Solutions of Cauchy and periodic problems for evolution inclusions with multi-valued $w_{\lambda_0}$-pseudomonotone maps

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\textbf{Abstract}
We consider differential-operator inclusions with $w_{\lambda_0}$-pseudomonotone multi-valued maps. The problem of investigation of solutions of Cauchy and periodic problems is solved by Faedo–Galerkin (FG) method. Important a priori estimates are obtained. Some topological descriptions of the resolvent operators are made.

\section{1. Introduction}

One of the most effective approaches to investigate nonlinear problems, represented by partial differential equations, inclusions and inequalities with boundary values, consists in the reduction of them into equations in Banach spaces governed by nonlinear operators. In order to study these objects the modern methods of nonlinear analysis have been used [14,15,17,19,20,22,29,52,53,61,62,78]. In [58], by using a special basis, the Cauchy problem for a class of equations with operators of Volterra type has been studied. The important periodic problem for equations with monotone differential op-
operators of Volterra type has been studied in [22]. Periodic solutions for pseudomonotone operators were considered in [53], for \( w_{\lambda_0} \)-pseudomonotone single-valued operators they were studied in [40].

Convergence of approximate solutions to an exact solution of the differential-operator equation or inclusion is frequently proved on the basis of a monotonicity or a pseudomonotonicity of corresponding operator. If the given property of an initial operator takes place then it is possible to prove convergence of the approximate solutions under weaker a priori estimations than it is demanded while using embedding theorems. The monotonicity concept has been introduced in papers of M.M. Vainberg, R.I. Kachurovsky, G.J. Minty, E.H. Zarantonello and others. Significant generalization to the case of monotonicity was given by H. Brezis [7]. Namely, Brezis calls the operator \( A : X \to X^* \) pseudomonotone if

(a) the operator \( A \) is bounded;
(b) from \( u_n \to u \) weakly in \( X \) and from

\[
\lim_{n \to \infty} \langle A(u_n), u_n - u \rangle_X \leq 0
\]

it follows that

\[
\lim_{n \to \infty} \langle A(u_n), u_n - v \rangle_X \geq \langle A(u), u - v \rangle_X \quad \forall v \in X,
\]

where \( X \) be a Banach space. In applications the sum of radially continuous monotone bounded operator and strongly continuous operator was considered as a pseudomonotone operator [22]. Concrete examples of pseudomonotone operators were obtained by extension of elliptic differential operators when only their summands complying with highest derivatives satisfied the monotonicity property [53]. In books of H. Gajewski, K. Gröger, and K. Zacharias [22], J.L. Lions [53], G.E. Ladas and V. Lakshmikantham [52], V. Barbu [5] the main results of solvability theory for abstract operator equations and differential operator equations that are monotone or pseudomonotone in Brezis sense are described. Also the applications of given theorems to concrete equations of mathematical physics, and in particular, to free boundary problems were given.

The theory of monotone operators in reflexive Banach spaces is one of the major areas of nonlinear functional analysis. Its basis is so-called variational methods, and since sixties years of the last century the theory has been intensively developed in tight interaction with the theory of convex functions and the theory of partial differential equations. Papers of F. Browder and P. Hess [9,10] became classical in the given direction of investigation. In particular in the work of F. Browder and P. Hess [10] the class of generalized pseudomonotone operators that enveloped the class of monotone maps was introduced. Let \( W \) be some normed space continuously embedded into the normed space \( Y \). A multi-valued map \( A : Y \to 2^{Y^*} \) is said to be "generalized pseudomonotone on \( W \)" if for each pair of sequences \( \{y_n\}_{n \geq 1} \subset W \) and \( \{d_n\}_{n \geq 1} \subset Y^* \) such that \( \forall n \geq 1, d_n \in A(y_n), y_n \to y \) weakly in \( W \), \( d_n \to d \) weakly star in \( Y^* \), from the inequality

\[
\lim_{n \to \infty} \langle d_n, y_n \rangle_Y \leq \langle d, y \rangle_Y
\]

it follows that \( d \in A(y) \) and \( \langle d_n, y_n \rangle_Y \to \langle d, y \rangle_Y \).

The grave disadvantage of the given theory is the fact that in general case it appeared problematic to prove the closedness with respect to the sum of pseudomonotone (in the classical sense) maps. This disadvantage becomes more substantial in investigation of differential-operator inclusions and evolution variation and hemivariation inequalities when we need to consider the sum of the classical pseudomonotone map and the subdifferential (of Gateaux or of Clarke) for multi-valued map that is generalized pseudomonotone map [12]. I.V. Skrypnik's idea of passing to subsequences in classical definitions [72], which was realized for stationary inclusions in papers of M.Z. Zgurovskii and V.S. Mel'nik [54,57,80,81], enabled to consider essentially wider class of \( \lambda_0 \)-pseudomonotone maps, closed
with respect to the sum of maps [81]. In papers of P.O. Kas'yanov, V.S. Mel'nik, and S. Toscano [40] there was introduced the class of $w_{λ_0}$-pseudomonotone maps which includes, in particular, a class of generalized pseudomonotone multi-valued operators and it is closed with respect to the sum also. A multi-valued map $A: Y → 2^{Y^*}$ with the nonempty, convex, bounded, closed values is said to be "$w_{λ_0}$-pseudomonotone" ("$λ_0$-pseudomonotone on $W$"), if for any sequence $\{y_n\}_{n\geq0} \subset W$ such that $y_n → y_0$ weakly in $W$, $d_n → d_0$ weakly star in $Y^*$ as $n → +∞$, where $d_n ∈ A(y_n)$ $\forall n \geq 1$, the inequality

$$\lim_{n → +∞} (d_n, y_n - y_0) ≤ 0$$

(1)

implies the existence of such subsequences $\{y_{n_k}\}_{k \geq 1} \subset \{y_n\}_{n \geq 1}$ and $\{d_{n_k}\}_{k \geq 1} \subset \{d_n\}_{n \geq 1}$ for which

$$\lim_{k → +∞} (d_{n_k}, y_{n_k} - w) ≤ \inf_{d_0 ∈ A(y_0)} (d_0, y_0 - w) ∀ w ∈ Y$$

follows. Now we must prove the resolvability for differential-operator inclusions with $λ_0$-pseudomonotone on $D(L)$ multi-valued maps in Banach spaces:

$$Lu + A(u) + B(u) ≥ f, \quad u ∈ D(L),$$

(2)

where $A: X_1 → 2^{X_1^*}, B: X_2 → 2^{X_2^*}$ are multi-valued maps of $D(L)_{λ_0}$-pseudomonotone type with nonempty, convex, closed, bounded values, $X_1, X_2$ are Banach spaces continuously embedded in some Hausdorff linear topological space, $X = X_1 ∩ X_2$, $L: D(L) ⊂ X → X^*$ is linear, monotone, closed, densely defined operator with a linear domain $D(L)$.

Let us remark that any multi-valued map $A: Y → 2^{Y^*}$ naturally generates upper and, respectively, lower form:

$$[A(y), ω]^+ = \sup_{d ∈ A(y)} (d, w)Y, \quad [A(y), ω]^- = \inf_{d ∈ A(y)} (d, w)Y, \quad y, ω ∈ X.$$ 

Properties of the given objects were investigated in the works of M.Z. Zgurovsky, V.S. Mel'nik, and P.O. Kasyanov [54,79–81,83,36]. Thus, together with the classical coercivity condition for the operator $A$:

$$\frac{(A(y), y)_Y}{\|y\|_Y} → +∞, \quad \text{as } \|y\|_Y → +∞,$$

which ensures the important a priori estimations, the properties of "+-coercivity" (and, respectively, "−-coercivity") arise as

$$\frac{[A(y), y]^+(-)}{\|y\|_Y} → +∞, \quad \text{as } \|y\|_Y → +∞.$$

Notice that +-coercivity is much weaker condition than −-coercivity.

When investigating multi-valued maps of $w_{λ_0}$-pseudomonotone type it was found out that even for subdifferentials of convex lower semicontinuous functionals the boundedness condition is not natural [38]. Thus it was necessary to introduce an adequate relaxation of the boundedness condition which would have enveloped at least the class of monotone multi-valued maps. In paper [27] the following definition was introduced: a multi-valued map $A: Y → 2^{Y^*}$ satisfies "Condition (1)"; for a bounded set $B ⊂ Y$, $y_0 ∈ Y$, $k > 0$, $d ∈ A$ (i.e. $d(y) ∈ A(y)$ $\forall y ∈ B$) the next inequality is true

$$\langle d(y), y - y_0 \rangle ≤ k \quad \text{for all } y ∈ B,$$
then there exists such $C > 0$ that

$$
\|d(y)\|_{Y^*} \leq C \quad \text{for all } y \in B.
$$

Recent development of the monotonicity method in the theory of differential-operator inclusions and evolution variational inequalities [4,11,12,15,19,20,23–25,43,62–70] ensures resolvability of the given objects under the conditions of $-$-coercivity, boundedness and the generalized pseudomonotonicity (it is necessary to notice, that the proof is not constructive). With relation to applications it would be actual to weaken some conditions for multi-valued maps in problem (2) replacing $-$-coercivity by $+$-coercivity, boundedness by Condition $(I)$ and pseudomonotonicity in classical sense or generalized pseudomonotonicity by $w_{\lambda_0}$-pseudomonotonicity.

