Unicity of minimal rank completions for tri-diagonal partial block matrices

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Abstract

In this paper, we establish necessary and sufficient conditions for a minimal rank completion of a tri-diagonal partial block matrix to be unique. The case of 4 × 4 block matrices with four consecutive diagonals is also considered, along with its application to the problem of completing a matrix and its inverse. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

We consider the minimal rank completion problem for partial matrices over a field \( F \) not equal to \([0, 1]\). Recall that a partial matrix over \( F \) is a matrix of which some entries are specified elements of \( F \) and the remaining entries (the unknowns) are free to be chosen from \( F \). A particular choice for the unknowns results into a completion of the partial matrix. A minimal rank completion is a completion with the lowest possible rank.

Minimal rank completions are of interest for the minimal representation problem for linear input/output systems [3,6,7], the partial realization problem [4,5,8,10], and for the problem of completing a matrix and its inverse [1,9]. In one of the early
papers on the subject [6], the question of unicity of minimal rank completions of triangular partial matrices was addressed. There it was proven that given matrices $A_{ij}, 1 \leq j < i \leq n$, there exist unique $A_{ij}, 1 \leq i < j \leq n$, so that \( \text{rank}[A_{ij}] \) is minimal among all possible choices for $\{A_{ij}: 1 \leq i < j \leq n\}$ if and only if

\[
\text{rank}[A_{n1}] = \text{rank} \begin{bmatrix} A_{11} & \cdots & A_{1i} \\ \vdots & \ddots & \vdots \\ A_{ni} & \cdots & A_{ni} \end{bmatrix}, \quad i = 1, \ldots, n.
\]

Here it is assumed that $A_{n1}$ is of non-trivial size. Similar results were proven in [6] for general operators that are triangular with respect to chains of orthogonal projections, including Volterra operators. In this paper, we consider the uniqueness question for some banded patterns.

Before we state our main result, we consider the following simple examples. For any $r \neq 0$,

\[
\begin{bmatrix} 1 & r \\ r^{-1} & 1 \end{bmatrix}
\]

is a minimal rank completion of

\[
\begin{bmatrix} 1 & ? \\ ? & 1 \end{bmatrix},
\]

since $F$ contains more than one non-zero element, minimal rank completions are not unique. Similarly,

\[
\begin{bmatrix} 0 & 1 & r \\ 1 & 0 & 1 \\ -r & 1 & 0 \end{bmatrix}, \quad r \in F,
\]

is a minimal rank completion of

\[
\begin{bmatrix} 0 & 1 & ? \\ 1 & 0 & 1 \\ ? & 1 & 0 \end{bmatrix}.
\]

In this case, there is also non-uniqueness. Examples where there is uniqueness are

\[
\begin{bmatrix} 1 & 1 & ? \\ 1 & 1 & 1 \\ ? & 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & ? \\ 1 & 0 & 1 \\ ? & 0 & 1 \end{bmatrix}.
\]

**Theorem 1.1.** The tri-diagonal partial block matrix

\[
\begin{bmatrix} A_{11} & A_{12} & \cdots & ? \\ A_{21} & A_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{n-1,n} \\ ? & \cdots & A_{n-1,n} & A_{nn} \end{bmatrix},
\] (1.1)
where both $A_{11}$ and $A_{nn}$ have a positive number of rows and columns, has a unique minimal rank completion if and only if the following four conditions hold:

(I) $s := \text{rank} \begin{bmatrix} A_{i,i} & A_{i,i+1} \\ A_{i+1,i} & A_{i+1,i+1} \end{bmatrix}$, $i = 1, \ldots, n-1$, are all equal.

(II) $v_i := \text{rank} \begin{bmatrix} A_{i-1,i} \\ A_{i,i} \\ A_{i+1,i} \end{bmatrix}$

$= \text{rank} \begin{bmatrix} A_{i-1,i} \\ A_{i,i} \end{bmatrix}$

$= \text{rank} \begin{bmatrix} A_{i,i} \\ A_{i+1,i} \end{bmatrix}$, $i = 2, \ldots, n-1$.

(III) $h_i := \text{rank} \begin{bmatrix} A_{i,i-1} & A_{i,i} & A_{i,i+1} \\ A_{i,i} \end{bmatrix}$

$= \text{rank} \begin{bmatrix} A_{i,i-1} \\ A_{i,i} \end{bmatrix}$

$= \text{rank} \begin{bmatrix} A_{i,i} & A_{i,i+1} \end{bmatrix}$, $i = 2, \ldots, n-1$.

(IV) $s = \max \{v_i, h_i\}$, $i = 2, \ldots, n-1$.

In that case, the unique minimal rank completion has rank equal to $s$.

Note that the result implies that if the $n \times n$ tri-diagonal partial block matrix has a unique minimal rank completion, then the $(n-1) \times (n-1)$ tri-diagonal partial block matrix that is obtained by removing the last block row and last block column also has a unique minimal rank completion. The converse is not true; that is, it may happen that an $n \times n$ tri-diagonal partial block matrix has several minimal rank completions while the smaller $(n-1) \times (n-1)$ block matrix has only one. For example,

\[
\begin{bmatrix}
1 & 1 & ? \\
1 & 1 & 0 \\
? & 0 & 0
\end{bmatrix}
\]

has a unique minimal rank completion, but

\[
\begin{bmatrix}
1 & 1 & ? & ? \\
1 & 1 & 0 & ? \\
? & 0 & 0 & 0 \\
? & ? & 0 & 0
\end{bmatrix}
\]

does not.

As an application of the minimal rank results we derive conditions for uniqueness in the problem of completing a $2 \times 2$ block matrix and its inverse.

The paper is organized as follows. In Section 2, we prove the main result for the $3 \times 3$ block matrix case. In Section 3, we prove the main result in the general case,
and show this result may be applied to other banded partial matrices; in particular, a 4 × 4 block matrix example is treated. Finally, Section 4 is devoted to the application of completing a matrix and its inverse using the 4 × 4 result.

2. The main theorem for 3 × 3 block matrices

In this section, we prove Theorem 1.1 for the case of 3 × 3 block matrices. We shall consider the partial matrix

\[
\begin{bmatrix}
A & B & X \\
C & D & E \\
Y & F & G
\end{bmatrix},
\]

(2.1)

where \(X\) and \(Y\) are the unknown blocks. The matrices both \(A\) and \(G\) have a positive number of rows and columns.

We first need the following auxiliary result.

**Lemma 2.1.** Consider the partial matrix (2.1). Write \(D\) as

\[
D = S \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} T,
\]

where \(S\) and \(T\) are invertible, and denote

\[
BT^{-1} = [B_1 \ B_2], \quad S^{-1}C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},
\]

\[
S^{-1}E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \quad FT^{-1} = [F_1 \ F_2],
\]

where \(B_1\) and \(F_1\) have \(d\) columns and \(C_1\) and \(E_1\) have \(d\) rows. Then (2.1) has a unique minimal rank completion if and only if

\[
\begin{bmatrix}
A - B_1 C_1 & B_2 & ? \\
C_2 & 0 & E_2 \\
? & F_2 & G - F_1 E_1
\end{bmatrix}
\]

(2.2)

has a unique minimal rank completion. Moreover, conditions (I)–(IV) of Theorem 1.1 hold for (2.1) if and only if conditions (I)–(IV) of Theorem 1.1 hold for (2.2).

**Proof of Lemma 2.1.** Multiplying (2.1) by

\[
\begin{bmatrix} I \\ S^{-1} \\ I \end{bmatrix}
\]

on the left-hand side and by
on the right-hand side (where blank entries are zeros), we obtain the following matrix:

\[
\begin{bmatrix}
A & B_1 & B_2 & X \\
C_1 & I & 0 & E_1 \\
C_2 & 0 & 0 & E_2 \\
Y & F_2 & F_2 & G
\end{bmatrix}
\]  \hspace{1cm} (2.3)

Now, perform the following row and column operations on (2.3):

\[
\begin{bmatrix}
I & -B_1 & I \\
I & I & I \\
-F_1 & I & I \\
-C_1 & I & I
\end{bmatrix}
\begin{bmatrix}
A & B_1 & B_2 & X \\
C_1 & I & 0 & E_1 \\
C_2 & 0 & 0 & E_2 \\
Y & F_1 & F_2 & G
\end{bmatrix}
\]

\[
\times
\begin{bmatrix}
I & I & I & I \\
I & I & I & I \\
I & I & I & I \\
I & I & I & I
\end{bmatrix}
\]

These operations yield the matrix

\[
\begin{bmatrix}
A-B_1C_1 & 0 & B_2 & X-B_1E_1 \\
0 & I & 0 & 0 \\
C_2 & 0 & 0 & E_2 \\
Y-F_1C_1 & 0 & F_2 & G-F_1E_1
\end{bmatrix}
\]  \hspace{1cm} (2.4)

which has rank equal to

\[
\text{rank } D + \text{rank } \begin{bmatrix}
A-B_1C_1 & B_2 & X-B_1E_1 \\
C_2 & 0 & E_2 \\
Y-F_1C_1 & F_2 & G-F_1E_1
\end{bmatrix}
\]

Clearly, the rank of (2.1) is made as low as possible if and only if the rank of

\[
\begin{bmatrix}
A-B_1C_1 & B_2 & X-B_1E_1 \\
C_2 & 0 & E_2 \\
Y-F_1C_1 & F_2 & G-F_1E_1
\end{bmatrix}
\]

is as low as possible. This proves the first part.

