uw \in K_0$. Thus for $h \in H_0$, $uw \in K_0$, and so $wuw = wK_0u = 0$. Given $a \in A_0$, $a = h + k$, $h \in H_0$, $k \in K_0$; whence $wuw = 0$. The right ideal $wA_0$ of $A_0$ is such that every element in it has cube 0. Since $A_0$ is semiprime, by Lemma 1.1 of [5] this forces $w = 0$; that is, $U_0U_0 = 0$.

Take elements $w \in U_1$, $k \in K_0$; then $[w, [w, k]] = w(ww - kw) + (wk - kw)w = w^2k - kw^2 = [w^2, k] \in [U_1, [U_1, K_0]] \subseteq U_0$. On the other hand, $w^2 = [w, w] \in [U_1, U_1] \subseteq U_0$. However, $U_0U_0 = 0$; hence $0 = w^2[w^2, k] = w^2(ww^2 - k^2) = w^2w^2k - w^2kw^2$. Notice that $w^2w^2 \in U_2U_0 = 0$; hence $w^2kw^2 = 0$. That is, $w^2K_0w^2 = 0$. Let $h \in H_0$. It is easy to see that $hw^2h \in K_0$, therefore $w^2h^2w^2 = 0$. As above, $A_0\omega^2$ is a nil left ideal of $A_0$ in which every element has cube 0, from which we deduce that $w^2 = 0$. If also $x \in U_1$, then $0 = (w + x)^2 = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx = w^2 + x^2 + wx + wx
Let \( u \in U_0, \ w, x \in U_1; \) then \([w, u] = wu - uw \in [U_1, U_0] \subseteq U_1. \) Hence \([w, u]x = (wu - uw)x = wux - wux = wux \in U_1 U_1 = 0. \) This proves that \( U_1 U_0 U_1 = 0. \) Therefore, \( U^3 = U_0 U_1 U_0 \) and \( U^4 = U_0^3 + U_1^3. \)

**Theorem 5.4.** Let \( K' \) be a Lie ideal of \( K \) such that \( K' = A \) and \( U \) a Lie ideal of \( K' \) satisfying the conclusions (i)–(iii) of Theorem 5.3. If \( u^2 = 0 \) for all \( u \in U_0, \) then \( U = 0. \)

**Proof.** Observe first that \( K' \supseteq [K, K] \) by Lemma 5.3. Therefore, \([U, [U, K]] \subseteq [U, [K, K]] \subseteq [U, K'] \subseteq U, \) and \( U \) satisfies the conditions of Lemma 5.7. By this lemma we have \( U^4 = 0. \) If \( U \) were nonzero, then by Lemma 5.4(i) applied to the subsuperspaces \( M = K' \) and \( S = U, \) we would get, consequently, \( U^3 = 0, \) \( U^2 = 0, \) and \( U = 0, \) a contradiction.

Theorem 5.4 completes the case in which \( u^2 = 0 \) for all \( u \in U_0. \) We will henceforth assume that \( U \) satisfies the conclusions of Theorem 5.3 and that there exists an element \( u \in U_0 \) such that \( 0 \neq u^2 = \lambda \in Z \) (notice that \( u^2 = \frac{1}{2}(u \circ u) \in Z \) by part (i) of Theorem 5.3). Since \( * \) is a \( Z \)-superinvolution (for it is of the first kind), we may also assume, without loss of generality, that \( U = ZU, \) i.e., that \( U \) is also a \( Z \)-subspace of \( K. \)

**Lemma 5.8.** Suppose that \( K' \) is a Lie ideal of \( K \) such that \( K' = A \) and \( U \) is a Lie ideal of \( K' \) satisfying conclusions (i)–(iii) of Theorem 5.3. Assume also that there exists an element \( u \in U_0 \) such that \( 0 \neq u^2 = \lambda \in Z. \) Then \( U_0 \) has a \( Z \)-basis of the form \( \{u, v, [u, v]\}, \) where \( 0 \neq u^2 = \lambda \in Z, \) \( 0 \neq v^2 = \mu \in Z, \) and \( u \circ v = 0. \)

**Proof.** Consider the symmetric bilinear form \( f \) on the \( Z \)-vector space \( U_0 \) given by \( f(u \circ w) = u \circ w = uw + wu. \) Suppose first that there exists an element \( v \in U_0 \) such that

\[
\det \begin{bmatrix} u \circ u & u \circ v \\ v \circ u & v \circ v \end{bmatrix} \neq 0.
\]

Then we can choose \( v \in U_0 \) satisfying \( 0 \neq v^2 = \mu \in Z \) and \( u \circ v = 0. \) We claim that \( \{u, v, [u, v]\} \) is a \( Z \)-basis of \( U_0. \) Indeed, since \( [u, v]^2 = -4\lambda\mu \in Z - \{0\} \) (in particular, \( [u, v] \neq 0), \) \( u \circ [u, v] = [u^2, v] = 0, \) and \( v \circ [u, v] = [u, v^2] = 0, \) the matrix of \( f \) on the subspace \( V_0 \) generated by \( u, v \) and \( [u, v] \) is

\[
A = 2 \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & -4\lambda\mu \end{bmatrix}.
\]
Therefore, \( u, v \) and \([u, v]\) are linearly independent over \( Z \), since \( \det A = -8\lambda^2 \neq 0 \). Let \( U_0 = V_0 \oplus (V_0)^\perp \) be the orthogonal decomposition of \( U_0 \) with respect to \( V_0 \) and let \( x \in (V_0)^\perp \). Then \( 0 = x \circ u = x \circ v = x \circ [u, v] \).