At the present time operator, differential-operator equations, inclusions and evolution variational inequalities are studied intensively by many authors: J.-P. Aubin, V. Barbu, Yu.G. Borisovich, S. Carl, H. Frankowska, B.D. Gelman, M.F. Gorodnii, S. Hu, M.I. Kamenskii, P.I. Kogut, O.A. Kovalevsky, A.D. Mishkis, D. Motreanu, V.V. Obukhovskii, N.S. Papageorgiou, V.Yu. Slusarchuk, O.M. Solonucha, A.N. Vakulenko, M.Z. Zgurovsky and others (see [1–6,19–25,35,43–50,73,79–83] and citations there). Similarly to differential-operator equations at least four approaches are well known: the Faedo–Galerkin (FG) method, the elliptic regularization, the theory of semigroups, the difference approximations. The extension of these approaches on evolutionary inclusions encounters a series of fundamental difficulties. For differential-operator inclusions the method of semigroups is realized in works of V. Barbu [5], M. Kamenskii, V.V. Obukhovskii, P. Zecca [29], A.A. Tolstonogov, Yu.I. Umanskii [75,76]. The method of finite differences firstly was extended on evolutionary inclusions and variational inequalities in the work of P.O. Kasyanov, V.S. Mel’nik and L. Toscano [39]. The method of singular perturbations (H. Brezis [8] and Yu.A. Dubinski [16]) and the FG method for differential-operator inclusions for $w_{\lambda_0}$-pseudomonotone multi-valued maps have not been systematically investigated up to now. It is required to justify the singular perturbations method and the FG method for differential-operator inclusions with $w_{\lambda_0}$-pseudomonotone $+$-coercive multi-valued maps in Banach spaces.

In the present paper we introduce a new construction to prove the existence of periodic and Cauchy problems solutions for differential-operator inclusions by the FG method for $w_{\lambda_0}$-pseudomonotone weakly $+$-coercive multi-valued operators, that satisfy property $(I)$ and different conditions on basis in the phase space (see Corollary 2, Remark 10 and Corollary 3). This paper is organized in the following way. At Section 2 we set the main problem (4). Section 3 is devoted to examination of some classes of $w_{\lambda_0}$-pseudomonotone type maps (see Lemmas 1, 2, Propositions 1 and 2). This properties are sufficiently used during the justification of the FG method for different classes of problems. In Section 4 we consider some technical axillary statements concerned with the approximation method of investigation for the main problem (see Propositions 3–6). At Section 5 we have described the FG method for main problem (4) (see Definition 6). This method is justified at Section 6 (see Theorem 1) for evolution inclusions with $w_{\lambda_0}$-pseudomonotone weakly $+$-coercive multi-valued operators, that satisfy Condition $(I)$ and some axillary condition on basis in the phase space. At Section 7 we consider some corollaries for periodic solution problem (46) (see Corollary 3) and for Cauchy problem (24) (see Corollary 2 and Remark 10) under different basis conditions. In this section we are also consider some anisotropic problem (49)–(50) perturbed by Clarke’s generalized gradient. From the point of view of the applications we have essentially extended the class of the operators, considered by other authors (see [1–6,19–25,35,43–50,73,79–83] and references therein).

2. The setting of the problem

Let $(V_i, H, V_i^*)$, $i = 1, 2$, be some evolution triple [37,77] such that

$$
V := V_1 \cap V_2 \quad \text{is dense in spaces } V_1, V_2 and H.
$$

(3)
For $i = 1, 2$ we consider the functional spaces \[22,37,\]

\[
X_i = L_{q_i}(S; H) \cap L_{p_i}(S; V_i), \quad X_i^* = L_{q_i}(S; V_i^*) + L_{r_i}(S; H),
\]

\[
X = X_1 \cap X_2, \quad X^* = X_1^* + X_2^*, \quad \mathscr{H} = L_2(S; H)
\]

with corresponding norms \[22,37,\], where $S = [0, T]$, $0 \leq T < +\infty$, $1 < p_i \leq r_i < +\infty$, $r_i^{-1} + p_i^{-1} = 1$. Let $\langle \cdot, \cdot \rangle$ be the duality form on $X^* \times X$, that coincides on $\mathscr{H} \times X$ with the inner product in $\mathscr{H} \[22,37,\]$.

Let $A : X_1 \rightrightarrows X_1^*$ and $B : X_2 \rightrightarrows X_2^*$ be multi-valued maps with nonempty closed convex values, $L : D(L) \subset X \to X^*$ be linear closed densely defined operator. We consider the problem:

\[
\begin{aligned}
& L y + A(y) + B(y) \ni f, \\
& y \in D(L),
\end{aligned}
\]

(4)

where $f \in X^*$ is an arbitrary fixed element.

3. Classes of maps

Let $Y$ be some real reflexive Banach space, $Y^*$ its dual space,

\[
\langle \cdot, \cdot \rangle_Y : Y^* \times Y \to \mathbb{R}
\]

be the duality form on $Y$. For each nonempty subset $B \subset Y^*$ let us consider its weak closed convex hull $\overline{cl}(B) := \text{cl} X_1^* (\text{co}(B))$. Further by $C_v(Y^*)$ we will denote the class of all nonempty convex weakly compact in $Y^*$ subsets.

For each multi-valued map $A : Y \rightrightarrows Y^*$ we can consider its upper and lower norms:

\[
\|A(y)\|_+ = \sup_{d \in A(y)} \|d\|, \quad \|A(y)\|_- = \inf_{d \in A(y)} \|d\|.
\]

The main properties of the given maps are considered in \[36,61,79–83,\]. In particular, the next properties hold true.

Let $A, B : Y \to C_v(Y^*)$. Then for arbitrary $y, v, v_1, v_2 \in Y$, $\alpha \in \mathbb{R}$:

1) the functional $Y \ni v \mapsto [A(y), v]_+$ is convex positively homogeneous and lower semicontinuous;
2) $[A(y), v_1 + v_2]_+ \supseteq [A(y), v_1]_+ + [A(y), v_2]_-$, $[A(y), v_1 + v_2]_- \subseteq [A(y), v_1]_+ + [A(y), v_2]_-;
3) [A(y) + B(y), v]_+ = [A(y), v]_+ + [B(y), v]_+$, $[A(y) + B(y), v]_- = [A(y), v]_- + [B(y), v]_-;
4) [A(y), v]_+ \subseteq \|A(y)\|_+ v, [A(y), v]_- \subseteq \|A(y)\|_- v;\]
5) the functional $\| \cdot \|_+ : C_v(Y^*) \to \mathbb{R}_+$ satisfies the following conditions:
   a) $\hat{0} = A(y) \iff \|A(y)\|_+ = 0$,
   b) $\|\alpha A(y)\|_+ = |\alpha|\|A(y)\|_+$,
   c) $\|A(y) + B(y)\|_+ \leq \|A(y)\|_+ + \|B(y)\|_+$;
6) $d_H(A(y), B(y)) \geq \|A(y)\|_+ - \|B(y)\|_+),$ where $d_H(\cdot, \cdot)$ is the Hausdorff metric;
7) $d \in A(y) \iff \forall \omega \in Y$, $[A(y), \omega]_+ \geq \langle d, \omega \rangle_Y$.

Now we consider the main classes of maps of $w_{\alpha}$-pseudomonotone type. Further, $y_n \to y$ in $Y$ will mean that $y_n$ weakly converges to $y$ in a reflexive Banach space $Y$. 

Definition 1. A multi-valued map $A : Y \to C(Y^*)$ is said to be:

- weakly $+(-)$-coercive if for each $f \in Y^*$ there exists $R > 0$ such that
  \[ [A(y) - f, y]_{+(-)} \geq 0 \quad \text{as } \|y\|_Y = R, \ y \in Y; \]

- bounded if for any $L > 0$ there exists $l > 0$ such that
  \[ \|A(y)\|_+ \leq l \quad \forall y \in Y: \|y\|_Y \leq L; \]

- locally bounded if for any fixed $y \in Y$ there exist constants $m > 0$ and $M > 0$ such that
  \[ \|A(\xi)\|_+ \leq M \quad \text{when } \|y - \xi\|_Y \leq m, \ \xi \in Y; \]

- finite-dimensionally locally bounded if for each finite-dimensional subspace $F \subset Y$ the restriction $A|_F$ is locally bounded on $(F, \|\cdot\|_Y)$.

Let $W$ be a normed space with the norm $\|\cdot\|_W$. We consider $W \subset Y$ with continuous embedding.

Definition 2. A multi-valued map $A : Y \to C(Y^*)$ satisfies property $(\kappa)$ if for each bounded set $D$ in $Y$ there exists $c \in \mathbb{R}$ such that

\[ [A(y), v]_{+} \geq -c\|v\|_Y \quad \forall v \in D. \]

Let $Y = Y_1 \cap Y_2$, $W = W_1 \cap W_2$, where $Y_1$, $Y_2$, $W_1$, $W_2$ are real reflexive Banach spaces such that $W_i \subset Y_i$ with the continuous embedding, $i = 1, 2$.

Lemma 1. Let $A : Y_1 \to C(Y_1^*)$ and $B : Y_2 \to C(Y_2^*)$ be $\lambda_0$-pseudomonotone on $W_1$ and respectively on $W_2$ multivalued maps, which satisfy Condition $(I)$. Then $C := A + B : Y \to C(Y^*)$ is $\lambda_0$-pseudomonotone on $W$.

The proof is similar to the proof of Lemma 33 from [36].

