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Note

Disjoint Finite Partial Steiner Triple Systems Can Be Embedded in Disjoint Finite Steiner Triple Systems*

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This paper shows that a pair of disjoint finite partial Steiner triple systems can be embedded in a pair of disjoint finite Steiner triple systems.

1. INTRODUCTION

A *Steiner triple system* (or triple system) is a pair (S, t) where S is a finite set and t is a collection of three element subsets of S (called triples) such that each pair of distinct elements of S belong to exactly one triple of t. The number $|S|$ is called the order of the triple system (S, t). It is well known that there is a Steiner triple system of order n if and only if $n \equiv 1$ or $3 \pmod 6$ [7]. Two Steiner triple systems (S, t_1) and (S, t_2) are said to be *disjoint* provided that they have no triples in common; i.e., $t_1 \cap t_2 = \varnothing$ [1]. A *partial* Steiner triple system is a pair (P, p) where P is a finite set and p is a collection of three element subsets of P such that each pair of distinct elements of P belong to *at most* one triple of p. Whereas there is the cardinality restriction that a Steiner triple system have order $n \equiv 1$ or $3 \pmod 6$ there is no such restriction on partial triple systems. Not too surprisingly, two partial Steiner triple systems (P, p_1) and (P, p_2) are *disjoint* provided $p_1 \cap p_2 = \varnothing$. It is important to note at this point that the definition of disjoint (partial) triple sytems *requires* that they be defined on the *same set*. The partial triple system (P, p) is said to be *embedded* in the triple system (S, t) provided that $P \subseteq S$ and $p \subseteq t$. In [8], C. Treash proved the following theorem.

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THEOREM 1 (C. Treash [8]). *A finite partial Steiner triple system can be embedded in a finite Steiner triple system.*

The purpose of this paper is to show that a pair of finite *disjoint partial* Steiner triple systems can be embedded in a pair of finite *disjoint* Steiner triple systems. That is, if (P, p_1) and (P, p_2) are a pair of finite partial disjoint triple systems, then there is a pair of finite disjoint Steiner triple systems (S, t_1) and (S, t_2) such that (P, p_1) is embedded in (S, t_1) and (P, p_2) is embedded in (S, t_2). The proof is based on a recent result of B. Ganter [2] that a finite partial Steiner quadruple system can be finitely embedded along with a construction, due to the author, of disjoint triple systems from quadruple systems [5]. The reader is referred to [5] and [6] for a more detailed account of the techniques used in what follows.

2. STEINER QUADRUPLE SYSTEMS

A *Steiner quadruple system* (or more simply a quadruple system) is a pair (Q, q) where Q is a set and q is a collection of four element subsets of Q (called *blocks*) such that every three element subset of Q belongs to exactly one block of q. The number $|Q|$ is called the order of the quadruple system (Q, q). In [3] H. Hanani has shown that the spectrum for Steiner quadruple systems consists of all positive integers $n \equiv 2$ or $4 \pmod 6$. There is a very useful connection between quadruple systems and triple systems. The connection is as follows. Let (Q, q) be a quadruple system and x any element of Q. Denote $Q \backslash \{x\}$ by Q_x and the set of all triples $\{a, b, c\}$ such that $\{x, a, b, c\} \in q$ by $q(x)$. It is a routine matter to see that $(Q_x, q(x))$ is a Steiner triple system. By a *partial* quadruple system is meant a pair (V, v) where V is a finite set and v is a collection of four element subsets of V such that every three element subset of V belongs to *at most* one of the blocks of v. The following important theorem was recently obtained by B. Ganter [2].

THEOREM 2 (B. Ganter [2]). *A finite partial quadruple system can be embedded in a finite quadruple system.*

Treash's Theorem [8] that a finite partial Steiner triple system is embeddable in a finite Steiner triple system is a very easy corollary of Ganter's Theorem. To see this, let (P, p) be a finite partial triple system. Let x be a symbol not belonging to P. Denote by $x(p)$ the set of all four element sets obtained by adding x to each triple in p. Then $(P \cup \{x\}, x(p))$ is a partial quadruple system and so can be embedded in a finite quadruple system (Q, q). Trivially (P, p) is embedded in $(Q_x, q(x))$.