For the second part, observe the following:

\[
\text{rank } \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \text{rank } D + \text{rank } \begin{bmatrix}
A-B_1C_1 & B_2 \\
C_2 & 0
\end{bmatrix}.
\]
rank[C  D  E] = \text{rank} D + \text{rank}[C_2  E_2],
rank[C  D] = \text{rank} D + \text{rank} C_2,
rank[D  E] = \text{rank} D + \text{rank} E_2,
\begin{equation*}
\text{rank}
\begin{bmatrix}
B \\
D \\
F
\end{bmatrix}
= \text{rank} D + \text{rank}
\begin{bmatrix}
B_2 \\
F_2
\end{bmatrix}.
\end{equation*}
\begin{equation*}
\text{rank}
\begin{bmatrix}
B \\
D
\end{bmatrix}
= \text{rank} D + \text{rank} B_2,
\end{equation*}
\begin{equation*}
\text{rank}
\begin{bmatrix}
D \\
F
\end{bmatrix}
= \text{rank} D + \text{rank} F_2.
\end{equation*}
\begin{equation*}
\text{rank}
\begin{bmatrix}
D  E \\
F  G
\end{bmatrix}
= \text{rank} D + \text{rank}
\begin{bmatrix}
0 & E_2 \\
F_2 & G - E_1 F_1
\end{bmatrix}.
\end{equation*}

The second statement of the lemma now follows. \(\square\)

**Proof of Theorem 1.1** *(For the 3 \times 3 case).* We will prove the “only if” part first.

Suppose that
\[
\begin{bmatrix}
A & B & ? \\
C & D & E \\
? & F & G
\end{bmatrix}
\]
has the unique minimal rank completion
\[
\begin{bmatrix}
A & B & X \\
C & D & E \\
Y & F & G
\end{bmatrix}.
\]
Then
\[
\begin{bmatrix}
A & B & ? \\
C & D & E \\
Y & F & G
\end{bmatrix}
\]
must also have a unique minimal rank completion which equals
\[
\begin{bmatrix}
A & B & X \\
C & D & E \\
Y & F & G
\end{bmatrix}.
\]
Theorem 5.1 in [6] implies that
\begin{equation*}
\text{rank}
\begin{bmatrix}
A & B \\
C & D \\
Y & F
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
C & D \\
Y & F
\end{bmatrix}.
\end{equation*}
where we use the fact that \( A \) has a positive number of rows and \( G \) a positive number of columns. If we interchange the roles of \( X \) and \( Y \), we get from the same theorem that

\[
\begin{align*}
\text{rank} \begin{bmatrix} A & B & X \\ C & D & E \end{bmatrix} &= \text{rank} \begin{bmatrix} B & X \\ D & E \end{bmatrix} \\
&= \text{rank} \begin{bmatrix} B & X \\ F & G \end{bmatrix} \\
&= \text{rank} \begin{bmatrix} A & B & X \\ C & D & E \end{bmatrix},
\end{align*}
\]

(2.6)

where we use the fact that \( A \) has a positive number of columns and \( G \) a positive number of rows. Now, if (2.5) holds, then the rows of \( [A \quad B] \) must be linear combinations of the rows of

\[
\begin{bmatrix} C & D \\ Y & F \end{bmatrix}.
\]

In particular, the rows of \( B \) are linear combinations of the rows of

\[
\begin{bmatrix} D \\ F \end{bmatrix}.
\]

So

\[
\text{rank} \begin{bmatrix} B \\ D \\ F \end{bmatrix} = \text{rank} \begin{bmatrix} D \\ F \end{bmatrix}.
\]

By similar logic,

\[
\begin{align*}
\text{rank}[C \quad D \quad E] &= \text{rank}[C \quad D], \\
\text{rank}[B \quad D \\ F] &= \text{rank}[B \quad D],
\end{align*}
\]

and

\[
\text{rank}[C \quad D \quad E] = \text{rank}[D \quad E].
\]

Thus, both conditions (II) and (III) hold. Eq. (2.6) implies that the rows of \( [Y \quad F \quad G] \) are linear combinations of the rows of
In particular, the rows of \([Y \ F]\) are linear combinations of the rows of
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}.
\]
Thus,
\[
\text{rank}\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \text{rank}\begin{bmatrix}
A & B
\end{bmatrix} = \text{rank}\begin{bmatrix}
C & D
\end{bmatrix}.
\]
likewise, by (2.5),
\[
\text{rank}\begin{bmatrix}
C & D & E \\
Y & F & G
\end{bmatrix} = \text{rank}\begin{bmatrix}
D & E \\
F & G
\end{bmatrix}.
\]
Combining these two observations with (2.5), we obtain condition (I).

Suppose next that \(s \neq h_2\) and that \(s \neq v_2\). By Lemma 2.1, we may, without loss of generality, assume that \(D = 0\). Now, if (2.1) is multiplied by invertible matrices \(S \oplus T \oplus I\) on the left-hand side and \(Q \oplus R \oplus I\) on the right-hand side, the submatrices \(B\) and \(C\) may be reduced to the forms
\[
\begin{bmatrix}
I_b & 0 \\
0 & 0
\end{bmatrix}
\]
respectively. Also, if (2.1) is multiplied by an invertible matrix of the form \(I \oplus I \oplus M\) on the right-hand side and \(I \oplus I \oplus N\) on the left-hand side, the submatrices \(E\) and \(F\) may be reduced to the forms
\[
\begin{bmatrix}
E_1 & 0 \\
0 & 0
\end{bmatrix}
\]
where \(E_1\) and \(F_1\) are invertible. The row of zeros in the transformed \(E\) matrix and the column of zeros in the transformed \(F\) matrix come from conditions (II) and (III), respectively. Thus, matrix (2.1) has been reduced into the following form:
\[
\begin{bmatrix}
A_{11} & A_{12} & I_b & 0 & ? & ? \\
A_{21} & A_{22} & 0 & 0 & ? & ? \\
I_c & 0 & 0 & E_1 & 0 & 0 \\
? & ? & F_1 & 0 & G_{11} & G_{12} \\
? & ? & 0 & 0 & G_{21} & G_{22}
\end{bmatrix}.
\]
The identity matrices and the invertible matrices \(E_1\) and \(F_1\) may be used to make some other elements in the matrix equal to zero without changing the rank. The matrix after equivalence may be rewritten as
The matrices $A_{22}$ and $G_{22}$ may be further reduced by multiplying them on the right and on the left by invertible matrices. They can be rewritten as

\[
\begin{bmatrix}
I_a & 0 & 0 & 0 & X_{11} & X_{12} & X_{13} \\
I_a & 0 & 0 & 0 & X_{21} & X_{22} & X_{23} \\
I_a & 0 & 0 & 0 & X_{31} & X_{32} & X_{33} \\
I_c & 0 & 0 & 0 & E_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Y_{11} & Y_{12} & Y_{13} & F_1 & 0 & 0 & 0 \\
Y_{21} & Y_{22} & Y_{23} & 0 & 0 & 0 & I_g \\
Y_{31} & Y_{32} & Y_{33} & 0 & 0 & 0 & 0
\end{bmatrix}
\]