Since \( x \) anticommutes with \( u \) and \( v \), it commutes with \([u, v]\); i.e., \([x, [u, v]] = 0 \). Thus we end up with \( x[u, v] = 0 \). But \([u, v]\) is invertible; hence \( x = 0 \). This proves that \([u, v, [u, v]]\) is a \( Z \)-basis of \( U_0 \) with the desired properties.

Suppose now that

\[
\det \begin{bmatrix} u \circ u & u \circ v \\ v \circ u & v \circ v \end{bmatrix} = 0.
\]

for every element \( v \in U_0 \). Let \( U_0 = Zu \oplus (Zu)^\perp \) be the orthogonal decomposition of \( U_0 \) with respect to \( Zu \) and let us set \( V_0 = (Zu)^\perp \).

Let \( v, w \in V_0 \). If \( v^2 \neq 0 \), then \( v \) does not satisfy the above condition; hence \( v^2 = 0 \) and linearizing \( v \circ w = 0 \). On the other hand, since \( u \) anticommutes with \( v \) and \( w \), it commutes with \([v, w]\); i.e., \([u, [v, w]] = 0 \).

But \( u \circ [v, w] = [u \circ v, w] - v \circ [u, w] = -v \circ [u, w] \in v \circ U_0 = 0 \); hence \( 2u \circ [v, w] = 0 \) and, \( u \) being invertible, \( v \circ w = 0 \). This proves that \( V_0^2 = 0 \).

Moreover, \( u \circ [u, v] = [u^2, v] = 0 \), since \( u^2 \in Z \); hence \( 2uv = [u, v] \in (Zu)^\perp = V_0 \). This proves that \( uV_0 \subseteq V_0 \).

We claim that \( V_0 = [V_0, u] = [K_0', u] \). Indeed, let \( v \in V_0 \); then \([u, v], u] = 2[u, v]u = -4\lambda u \in [K_0', u] \), which implies that \( V_0 \subseteq [K_0', u] \). On the other hand, if \( k \in K_0', \) then \( u \) anticommutes with \( [k, u] \in U_0 \); hence \([K_0', u] \subseteq V_0 \). In particular, \([V_0, u] \subseteq V_0 \). Finally, since \([u, v] \in [K_0', u] \subseteq V_0 \), \( -4\lambda v \in [K_0', u] \). This proves that \( V_0 \subseteq [V_0, u] \).

Now let \( u_1 \in U_1 \); then \( 2u_1^2 = [u_1, u_1] \subseteq U_0 \), and by conclusion (iii) of Theorem 5.3 we have \( 4u_1^6 = (u_1^2 \circ u_1)^2 = 0 \). Consider \( u_1^4 = (u_1^2)^2 = \xi \in Z \).

If \( \xi \neq 0 \), then \( u_1^2 = 0 \) (since \( 0 = u_1^5 = \xi u_1^2 \)), a contradiction. Thus \( \xi = 0 \).

Let us write \( u_1^2 = zu + v \) with \( z \in Z \), \( v \in V_0 \). We have \( 0 = u_1^4 = (zu + v)^2 = z^2u^2 + v^2 + z(u \circ v) = z^2 \), since \( v^2 \in V_0^2 = 0 \) and \( u \circ v = 0 \). But \( z^2 = 0 \) and \( z \in Z \) imply \( z = 0 \); hence \( u_1^2 = v \in V_0 \). Let also \( u_2 \in U_1 \).

Then \( 0 = u_1 \circ u_2 \) (Theorem 5.3(ii)); therefore \((u_1 + u_2)^2 = u_1^2 + u_2^2 + 2u_1u_2 \in V_0 \) with \( u_1^2, u_2^2 \in V_0 \) and hence \( u_1u_2 \in V_0 \). This proves that \( U_1 u_2 \subseteq V_0 \).

We now claim that \([U, U] = 0 \). Indeed, consider \( W = [U, U] \), which is obviously a Lie ideal of \( K' \). Using the preceding paragraphs, we deduce that \( W_0 = [U_0, U_0] + [U_1, U_1] \subseteq [Zu + V_0, Zu + V_0] + V_0 \subseteq [u, V_0] + V_0 \subseteq V_0 \). However, \( V_0^2 = 0 \); hence \( w^2 = 0 \) for all \( w \in W_0 \). Clearly \( W \) satisfies the hypotheses of Theorem 5.4; hence \( W = 0 \).

We arrive then at \([K_0', u] = V_0 = [V_0, u] \subseteq [U, U] = 0 \); hence \( U = Zu + U_1 \) with \( U_1^2 = 0 \), \([u, U_1] = 0 \) and \([u, [u, K_1]] \subseteq \).
[u, U₁] = 0, and if k ∈ K₁, then u ∙ [u, k] = [u², k] = 0. Therefore, u(αu, k) = 0, and since u is invertible, [u, k] = 0; i.e., [u, K'] = 0.

Finally, since K' = A, we have [u, A] = 0; that is, u ∈ Z, a contradiction. □

**Corollary 5.2.** Under the conditions of Lemma 5.8, Q = Z + U₀ is a Z-subalgebra of A such that Q = Q(λ, μ), a generalized quaternion algebra over Z. Moreover, the superinvolution * restricts to the standard involution on Q.