This method of obtaining Treash's theorem via the use of Ganter's theorem is extremely important in proving the main result in this paper.

3. Partial Disjoint Triple Systems Can be Embedded in Disjoint Triple Systems

In what follows, in order to avoid excessive use of the word finite, everything will be finite. Let (P, p_1) and (P, p_2) be a pair of disjoint partial triple systems. We will assume that P does not contain the symbols 1, 2, 3, or 4. Consider the partial quadruple system (P^*, p^*) where $P^* = P \cup \{1, 2, 3, 4\}$ and $p^* = 1(p_1) \cup 2(p_2) \cup \{1, 2, 3, 4\}$, where $1(p_1)$ is the set of quadruples obtained by adding 1 to each triple in p_1 and $2(p_2)$ is the set of quadruples obtained by adding 2 to each triple in p_2. Embed (P^*, p^*) in a *finite* quadruple system (Q, q) of order n. Since (Q, q) can be embedded in arbitrarily large quadruple systems (take enough direct products) we can assume $(n - 2)/2 - 4 > |P|$. This is IMPORTANT. Let $(Q_1, q(1))$ be the triple system obtained by deleting 1 and $(Q_{21}, q(21))$ the triple system obtained by first deleting 2 and then renaming 1 with 2. Then both Q_1 and Q_{21} are based on the same symbols. Evidently, (P, p_1) is embedded in $(Q_1, q(1))$ and (P, p_2) is embedded in $(Q_{21}, q(21))$. In [5] it is shown that if $(Q_1{}^*, q(1)^*)$ is the triple system obtained from $(Q_1, q(1))$ by interchanging 2 and 3 that $(Q_1{}^*, q(1)^*)$ and $(Q_{21}, q(21))$ have exactly the triple $\{2, 3, 4\}$ in common. Since interchanging 2 and 3 does not involve any symbols in P we still have (P, p_1) embedded in $(Q_1{}^*, q(1)^*)$ and (P, p_2) embedded in $(Q_{21}, q(21))$. At this point the triple systems $(Q_1{}^*, q(1)^*)$ and $(Q_{21}, q(21))$ are *almost disjoint* (have exactly one triple in common) [4], [5], and [6]. This situation is easily rectified. In [5] it is shown that there is a subset X of Q containing at least $(n - 2)/2 - 4$ elements such that if $x \in X$ and $(Q_{21}^*, q(21)^*)$ is the triple system obtained from $(Q_{21}, q(21))$ by interchanging 3 and x then $(Q_1{}^*, q(1)^*)$ and $(Q_{21}^*, q(21)^*)$ are disjoint. Since $|X| \geqslant (n - 2)/2 - 4 > |P|$ we can choose an element x which does not belong to P. Hence 3 and x do not belong to any of the triples in p_2 and so (P, p_2) is embedded in $(Q_{21}^*, q(21)^*)$. Since $(Q_1{}^*, q(1)^*)$ and $(Q_{21}^*, q(21)^*)$ are disjoint and (P, p_1) is embedded in $(Q_1{}^*, q(1)^*)$ and (P, p_2) is embedded in $(Q_{21}^*, q(21)^*)$ we have the following theorem.

THEOREM 3. *A pair of disjoint finite partial Steiner triple systems can always be embedded in a pair of finite disjoint Steiner triple systems.*

4. Remarks

The amount of space necessary to give a nontrivial example of the construction used in Theorem 3 is somewhat prohibitive and so we will not give such an example here. The reader is referred to [5] for an example of constructing a pair of disjoint triple systems from a quadruple system. This is the construction used in this paper once the embedding of the disjoint partial triple systems (P, p_1) and (P, p_2) into the quadruple system (Q, q) is achieved.

Finally, a rather obvious problem remains: namely, the embedding of a collection of $t > 2$ mutually disjoint partial Steiner triple systems in t mutually disjoint Steiner triple systems. It seems doubtful that the techniques developed in this paper can be used to attack this more general problem.

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