respectively. Thus, (2.1) ultimately reduces to the following:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I_b & 0 & 0 & 0 & X_{11} & X_{12} & X_{13} \\
I_a & 0 & 0 & 0 & X_{21} & X_{22} & X_{23} \\
I_a & 0 & 0 & 0 & X_{31} & X_{32} & X_{33} \\
I_c & 0 & 0 & 0 & E_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Y_{11} & Y_{12} & Y_{13} & F_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Y_{21} & Y_{22} & Y_{23} & 0 & 0 & 0 & I_g \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Now, let the $e = \text{size of } E_1$ and $f = \text{size of } F_1$. Note that $s = a + b + c = g + e + f$, $h_2 = c = e$, and $v_2 = b = f$. Since $s \neq h_2$ and $s \neq v_2$, we get that $a + b > 0$ and $e + g > 0$, and that $a + c > 0$ and $f + g > 0$. If $a > 0$ and $g > 0$, then choose $X_{22}$ to be any invertible matrix such that $Y_{22} = X_{22}^{-1}$, and choose all other submatrices of $X$ and $Y$ as zero. Since $F \neq \{0, 1\}$, this yields more than one minimal rank completion, giving non-uniqueness. If $a = 0$ or $g = 0$, then $b = f > 0$ by condition (III) and $c = e > 0$ by condition (II). Then, choose $X_{11}$ to be any matrix, choose $Y_{11} = -F_1 X_{11} E_1^{-1}$, and choose all other submatrices of $X$ and $Y$ equal to zero. This yields more than one minimal rank completion because this choice of $X$ and $Y$ gives

\[
\begin{bmatrix}
A & B & X \\
C & D & E \\
Y & F & G
\end{bmatrix}
\]

\[
\begin{bmatrix}
A & B \\
C & D \\
? & F
\end{bmatrix}
\]

according to Theorem 1.1 in [9]. Indeed, the minimal rank of (2.7) is the maximum of

\[
\{ \text{rank } \begin{bmatrix} 0 & 0 & I_b \\ A_{22} & 0 & 0 \\ I_c & 0 & 0 \end{bmatrix}, \text{rank } \begin{bmatrix} 0 & E_1 \\ F_1 & 0 \\ 0 & 0 & G_{22} \end{bmatrix} \},
\]

and is equal to $r + \text{rank } A_{22} + \text{rank } I_b$, since the rank of $A_{22}$ equals the rank of $G_{22}$ by condition (I). Again, this gives non-uniqueness. Thus, uniqueness implies condition (IV). This proves the "only if" part.
Next, we will prove the “if” part. Suppose that conditions (I)–(III) hold and that $s = v_2$. Then by Theorem 1.1 in [9] the minimal rank of
\[
\begin{bmatrix}
A & B & ? \\
C & D & E \\
? & F & G
\end{bmatrix}
\]
equals the rank of
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]
Also
\[
\text{rank} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \text{rank} \begin{bmatrix} B \\ D \end{bmatrix} = \text{rank} \begin{bmatrix} B \\ D \end{bmatrix},
\]
because $s = v_2$. So
\[
\begin{bmatrix}
A & B \\
C & D \\
? & F
\end{bmatrix}
\]
satisfies the unique minimal rank condition. Thus, there is a unique $Y$ so that
\[
\begin{bmatrix}
A & B \\
C & D \\
Y & F
\end{bmatrix}
\]
is the minimal rank completion of
\[
\begin{bmatrix}
A & B \\
C & D \\
? & F
\end{bmatrix},
\]
and
\[
\text{rank} \begin{bmatrix} A & B \\ C & D \\ Y & F \end{bmatrix} = \text{rank} \begin{bmatrix} B \\ D \end{bmatrix}
\]
holds. In particular, the columns of
\[
\begin{bmatrix}
A \\
C \\
Y
\end{bmatrix}
\]
are linear combinations of the columns of
\[
\begin{bmatrix}
B \\
D \\
F
\end{bmatrix}.
\]
This implies that
\begin{align*}
\text{rank} \begin{bmatrix} C & D & E \\ Y & F & G \end{bmatrix} &= \text{rank} \begin{bmatrix} D & E \\ F & G \end{bmatrix} \\
&= \text{rank} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\
&= \text{rank} \begin{bmatrix} A & B \\ C & D \\ Y & F \end{bmatrix}
\end{align*}

and
\begin{align*}
\text{rank} \begin{bmatrix} C & D \\ Y & F \end{bmatrix} &= \text{rank} \begin{bmatrix} D \end{bmatrix} \\
&= \text{rank} \begin{bmatrix} B \\ D \end{bmatrix} = \text{rank} \begin{bmatrix} A & B \\ C & D \\ Y & F \end{bmatrix}.
\end{align*}

Thus
\[
\begin{bmatrix}
A & B & ? \\
C & D & E \\
Y & F & G
\end{bmatrix}
\]
satisfies the triangular minimal rank uniqueness condition from Theorem 5.1 in [6]. Consequently, there is a unique \( X \) that will make the rank of
\[
\begin{bmatrix}
A & B & X \\
C & D & E \\
Y & F & G
\end{bmatrix}
\]
as small as possible, which equals \( s \). The case in which \( s = h_2 \) follows in a similar way. \( \Box \)

It should be observed that the condition of non-triviality of the sizes of \( A \) and \( G \) were necessary solely for the “only if” part.

**Remark 2.2.** In order to construct the unique minimal rank completion of (2.1) in case conditions (I)–(IV) of Theorem 1.1 hold, one proceeds as follows. If \( s = h_2 \), construct the unique minimal rank completion of the triangular partial matrix
\[
\begin{bmatrix}
A & B & ? \\
C & D & E
\end{bmatrix}
\]
by using the technique of [8] or, equivalently, by setting
\[
? = X := [A & B] \begin{bmatrix} S \\ R \end{bmatrix},
\]
where
\[
\begin{bmatrix} S \\ R \end{bmatrix}
\]
is such that $CS + DR = E$. Next, use again the technique of triangular arrays to complete the matrix
\[
\begin{bmatrix}
A & B & X \\
C & D & E \\
? & F & G
\end{bmatrix}
\] (2.8)
with minimal rank.

When $s = v_2$, one proceeds in a similar fashion, but with first completing
\[
\begin{bmatrix}
B & ? \\
D & E \\
F & G
\end{bmatrix}
\]
and subsequently completing (2.8). Other variations, such as starting with
\[
\begin{bmatrix}
A & B \\
D & E \\
? & F
\end{bmatrix}
\]
when $s = v_2$, or starting with
\[
\begin{bmatrix}
C & D & E \\
? & F & G
\end{bmatrix}
\]
when $s = h_2$, are also possible.

Let us also draw the following consequence.

**Corollary 2.3.** Suppose that $P$ and $S$ both have a positive number of rows. Then the partial block matrix
\[
\begin{bmatrix}
P & ? \\
Q & R \\
? & S
\end{bmatrix}
\]
has a unique minimal rank completion if and only if the following condition holds:
\[
t := \text{rank} \begin{bmatrix} P \\ Q \end{bmatrix} = \text{rank} [Q] = \text{rank} [Q R] = \text{rank} [R] = \text{rank} \begin{bmatrix} R \\ S \end{bmatrix}.
\] (2.9)

In that case, the minimal rank completion has rank equal to $t$.

**Proof of Corollary 2.3.** Clearly, the statement is true when either $P$ or $S$ has zero columns, so assume that they have a positive number of columns. We may add a column of (possibly trivial size) zeros to the above matrix without changing the uniqueness of the minimal rank completion, leading to the partial matrix
\[
\begin{bmatrix}
P & 0 & ? \\
Q & 0 & R \\
? & 0 & S
\end{bmatrix}.
\]
Applying the $3 \times 3$ case of Theorem 1.1 to this new matrix yields the following three conditions for uniqueness:

1. $t_1 = \text{rank} \begin{bmatrix} P \\ Q \end{bmatrix} = \text{rank} \begin{bmatrix} R \\ S \end{bmatrix}$.

2. $t_2 = \text{rank} \begin{bmatrix} Q & R \end{bmatrix} = \text{rank} \begin{bmatrix} Q \end{bmatrix} = \text{rank} \begin{bmatrix} R \end{bmatrix}$.

3. $t_1 = t_2$ or $t_1 = 0$.

Note that both options in (3) lead to (2.9). Conversely, if (2.9) holds, then conditions (I)-(III) of Theorem 1.1 are satisfied. The corollary now follows.