Lemma 2. (See [36, Lemma 36].) Let $A : Y_1 \rightrightarrows Y_1^*$, $B : Y_2 \rightrightarrows Y_2^*$ be $+-coercive maps, which satisfy Condition $(\kappa)$. Then the map $C := A + B : Y \rightrightarrows Y^*$ is $+-coercive.$

Remark 1. Under the conditions of the last lemma it follows that the operator $C = A + B : Y \to C(Y^*)$ is weakly $+-coercive.$

Proposition 1. Let $A : X \rightrightarrows X^*$ be a $\lambda_0$-pseudomonotone operator on $W$, the embedding of $W$ into the Banach space $Y$ be compact and dense, the embedding of $X$ into $Y$ be continuous and dense, and let $\overline{C^*}B : Y \rightrightarrows Y^*$ be a locally bounded map such that the graph of $\overline{C^*}B$ is closed in $Y \times Y_w^*$ (i.e. with respect to the strong topology of $Y$ and the weakly star one on $Y^*$). Then $C = A + B$ is a $\lambda_0$-pseudomonotone on $W$ map.

Proof. Let (1) be fulfilled. The operator $\overline{C^*}B$ is locally bounded, i.e., for all $y \in Y$ there exist $N > 0$ and $\varepsilon > 0$ such that

\[ \|\overline{C^*}B(\xi)\|_+ \leq N, \quad \text{if } \|\xi - y\|_Y \leq \varepsilon. \]

Obviously, a locally bounded operator is bounded-valued. Therefore, $\overline{C^*}C(y) = \overline{C^*}A(y) + \overline{C^*}B(y)$ and $d_n = d^+_n + d^-_n$, $d^+_n \in \overline{C^*}A(y_n)$, $d^-_n \in \overline{C^*}B(y_n)$. Since the embedding $W \subset Y$ is compact, we have $y_n \to y$ strongly in $Y$ and by virtue of local boundedness of $\overline{C^*}B$ the sequence $\{d^+_n\}$ is bounded.
in $Y^*$ (and then also in $X^*$), implying that there will be a subsequence $\{d_m''\} \subset \{d_m'''\}$ such that $d_m'' \rightharpoonup d''$ weakly star in $Y^*$. The operator of embedding $I^*: Y^* \to X^*$ is continuous, so that $I^*$ remains continuous also in the weakly star topologies [71]. Hence, $d_m''' \rightharpoonup d''$ weakly star in $X^*$, so $d_m'' = d_m - d_m' \rightharpoonup d' = d - d''$ weakly star in $X^*$. Therefore

$$\langle d_m'', y_m - y \rangle_X \to 0.$$ 

From inequality (1), up to a subsequence $\{y_m\} \subset \{y_n\}$, we find

$$0 \geq \lim_{n \to \infty} \langle d_n, y_n - y \rangle_X$$
$$\geq \lim_{n \to \infty} \langle d_n', y_n - y \rangle_X + \lim_{n \to \infty} \langle d_n', y_n - y \rangle_X$$
$$\geq \lim_{m \to \infty} \langle d_m', y_m - y \rangle_X + \lim_{m \to \infty} \langle d_m', y_m - y \rangle_X.$$ 

Then we obtain $\lim_{m \to \infty} \langle d_m', y_m - v \rangle_X \leq 0$, whence after passing to a subsequence

$$\lim_{m_k \to \infty} \langle d_{m_k}', y_{m_k} - v \rangle_X \geq \left[ \mathcal{CO}^* A(y, y - v) \right]_- \forall v \in X.$$ 

Further, as the operator $\mathcal{CO}^* B$ is closed in $Y \times Y^*_w$, we have $d'' \in \mathcal{CO}^* B(y)$ and

$$\lim_{m_k \to \infty} \langle d_{m_k}', y_{m_k} - v \rangle_X = \lim_{m_k \to \infty} \langle d_{m_k}', y_{m_k} - v \rangle_X + \lim_{m_k \to \infty} \langle d_{m_k}'', y_{m_k} - v \rangle_X$$
$$\geq \left[ \mathcal{CO}^* A(y, y - v) \right]_- + \left[ \mathcal{CO}^* B(y), y - v \right]_-$$
$$= \left[ \mathcal{CO}^* C(y), y - v \right]_- \forall v \in X.$$ 

Proposition 1 is proved. □

Now we consider a functional $\varphi: X \mapsto \mathbb{R}$.

**Definition 3.** A functional $\varphi$ is said to be the locally Lipschitz, if for any $x_0 \in X$ there are $r, c > 0$ such that

$$|\varphi(x) - \varphi(y)| \leq c \|x - y\|_X \quad \forall x, y \in B_r(x_0) = \{x \in X \mid \|x - x_0\|_X < r\}.$$ 

For a locally Lipschitz functional $\varphi$, defined on a Banach space $X$, we consider the upper Clarke’s derivative [13],

$$\varphi^\uparrow_{Cl}(x, h) = \lim_{\alpha \downarrow 0+, \varphi \to x} \frac{1}{\alpha}(\varphi(\alpha h + \varphi) - \varphi(\varphi)) \in \mathbb{R}, \quad x, h \in X$$

and Clarke’s generalized gradient

$$\partial_{Cl} \varphi (x) = \{p \in X^* \mid \langle p, v - x \rangle_X \leq \varphi^\uparrow_{Cl}(x, v - x) \forall v \in X\}, \quad x \in X.$$ 

**Proposition 2.** Let $W$ be a Banach space compactly embedded into some Banach space $Y$, $\varphi: Y \mapsto \mathbb{R}$ be a locally Lipschitz functional. Then Clarke’s generalized gradient $\partial_{Cl} \varphi: Y \rightrightarrows Y^*$ is $\lambda_0$-pseudomonotone on $W$. 

**Proof.** From the calculus of Clarke’s generalized gradient (see [13, Chap. 2]) we know that \( \partial_{\text{Cl}} \varphi(x) \) is nonempty, closed, bounded and convex. Hence, for each \( x \in Y \), \( \partial_{\text{Cl}} \varphi(x) \in C_Y(Y^*) \).

Now let \( \{y_n\}_{n \geq 0} \subset W \) be a sequence such that \( y_n \to y_0 \) in \( W \), \( d_n \to d_0 \) in \( Y^* \), where \( d_n \in \partial_{\text{Cl}} \varphi(y_n) \) \( \forall n \geq 1 \) and inequality (1) is true. Due to the compact embedding \( W \subset Y \) we conclude that \( y_n \to y_0 \) in \( Y \). Since the mapping \( \partial_{\text{Cl}} \varphi : Y \to Y^* \) is weak-closed (see [13, p. 29]), we have \( d_0 \in \partial_{\text{Cl}} \varphi(y_0) \). Thus,

\[
\lim_{n \to \infty} (d_n, y_n - \omega)_Y \geq \lim_{n \to \infty} (d_n, y_n - y_0)_Y + \lim_{n \to \infty} (d_n, y_0 - \omega)_Y = 0 + (d_0, y_0 - \omega)_Y 
\geq \left[ \partial_{\text{Cl}} \varphi(y_0), y_0 - \omega \right] \quad \forall \omega \in Y,
\]

which completes the proof. □

**Definition 4.** An operator \( L : D(L) \subset Y \to Y^* \) is said to be

- monotone, if for each \( y_1, y_2 \in D(L) \) \( \langle Ly_1 - Ly_2, y_1 - y_2 \rangle_Y \geq 0 \);
- maximal monotone, if it is monotone and from \( \langle w - Lu, v - u \rangle_Y \geq 0 \) for each \( u \in D(L) \) it follows that \( v \in D(L) \) and \( Lv = w \).

**Remark 2.** If \( Y \) is strictly convex with its conjugate then [53, Lemma 3.1.1] the linear operator \( L : D(L) \subset Y \to Y^* \) is maximal monotone and densely defined if and only if \( L \) is a closed non-bounded operator such that

\[
\langle Ly, y \rangle_Y \geq 0 \quad \forall y \in D(L) \quad \text{and} \quad \{L^* y, y\}_Y \geq 0 \quad \forall y \in D(L^*),
\]

where \( L^* : D(L^*) \subset Y \to Y^* \) is the conjugate operator of \( L \) in the sense of unbounded operators theory (see [26]).

### 4. Auxiliary statements

Notice that \( V = V_1 \cap V_2 \subset H \) with continuous and dense embedding. Since \( V \) is a separable Banach space, then there exists a complete countable system of vectors \( \{h_i\}_{i \geq 1} \subset V \) (and it is complete in \( H \) consequently).

Let for each \( n \geq 1 \) \( H_n = \text{span} \{h_i\}_{i=1}^n \), on which we consider the inner product induced from \( H \) that we denote by \( \langle \cdot, \cdot \rangle \) again; \( P_n : H \to H_n \subset H \) be the operator of orthogonal projection from \( H \) on \( H_n \), i.e.,

\[
\forall h \in H, \quad P_n h = \arg \min_{h_n \in H_n} \|h - h_n\|_H.
\]

**Definition 5.** We say that the triple \( (\{h_i\}_{i \geq 1}; V; H) \) satisfies Condition \((\gamma)\) if \( \sup_{n \geq 1} \|P_n\|_{L(V, V)} < +\infty \), i.e. there exists \( C > 1 \) such that

\[
\forall v \in V, \forall n \geq 1 \quad \|P_n v\|_V \leq C \cdot \|v\|_V.
\]

Let us note that a construction of basis which satisfies (or not) the above condition was introduced in papers [18, 40, 41] and in book [77].

**Remark 3.** When the system of vectors \( \{h_i\}_{i \geq 1} \subset V \) is orthogonal in \( H \), Condition \((\gamma)\) means that the given system is a Schauder basis in the Banach space \( V \) [77].
**Remark 4.** Since $P_n \in \mathcal{L}(V, V)$, its conjugate operator $P_n^* \in \mathcal{L}(V^*, V^*)$ and $\|P_n\|_{\mathcal{L}(V, V)} = \|P_n^*\|_{\mathcal{L}(V^*, V^*)}$. It is clear that, for each $h \in H$, $P_nh = P_n^*h$. Hence, we identify $P_n$ with $P_n^*$. Then Condition (γ) means that for each $v \in V$ and $n \geq 1$, $\|P_nv\|_{V^*} \leq C \cdot \|v\|_{V^*}$.