**Proof.** Notice that (1, u, v, [u, v]) is a Z-basis of Q. Moreover, since u ∙ v = uv + vu = 0, [u, v] = uv − vu = 2uv; hence (1, u, v, w) is also a Z-basis of Q. We already have that 0 ≠ u² = λ ∈ Z, 0 ≠ v² = μ ∈ Z, and vu = −uv, hence Q = Q(λ, μ) is a generalized quaternion algebra over Z and a subalgebra of A (and hence a (graded) subalgebra of A, where A is considered as a superalgebra over Z).

With respect to the superinvolution, 1* = 1, u* = −u, v* = −v, and (uv)* = v*u* = vu = −uv; hence * restricts to the standard involution on Q. □

We can now prove the following complement to Theorem 5.4.

**Theorem 5.5.** Let K' be a Lie ideal of K such that K' = A and U be a Lie ideal of K' satisfying the conclusions (i)–(iii) of Theorem 5.3. If there exists an element u ∈ U₀ such that u² ≠ 0, then A is C(4), where dimₗ K = 8, dimₗ [K, K] = 7, dimₗ U = 6, and K' is K or [K, K].

**Proof.** By Lemma 5.8, U₀ contains a Z-basis of the form (u, v, [u, v]), where 0 ≠ u² = λ ∈ Z, 0 ≠ v² = μ ∈ Z, and u ∙ v = 0. Moreover, by the preceding corollary, Q = Z + U₀ is a Z-subalgebra of A such that Q = Q(λ, μ), and * restricts to the standard involution on Q.

Now let P = {p ∈ A | pq = qp, ∀q ∈ Q} be the centralizer of Q in A. Notice that P is a (graded) subalgebra of A. Moreover, since Q* ⊆ Q, it is easy to see that P* ⊆ P. Since Q is finite-dimensional central simple over Z, A = Q ⊗₂ P. In particular, Q and P being *-subalgebras of A, it is clear that * = * |ₗ Q ⊗ * |ₗ P. We will henceforth write the tensor product by juxtaposition, for q ⊗ p corresponds to qp (q ∈ Q, p ∈ P).

We will use the notation Pₜ = P ∩ H, Pₖ = P ∩ K, with which P = Pₜ ⊕ Pₖ. It is obvious that K = Pₖ + U₀Pₜ and H = Pₜ + U₀Pₖ. Now we will prove some facts about P.

We first claim that (P₀)₀ = Z. Observe that K' ⊇ [K, K] by Lemma 5.3; therefore, since U₀ = [U₀, U₀], we have U₀Pₜ = [U₀, U₀]Pₜ = [U₀, U₀]P₀ ⊆ [K, K] ⊆ K'. Let p ∈ (P₀)₀; then p[u, v] = [pu, v] ∈ [K₀, U₀] ⊆ U₀. Thus p[u, v] = αu + βv + γ[u, v], where α, β, γ ∈ Z. Now it is clear that p = γ ∈ Z.
We also claim that \( U_0(P_H)_1 \subseteq U_1 \). Let \( p \in (P_H)_2 \). Then \( 2 \mu up = [uv, v]p = [uv, vp] \subseteq [U_0, K_1] \subseteq U_1 \). Analogously, \( 2 \lambda vp \subseteq U_1 \). Finally, \( 2up = [u, v]p = [u, vp] \subseteq [U_0, K_1] \subseteq U_1 \). The desired containment is now easy to get.

We further claim that \( (P_H)_2^2 = 0 \). Indeed, let \( p_1, p_2 \in (P_H)_2 \). Then \( up_1, up_2, vp_2 \subseteq U_1 \) by the preceding paragraph. Since \( U_1 \circ U_1 = 0 \) by Theorem 5.3(ii), we have that

\[
0 = (up_1) \circ (up_2) = u^2 p_1 p_2 - u^2 p_2 p_1 = \lambda(p_1 p_2 - p_2 p_1);
\]

in the same way,

\[
0 = (up_1) \circ (vp_2) = wp_1 p_2 - wp_2 p_1 = w(p_1 p_2 + p_2 p_1).
\]

Since \( uw \) is invertible, we deduce from this that \( p_1 p_2 = 0 \); hence \( (P_H)_2^2 = 0 \).

We know that \( P \) is a unital associative superalgebra with superinvolution \(* \mid_P \). Since \( A \) is nontrivial and simple, necessarily so is \( P \). Moreover, it is easy to see that \( Z(P)_0 = Z \). Furthermore, notice that \(* \mid_P \) is of the first kind, for such is \(* \). However, \( P_H = Z + (P_H)_2 \) with \( (P_H)_2^2 = 0 \); hence \( P_H = P_H \) is not simple, and so \( P_H \neq P \). By Theorem 1 of [2], \( P \) must be \( C(2) \). In this situation, analogous to that in the commentaries after Theorem 4.2, we conclude by Theorems 3.1 and 3.2 that \( (P, * \mid_P) \) is isomorphic to \( (M(1), tr) \) as superalgebras over \( Z \); that is, we may assume that \( P = M(1) \) with superinvolution

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}^{tr} = \begin{bmatrix}
d & -b \\
c & a
\end{bmatrix}.
\]

Since \( A = Q \otimes Z P \), where \( Q \) is a quaternion algebra over \( Z \), and \( P \) is \( C(2) \), it is straightforward to check that \( A \) must be \( C(4) \). However, we will get further consequences in this case, which will prove useful in the following section.