3. Proof of the main theorem

**Proof of Theorem 1.1 (The general case).** We will prove this theorem using mathematical induction. From Section 2 we know that the theorem holds for $n = 3$. Now, let $n \in \{3, 4, 5, \ldots\}$ be arbitrary and suppose that the theorem holds for this particular value of $n$. Now, consider

\[
\begin{bmatrix}
A_{11} & A_{12} & \cdots & \cdots & A_{1,n+1} \\
A_{21} & A_{22} & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
A_{n+1,1} & A_{n+1,2} & \cdots & \cdots & A_{n+1,n+1}
\end{bmatrix}
\]  

(3.1)

We will prove the “if” part first. So, assume conditions (I)-(IV) hold for (3.1). Note that by Theorem 1.1 in [9], the minimal rank of (1.1) equals $s$. In that case, conditions (I)-(IV) also hold for (1.1). Thus, there exist unique $A_{ij}$, $|j - i| \geq 2$, $1 \leq i, j \leq n$, so that $(A_{ij})_{i,j=1}^n$ has rank $s$. Since a minimal rank completion of (3.1) has rank equal to $s$, its restriction to the first $n$ rows and columns must equal $(A_{ij})_{i,j=1}^n$. Now, consider the $3 \times 3$ problem

\[
\begin{bmatrix}
\alpha & \beta & ? \\
\gamma & A_{nn} & A_{n,n+1} \\
? & A_{n+1,n} & A_{n+1,n+1}
\end{bmatrix}
\]  

(3.2)

where $\alpha = (A_{ij})_{i,j=1}^{n-1}$, $\beta = (A_{ij})_{i=2}^n$, and $\gamma = (A_{ij})_{i=2}^n$. Observe that

\[
\text{rank} \begin{bmatrix} \alpha & \beta \\ \gamma & A_{nn} \end{bmatrix} = s = \text{rank} \begin{bmatrix} A_{nn} & A_{n,n+1} \\ A_{n+1,n} & A_{n+1,n+1} \end{bmatrix}
\]

and, in addition, that

\[
\text{rank} \begin{bmatrix} \alpha & \beta \\ \gamma & A_{nn} \end{bmatrix} = s = \text{rank} \begin{bmatrix} A_{n-1,n+1} & A_{n-1,n} \\ A_{n,n+1} & A_{nn} \end{bmatrix}
\]
Since

$$\begin{bmatrix}
A_{n-1,n-1} & A_{n-1,n} \\
A_{n,n-1} & A_{nn}
\end{bmatrix}$$

is a submatrix of

$$\begin{bmatrix}
\alpha & \beta \\
\gamma & A_{nn}
\end{bmatrix},$$

the rows of the first \((n-2)\) block rows must be linear combinations of the rows in the last two block rows. Thus, we may infer that

$$\text{rank} \begin{bmatrix} \beta \\ A_{nn} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{n-1} \\ A_{nn} \end{bmatrix}$$

and that

$$\text{rank} \begin{bmatrix} \gamma & A_{nn} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{n,n-1} & A_{nn} \end{bmatrix}.$$ 

Moreover, we see that

$$\text{rank} \begin{bmatrix} A_{n,n+1} \\ A_{nn} \\ A_{n,n+1} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{n-1,n} & A_{nn} \\ A_{nn} & A_{n,n+1} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{nn} & A_{n,n+1} \end{bmatrix}.$$ 

Since conditions (I)–(IV) hold for (3.1), at least one of these last two quantities equals \(s\). Thus, (3.2) satisfies the conditions from Theorem 1.1. The application of this theorem provides for a unique minimal rank completion of rank \(s\) for (3.2). Since the minimal rank completion of (3.1) has rank \(s\), this must form the unique minimal rank completion of (3.1).

Next, consider the “only if” part. We first address the case where \(n = 4\). Consider the matrix

$$\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33} & A_{34} \\
A_{43} & A_{44}
\end{bmatrix}$$

and let us assume that it has the unique minimal rank completion \(\left[ A_{ij} \right]_{i,j=1}^{4} \) with rank equal to \(R\). We consider the following four partial matrices which all have \(\left[ A_{ij} \right]_{i,j=1}^{4} \) as a unique minimal rank completion.

First, consider the partial matrix

$$\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33} & A_{34} \\
A_{43} & A_{44}
\end{bmatrix}$$.
Since $[A_{ij}]_{i,j=1}^4$ is the unique minimal rank completion, the $3 \times 3$ case of Theorem 1.1 provides the following four conditions:

1. $\rho_1 := \text{rank} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \text{rank} \begin{pmatrix} A_{33} & A_{34} \\ A_{43} & A_{44} \end{pmatrix} = R.$

2. $\rho_2 := \text{rank} [A_{31} \ A_{32} \ A_{33} \ A_{34}] = \text{rank} [A_{31} \ A_{32} \ A_{33}] = \text{rank} [A_{32} \ A_{33} \ A_{34}].$

3. $\rho_3 := \text{rank} \begin{pmatrix} A_{13} \\ A_{23} \\ A_{33} \\ A_{43} \end{pmatrix} = \text{rank} \begin{pmatrix} A_{13} \\ A_{23} \\ A_{33} \\ A_{43} \end{pmatrix} = \text{rank} \begin{pmatrix} A_{33} \\ A_{43} \end{pmatrix}.$

4. Either $\rho_1 = \rho_2$ or $\rho_1 = \rho_3$.

Moreover, the minimal rank completion has rank $R$, which equals $\rho_1$.

Next, consider the partial matrix

$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{32} & A_{33} & A_{34} \\ A_{42} & A_{43} & A_{44} \end{pmatrix}$.

Once again, the $3 \times 3$ case of Theorem 1.1 yields four conditions:

1. $\rho_4 := \text{rank} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \text{rank} \begin{pmatrix} A_{22} & A_{23} & A_{24} \\ A_{32} & A_{33} & A_{34} \\ A_{42} & A_{43} & A_{44} \end{pmatrix} = R.$

2. $\rho_5 := \text{rank} [A_{21} \ A_{22} \ A_{23} \ A_{24}] = \text{rank} [A_{21} \ A_{22} \ A_{23}] = \text{rank} [A_{22} \ A_{23} \ A_{24}].$

3. $\rho_6 := \text{rank} \begin{pmatrix} A_{12} \\ A_{22} \\ A_{32} \\ A_{42} \end{pmatrix} = \text{rank} \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} = \text{rank} \begin{pmatrix} A_{22} \\ A_{32} \\ A_{42} \end{pmatrix}.$

4. Either $\rho_4 = \rho_5$ or $\rho_4 = \rho_6$.

In that case, the minimal rank completion has rank $R$, which is equal to $\rho_4$. Likewise, consider
which has a unique minimal rank completion. Corollary 2.3 yields the condition

\[ \rho_7 := \text{rank} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \]

\[ = \text{rank} \begin{bmatrix} A_{11} & A_{12} & A_{23} & A_{24} \\ A_{21} & A_{22} & A_{33} & A_{34} \\ A_{31} & A_{32} \end{bmatrix} \]

\[ = \text{rank} \begin{bmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \]

\[ = \text{rank} \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} \]

\[ = R. \]

Finally, we may apply the transpose result of Corollary 2.3 to the matrix

\[ \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{32} & A_{33} & A_{34} \\ A_{42} & A_{43} & A_{44} \end{bmatrix} \]

to obtain one last condition

\[ \rho_8 := \text{rank} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{32} & A_{33} & A_{34} \\ A_{42} & A_{43} & A_{44} \end{bmatrix} \]

\[ = \text{rank} \begin{bmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \\ A_{32} & A_{33} \\ A_{42} & A_{43} \end{bmatrix} \]

\[ = \text{rank} \begin{bmatrix} A_{12} & A_{13} \end{bmatrix} \]

\[ = \text{rank} \begin{bmatrix} A_{22} & A_{23} \end{bmatrix} \]

\[ = \text{rank} \begin{bmatrix} A_{42} & A_{43} \end{bmatrix} \]

\[ = \text{rank} \begin{bmatrix} A_{32} & A_{33} & A_{34} \\ A_{42} & A_{43} & A_{44} \end{bmatrix} \]

\[ = R. \]
Combining some of the conditions above, we obtain that the following six conditions are satisfied:

1. \( \delta_1 := \text{rank} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{33} & A_{34} \\ A_{43} & A_{44} \end{bmatrix} = R. \)

2. \( \delta_2 := \text{rank} \begin{bmatrix} A_{12} \\ A_{22} \\ A_{32} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{22} \\ A_{32} \end{bmatrix}. \)

3. \( \delta_3 := \text{rank} [A_{21} \ A_{22} \ A_{23}] = \text{rank} [A_{21} \ A_{22}] = \text{rank} [A_{22} \ A_{23}]. \)

4. \( \delta_4 := \text{rank} \begin{bmatrix} A_{23} \\ A_{33} \\ A_{43} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{23} \\ A_{33} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{33} \\ A_{34} \end{bmatrix}. \)

5. \( \delta_5 := \text{rank} [A_{32} \ A_{33} \ A_{34}] = \text{rank} [A_{32} \ A_{33}] = \text{rank} [A_{33} \ A_{34}]. \)

6. Either \( \delta_1 = \delta_2 \) or \( \delta_1 = \delta_3 \), and either \( \delta_1 = \delta_4 \) or \( \delta_1 = \delta_5 \).