From the equivalence of $H^*$ and $H$ it follows that $H^*_n = H_n$. For each $n \geq 1$ we consider the Banach space $X_n = L_{p_0}(S; H_n) \subset X$, where $p_0 := \max\{r_1, r_2\}$, with the norm $\|\cdot\|_{X_n}$ induced by the space $X$. This norm is equivalent to the natural norm in $L_{p_0}(S; H_n)$ [22].

The space $L_{q_0}(S; H_n) (q_0^{-1} + p_0^{-1} = 1)$ with the norm

$$\|f\|_{X_n^*} := \sup_{x \in X_n \setminus \{0\}} \frac{|\langle f, x \rangle|}{\|x\|_X} = \sup_{x \in X_n \setminus \{0\}} \frac{|\langle f, x_n \rangle|}{\|x\|_{X_n}}$$

is isometrically isomorphic to the conjugate space $X_n^*$ of $X_n$ (further the given spaces will be identified); moreover, the map

$$X_n^* \times X_n \ni f, x \rightarrow \int_S \left(\int (f(\tau), x(\tau)) h_n d\tau\right) = \int_S \left(\int (f(\tau), x(\tau)) \right) d\tau = \langle f, x \rangle_{X_n}$$

is the duality form on $X_n^* \times X_n$. Let us note that $\langle \cdot, \cdot \rangle_{X_n^* \times X_n} = \langle \cdot, \cdot \rangle_{X_n}$.

**Proposition 3.** (See [40, Proposition 1].) For each $n \geq 1$, $X_n = P_n X$, i.e., $X_n = \{P_n y(\cdot) \mid y(\cdot) \in X\}$, and we have

$$\langle f, P_n y \rangle = \langle f, y \rangle \quad \forall y \in X \text{ and } f \in X_n^*.$$ 

Moreover, if the triples $\langle V_i; H \rangle$, $i = 1, 2$ satisfy Condition (γ) with $C = C_i$, then

$$\|P_n y\|_X \leq \max\{C_1, C_2\} \cdot \|y\|_X \quad \forall y \in X \text{ and } n \geq 1.$$ 

For each $n \geq 1$ we denote the canonical embedding of $X_n$ into $X$ by $I_n$ ($\forall x \in X_n$, $I_n x = x$), $I_n^* : X^* \to X_n^*$ its conjugate operator. We point out that

$$\|I_n\|_{\mathcal{L}(X_n, \|\cdot\|_X)} = \|I_n^*\|_{\mathcal{L}(X_n^*, \|\cdot\|_{X_n^*})} = 1.$$ 

**Proposition 4.** (See [40, Proposition 2].) For each $n \geq 1$ and $f \in X^*$, $(I_n^* f)(t) = P_n f(t)$ for a.e. $t \in S$. Moreover, if the triples $\langle V_i; H \rangle$ for $i = 1, 2$ satisfy Condition (γ) with $C = C_i$, then

$$\|I_n^*f\|_{X_n^*} \leq \max\{C_1, C_2\} \cdot \|f\|_{X^*} \quad \text{for each } f \in X^* \text{ and } n \geq 1,$$

i.e.

$$\sup_{n \geq 1} \|I_n^*\|_{\mathcal{L}(X_n^*, X^*)} \leq \max\{C_1, C_2\}.$$ 

From the last two propositions and the properties of $I_n^*$ it immediately follows the next

**Corollary 1.** For each $n \geq 1$, $X_n^* = P_n X^* = I_n^* X$, i.e.

$$X_n^* = \{P_n f(\cdot) \mid f(\cdot) \in X^*\} = \{I_n^* f \mid f \in X^*\}.$$
Proposition 5. (See [40, Proposition 3].) The set \( \bigcup_{n \geq 1} X_n \) is dense in \((X, \| \cdot \|_X)\).

For some linear densely defined operator \( L : D(L) \subset Y \to Y^* \) we consider the normed space \( D(L) \) with the graph norm

\[
\|y\|_{D(L)} = \|y\|_Y + \|Ly\|_{Y^*} \quad \forall y \in D(L).
\]

Proposition 6. Let \( Y \) be a reflexive Banach space, \( L : D(L) \subset Y \to Y^* \) be a linear maximal monotone operator. Then every bounded sequence of the space \( D(L) \) with the graph norm (5) has a weakly convergent subsequence.

5. The Faedo–Galerkin method

For each \( n \geq 1 \), let us set

\[
L_n := I_n^* L_n : D(L_n) = D(L) \cap X_n \subset X_n \to X_n^*, \quad f_n := I_n^* f \in X_n^*,
\]

\[
A_n := I_n^* A I_n : X_n \to C_\nu(X_n^*), \quad B_n := I_n^* B I_n : X_n \to C_\nu(X_n^*).
\]

Remark 5. We will denote the conjugate operators of the canonical embeddings of \( X_n \) in \( X_1 \) and of \( X_n \) in \( X_2 \) by \( I_n^* \) also, because these operators coincide with \( I_n^* \) on \( X_1^* \cap X_2^* \) which is dense in \( X_1^*, X_2^*, X^* \).

By the analogy with Proposition 6, we consider the normed space \( D(L) \) with the graph norm (5). We note that if the linear operator \( L \) is closed and densely defined then \( (D(L), \| \cdot \|_{D(L)}) \) is a Banach space continuously embedded in \( X \).

In addition to problem (4), we consider the following class of problems:

\[
\left\{ \begin{array}{l}
L_n y_n + A_n(y_n) + B_n(y_n) \ni f_n, \\
y_n \in D(L_n).
\end{array} \right.
\]

Remark 6. We consider on \( D(L_n) \) the graph norm

\[
\|y_n\|_{D(L_n)} = \|y_n\|_{X_n} + \|L_n y_n\|_{X_n^*} \quad \text{for each} \ y_n \in D(L_n).
\]

Definition 6. We say that the solution of (4) \( y \in D(L) \) is obtained by the FG method, if \( y \) is the weak limit of a subsequence \( \{y_n\}_{n \geq 1} \subset \{y_n\}_{n \geq 1} \) in \( D(L) \), where for each \( n \geq 1 \) \( y_n \) is a solution of problem (6).

6. The main resolvability theorem

Theorem 1. Let \( L : D(L) \subset X \to X^* \) be a linear operator, \( A : X_1 \to C_\nu(X_1^*) \) and \( B : X_2 \to C_\nu(X_2^*) \) be multi-valued maps such that

1. \( L \) is maximal monotone on \( D(L) \) and satisfies
   - Condition \( L_1 \): for each \( n \geq 1 \) and \( x_n \in D(L_n) \) \( L x_n \in X_n^* \);
   - Condition \( L_2 \): for each \( n \geq 1 \) the set \( D(L_n) \) is dense in \( X_n \);
   - Condition \( L_3 \): for each \( n \geq 1 \) \( L_n \) is maximal monotone on \( D(L_n) \);
2. there exist Banach spaces \( W_1 \) and \( W_2 \) such that \( W_1 \subset X_1 \), \( W_2 \subset X_2 \) and \( D(L) \subset W_1 \cap W_2 \) with continuous embedding;
3. \( A \) is \( \lambda_0 \)-pseudomonotone on \( W_1 \) and satisfies Condition (I);
4. \( B \) is \( \lambda_0 \)-pseudomonotone on \( W_2 \) and satisfies Condition (I);
5. the sum \( C = A + B : X \rightrightarrows X^* \) is finite-dimensionally locally bounded and weakly \( + \)-coercive.
Furthermore, let $\{h_j\}_{j \geq 1} \subset V$ be a complete system of vectors in $V_1, V_2, H$ such that $\forall i = 1, 2$ the triple $(\{h_j\}_{j \geq 1}; V_i; H)$ satisfies Condition $(\gamma)$.

Then for each $f \in X^*$ the set

$$KH(f) := \{ y \in D(L) \mid y \text{ is the solution of } (4), \text{ obtained by the FG method} \}$$

is nonempty and the representation

$$KH(f) = \bigcap_{n \geq 1} \left[ \bigcup_{m \geq n} K_m(f_m) \right]_{X^w}$$

is true, where for each $n \geq 1$,

$$K_n(f_n) = \{ y_n \in D(L_n) \mid y_n \text{ is the solution of } (6) \}$$

and $[\cdot]_{X^w}$ is the closure operator in the space $X$ with respect to the weak topology.

Moreover, if the operator $A + B : X \Rightarrow X^*$ is $--$-coercive, then $KH(f)$ is weakly compact in $X$ and in $D(L)$ with respect to the graph norm $(5)$.

**Remark 7.** The sufficient condition to obtain the weak $+$-coercivity of $A + B$ is the following one: $A$ is $+$-coercive and it satisfies Condition $(\kappa)$ on $X_1$, $B$ is $+$-coercive and it satisfies Condition $(\kappa)$ on $X_2$ (see Lemma 2).