Using the standard notation for unit matrices, we observe that \( P_H = Z + Ze_{21} \) and \( P_K = Z(e_{11} - e_{22}) + Ze_{12} \). Since \( K = P_K + U_0 + U_0 e_{21} \), \( \dim_Z K = 8 \). On the other hand, it is easy to check that \( [K, K] = [P_K, P_K] + U_0 + U_0 e_{21} = Ze_{12} + U_0 + U_0 e_{21} \); hence \( \dim_Z [K, K] = 7 \). Let us finally calculate the dimension of \( U \) over \( Z \). On one hand, since \( K_0 = Z(e_{11} - e_{22}) + U_0 \), \( \dim_Z K_0 = 4 \); hence \( U \neq K \) for \( \dim_Z U_0 = 3 \). As we have seen \( U_0 \subseteq U_1 \); hence \( \dim_Z U_1 \geq 3 \), and consequently, \( \dim_Z U \geq 6 \). If \( \dim_Z U = 7 \), then \( U_1 = K_1 = Ze_{12} + U_0 e_{21} \). In this case, since \( U_1 \circ U_1 = 0 \), \( 0 = e_{12} \circ (ue_{21}) = u(e_{11} - e_{22}) \), which is absurd. Moreover, \( \dim_Z U \neq 8 \) for \( U \neq K \). Hence \( \dim_Z U = 6 \). It is also clear that \( K \) coincides with \( K \) or \( [K, K] \).
We finally arrive at the main theorem on the Lie structure of $K$ when the superinvolution is of the first kind.

**Theorem 5.6.** If the superinvolution $*$ is of the first kind and $U$ is a nontrivial Lie ideal of $K$, then $U \supseteq [K, K]$, except if $A$ is $C(4)$.

**Proof.** Suppose first that $A$ is not $C(2)$. If $U \supseteq [K, K]$, we are finished. In the contrary case, $[u \circ v, w] = 0$ for all $u, v, w \in U$ by virtue of Theorem 5.2. In particular, $U$ satisfies the hypotheses (and so the conclusion) of Theorem 5.3. Furthermore, $K = A$ by Theorem 4.2, so in the case where $u^2 = 0$ for all $u \in U_0$, Theorem 5.4 with $K' = K$ implies $U = 0$, which contradicts the hypothesis that $U$ is nontrivial.

If there exists $u \in U_0$ such that $0 \neq u^2$, then by Theorem 5.5. (again with $K' = K$), we obtain that $A = C(4)$, which is an exceptional case. We have already seen that in this case the theorem is not true.

Suppose now that $A$ is $C(2)$. By Theorems 3.1 and 3.2, $(A, *)$ is isomorphic to $(M(1), trp)$ as superalgebras over $Z$. As in the commentaries after Theorem 4.2,

$$K = K(M(1), trp) = \left\{ \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix} \big| a, b \in Z \right\}.$$  

It is straightforward to check that $[K, K] = K_1$, and that the only nontrivial Lie ideals of $K$ are $K$ itself and $[K, K]$. Therefore, a superalgebra like this satisfies the statement of the theorem.

In the proof of Theorem 5.5 we have described a superalgebra $A$ that is $C(4)$ and is in fact a counterexample to the property expressed by Theorem 5.6. A more natural way of presenting a counterexample of this sort is to consider the superalgebra $A = M(2)$ with the superinvolution $* = trp$. Indeed, straightforward calculations show that the only nontrivial Lie ideals of $K = K(M(2), trp)$ are the following ones (here $tr$ denotes the trace of a matrix):

$$K = \left\{ \begin{bmatrix} A & B \\ C & -A^t \end{bmatrix} \big| B' = B, C' = -C \right\}, \quad \dim K = 8;$$

$$[K, K] = \left\{ \begin{bmatrix} A & B \\ C & -A^t \end{bmatrix} \big| trA = 0, B' = B, C' = -C \right\}, \quad \dim[K, K] = 7;$$

$$U = \left\{ \begin{bmatrix} A & B \\ 0 & -A^t \end{bmatrix} \big| trA = 0, B' = B \right\}, \quad \dim U = 6.$$  

As far as the superalgebra $A = M(2)$ with the superinvolution $* = osp$ is concerned, it is not a counterexample to this theorem, as we shall see later.
Finally, we may join Theorems 5.1 and 5.6 together, omitting the hypotheses on the superinvolution.

Theorem 5.7. Suppose that $A$ is not $C(n)$ for $n = 2$ or 4. If $U$ is a Lie ideal of $K$, then either $U \subseteq Z$ or $U \supseteq [K, K]$.

6. Lie Structure of $[K,K]$

As in the previous section, we begin our study of the Lie structure of $[K,K]$ in the case in which $*$ is of the second kind.

Theorem 6.1. If the superinvolutions $*$ is of the second kind and $U$ is a Lie ideal of $[K,K]$, then either $U \subseteq Z$ or $U = [K,K]$, except if $A$ is $C(n)$ for $n = 2$ or 3.


Let us consider the $Z$-subspace $ZU$ generated by $U$. We claim that $ZU$ is a Lie ideal of $[A,A]$. Indeed, $[ZU,[A,A]] = [ZU,[K,K]] + [ZU,q[K,K]] \subseteq Z[U,[K,K]] \subseteq ZU$.

Suppose that $A$ is not $C(n)$ for $n = 2$ or 3. Then we may apply Theorem 3.2 of [9] to conclude that either $ZU \subseteq Z$ or $ZU = [A,A]$. In the first case it is clear that $U \subseteq Z$.