To complete the proof the “only if” part of Theorem 1.1 when \( n = 4 \), we need to show that

\[ \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} = \delta_1 = R. \]

We observe that four possible combinations may be derived from condition (6) above, namely,

\( \delta_1 = \delta_2 = \delta_4, \)

\( \delta_1 = \delta_2 = \delta_5, \)

\( \delta_1 = \delta_3 = \delta_4, \)

\( \delta_1 = \delta_3 = \delta_5. \)

Suppose that \( \delta_1 = \delta_2 \). Then

\[ \text{rank} \begin{bmatrix} A_{11} & \cdots & A_{14} \\ \vdots & \vdots & \vdots \\ A_{41} & \cdots & A_{44} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = R. \]

In particular, the columns of

\[ \begin{bmatrix} A_{24} \\ A_{34} \end{bmatrix} \]

and

\[ \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} \]
must be linear combinations of the columns of
\[
\begin{bmatrix}
A_{22} \\
A_{32}
\end{bmatrix}.
\]
Thus,
\[
R = \text{rank} \begin{bmatrix}
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34}
\end{bmatrix} = \text{rank} \begin{bmatrix}
A_{22} & A_{23} \\
A_{32} & A_{33}
\end{bmatrix}.
\]

Similarly, if \( \delta_1 = \delta_3 \), then
\[
\text{rank} \begin{bmatrix}
A_{11} & \cdots & A_{14} \\
\vdots & \ddots & \vdots \\
A_{41} & \cdots & A_{44}
\end{bmatrix} = \text{rank} \begin{bmatrix}
A_{21} & A_{22} & A_{23}
\end{bmatrix} = R.
\]

In particular, the rows of \([A_{12} \ A_{13}]\) and \([A_{42} \ A_{43}]\) are linear combinations of the rows of \([A_{22} \ A_{23}]\). Thus,
\[
R = \text{rank} \begin{bmatrix}
A_{22} & A_{23} \\
A_{32} & A_{33}
\end{bmatrix} = \text{rank} \begin{bmatrix}
A_{22} & A_{23} \\
A_{32} & A_{33}
\end{bmatrix}.
\]

So conditions (I)–(IV) of Theorem 1.1 hold for \( n = 4 \).

Now, let \( n = \{4, 5, \ldots\} \) and suppose that the "only if" part holds for \( n \). Suppose (3.1) has a unique minimal rank completion \( A = (A_{ij})_{i,j=1}^{n+1} \). Consider
\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33} & A_{34} \\
& & & & & & & A_{43} & A_{44} & \ddots \\
& & & & & & & & & \vdots & \ddots & \ddots & \ddots \\
& & & & & & & & & & & A_{n,n+1} \\
? & & & & & & & & & & & ?
\end{bmatrix}, \quad (3.4)
\]

which we may view as an \( n \times n \) partial tri-diagonal block matrix when we take the first two block rows together and the first two block columns together. So, by the induction assumption, conditions (I)–(IV) from Theorem 1.1 hold for (3.4). Consider also the partial block matrix
\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
? & & & & & A_{n,n+1} \\
? & & & & & ?
\end{bmatrix}, \quad (3.5)
\]

which has a unique minimal rank completion. If this is viewed as an \( n \times n \) partial tri-diagonal block matrix, then conditions (I)–(IV) hold for (3.5).
From condition (I) for (3.4), we observe that
\[ \text{rank} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{33} & A_{34} \\ A_{43} & A_{44} \end{bmatrix} = \ldots = \text{rank} \begin{bmatrix} A_{nn} & A_{n,n+1} \\ A_{n+1,n} & A_{n+1,n+1} \end{bmatrix}. \]

Similarly, from condition (I) for (3.5), we observe that
\[ \text{rank} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} = \ldots = \text{rank} \begin{bmatrix} A_{n-2,n-2} & A_{n-2,n-1} \\ A_{n-1,n-2} & A_{n-1,n-1} \\ A_{n-1,n-1} & A_{n-1,n} \\ A_{n,n-1} & A_{n,n+1} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{n-1,n-1} & A_{n-1,n} & A_{n-1,n+1} \\ A_{n,n-1} & A_{nn} & A_{n,n+1} \\ A_{n+1,n-1} & A_{n+1,n} & A_{n+1,n+1} \end{bmatrix}. \]

Combining the two, we get that
\[ \text{rank} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \ldots = \text{rank} \begin{bmatrix} A_{nn} & A_{n,n+1} \\ A_{n+1,n} & A_{n+1,n+1} \end{bmatrix}, \]

thus establishing condition (I) for (3.1). When \( n = 4 \), we use
\[ \text{rank} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \leq \text{rank} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{33} & A_{34} \\ A_{43} & A_{44} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{44} & A_{45} \\ A_{54} & A_{55} \end{bmatrix} \leq \text{rank} \begin{bmatrix} A_{33} & A_{34} & A_{35} \\ A_{43} & A_{44} & A_{45} \\ A_{53} & A_{54} & A_{55} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \]

From condition (II) for (3.4), we observe that
\[
\begin{align*}
\text{rank } \begin{bmatrix} A_{13} \\ A_{23} \\ A_{33} \\ A_{43} \end{bmatrix} &= \text{rank } \begin{bmatrix} A_{13} \\ A_{23} \\ A_{33} \end{bmatrix} = \text{rank } \begin{bmatrix} A_{33} \\ A_{43} \end{bmatrix} \\
\text{and} \\
\text{rank } \begin{bmatrix} A_{i-1,i} \\ A_{ii} \\ A_{i+1,i} \end{bmatrix} &= \text{rank } \begin{bmatrix} A_{i-1,i} \\ A_{ii} \end{bmatrix} = \text{rank } \begin{bmatrix} A_{ii} \\ A_{i+1,i} \end{bmatrix} \\
\end{align*}
\]
for \(i = 4, \ldots, n\). Similarly, condition (II) for (3.5) yields
\[
\begin{align*}
\text{rank } \begin{bmatrix} A_{n-2,n-1} \\ A_{n-1,n-1} \\ A_{n+1,n-1} \end{bmatrix} &= \text{rank } \begin{bmatrix} A_{n-2,n-1} \\ A_{n-1,n-1} \end{bmatrix} = \text{rank } \begin{bmatrix} A_{n-1,n-1} \\ A_{n+1,n-1} \end{bmatrix} \\
\text{and} \\
\text{rank } \begin{bmatrix} A_{i-1,i} \\ A_{ii} \\ A_{i+1,i} \end{bmatrix} &= \text{rank } \begin{bmatrix} A_{i-1,i} \\ A_{ii} \end{bmatrix} = \text{rank } \begin{bmatrix} A_{ii} \\ A_{i+1,i} \end{bmatrix} \\
\end{align*}
\]
for \(i = 2, \ldots, n - 2\). Thus, we obtain
\[
\begin{align*}
\text{rank } \begin{bmatrix} A_{n-2,n-1} \\ A_{n-1,n-1} \\ A_{n+1,n-1} \end{bmatrix} &= \text{rank } \begin{bmatrix} A_{n-2,n-1} \\ A_{n-1,n-1} \end{bmatrix} = \text{rank } \begin{bmatrix} A_{n-1,n-1} \\ A_{n+1,n-1} \end{bmatrix} \\
\text{and} \\
\text{rank } \begin{bmatrix} A_{i-1,i} \\ A_{ii} \\ A_{i+1,i} \end{bmatrix} &= \text{rank } \begin{bmatrix} A_{i-1,i} \\ A_{ii} \end{bmatrix} = \text{rank } \begin{bmatrix} A_{ii} \\ A_{i+1,i} \end{bmatrix} \\
\end{align*}
\]
for \(i = 2, 3, \ldots, n\) as condition (II) for (3.1). As with condition (I), a separate argument is needed for \(n = 4\).