**Remark 8.** From Condition $L_2$ on the operator $L$ and from Proposition 5 it follows that $L$ is densely defined.

**Proof.** By Lemma 1 we consider the $\lambda_0$-pseudomonotone on $W_1 \cap W_2$ (and hence on $D(L)$), finite-dimensionally locally bounded, weakly $+$-coercive map

$$X \ni y \to C(y) := A(y) + B(y) \in C_v(X^*),$$

which satisfies Condition $(II)$. Let $f \in X^*$ be fixed. Now let us use the weak $+$-coercivity condition for $C$. There exists $R > 0$ such that

$$[C(y) - f, y]_+ \geq 0 \quad \forall y \in X: \|y\|_X = R. \quad (8)$$

6.1. Resolvability of approximating problems

**Lemma 3.** For each $n \geq 1$ there exists a solution of problem $(6)$ $y_n \in D(L_n)$ such that $\|y_n\|_X \leq R$.

**Proof.** Let us prove that for each $n \geq 1$,

$$C_n := A_n + B_n = I_n^*(A + B) : X_n \rightarrow C_v(X_n^*)$$

has the following property

$(i_1)$ $C_n$ satisfies Condition $(II)$;

$(i_2)$ $C_n$ is $\lambda_0$-pseudomonotone on $D(L_n)$, locally finite-dimensionally bounded;

$(i_3)$ $[C_n(y_n) - f_n, y_n]_+ \geq 0 \quad \forall y_n \in X_n: \|y_n\|_{X_n} = R.$
Let us consider \((i_1)\). Let \(B \subset X_n\) be some nonempty bounded subset, \(k > 0\) be a constant and \(d_n = I^*_n d \in C_n\) (where \(d \in C\) is a selector) is such that
\[
\langle d_n(y), y \rangle \leq k \quad \text{for each } y \in B.
\]
Since for each \(y \in X_n\) \(\langle d_n(y), y \rangle = \langle I^*_n d(y), y \rangle = \langle d(y), y \rangle\) then
\[
\langle d(y), y \rangle \leq k \quad \text{for each } y \in B.
\]
Since \(C\) satisfies Condition \((II)\) there exists \(K > 0\) such that
\[
\|d(y)\|_{X^*} \leq K \quad \text{for all } y \in B.
\]
Consequently,
\[
\sup_{y \in B} \|d_n(y)\|_{X^*} \leq K \|I^*_n\|_{\mathcal{L}(X^*; X^*_n)} < +\infty.
\]
Now we consider \((i_2)\). From the boundedness of \(I_n \in \mathcal{L}(X_n; X)\), \(I^*_n \in \mathcal{L}(X^*; X^*_n)\) and the locally finite-dimensional boundedness of \(C : X \to C_y(X^*)\) it follows the locally finite-dimensional boundedness of \(C_n\) on \(X_n\).

Now we prove the \(\lambda_0\)-pseudomonotonicity of \(C_n\) on \(D(L_n)\). Let \(\{y_m\}_{m \geq 1} \subset D(L_n)\) be an arbitrary sequence such that
\[
y_m \rightharpoonup y_0 \quad \text{in } D(L_n), \quad d_n(y_m) = I^*_n d(y_m) \in C_n(y_m) \rightharpoonup d \in X^*_n \quad \text{as } m \to +\infty,
\]
where \(d(y_m) \in C(y_m)\) is a selector, and the inequality (1) holds true. Since \(D(L_n) \subset D(L)\) with continuous embedding then
\[
y_m \to y_0 \quad \text{in } D(L) \text{ as } m \to +\infty. \tag{9}
\]
Since \(\forall m \geq 1,\)
\[
\langle I^*_n d(y_m), y_m - y_0 \rangle = \langle d(y_m), y_m - y_0 \rangle
\]
then
\[
\lim_{m \to \infty} \langle d(y_m), y_m - y_0 \rangle = \lim_{m \to \infty} \langle d_n(y_m), y_m - y_0 \rangle \leq 0. \tag{10}
\]
Hence
\[
\lim_{m \to \infty} \langle d(y_m), y_m \rangle = \lim_{m \to \infty} \langle d_n(y_m), y_m - y_0 \rangle + \lim_{m \to \infty} \langle d_n(y_m), y_0 \rangle \leq \langle d, y_0 \rangle < +\infty.
\]
Since \(C\) satisfies Condition \((II)\) we have that the sequence \(\{d(y_m)\}_{m \geq 1}\) is bounded in \(X^*\). Hence, without loss of generality we may assume that
\[
d(y_m) \rightharpoonup g \quad \text{in } X^* \text{ as } m \to \infty.
for some \( g \in X^* \). Consequently from (9) and (10), we obtain the existence of the subsequence \( \{ y_{m_k} \}_{k \geq 1} \subset \{ y_m \}_{m \geq 1} \) such that \( \forall w \in X, \)
\[
\lim_{k \to \infty} \langle d( y_{m_k} ), y_{m_k} - w \rangle \geq [ C(y_0), y_0 - w ]_.
\]
This means that for each \( w \in X_n \),
\[
\lim_{k \to \infty} \langle d_n( y_{m_k} ), y_{m_k} - w \rangle \geq [ C_n(y_0), y_0 - w ]_.
\]
So, \( C_n \) is \( \lambda_0 \)-pseudomonotone on \( D(L_n) \).

Condition \( (i_3) \) holds true by virtue of (8).

Now let us continue the proof of the given lemma. From [42, Theorem 3.1] with \( X = X_n, A = C_n, B \equiv 0, L = L_n, D(L) = D(L_n), f = f_n, R = R \) and with properties \( (i_1)-(i_3) \) for \( C_n, L_2 - L_3 \) for \( L_n \), it follows that the problem (6) has at least one solution \( y_n \in D(L_n) \) such that \( \| y_n \|_X \leq R \). We remark that the proof of Theorem 3.1 from [42] is based on one multivalued analogue of the “acute angle lemma” (see for example [57] and citations there).

Lemma 3 is proved. \( \square \)

6.2. Passing to the limit

Due to Lemma 3 we have a sequence of Galerkin approximate solutions \( \{ y_n \}_{n \geq 1} \) that satisfies the next conditions
\[
\forall n \geq 1, \quad \| y_n \|_X \leq R; \tag{11}
\]
\[
\forall n \geq 1, \quad y_n \in D(L_n) \subset D(L), \quad L_n y_n + d_n(y_n) = f_n, \tag{12}
\]
where \( d_n(y_n) = I_n^* d(y_n), d(y_n) \in C(y_n) \) is a selector.

In order to prove the given theorem we need to prove the next lemma.

Lemma 4. Let for some subsequence \( \{ n_k \}_{k \geq 1} \subset \mathbb{N} \) the sequence \( \{ y_{n_k} \}_{k \geq 1} \) satisfies the next conditions:

- \( \forall k \geq 1, y_{n_k} \in D(L_{n_k}) = D(L) \cap X_{n_k}; \)
- \( \forall k \geq 1, L_{n_k} y_{n_k} + d_{n_k}(y_{n_k}) = f_{n_k}, d_{n_k}(y_{n_k}) = I_{n_k}^* d(y_{n_k}), d(y_{n_k}) \in C(y_{n_k}); \)
- \( y_{n_k} \rightharpoonup y \) in \( X \) as \( k \to +\infty \) for some \( y \in X. \)

Then, \( y \in K_H(f). \)

Proof. From the definitions of \( L_{n_k}, d_{n_k} \) and \( f_{n_k} \) for each \( k \geq 1, \)
\[
\langle d_{n_k}(y_{n_k}), y_{n_k} \rangle = \langle f_{n_k} - L_{n_k} y_{n_k}, y_{n_k} \rangle = \langle f - L y_{n_k}, y_{n_k} \rangle \leq \| f \|_{X^*} \sup_{k \geq 1} \| y_{n_k} \|_X =: K_1 < +\infty,
\]
where \( K_1 \) is a constant which does not depend on \( k \geq 1 \). Hence, due to Property \( (II) \) for operator \( C \) it follows that there exists \( K_2 > 0 \) such that for each \( k \geq 1, \)
\[
\| d(y_{n_k}) \|_{X^*} \leq K_2 < +\infty. \tag{13}
\]
From Condition \( L_1 \) for \( L \) and Proposition 4 it follows that for all \( k \geq 1, \)
\[ \|\text{Ly}_n\|_{X^*} = \|I_n^*(f - d(y_n))\|_{X^*} \leq \max\{C_1, C_2\}(\|f\|_{X^*} + K_2) =: K_3 < +\infty, \] (14)

where \( K_3 \) is a constant which does not depend on \( k \geq 1 \). Hence, for each \( k \geq 1 \),
\[ \|y_n\|_{D(L)} = \|y_n\|_X + \|\text{Ly}_n\|_{X^*} \leq \sup_{k \geq 1} \|y_n\|_X + K_3 =: K_4 < +\infty, \]
where \( K_4 \) is a constant which does not depend on \( k \geq 1 \). Consequently, due to (13), Proposition 6 and the Banach–Alaoglu theorem there exists a subsequence \( \{y_m\} \) from \( \{y_n\} \) such that for some \( y \in D(L) \) and \( d \in X^* \) the next convergence takes place:
\[ y_m \rightharpoonup y \quad \text{in} \quad D(L), \quad d(y_m) \rightharpoonup d \quad \text{in} \quad X^*. \] (15)

a) Let us prove that
\[ \lim_{m \to \infty} \langle \text{Ly}_m + d(y_m), y_m - y \rangle = 0. \] (16)

Since the set \( \bigcup_{n \geq 1} X_n \) is dense in \( X \) then for each \( m \) there exists \( u_m \in X_m \) (for example \( u_m \in \arg\min_{v_m \in X_m} \|y - v_m\|_X \)) such that \( u_m \to y \) in \( X \). So, due to (14), (13) we obtain that for each \( m \),
\[ \|\text{Ly}_m + d(y_m), y_m - y\| \leq \|\text{Ly}_m + d(y_m), y_m - u_m\| + \|\text{Ly}_m + d(y_m), u_m - y\| \leq \|f, y_m - u_m\| + (K_3 + K_2) \cdot \|y - u_m\|_X \to \|f, y - y\| = 0. \]

b) Now we obtain that
\[ \overline{\lim}_{m \to \infty} \langle d(y_m), y_m - y \rangle \leq 0. \] (17)