Assume that $A_0$ is not commutative; then by the corollary of Theorem 5.1, $[K,K] = [[K,K],[K,K]]$. Therefore, we have in this case $U \supseteq [U,[K,K]] = [U,Z_h [K,K]] = [Z_h U,[K,K]] = [[K,K],[K,K]] = [K,K]$, that is, $U = [K,K]$.

Finally, suppose that $A_0$ is commutative; then again by the corollary of Theorem 5.1, we have $[K,K] = Z$. It is clear in this case that every Lie ideal of $[K,K]$ is contained in $Z$.

With this theorem out of the way, we can focus our attention on the case of superinvolutions of the first kind. While many of the results to follow hold irrespective of the nature of the superinvolution, we henceforth assume in this section that the superinvolution $*$ is of the first kind.

Lemma 6.1. If $A$ is not $C(2)$, then $[K,K] \neq 0$.

Proof. Since $A$ is not $C(2)$, $\bar{K} = A$ by Theorem 4.2. Suppose, contrary to our claim, that $[K,K] = 0$. Let $0 \neq k \in K_1$; then the subspaces $S = Fk$ and $M = K$ satisfy the hypothesis of Lemma 5.4. We have $2k^2 = [k,k] = 0$; hence by Lemma 5.4(i) $k = 0$, a contradiction. Thus $K_1 = 0$ and $K = K_0 \subseteq Z$. Since $A = \bar{K}$, we also have $A = A_0$, which contradicts our assumption that the superalgebra $A$ is nontrivial.
Theorem 4.2. deduced from the last paragraph of the proof of Theorem 5.6.

Theorem 6.2. Suppose that $A$ is not $C(2)$. Then $[K, K] = A$.

Proof. We first notice that $[K, K]$ is a Lie ideal of $K$, which is nontrivial by Lemma 6.1, since $A$ is not $C(2)$.

By Lemma 5.5, either $[K, K] = A$ or $[u, v, w] = 0$ for all $u, v, w \in [K, K]$. In the first case we are finished, so suppose the second. If we write $U = [K, K]$, then by Theorems 5.2 and 5.3 $U$ satisfies conclusions (i)–(iii) of Theorem 5.3. Since $U \neq 0$, by Theorems 5.4 and 5.5 applied to $U$ and $K'$, we conclude that $A = C(4)$ and $\dim U = 7$, $\dim U = 6$. By Lemma 6.2, $[K, K] = A$, a contradiction.

Notice also that if $A$ is not $C(2)$, then $[K, K] \neq A$ for $[K, K] \subseteq K$ and $K \neq A$ (see the commentaries after Theorem 4.2).

The remain of the section is devoted to proving that $[K, K]$ is a simple Lie superalgebra.

Lemma 6.2. If $A$ is not $C(2)$, then $[K, A] = [H, A] = [A, A]$.

Proof. Let $B$ be a subsuperspace of $A$ such that $B = A$. It is straightforward to check, by induction on $n$, that if $a \in A$ and $b_1, \ldots, b_n \in B$ (n ≥ 1), then

$$[a, b_1 \cdots b_n] = \sum_{i=1}^{n} (-1)^{(\sigma_i + \cdots + \sigma_n)\pi + \cdots + \sigma_n} b_i b_{i+1} \cdots b_n a b_{i-1}.$$

We deduce from this that $[B^n, A] \subseteq [B, A] \subseteq A$ for $n \geq 1$. Consequently, $[A, A] = [B, A] \subseteq [B, A]$, and equality holds.

In particular, since $A$ is not $C(2)$, $K = H = A$ by Theorem 4.2 and Theorem 1 of [2]; hence $[K, A] = [H, A] = [A, A]$.

Lemma 6.3. If $A$ is not $C(2)$, then $[K, K] = [H, H]$ and $[A, A] = [H, K] + [K, K]$. In particular, $[K, K] = [A, A] \cap K$.

The equality $[A, A] = [H, K] + [K, K]$ was already proved. Moreover, notice that $[H, K] \subseteq H$ and $[K, K] \subseteq K$; hence the above sum is direct, and evidently $[K, K] = [A, A] \cap K$. 

**Lemma 6.4.** Suppose that $A$ is not $C(n)$ for $n = 2$ or 4. If $U$ is a Lie ideal of $[K, K]$, then either $\bar{U} = A$ or $[U, U] = 0$.

**Proof.** Let $u \in [U, U]$, $k \in K$. Notice that $u$ is a sum of elements of the form $[x, y]$ with $x, y \in U$; moreover, $[[x, y], k] = [x, [y, k]] + (-1)^y[[x, k], y] \in U$, since $[x, k], [y, k] \in [U, K] \subseteq [K, K]$. This proves that $[u, K] \subseteq U$. Let also $v \in [U, U]$. Then $[u \circ v, k] = u \circ [v, k] + (-1)^y[u, k] \circ v \in U$ since $[u, k], [v, k] \in U$. We have then that $[u \circ v, K] \subseteq U$. Now take $h \in H$ and notice that $u \circ h, v \circ h \in K \circ H \subseteq K$. Taking this into account, we obtain that $[u \circ v, h] = [u, v \circ h] + (-1)^y[u \circ v, h] \in U$. Thus $[u \circ v, H] \subseteq U$. In fact, since $A + H \oplus K, [u \circ v, A] \subseteq U$. Finally, let $w \in U$, $a \in A$. Then $[u \circ v, aw] = [u \circ v, a]w + (-1)^{y + y}u[a \circ v, w] \in U$; but clearly, $[u \circ v, aw] \in U$. Hence also $a[u \circ v, w] \in U$; that is, $A[u \circ v, w] \subseteq U$.