Finally, from condition (III) for (3.4), we observe that
\[
\begin{align*}
\text{rank } [A_{31} \ A_{32} \ A_{33} \ A_{34}] &= \text{rank } [A_{31} \ A_{32} \ A_{33}] \\
&= \text{rank } [A_{33} \ A_{34}] \\
\text{and} \\
\text{rank } [A_{i,i-1} \ A_{ii} \ A_{i,i+1}] &= \text{rank } [A_{i,i-1} \ A_{ii}] = \text{rank } [A_{ii} \ A_{i,i+1}] \\
\end{align*}
\]
for \(i = 4, \ldots, n\). Applying condition (III) to (3.5), we observe that
\[
\begin{align*}
\text{rank } [A_{n-2,n-2} \ A_{n-1,n-1} \ A_{n-1,n+1} \\ A_{n-1,n-1} \ A_{n-1,n} \ A_{n-1,n+1}] \\
&= \text{rank } [A_{n-1,n-2} \ A_{n-1,n-1}] \\
&= \text{rank } [A_{n-1,n-1} \ A_{n-1,n} \ A_{n-1,n+1}] \\
\text{and} \\
\text{rank } [A_{i,i-1} \ A_{ii} \ A_{i,i+1}] &= \text{rank } [A_{i,i-1} \ A_{ii}] = \text{rank } [A_{ii} \ A_{i,i+1}] \\
\end{align*}
\]
for \(i = 2, \ldots, n - 2\). Thus, we obtain
rank\[A_{i,i-1} \ A_{ii} \ A_{i,i+1}\] = rank\[A_{i,i+1} \ A_{ii}\] = rank\[A_{ii} \ A_{i,i+1}\],
i = 2, 3, \ldots, n, as condition (III) for (3.3). Lastly, condition (IV) for (3.3) follows
in a similar fashion from combining condition (IV) for (3.3) and (3.4). \(\square\)

**Remark.** By repeatedly applying Remark 2.2, one may construct the unique minimal
rank completion of (1.1) in the case that Theorem 1.1 is valid.

By using the \(3 \times 3\) case repeatedly, one may obtain results for other banded patterns. The following theorem provides one such example.

**Corollary 3.1.** The banded partial block matrix
\[
\begin{bmatrix}
A & B & \? & \? \\
C & D & E & \? \\
F & G & H & J \\
? & K & L & M
\end{bmatrix},
\] (3.6)
where both \(A\) and \(M\) have a positive number of rows and columns, has a unique
minimal rank completion if and only if the following six conditions hold:

1. \(s_1 := \text{rank}\[F \ G \ H \ J\] = \text{rank}\[F \ G \ H\] = \text{rank}\[G \ H \ J\].

2. \(s_2 := \text{rank}\begin{bmatrix} B \\ D \\ G \\ K \end{bmatrix} = \text{rank}\begin{bmatrix} B \\ D \\ G \\ K \end{bmatrix} = \text{rank}\begin{bmatrix} D \\ G \\ K \end{bmatrix}.

3. \(s_3 := \text{rank}\begin{bmatrix} C & D & E \\ F & G & H \\ G & H & J \end{bmatrix} = \text{rank}\begin{bmatrix} C & D \\ F & G \end{bmatrix}.

4. \(s_4 := \text{rank}\begin{bmatrix} D & E \\ G & H & J \end{bmatrix} = \text{rank}\begin{bmatrix} G & H \\ K & L \end{bmatrix}.

5. \(s_5 := \text{rank}\begin{bmatrix} A & B \\ C & D \\ F & G \end{bmatrix} = \text{rank}\begin{bmatrix} G & H & J \\ K & L & M \end{bmatrix} \geq \text{rank}\begin{bmatrix} D & E \\ G & H \\ K & L \end{bmatrix} + \text{rank}\begin{bmatrix} C & D & E \\ F & G & H \end{bmatrix} = \text{rank}\begin{bmatrix} D & E \\ G & H \end{bmatrix}.

6. Either \(s_5 = s_2\) or \(s_5 = s_3\), and either \(s_5 = s_1\) or \(s_5 = s_4\).
In that case, the minimal rank completion has rank equal to $s_5$.

**Proof of Corollary 3.1.** Assume that (3.6) has the unique minimal rank completion
\[
\begin{bmatrix}
A & B & Y & Z \\
C & D & E & W \\
F & G & H & J \\
X & K & L & M
\end{bmatrix},
\]
(3.7)
In that case, the matrix
\[
\begin{bmatrix}
A & B & ? & ? \\
C & D & E & ? \\
F & G & H & J \\
X & K & L & M
\end{bmatrix}
\]
also has a unique minimal rank completion equal to (3.7). Theorem 5.1 in [6] yields that
\[
\text{rank}
\begin{bmatrix}
A & B \\
C & D \\
F & G \\
X & K
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
C & D & E \\
F & G & H \\
X & K & L \\
M
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
F & G & H & J \\
X & K & L & M
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
C & D \\
F & G \\
X & K
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
F & G & H \\
X & K & L
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
A & B & Y & Z \\
C & D & E & W \\
F & G & H & J \\
X & K & L & M
\end{bmatrix}.
\]
We observe that certain rows and columns are linear combinations of each other, just as in the previous section. From this, we may infer the following:
\[
\text{rank}[F \ G \ H \ J] = \text{rank}[F \ G \ H],
\] (3.8)
\[
\text{rank}
\begin{bmatrix}
B \\
D \\
G \\
K
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
D \\
G \\
K
\end{bmatrix},
\] (3.9)
\[
\text{rank}
\begin{bmatrix}
C & D & E \\
F & G & H
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
C & D \\
F & G
\end{bmatrix}.
\] (3.10)
Thus, equalities (3.8)–(3.11) are necessary conditions for a minimal rank completion.
Now, the matrix
\[
\begin{bmatrix}
A & B \\
C & D \\
F & G \\
? & ?
\end{bmatrix}
\begin{bmatrix}
E \\
W \\
H \\
J
\end{bmatrix}
\]
must also have a unique minimal rank completion equal to (3.7). By drawing from Theorem 1.1, we obtain four additional conditions that must be necessary for a unique minimal rank completion:

\[
\text{rank}
\begin{bmatrix}
A & B \\
C & D \\
F & G
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
D & E & W \\
G & H & J \\
K & L & M
\end{bmatrix},
\quad (3.12)
\]

\[
\text{rank}
\begin{bmatrix}
A & B \\
C & D \\
F & G
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
D & E \\
G & H
\end{bmatrix},
\quad (3.13)
\]

\[
\text{rank}
\begin{bmatrix}
B \\
D \\
G
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
D \\
G
\end{bmatrix},
\quad (3.14)
\]

Either (3.12) = (3.13) or (3.12) = (3.14). (3.15)

Moreover, the minimal rank completion (3.7) has rank equal to the rank of (3.12). Likewise, the matrix
\[
\begin{bmatrix}
A & B & Y \\
C & D & E \\
F & G & H \\
? & K & L \\
? & ? & ?
\end{bmatrix}
\]
has a unique minimal rank completion equal to (3.7). Thus, we obtain four more conditions necessary for a unique minimal rank completion:

\[
\text{rank}
\begin{bmatrix}
A & B & Y \\
C & D & E \\
F & G & H
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
G & H & J \\
K & L & M
\end{bmatrix},
\quad (3.16)
\]

\[
\text{rank}[F & G & H & J] = \text{rank}[F & G & H] = \text{rank}[G & H & J],
\quad (3.17)
\]
rank \begin{bmatrix} B & Y \\ D & E \\ G & H \\ K & L \end{bmatrix} = \text{rank} \begin{bmatrix} D & E \\ G & H \end{bmatrix}, \quad (3.18)

Either (3.16) = (3.17) or (3.16) = (3.18). \quad (3.19)

In addition, the minimal rank completion (3.7) has rank equal to the rank of (3.16).

Now, equality (3.17) gives us condition (1), equality (3.14) yields condition (2),
equality (3.10) gives condition (3), and equality (3.11) gives condition (4). In addition,
equality (3.15) gives the first part of condition (6), and equality (3.19) gives the
second part.

To obtain condition (5), we observe that the banded matrix (3.6) has three maxi-
mal triangular subpatterns (using the terminology of [9]). They are

\begin{align*}
T_1 &:= \begin{bmatrix} K & B & A \\ D & C & E \\ G & F & H \\ J \end{bmatrix}, \\
T_2 &:= \begin{bmatrix} B & D & E \\ K & L & M \\ G & H & J \\ F \end{bmatrix}, \\
T_3 &:= \begin{bmatrix} B & K & L \\ D & E & C \\ G & H & F \\ J \end{bmatrix}.
\end{align*}