From (16), (15) and from the monotonicity of \( L \) we have
\[ \overline{\lim}_{m \to \infty} \langle d(y_m), y_m - y \rangle = \lim_{m \to \infty} \langle \text{Ly}_m + d(y_m), y_m - y \rangle - \lim_{m \to \infty} \left( \langle \text{Ly}_m - Ly, y_m - y \rangle + \langle Ly, y_m - y \rangle \right) \leq 0 + \overline{\lim}_{m \to \infty} \left( -\langle \text{Ly}_m - Ly, y_m - y \rangle \right) + \overline{\lim}_{m \to \infty} \langle Ly, y - y_m \rangle \leq 0. \]

From (15) and (17) we can use the \( \lambda_0 \)-pseudomonotonicity of \( C \) on \( D(L) \). Hence, there exists a subsequence \( \{y_k\}_k \) of \( \{y_m\}_m \) such that
\[ \forall \omega \in X, \quad \lim_{k \to \infty} \langle d(y_k), y_k - \omega \rangle \geq [C(y), y - \omega]. \] (18)

In particular, from (17) and (18) it follows that
\[ \lim_{k \to \infty} \langle d(y_k), y_k - y \rangle = 0. \]
c) Let us prove that
\[ \forall u \in D(L) \cap \left( \bigcup_{n \geq 1} X_n \right), \quad (f - d - Ly + Lu, u) \geq 0. \tag{19} \]

In order to prove (19) it is necessary to obtain that
\[ \forall u \in D(L) \cap \left( \bigcup_{n \geq 1} X_n \right), \quad \lim_{k \to \infty} \langle Ly_k - Ly + Lu, u \rangle \geq 0. \tag{20} \]

From the monotonicity of \( L \) and from (15), for each \( u \in D(L) \cap \left( \bigcup_{n \geq 1} X_n \right) \) we have
\[ \lim_{k \to \infty} \langle Ly_k - Ly + Lu, u \rangle = \lim_{k \to \infty} \langle Ly_k - Ly, u \rangle = 0. \tag{21} \]

So, (19) directly follows from (20) and (21).

d) Now we prove that \( Ly = f - d \). Let us use (19). We obtain that for each \( t > 0 \) and \( u \in D(L) \cap \left( \bigcup_{n \geq 1} X_n \right) \),
\[ \langle f - d - Ly, t \cdot u \rangle \geq - \langle t \cdot Lu, t \cdot u \rangle \]
that is equivalent to
\[ \langle f - d - Ly, u \rangle \geq - t \cdot \langle Lu, u \rangle. \]

Hence,
\[ \forall u \in D(L) \cap \left( \bigcup_{n \geq 1} X_n \right), \quad (f - d - Ly, u) \geq 0 \]

and, due to Proposition 5, the last relation is equivalent to \( Ly = f - d \).

e) In order to prove that \( y \in D(L) \) is the solution of (4) it is sufficiently to show that \( d \in C(y) \). From (17), (18) and (15) it follows that for each \( \omega \in X \),
\[ \left[ C(y), y - \omega \right] \leq \lim_{k \to \infty} \langle d(y_k), y_k - \omega \rangle \leq \lim_{k \to \infty} \langle d(y_k), y_k - y \rangle + \lim_{k \to \infty} \langle d(y_k), y - \omega \rangle \leq \langle d, y - \omega \rangle, \]
that is equivalent to the desired statement. So, \( y \in K_H(f) \).

Lemma 4 is proved. \( \square \)
From (11), (12), Lemma 4, the Banach–Alaoglu theorem and the topological property of the upper limit [51, Property 2.29.IV.8] it follows that
\[ \emptyset \neq \bigcap_{n \geq 1} \left[ \bigcup_{m \geq n} K_m(f_m) \right]_{\text{w}} \subset K_H(f). \]

The converse inclusion is obvious; it follows from the same topological property of upper limit and from \( D(L) \subset X \) with continuous embedding.

Now let us prove that \( K_H(f) \) is weakly compact in \( X \) and in \( D(L) \) under the \(-\)coercivity condition of the operator \( C = A + B : X \to C_v(X^*) \). From (7) and \( D(L) \subset X \) with continuous embedding it follows that it is sufficient to show that the given set is bounded in \( D(L) \). Let \( \{y_n\}_{n \geq 1} \subset K_H(f) \) be an arbitrary sequence. Then for some \( d_n \in C(y_n) \),
\[ Ly_n + d(y_n) = f. \]

If \( \{y_n\}_{n \geq 1} \) is such that
\[ \|y_n\|_X \to +\infty \quad \text{as } n \to \infty, \]
we obtain the contradiction
\[
+\infty \leftarrow \frac{1}{\|y_n\|_X} \left[ C(y_n), y_n \right]_1 \leq \frac{1}{\|y_n\|_X} \left( d(y_n), y_n \right)
\leq \frac{1}{\|y_n\|_X} \left( Ly_n + d(y_n), y_n \right)
= \frac{1}{\|y_n\|_X} (f, y_n) \leq \|f\|_X^* < +\infty.
\]

Hence, for some \( k > 0 \),
\[ \|y_n\|_X \leq k \quad \forall n \geq 1. \] (23)

Due to Condition (P) for \( C \), from (22)–(23) it follows there exists \( K > 0 \) such that \( \|d_n\|_{X^*} \leq K \). Hence, \( \|Ly_n\|_{X^*} \leq K + \|f\|_{X^*} \) and \( \|y_n\|_{D(L)} \leq k + K + \|f\|_{X^*} \).

Theorem 1 is proved. \( \square \)

7. An application

7.1. On the solvability for one Cauchy problem by the Faedo–Galerkin method

Let \( A : X_1 \to C_v(X_1^*) \) and \( B : X_2 \to C_v(X_2^*) \) be multi-valued maps. We consider the problem:
\[
\begin{cases}
  y' + A(y) + B(y) \ni f, \\
  y(0) = 0
\end{cases}
\]
(24)
in order to find the solutions by the FG method in the class
\[ W = \{ y \in X \mid y' \in X^* \}. \]
where the derivative $y'$ of an element $y \in X$ is considered in the sense $D^*(S; V^*)$ (see [22]). We consider the norm on $W$ given as
\[
\|y\|_W = \|y\|_X + \|y'\|_{X^*} \quad \text{for each } y \in W.
\]
We also consider spaces $W_i = \{y \in X_i \mid y' \in X_i^*\}$, $i = 1, 2$.

**Remark 9.** The space $W$ is continuously embedded in $C(S; H)$ (see [37]). Hence, the initial condition in (24) is well defined.

Along with the problem (24) we consider the next class of problems in order to search solutions in $W_n = \{y \in X_n \mid y' \in X_n^*\}$:
\[
\begin{cases}
  y'_n + A_n(y_n) + B_n(y_n) \ni f_n, \\
  y_n(0) = \bar{0},
\end{cases}
\]  
(25)
where maps $A_n$, $B_n$, $f_n$ were introduced in Section 5, the derivative $y'_n$ of an element $y_n \in X_n$ is considered in the sense of $D^*(S; H_n)$.

Let $W_\bar{0} := \{y \in W \mid y(0) = \bar{0}\}$, let us introduce the map
\[
L : D(L) = W_\bar{0} \subset X \rightarrow X^*
\]
as $Ly = y'$ for each $y \in W_\bar{0}$.

From the main resolvability theorem the next assertion follows:

**Corollary 2.** Let $A : X_1 \rightarrow C_v(X_1^*)$ and $B : X_2 \rightarrow C_v(X_2^*)$ be multi-valued maps such that

1. $A$ is $\lambda_0$-pseudomonotone on $W_1$ and it satisfies Condition (I);
2. $B$ is $\lambda_0$-pseudomonotone on $W_2$ and it satisfies Condition (II);
3. the sum $C = A + B : X \rightrightarrows X^*$ is finite-dimensionally locally bounded and weakly $++$-coercive.

Furthermore, let $\{h_j\}_{j \geq 1} \subset V$ be a complete system of vectors in $V_1$, $V_2$, $H$ such that the triple $\{(h_j)_{j \geq 1}; V_i; H\}$ for $i = 1, 2$ satisfies Condition (γ).

Then for each $f \in X^*$ the set
\[
K_{H}(f) := \{y \in W \mid y \text{ is the solution of (24), obtained by the FG method}\}
\]
is nonempty and the representation
\[
K_{H}(f) = \bigcap_{n \geq 1} \left[ \bigcup_{m \geq n} K_{H}^{0}(f_m) \right]_{X^w}
\]
is true, where for each $n \geq 1$,
\[
K_{H}^{0}(f_n) = \{y_n \in W_n \mid y_n \text{ is the solution of (25)}\}.
\]

Moreover, if the operator $A + B : X \rightrightarrows X^*$ is $--$-coercive, then $K_{H}(f)$ is weakly compact in $X$ and in $W$.