Since $A$ is not $C(2)$, $[K, K] = A$ by Theorem 6.2. Putting in Lemma 5.4(ii) $M = [K, K], S = U, a = [u \circ v, w]$, we conclude that either $\bar{U} = A$ or $[u \circ v, w] = 0$ for all $u, v \in [U, U], w \in U$.

If $\bar{U} = A$, then we are finished. So suppose that $[u \circ v, w] = 0$ for all $u, v \in [U, U], w \in U$. Then it is obvious that $[u \circ v, w] = 0$ for all $u, v \in [U, U], w \in \bar{U}$; that is, if $u, v \in [U, U]$, then $[u \circ v, \bar{U}] = 0$. However, as we have seen, $[u \circ v, A] \subseteq \bar{U}$; hence $[u \circ v, [u \circ v, A]] = 0$. By Lemma 5.6, we deduce from here that if $u$ and $v$ are of the same parity, then $u \circ v \in Z$; and if $u$ and $v$ are of distinct parity, then $(u \circ v)^2 = 0$. Moreover, the final part of the conclusion of Theorem 5.3 implies that $u \circ v = 0$ for any odd elements $u, v \in [U, U]$; hence the graded space $[U, U]$ satisfies conclusions (i)–(iii) of Theorem 5.3.

It is clear that $[U, U]$ is a Lie ideal of $[K, K]$; so, putting $K' = [K, K]$, $U = [U, U]$ in Theorems 5.4 and 5.5, we obtain that either $[U, U] = 0$ or $A = C(4)$. But the last possibility contradicts our assumptions; hence, finally, $[U, U] = 0$.

**Lemma 6.5.** Suppose that $A$ is not $C(n)$ for $n = 2$ or 4. If $U$ is a proper Lie ideal of $[K, K]$, then $[[U, U], [U, U]] = 0$.

**Proof.** Suppose, on the contrary, that $[[U, U], [U, U]] \neq 0$. Then, by Lemma 6.4, $\bar{U} = A$. Now take $B = [U, U]$ in the proof of Lemma 6.2 to obtain $[[U, U], A] = [A, A]$. Therefore, $[[U, U], K] + [[U, U], H] = [[U, U], A] = [A, A] = [K, K] + [H, K]$ (Lemma 6.3). Comparing the skew parts in this expression, we get $[[U, U], K] = [K, K]$. However, fol-
owing the proof of Lemma 6.4, we get, as in the proof, that $[[U, U], K] \subseteq U$. Hence $[K, K] \subseteq U$, contradicting the hypothesis that $U$ is proper.

To prove now that $[K, K]$ is simple, it suffices to prove that if $V$ is a Lie ideal of $[K, K]$ such that $[V, V] = 0$, then $V = 0$.

**Lemma 6.6.** Suppose that $A$ is not $C(2)$. If $V$ is a Lie ideal of $[K, K]$ such that $[V, V] = 0$, then $V = 0$.

**Proof.** We sketch the proof in 12 steps.

1. $v^2 = 0$ for all $v \in V$. Indeed, if $v \in V$, then $0 = [v, v] = v^2 + v^2 = 2v^2$; hence $v^2 = 0$.

2. $v_i v_j [K, K] \subseteq Av_1 + Av_2$ for all $v_1, v_2 \in V$. Indeed, let $v_1, v_2 \in V$ and let $k \in [K, K]$. $0 = [v_j, k] = v_j v_2 k - (-1)^{|v_j|} v_2 k v_j - (-1)^{|k|} v_j k v_j$, hence $v_i v_j [K, K] \subseteq Av_1 + Av_2$.

3. $v_i v_j v_k [K, K] \subseteq Av_1 + Av_2 + Av_3$ for all $v_1, v_2, v_3 \in V$. Indeed, let $v_1, v_2, v_3 \in V$ and let $k \in K$. By step 2, $v_j v_k [K, K] = v_i v_j v_k v_l k - (-1)^{|v_j|} v_j v_k v_l k v_j - (-1)^{|k|} v_j v_k v_l k v_j$, hence $v_i v_j v_k [K, K] \subseteq Av_1 + Av_2 + Av_3$.

4. $v_i v_j v_k v_l [K, K] \subseteq Av_1 + Av_2 + Av_3 + Av_4$ for all $v_1, v_2, v_3, v_4 \in V$. Indeed, let $v_1, v_2, v_3, v_4 \in V$ and let $k \in K$. By step 3, $v_i v_j v_k v_l [K, K] = v_i v_j v_k v_l v_m k - (-1)^{|v_j|} v_j v_k v_l v_m k v_j - (-1)^{|k|} v_j v_k v_l v_m k v_j$, hence $v_i v_j v_k v_l [K, K] \subseteq Av_1 + Av_2 + Av_3 + Av_4$.

5. For every $v \in V$, either $v^3 = 0$ or $v$ is invertible. Indeed, given $v \in V$, $v^4 A \subseteq Av$ by step 4. If $v^4 \neq 0$, then, since $A$ is simple, $A = Av^4 A \subseteq Av$; hence $A = Av$ and $v$ is invertible. Now suppose that $v^4 = 0$. By step 2, $v^4 [K, K] \subseteq Av$; hence $v^4 [K, K] \subseteq Av^4 = 0$. Thus, if $k \in K$, $0 = v^4 [v, k] = v^4 (vk - kv) v^3 = v^3 kv^3 - v^2 k v^4 = v^3 k v^3$. Since $v^3 \in K$, we finally deduced from the remark on page 43 of [5] that $v^3 = 0$.