Now, according to Woerdeman [9], the minimal rank of $T_1$ is equal to the rank of

\begin{bmatrix} B & A \\ D & C \\ G & F \end{bmatrix},

which is equal to the rank of (3.7) by equality (3.12). Similarly, the minimal rank of
$T_2$ is equal to the rank of

\begin{bmatrix} K & L & M \\ G & H & J \end{bmatrix},

which is equal to the rank of (3.7) by equality (3.16). The minimal rank of $T_3$ is

\begin{align*}
\text{rank} \begin{bmatrix} K & L \\ D & E \\ G & H \end{bmatrix} + \text{rank} \begin{bmatrix} D & E & C \\ G & H & F \end{bmatrix} - \text{rank} \begin{bmatrix} D & E \\ G & H \end{bmatrix},
\end{align*}

which must be less than or equal to the rank of (3.7). So condition (5) holds. This
proves the “only if” part of the theorem.
To prove the "if" part, we note that four possible combinations may be derived from condition (6). They are
\[ s_5 = s_2 = s_1, \]
\[ s_5 = s_3 = s_1, \]
\[ s_5 = s_4 = s_2, \]
\[ s_5 = s_4 = s_3. \]
We will prove this part by examining each case in turn. When \( s_5 = s_2 = s_1 \), it must be the case that
\[
\begin{bmatrix}
F & G & H & J \\
? & K & L & M
\end{bmatrix}
\]
has a unique minimal rank completion equal to \( s_5 \). Since we desire a minimal rank completion of (3.6) equal to \( s_5 \), \( X \) must be the unique matrix so that
\[
\text{rank} \begin{bmatrix}
F & G & H & J \\
X & K & L & M
\end{bmatrix} = s_5.
\]
Likewise, \( X \) must be the unique matrix so that
\[
\text{rank} \begin{bmatrix}
A & B \\
C & D \\
F & G \\
X & K
\end{bmatrix}.
\]
Thus, we may infer that
\[
\text{rank} \begin{bmatrix}
A & B \\
C & D \\
F & G \\
X & K
\end{bmatrix} = \text{rank} \begin{bmatrix}
F & G & H & J \\
X & K & L & M
\end{bmatrix} = \text{rank} \begin{bmatrix}
F & G & H \\
X & K & L
\end{bmatrix} = \text{rank} \begin{bmatrix}
C & D \\
F & G \\
X & K
\end{bmatrix} = s_5,
\]
and that
\[
s_5 \leq \text{rank} \begin{bmatrix}
C & D & E \\
F & G & H \\
X & K & L
\end{bmatrix} = \text{rank} \begin{bmatrix}
D & E \\
G & H \\
K & L
\end{bmatrix} = \text{rank} \begin{bmatrix}
G & H \\
K & L
\end{bmatrix} \leq s_5.
\]
So
\[
\begin{bmatrix}
A & B & ? & ? \\
C & D & E & ? \\
F & G & H & J \\
X & K & L & M
\end{bmatrix}
\]
has a unique minimal rank completion, giving uniqueness for the minimal rank completion of (3.6).

Let now $s_5 = s_3 = s_1$. The fact that $s_5 = s_3$ allows us to cancel in the inequality of condition (5) the entire left-hand side along with

$$\text{rank} \begin{bmatrix} C & D & E \\ F & G & H \end{bmatrix}$$

from the right-hand side. Thus, we are left with the inequality

$$0 \geq \text{rank} \begin{bmatrix} D & E \\ G & H \\ K & L \end{bmatrix} - \text{rank} \begin{bmatrix} D & E \\ G & H \end{bmatrix}.$$

But,

$$\text{rank} \begin{bmatrix} D & E \\ G & H \\ K & L \end{bmatrix} \geq \text{rank} \begin{bmatrix} D & E \\ G & H \end{bmatrix},$$

so it must be the case that

$$\text{rank} \begin{bmatrix} D & E \\ G & H \\ K & L \end{bmatrix} = \text{rank} \begin{bmatrix} D & E \\ G & H \end{bmatrix}.$$

We may now apply Theorem 1.1 to obtain that

$$\begin{bmatrix} C & D & E & ? \\ F & G & H & J \\ ? & K & L & M \end{bmatrix}$$

has a unique minimal rank completion

$$\begin{bmatrix} C & D & E & W \\ F & G & H & J \\ X & K & L & M \end{bmatrix}$$

with rank equal to $s_5$. Now we check that

$$\begin{bmatrix} A & B & ? & ? \\ C & D & E & W \\ F & G & H & J \\ X & K & L & M \end{bmatrix}$$

satisfies the uniqueness completion of Theorem 5.1 in [6] to arrive at the unique minimal rank completion of (3.6).

When $s_5 = s_4 = s_2$, we find that

$$\begin{bmatrix} A & B & ? \\ C & D & E \\ F & G & H \\ ? & K & L \end{bmatrix}$$
must have a unique minimal rank completion equal to $s_5$. Indeed, recalling condition (5), we cancel the entire left-hand side of the inequality along with

\[
\begin{bmatrix}
D & E \\
G & H \\
K & L
\end{bmatrix}
\]

from the right-hand side since $s_5 = s_4$, which yields the inequality

\[
0 \geq \text{rank}
\begin{bmatrix}
C & D & E \\
F & G & H
\end{bmatrix}
- \text{rank}
\begin{bmatrix}
D & E \\
G & H
\end{bmatrix}.
\]

But,

\[
\text{rank}
\begin{bmatrix}
C & D & E \\
F & G & H
\end{bmatrix} \geq \text{rank}
\begin{bmatrix}
D & E \\
G & H
\end{bmatrix},
\]

so it must be the case that

\[
\text{rank}
\begin{bmatrix}
C & D & E \\
F & G & H
\end{bmatrix} = \text{rank}
\begin{bmatrix}
D & E \\
G & H
\end{bmatrix}.
\]

Thus, the various conditions and Theorem 1.1 yield that

\[
\begin{bmatrix}
A & B & ? \\
C & D & E \\
F & G & H \\
? & K & L
\end{bmatrix}
\]

has a unique minimal rank completion

\[
\begin{bmatrix}
A & B & Y \\
C & D & E \\
F & G & H \\
X & K & L
\end{bmatrix}.
\]

As before, we check that

\[
\begin{bmatrix}
A & B & Y & ? \\
C & D & E & ? \\
F & G & H & J \\
X & K & L & M
\end{bmatrix}
\]

satisfies the uniqueness condition from Theorem 5.1 in [6], leading to the desired result.

When $s_5 = s_4 = s_3$, we obtain as before that

\[
\text{rank}
\begin{bmatrix}
C & D & E \\
F & G & H
\end{bmatrix} = \text{rank}
\begin{bmatrix}
D & E \\
G & H
\end{bmatrix} = \text{rank}
\begin{bmatrix}
D & E \\
G & H \\
K & L
\end{bmatrix}.
\]

Thus, there is a unique $X$ so that

\[
\text{rank}
\begin{bmatrix}
C & D & E \\
F & G & H \\
X & K & L
\end{bmatrix} = s_5.
\]
We use arguments as before to show that
\[
\begin{bmatrix}
A & B & ? & ? \\
C & D & E & ? \\
F & G & H & J \\
? & K & L & M
\end{bmatrix}
\]
has a unique minimal rank completion of rank $s_5$. □

Again we observe that an appropriate repeated application of Remark 2.2 leads to a construction of the unique minimal rank completion of (3.6) if Corollary 3.1 applies.

4. Uniqueness in the problem of completing a matrix and its inverse

By using the connection between the minimal rank completion problem and the problem of simultaneously completing a matrix and its inverse (see [9]), we may also obtain unique results for the latter question. We will illustrate this in three specific cases.

**Theorem 4.1.** Given matrices $A$, $B$, $C$, and $D$ of sizes $r \times q$, $s \times q$, $r \times p$, and $q \times s$, respectively, where $p + q = r + s$, there exist matrices $X$, $Y$, $Z$, and $W$ of sizes $s \times p$, $q \times r$, $p \times r$, and $p \times s$, respectively, such that
\[
\begin{bmatrix}
A & C \\
B & X
\end{bmatrix}^{-1} = \begin{bmatrix} Y & D \\
Z & W
\end{bmatrix}
\]
if and only if the following four conditions hold:

1. $\text{rank} \left[ I_q - DB \right] = \text{rank}[A],$
2. $\text{rank}[A \ C] = r,$
3. $\text{rank}[AD \ C] = \text{rank}[C],$
4. $\text{rank}[A] - r + \text{rank}[C] - \text{rank}[AD] \leq 0.$

If in addition $p, r, s > 0$, the matrices $X$, $Y$, $Z$, and $W$ are unique if and only if the following four conditions hold:

5. $\text{rank}[A] = r \leq q,$
6. $\text{rank}[C] = p = \text{rank}[AD],$
7. $\text{rank} \left[ I_q - DB \right] = \text{rank}[A].$
In that case, let $P$, $Q$, and $Y$ be such that
\[ ADP = C, \]
\[ QC = I_p, \]
\[ YA = I - DB. \]
Then the unique solution to (4.1) is given by
\[ X = -P + BDP, \]
\[ Y = Y, \]
\[ Z = Q - QAY, \]
\[ W = -QAD. \]