**Proof.** At first let us prove the maximal monotonicity of $L$ on $W_\bar{0}$. For $v \in X$, $w \in X^*$ such that for each $u \in W_\bar{0}$ ($w - Lu, v - u \geq 0$ is true, let us prove that $v \in W_\bar{0}$ and $v' = w$. If we take $u = h\varphi x \in W_\bar{0}$ with $\varphi \in D(S)$, $x \in V$ and $h > 0$ we obtain
0 \leq \langle w - \varphi'hx, v - \varphi hx \rangle \\
= \langle w, v \rangle - \left( \int_S \left( \varphi'(s)v(s) + \varphi(s)w(s) \right) ds, hx \right) + \langle \varphi'hx, \varphi hx \rangle \\
= \langle w, v \rangle + h\langle v'((\varphi) - w(\varphi), x) \rangle,

where \( v'(\varphi) \) and \( w(\varphi) \) are the values of the distributions \( v' \) and \( w \) on \( \varphi \in \mathcal{D}(S) \). So, for each \( \varphi \in \mathcal{D}(S) \) and \( x \in V \), \( \langle v'(\varphi) - w(\varphi), x \rangle \geq 0 \) is true. Thus we obtain \( v'(\varphi) = w(\varphi) \) for all \( \varphi \in \mathcal{D}(S) \). It means that \( v' = w \in X^* \).

Now we prove \( v(0) = \bar{0} \). If we use [22, Theorem IV.1.17] with \( u(t) = v(T) \frac{t}{T} \in W_0 \), we obtain that

\[
0 \leq \langle v' - Lu, v - u \rangle = \langle v' - u', v - u \rangle \leq \frac{1}{2} \left( \| v(T) - v(T) \|_H^2 - \| v(0) \|_H^2 \right) \\
= -\frac{1}{2} \| v(0) \|_H^2 \leq 0
\]

and then \( v(0) = \bar{0} \).

In order to prove the given statement, it is sufficient to show that \( L \) satisfies Conditions \( L_1-L_3 \). Condition \( L_1 \) follows from the

**Proposition 7.** (See [40, Proposition 6.]) For each \( y \in X, n \geq 1 \), \( P_n y' = (P_n y)' \), where the derivative of an element \( x \in X \) is considered in the sense of \( \mathcal{D}^*(S; V^*) \).

Condition \( L_2 \) follows from [22, Lemma VI.1.5] and from the fact that the set \( C^1(S; H_n) \) is dense in \( L_{p_0}(S, H_n) = X_n \). Condition \( L_3 \) follows from the previous reasonings with \( V = H = H_n \) and \( X = X_n \). Corollary 2 is proved. \( \square \)

**Remark 10.** In Corollary 2 we may retract from Condition \( (\gamma) \) in such way:

Following by [53], we may assume that there is a separable Hilbert space \( V_{\sigma} \) such that \( V_{\sigma} \subset V_1, V_{\sigma} \subset V_2 \) with continuous and dense embedding, \( V_{\sigma} \subset H \) with compact and dense embedding. Then

\[
V_{\sigma} \subset V_1 \subset H \subset V_1^* \subset V_{\sigma}^*, \quad V_{\sigma} \subset V_2 \subset H \subset V_2^* \subset V_{\sigma}^*
\]

with continuous and dense embedding. For \( i = 1, 2 \) let us set

\[
X_{i,\sigma} = L_{r_i}(S; H) \cap L_{p_i}(S; V_{\sigma}), \quad X_{\sigma} = X_{1,\sigma} \cap X_{2,\sigma}, \\
X_{i,\sigma}^* = L_{r_i}^*(S; H) + L_{q_i}(S; V_{\sigma}^*), \quad X_{\sigma}^* = X_{1,\sigma}^* + X_{2,\sigma}^*, \\
W_{i,\sigma} = \{ y \in X_i \mid y' \in X_{i,\sigma}^* \}, \quad W_{\sigma} = W_{1,\sigma} \cap W_{2,\sigma}.
\]

Let us take a special basis [74] as complete system of vectors \( \{ h_1 \}_{1 \geq 1} \subset V_{\sigma} \), i.e.

\[
(i) \{ h_i \}_{i \geq 1} \text{ orthonormal in } H; \\
(ii) \{ h_i \}_{i \geq 1} \text{ orthogonal in } V_{\sigma}; \\
(iii) \forall j \geq 1, (h_j, v)_{V_{\sigma}} = \lambda_j(h_j, v) \forall v \in V_{\sigma},
\]

where \( 0 \leq \lambda_1 \leq \lambda_2, \ldots, \lambda_j \rightarrow \infty \) as \( j \rightarrow \infty \), \( (\cdot, \cdot)_{V_{\sigma}} \) is the natural inner product in \( V_{\sigma} \).
Then

$$\sup_{n \geq 1} \| I_n^* \|_{L(X^n_\sigma; X^*_\sigma)} = 1. \quad (26)$$

In order to use this construction we need to consider some more strong condition for $A$, $B$:

- $A$ is $\lambda_0$-pseudomonotone on $W_{1,\sigma}$;
- $B$ is $\lambda_0$-pseudomonotone on $W_{2,\sigma}$.

So, in the proof of Theorem 1 we need to modify only “Passing to the limit”.

Due to Lemma 3 we have a sequence of Galerkin approximate solutions $\{y_n\}_{n \geq 1}$, that satisfies the following conditions:

a) $\forall n \geq 1$: $\| y_n \|_X \leq R$; \hfill (27)
b) $\forall n \geq 1$: $y_n \in W_n \subseteq W$, $y'_n + C_n(y_n) \ni f_n$; \hfill (28)
c) $\forall n \geq 1$: $y_n(0) = \bar{0}$. \hfill (29)

From inclusion (28) we have that

$$\forall n \geq 1 \exists d_n \in C(y_n): \quad I_n^* d_n = : d^1_n = f_n - y'_n \in C_n(y_n) = I_n^* C(y_n). \quad (30)$$

**Lemma 5.** From sequences $\{y_n\}_{n \geq 1}$, $\{d_n\}_{n \geq 1}$, which satisfy (27)-(30), we can choose some subsequences $\{y_{nk}\}_{k \geq 1} \subseteq \{y_n\}_{n \geq 1}$ and $\{d_{nk}\}_{k \geq 1} \subseteq \{d_n\}_{n \geq 1}$ such that for some $y \in W_0$, $d \in X^*$, $z \in H$ the following types of convergence take place:

1) $y_{nk} \rightharpoonup y$ in $X$ as $k \to \infty$; \hfill (31)
2) $y'_{nk} \rightharpoonup y'$ in $X^*_\sigma$ as $k \to \infty$; \hfill (32)
3) $d_{nk} \rightharpoonup d$ in $X^*$ as $k \to \infty$; \hfill (33)
4) $y_{nk}(T) \to z$ in $H$ as $k \to \infty$. \hfill (34)

Moreover in (34),

$$z = y(T). \quad (35)$$

**Proof.** 1°. The boundedness of $\{d_n\}_{n \geq 1}$ in $X^*$ can be proved similarly to (13). So,

$$\exists c_1 > 0: \forall n \geq 1, \quad \| d_n \|_{X^*} \leq c_1. \quad (36)$$

2°. Let us prove boundedness of $\{y'_{n}\}_{n \geq 1}$ in $X^*_\sigma$. Due to (30) it follows that $\forall n \geq 1$, $y'_{n} = I_n^* (f - d_n)$, and therefore taking into account (26)-(28) and (36) we have:

$$\| y'_{n} \|_{X^*_\sigma} \leq \| y_n \|_{W_{1,\sigma}} \leq R + c(\| f \|_{X^*} + c_1) =: c_2 < +\infty, \quad (37)$$

where $c > 0$ is the constant from the inequality:

$$\| f \|_{X^*_\sigma} \leq c \| f \|_{X^*} \quad \forall f \in X^*.$$
3°. From (30) and from \( W_n \subset C(S; H) \) with continuous embedding we obtain that for each \( t \in S \), \( n \geq 1 \),
\[
\| y_n(t) \|_H^2 = 2 \int_0^t (y'_n(s), y_n(s)) \, ds = 2 \int_0^t (f(s) - d_n(s), y_n(s)) \, ds \leq \| f \|_{X^*} + c_1) R.
\]
Hence there exists \( c_3 > 0 \):
\[
\| y_n(t) \|_H \leq c_3 < +\infty \quad \forall n \geq 1 \text{ for all } t \in S.
\]

In particular,
\[
\| y_n(T) \|_H \leq c_3 \quad \forall n \geq 1. \tag{38}
\]

4°. From estimates (27), (36)–(38), due to the Banach–Alaoglu theorem, it follows the existence of subsequences
\[
\{ y_{nk} \}_{k \geq 1} \subset \{ y_n \}_{n \geq 1}, \quad \{ d_{nk} \}_{k \geq 1} \subset \{ d_n \}_{n \geq 1}
\]
and of elements \( y \in W_\sigma \), \( d \in X^* \) and \( z \in H \), for which convergence of types (31)–(34) take place.

5°. Let us prove that
\[
y' = f - d. \tag{39}
\]
Let \( \varphi \in D(S) \), \( n \in \mathbb{N} \) and \( h \in H_n \). Then \( \forall k \colon n_k \geq n \) we have:
\[
\left( \int_S \varphi(\tau)(y'_{nk}(\tau) + d_{nk}(\tau)) \, d\tau, h \right) \, \int_S (\varphi(\tau)(y'_{nk}(\tau) + d_{nk}(\tau)), h) \, d\tau\leq
\int_S (y'_{nk}(\tau) + d_{nk}(\tau), \varphi(\tau)h) \, d\tau
\]
\[
= \langle y'_{nk} + d_{nk}, \psi \rangle,
\]
where \( \psi(\tau) = h \cdot \varphi(\tau) \in X_n \subset X \).