6. If char $F = p \geq 3$ and $v \in V$, then $v^3 = 0$. Indeed, let $v \in V$ and $k \in [K, K]$. It is clear that $0 = [v, [v, k]]$; hence $v [v, k] = [v, k] v$.

Using this, it is easily checked by induction on $n$ that $[v^n, k] = n [v, k]$ for all $n \geq 1$. Since char $F = p$, $[v^p, k] = p [v^p, k] = 0$ for all $k \in [K, K]$. However, by Theorem 6.2, $[K, K] = A$; hence $[v^p, a] = 0$ for all $a \in A$ and thus $v^p \in Z$. On the other hand, $(v^p)^p = (v^p)^p = (-v^p)^p = (-1)^p v^p = -v^p$ for $p$ is odd. Consequently, $v^p \in Z \cap K = 0$; since $v$ is nilpotent, it must be $v^3 = 0$ by step 5.
7. If char $F = 0$ and $\mathcal{A}$ is a simple algebra, the $v^3 = 0$ for all $v \in V_0$. Indeed, let $v \in V_0$ and consider the derivation $d = [v, \cdot]$ of $\mathcal{A}$. It is evident, by the hypothesis on $V$, that $[V, [V, [V, K]]]] = 0$; in particular $d^3(K) = 0$. A straightforward calculation using Leibnitz' formula shows that $d^3(K \circ K) = 0$. Equivalently, $d^3(H) = 0$ by Theorem 4.3; thus $d^3(A) = 0$. Applying now the fundamental theorem of [4], we conclude that there exists $z \in Z(A) = Z$ such that $(v - z)^3 = 0$ (notice that $Z(A) = Z$, since if $Z(A) \neq 0$, then $*$ is of the second kind by Lemma 3.2). Spreading this expression, we obtain $0 = v^3 + 3vz^2 - (3v^2z + z^3)$ with $v^3 + 3vz^2 \in K$ and $3v^2z + z^3 \in H$; therefore $0 = v^3 + 3vz^2 = 3v^2z + z^3$. If $z = 0$, then $v^3 = 0$, and we are finished. So suppose that $z \neq 0$. Since $3v^2z = -z^3 \in Z$, we have $v^3 \in Z$. Now let $k \in [K, K]$. Then $v[v, k] + [v, k]v = [k^2, k] = 0$ and $0 = [v_v^3, k] = [v, k] - [v, k]v$; hence $v[v, k] = 0$. If $v$ is nilpotent, then $v^3 = 0$ by step 5. On the contrary, $v$ is invertible by step 5 again; hence $[v, k] = 0$. Since this happens for every $k \in [K, K]$ and $[K, K] = \mathcal{A}$ by Theorem 6.2, it is obvious that $v \in Z$. But $v$ is skew; hence $v = 0$, a contradiction.

8. If $\mathcal{A}_0$ is simple, then $V_0 = 0$. Simply notice that the hypotheses of Lemma 2.11 of [5] hold for the Lie ideal $V_0 = K(A_0, *|A_0)$.

9. $V_0^8 = 0$. Indeed, let us first take into account that if $v \in V_0$, then $v^3 = 0$ by steps 6, 7, and 8 (use Lemma 3 of [10]). Let $v_1, v_2, v_3, v_4 \in V_0$. By step 4, assuming that $v_1 v_2 v_3 v_4 \neq 0$ and taking into account the simplicity of $\mathcal{A}$, $\mathcal{A} = Av_1 v_2 v_3 v_4 \subseteq \mathcal{A}v_1 + Av_2 + Av_3 + Av_4$. In particular, there exist $a_1, a_2, a_3, a_4 \in \mathcal{A}$ such that $1 = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$. We deduce from this, since the $v_i$ commute and $v_i^3 = 0$ for all $i$, that $v_1 v_2 v_3 v_4^2 = 1 v_1^2 v_2^2 v_3^2 v_4^2 = 0$. It is obvious that the same occurs if $v_i v_2 v_3 v_4 = 0$. Linearizing this identity, we finally obtain that $v_1 w_1 v_2 w_2 v_3 w_3 v_4 w_4 = 0$ for all $v_1, v_2, v_3, v_4 w_1, w_2, w_3, w_4 \in V_0$, that is, $V_0^8 = 0$.

10. $V_1^4 = 0$. Suppose, on the contrary, that there exist $v_1, v_2, v_3, v_4 \in V_1$ such that $v_1 v_2 v_3 v_4 \neq 0$. As in step 9, we deduce from step 4 that there exist $a_1, a_2, a_3, a_4 \in \mathcal{A}$ such that $1 = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$. We conclude from this, since the $v_i$ anticommute and $v_i^2 = 0$ for all $i$ (by step 1), that $v_1 v_2 v_3 v_4 = 1 v_1^2 v_2^2 v_3 v_4 = 0$, a contradiction.

11. $V_1^{11} = 0$. Indeed, it suffices to note that $[V_0, V_1] = 0$ and that, by steps 9 and 10, $V_0^8 = V_1^4 = 0$.