**Proof of Theorem 4.1.** According to [9], (4.1) has a solution if and only if
\[ \min \text{ rank} \begin{bmatrix} I_r & 0 & A & C \\ 0 & I_s & B \\ D & I_q & 0 \\ 0 & I_p \end{bmatrix} = p + q = r + s. \]
In addition, (4.1) has a unique solution if and only if the above matrix has a unique minimal rank completion. By permuting the first and second rows, we obtain
\[ \begin{bmatrix} 0 & I_r & B \\ I_r & 0 & A & C \\ D & I_q & 0 \\ 0 & I_p \end{bmatrix}, \]
which is a version of the $4 \times 4$ case discussed in Section 3. Drawing from [8], we observe three maximal triangular subpatterns in (4.2):
\[
\begin{bmatrix}
0 & I_q & D \\
I_q & I_s & 0 \\
A & 0 & I_r
end{bmatrix},
\begin{bmatrix}
0 & B & I_s \\
B & I_s & D \\
A & 0 & C
end{bmatrix},
\begin{bmatrix}
B & 0 & I_p \\
0 & I_p & I_q \\
A & C & 0
end{bmatrix}.
\]
which have minimal ranks equal to
\[ \text{rank} \begin{bmatrix} I_q & D \\ B & I_s \\ A & 0 \end{bmatrix} - \text{rank} \begin{bmatrix} B & I_s \\ A & 0 \end{bmatrix} + \text{rank} \begin{bmatrix} B & I_s & 0 \\ A & 0 & I_r \end{bmatrix}, \]
\[ \text{rank} \begin{bmatrix} B & I_s \\ I_q & D \\ A & 0 \end{bmatrix} - \text{rank} \begin{bmatrix} I_q & D \\ A & 0 \end{bmatrix}, \]
\[ + \text{rank} \begin{bmatrix} I_q & D \\ A & 0 & C \end{bmatrix} - \text{rank} \begin{bmatrix} A & 0 & C \end{bmatrix} + r. \]
respectively. If we apply row and column operations to some of these terms, we notice that
\[
\begin{align*}
\text{rank} \begin{bmatrix} I_q & D \\ B & I_s \\ A & 0 \end{bmatrix} &= \text{rank} \begin{bmatrix} I_q - DB & 0 \\ 0 & I_s \\ A & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} I_q - DB \\ A \end{bmatrix} + s, \\
\text{rank} \begin{bmatrix} I_q & D \\ A & 0 \\ 0 & C \end{bmatrix} &= \text{rank} \begin{bmatrix} I_q D & 0 \\ 0 & -AD \\ C \end{bmatrix} = q + \text{rank}[AD], \\
\text{rank} \begin{bmatrix} I_q & D & 0 \\ A & 0 & C \end{bmatrix} &= \text{rank} \begin{bmatrix} I_q D & 0 \\ 0 & -AD \\ C \end{bmatrix} = q + \text{rank}[AD \ C].
\end{align*}
\]

Theorem 1.1 in [9] yields that the minimal rank of (4.2) is
\[
\begin{align*}
\max \left\{ \begin{array}{c}
\text{rank} \begin{bmatrix} I_q - DB \\ A \end{bmatrix} + s + r - \text{rank}[A] \\
\text{rank} \begin{bmatrix} I_q - DB \\ A \end{bmatrix} + s - \text{rank}[AD] + \text{rank}[AD \ C] - \text{rank}[A \ C] + r \\
p - \text{rank}[C] + q + \text{rank}[AD \ C] - \text{rank}[A \ C] + r
\end{array} \right\}.
\end{align*}
\]
Furthermore, these three ranks are less than or equal to \( p + q = r + s \) if and only if
\[
\begin{align*}
\text{rank} \begin{bmatrix} I_q - DB \\ A \end{bmatrix} &= \text{rank}[A], \quad \text{rank}[A \ C] = r, \\
\text{rank}[AD \ C] &= \text{rank}[C], \quad \text{rank}[A] - r + \text{rank}[C] - \text{rank}[AD] \leq 0.
\end{align*}
\]
This proves the first part of the theorem.

Next, we observe that Corollary 3.1 provides a unique solution for (4.2) if and only if in addition
\[
\begin{align*}
\text{rank} \begin{bmatrix} I_s & B \\ 0 & A \end{bmatrix} &= \text{rank} \begin{bmatrix} I_s & B \\ 0 & A \end{bmatrix} = \text{rank} \begin{bmatrix} I_s & B \\ 0 & A \end{bmatrix} = p + q = r + s, \\
\text{rank} \begin{bmatrix} 0 & A \\ D & I_q \end{bmatrix} &= \text{rank} \begin{bmatrix} A & C \\ I_q & 0 \end{bmatrix} = p + q = r + s, \\
\text{rank} \begin{bmatrix} 0 & A \\ D & I_q \end{bmatrix} &= p + q.
\end{align*}
\]
But, this will occur if and only if in addition
\[
\begin{align*}
\text{rank}[A] &= r \leq q, \quad \text{rank}[C] = p = \text{rank}[AD \ C], \quad \text{rank}[AD] = p.
\end{align*}
\]
Thus, we obtain conditions (5)–(8).

Finally, we show the validity of the formulas of the unique solution. Indeed, if $P, Q, X, Y, Z,$ and $W$ are as stated in the theorem, we get that

\begin{align*}
YA + DB &= I, \\
YC + DX &= YC - DP + DBDP = YC - YADP = 0, \\
ZA + WB &= QA - QAYA - QADB = QA(I - YA - DB) = 0, \\
ZC + WX &= QC - QAYC + QADP - QADBDP \\
&= QC - QAYC + QC + QADP + QAYADP = I_p.
\end{align*}

Thus, (4.1) holds. $\square$

Similar results hold for other configurations. Below we state the results. The proofs are omitted as they are of the same nature as the proof of Theorem 4.1, but they can easily be constructed by the interested reader.

**Theorem 4.2.** Given matrices $A, B, C,$ and $D$ of sizes $m \times n,$ $m \times k,$ $l \times n,$ and $k \times l,$ respectively, where $m + l = k + n,$ there exist matrices $X, Y, Z,$ and $W$ of sizes $l \times k,$ $n \times l,$ $n \times m,$ and $k \times m,$ respectively, such that

\begin{equation}
\begin{bmatrix}
A & B \\
C & X
\end{bmatrix}^{-1} = \begin{bmatrix}
Y & Z \\
W & D
\end{bmatrix} (4.3)
\end{equation}

if and only if

\begin{align*}
\text{rank} \begin{bmatrix} A \\ C \end{bmatrix} &= n, \\
\text{rank} \begin{bmatrix} A \\ DC \end{bmatrix} &= \text{rank} [A] = \text{rank} [A \ B \ D], \\
\text{rank} [A \ B] &= m, \\
k + \text{rank} [A] &\leq m + \text{rank} [D].
\end{align*}

If in addition $k, l, m, n > 0,$ then a unique solution to (4.3) exists if and only if $A$ and $D$ are square and invertible. In that case, the unique solution is given by

\begin{align*}
W &= -DCA^{-1}, \\
X &= CA^{-1}B + D^{-1}, \\
Y &= A^{-1} - A^{-1}BDCA^{-1}.
\end{align*}
\[ Z = -A^{-1}BD. \]

The matrix completion problem (4.3) was treated in [2], and necessary and sufficient conditions for a solution to exist are given there, in slightly different terms.

**Theorem 4.3.** Let \( p \) and \( s \) be positive integers. Given matrices \( A, B, C, \) and \( D \) of sizes \( r \times q, s \times q, r \times p, \) and \( q \times r, \) respectively, where \( p + q = r + s, \) there exist unique matrices \( X, Y, Z, \) and \( W \) of sizes \( s \times p, q \times s, p \times r, \) and \( q \times r, \) respectively, such that

\[
\begin{bmatrix}
A & C \\
B & X
\end{bmatrix}^{-1} = 
\begin{bmatrix}
D & Y \\
Z & W
\end{bmatrix}
\tag{4.4}
\]

if and only if

\[
\text{rank}[I_q - DA] = \text{rank}\begin{bmatrix}
I_q - DA \\
B
\end{bmatrix} = \text{rank}[B] = s (\leq q)
\]

and

\[
\text{rank}[I_r - DA] = \text{rank}\begin{bmatrix}
C & I_r - AD
\end{bmatrix} = \text{rank}[C] = p (\leq r).
\]

In that case, let \( R, S, \) and \( T \) be such that

\[ BR = I_s, \quad SC = I_p, \quad (I_r - AD)T = C. \]

Then the unique solution to (4.1) is given by

\[
\begin{align*}
W &= -SAR + SADAR, \\
X &= -BDT, \\
Y &= R - DAR, \\
Z &= S - SAD.
\end{align*}
\]

Note that Theorem 2.2 in [9] states the necessary and sufficient conditions for a solution to (4.4) to exist.

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**References**