12. $V = 0$. Indeed, let $v \in V$ and $k \in [K, K]$. Notice that $vk = (-1)^{k_v} v_k + [v, k] \in \mathcal{A}V$; hence $\mathcal{A}V \subseteq \mathcal{A}V$ for $[K, K] = A$. It is easy to obtain from this by induction on $n$ that $\mathcal{A}V^n \subseteq \mathcal{A}V^n$ for all $n \geq 1$, and hence that $(\mathcal{A}V)^n \subseteq \mathcal{A}V^n$ for all $n \geq 1$. In particular, for $n = 11$, $(\mathcal{A}V)^{11} \subseteq \mathcal{A}V^{11} = 0$ by step 11. Since $A$ is semiprime, we conclude from this that $\mathcal{A}V = 0$, and hence that $V = 0$.  

We arrive finally at the fundamental theorem on the ideal structure of 
\([K, K]\) in the case in which \(\ast\) is of the first kind.

**Theorem 6.3.** If the superinvolution \(\ast\) is of the first kind, then \([K, K]\) is a simple Lie superalgebra, except if \(A\) is \(C(n)\) for \(n = 2\) or \(4\).

Proof. Let \(U\) be a proper Lie ideal of \([K, K]\). By Lemma 6.5, 
\([[U, U], [U, U]] = 0\). Now applying Lemma 6.6 to the ideal \([U, U]\) of \([K, K]\), we obtain that \([U, U] = 0\). Finally, applying the same lemma again, now to the ideal \(U\), we obtain that \(U = 0\).

Further notice that \([K, K] \neq 0\) by Lemma 6.1; hence \([[K, K], [K, K]] \neq 0\) by Lemma 6.6. That is, \([K, K]\) does not have trivial multiplication.

If \(A\) is \(C(2)\), then we may assume that \((A, \ast) = (M(1), \text{trp})\), considering \(A\) as a superalgebra over \(\mathbb{Z}\). As was seen in the last paragraph of the proof of Theorem 5.6, in this case \([K, K] = K_1\); hence \([K, K]\) has trivial multiplication (although it is one-dimensional over \(\mathbb{Z}\) and hence it has no nontrivial proper ideals).

On the other hand, there are in fact examples in which \(A\) is \(C(4)\) and \([K, K]\) is not a simple Lie superalgebra, as may easily be checked from the commentaries that follow Theorem 5.6.

As in the previous section, we may join Theorems 6.1 and 6.3 together, omitting the hypotheses on the superinvolution. We arrive in this way at the main theorem of the paper.

**Theorem 6.4.** Suppose that \(A\) is not \(C(n)\) for \(n = 2, 3,\) or \(4\). If \(U\) is a proper Lie ideal of \([K, K]\), then \(U \subseteq Z\).

As a corollary of Theorems 5.6 and 6.3, we obtain that the Lie superalgebras of skew elements that appear in Kac's classification [8] are always simple, irrespective of the characteristic of the ground field (different from 2) and of whether it is algebraically closed. That is, we have the following result.

**Theorem 6.5.** The following Lie superalgebras are simple:

(i) \(K(M(r|s), osp) = [K(M(r|s), osp), K(M(r|s), osp)]\) for all \(r, s \geq 1\) (\(s = 2t\) is necessarily even).

(ii) \([K(M(r), \text{trp}), K(M(r), \text{trp})]\) for all \(r \geq 3\).

Proof. Let us write \(A = M(r|s)\); in particular, \(A = M(r)\) is obtained for \(s = r\). It may be proved [8] that

\[
[A, A] = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mid trA = trD \right\},
\]
where \( tr \) denotes the trace of a matrix. \([A, A]\) consists exactly of the matrices of supertrace 0, the supertrace \( str \) of a matrix being defined by

\[
str\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = trA - trD.
\]

Let us consider the superinvolution \( * = osp \) on \( A \). \( s = 2t \) necessarily is even. By Lemma 6.3, \([K, K] = [A, A] \cap K\). However, because of the definition of the orthosymplectic superinvolution (see Example 3.1), every matrix

\[
 \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in K
\]

satisfies \( A = -H^{-1}A'H, D = -K^{-1}D'K \). In particular, \( trA = \text{tr}(-H^{-1}A'H) = -\text{tr}(HH^{-1}A') = -\text{tr}(A') = -trA; \) hence \( trA = 0 \), and analogously, \( trD = 0 \). This proves that \( K \subseteq [A, A] \) and hence that \([K, K] = K\). Now, by virtue of Theorem 5.6, \([K, K] = K\) has no nontrivial proper ideals, except if \( A \) is \( C(4) \) (which may happen only if \( r = s = 2 \) for \( s \) is even). However, in the exceptional case of the above-mentioned theorem, we must have \( \dim_{\mathbb{F}} K = 8 \) and \( \dim_{\mathbb{F}}[K, K] = 7 \) which is a contradiction. This proves (i).

Now let us consider the superinvolution \( * = trp \) on \( A \) (in particular, \( s = r \)). By Lemma 6.3 again, \([K, K] = [A, A] \cap K\); hence

\[
[K, K] = \left\{ \begin{bmatrix} A & B \\ C & -A' \end{bmatrix} \bigg| trA = 0, B' = B, C' = -C \right\}.
\]

If \( r \geq 3 \), then \( A \) is not \( C(n) \) for \( n = 2 \) or 4, and, by Theorem 6.3, \([K, K] \) is a simple Lie superalgebra. This proves (ii).

Notice also that \([K(M(r), trp), K(M(r), trp)] \) is not a simple Lie superalgebra if \( r = 1 \) or 2, as may be deduced from the proof of Theorem 5.6 and the commentaries which follow it.

Finally, it is worth mentioning the Clifford flavor of all of the counterexamples to the theorems in this paper. Perhaps a classification of superinvolutions in Clifford superalgebras might throw some light on this.

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REFERENCES