Let us note that we use the property of Bochner’s integral here [22, theorem IV.1.8, c.153]. Since \( H_{nk} \supset H_n \) for \( n_k \geq n \), then
\[
\langle y'_{nk} + d_{nk}, \psi \rangle = \langle f_{nk}, \psi \rangle.
\]

So, \( \forall k \geq 1 \colon n_k \geq n \) it follows that
\[
\langle f_{nk}, \psi \rangle = \langle f, I_{nk} \psi \rangle = \int_S (f(\tau), \varphi(\tau)h) \, d\tau = \int_S (\varphi(\tau)f(\tau), h) \, d\tau = \left( \int_S \varphi(\tau)f(\tau) \, d\tau, h \right).
\]
Therefore, for all $k$: $n_k \geq n$,

$$
\left( \int_S \varphi(\tau) y'_{n_k}(\tau) d\tau, h \right) = \left( \int_S \varphi(\tau) (f(\tau) - d_{n_k}(\tau)) d\tau, h \right)
$$

$$
= \int_S ((f(\tau) - d_{n_k}(\tau)) \cdot \varphi(\tau)h) d\tau
$$

$$
= (f - d_{n_k}, \psi) \quad \to \quad (f - d, \psi)
$$

$$
= \left( \int_S \varphi(\tau) (f(\tau) - d(\tau)) d\tau, h \right) \quad \text{as } k \to \infty. \quad (40)
$$

The latter follows from the weak convergence of $d_{n_k}$ to $d$ in $X^*$.

From the convergence (32) we have:

$$
\left( \int_S \varphi(\tau) y'_{n_k}(\tau) d\tau, h \right) \to \left( \int_S \varphi(\tau) y'(\tau) d\tau, h \right) = (y'(\varphi), h) \quad \text{as } k \to +\infty, \quad (41)
$$

where

$$
\forall \varphi \in D(S), \quad y'(\varphi) = -y(\varphi') = -\int_S y(\tau) \varphi'(\tau) d\tau
$$

is the derivative of an element $y$ considered in the sense of $D^*(S, V^*)$.

Hence, due to (40) and (41) it follows that

$$
\forall \varphi \in D(S), \forall h \in \bigcup_{n \geq 1} H_n, \quad (y'(\varphi), h) = \left( \int_S \varphi(\tau) (f(\tau) - d(\tau)) d\tau, h \right).
$$

Since $\bigcup_{n \geq 1} H_n$ in dense in $V$ then

$$
\forall \varphi \in D(S), \quad y'(\varphi) = \int_S \varphi(\tau) (f(\tau) - d(\tau)) d\tau.
$$

So, $y' = f - d \in X^*$ and $y \in W$.

6°. Let us prove that $y(0) = \bar{0}$. Let $h \in H_n$, $\varphi \in D(S)$, $n \in \mathbb{N}$, $\psi(\tau) := (T - \tau)h \in X_n$. From (39) it follows:

$$
\langle y', \psi \rangle = \int_S (y'(\tau), \varphi(\tau)) d\tau = \int_S (f(\tau) - d(\tau), \psi(\tau)) d\tau
$$

$$
= \lim_{k \to \infty} \int_S (f(\tau) - d_{n_k}(\tau), \psi(\tau)) d\tau = \lim_{k \to \infty} (f - d_{n_k}, I_{n_k}\psi)
$$

$$
= \lim_{k \to \infty} (I_{n_k}^* (f - d_{n_k}), \psi) = \lim_{k \to \infty} \langle f_{n_k} - d_{n_k}^1, \psi \rangle = \lim_{k \to \infty} \langle y'_{n_k}, \psi \rangle.
$$
Noting that $\psi'(\tau) = -h$, $\tau \in S$, we obtain:

$$
\lim_{k \to \infty} \langle y'_{nk}, \psi \rangle = \lim_{k \to \infty} \left\{ -\langle \psi', y_{nk} \rangle + (y_{nk}(T), \psi(T)) \right\} = \lim_{k \to \infty} \int_{S} (y_{nk}(\tau), h) \, d\tau
$$

$$
= \int_{S} (y(\tau), h) \, d\tau = -\langle \psi', y \rangle.
$$

The latter holds true due to $y_{nk} \rightharpoonup y$ in $X$. On the other hand,

$$
-\langle \psi', y \rangle = \langle y', \psi \rangle - (y(T), \psi(T)) + (y(0), \psi(0)) = \langle y', \psi \rangle + T(y(0), h).
$$

Hence, $\forall h \in \bigcup_{n \geq 1} H_n$,

$$
\langle y', \psi \rangle = \langle y', \psi \rangle + T(y(0), h) \iff (y(0), h) = 0.
$$

Due to density of $\bigcup_{n \geq 1} H_n$ in $H$ it follows that $y(0) = \bar{0}$ and $y \in W_{0}$.  

7°. In order to complete the proof we must show that $y(T) = z$. The proof is similar to that of 6°. Lemma 5 is proved. $\square$

Now, in order to prove that $y$ is a solution of problem (25) it is necessary to show that $y$ satisfies the inclusion from (25). Due to identity (39), it is sufficient to prove that $d \in C(y)$. 

At first let us make sure that

$$
\lim_{k \to \infty} \langle d_{nk}, y_{nk} - y \rangle \leq 0.
$$

Indeed, due to (39), $\forall k \geq 1$ we have:

$$
\langle d_{nk}, y_{nk} - y \rangle = \langle d_{nk}, y_{nk} \rangle - \langle d_{nk}, y \rangle
$$

$$
= \langle d_{nk}^1, y_{nk} \rangle - \langle d_{nk}, y \rangle
$$

$$
= \langle f_{nk} - y'_{nk}, y_{nk} \rangle - \langle d_{nk}, y \rangle
$$

$$
= \langle f_{nk}, y_{nk} \rangle - \langle y'_{nk}, y_{nk} \rangle - \langle d_{nk}, y \rangle
$$

$$
= \langle f, y_{nk} \rangle - \langle d_{nk}, y \rangle - \frac{1}{2} \| y_{nk}(T) \|_H^2.
$$

Further in left and right sides of equality (43) we pass to the upper limit as $k \to \infty$. We have:

$$
\lim_{k \to \infty} \langle d_{nk}, y_{nk} - y \rangle \leq \lim_{k \to \infty} \langle f, y_{nk} \rangle + \lim_{k \to \infty} \langle d_{nk}, y \rangle - \lim_{k \to \infty} \frac{1}{2} \| y_{nk}(T) \|_H^2
$$

$$
\leq \langle f, y \rangle + \langle d, y \rangle - \frac{1}{2} \| y(T) \|_H^2
$$

$$
= \langle f - d, y \rangle - \langle y', y \rangle = 0.
$$
From conditions (31), (32), (33), (42) and $\lambda_0$-pseudomonotonicity of $C$ on $W_\sigma$ (it holds due to Lemma 1) it follows that there exist $\{d_m\} \subset \{d_{n_k}\}_{k \geq 1}, \{y_m\} \subset \{y_{n_k}\}_{k \geq 1}$, such that

$$\forall \omega \in X, \quad \lim_{m \to \infty} \langle d_m, y_m - \omega \rangle \geq [C(y), y - \omega].$$  \hfill (44)

If we prove that

$$\langle d, y \rangle \geq \lim_{m \to \infty} \langle d_m, y_m \rangle,$$  \hfill (45)

hence from (44) and from convergence of (36) we have:

$$\forall \omega \in X, \quad [C(y), y - \omega] \leq \langle d, y - \omega \rangle,$$

and we obtain that this is equivalent to the inclusion $y \in C(y) \in C_v(X^*)$. Therefore, $y$ is a solution of problem (25).

Let us prove (45):

$$\lim_{m \to \infty} \langle d_m, y_m \rangle = \lim_{m \to \infty} \langle f_m - y'_m, y_m \rangle$$

$$\leq \lim_{m \to \infty} \langle f_m, y_m \rangle + \lim_{m \to \infty} \langle -y'_m, y_m \rangle$$

$$= \lim_{m \to \infty} \langle f, y_m \rangle - \frac{1}{2} \lim_{m \to \infty} \| y_m(T) \|_H^2$$

$$\leq \langle f, y \rangle - \frac{1}{2} \| y(T) \|_H^2$$

$$= \langle f, y \rangle - \langle y', y \rangle = \langle d, y \rangle.$$

So, $y \in W$ is a solution of problem (25).

7.2. On searching the periodic solutions for differential-operator inclusions by the Faedo–Galerkin method

Let $A : X_1 \to C_v(X_1^*)$ and $B : X_2 \to C_v(X_2^*)$ be multi-valued maps. We consider the next problem:

$$\begin{cases}
y' + A(y) + B(y) \ni f, \\
y(0) = y(T)
\end{cases}$$ \hfill (46)

in order to find solutions by the FG method in the class

$$W = \{ y \in X \mid y' \in X^* \},$$

where the derivative $y'$ of an element $y \in X$ is considered in the sense of scalar distributions space $D^*(S; V^*) = \mathcal{L}(D(S); V^*_w)$, with $V = V_1 \cap V_2$, $V^*_w$ equals to $V^*$ with the topology $\sigma(V^*, V)$ [71]. We consider the norm on $W$:

$$\| y \|_W = \| y \|_X + \| y' \|_{X^*} \quad \text{for each } y \in W.$$

We also consider spaces $W_i = \{ y \in X_i \mid y' \in X^* \}, i = 1, 2.$
Remark 11. It is clear that the space $W$ is continuously embedded in $C(S; V^*)$. Hence, the condition in (46) is well defined.

Parallel to the problem (46) we consider the following class of problems in order to search the solutions in $W_n = \{ y \in X_n \mid y' \in X_n^* \}$:

\[
\begin{align*}
\begin{cases}
y'_n + A_n(y_n) + B_n(y_n) \ni f_n, \\
y_n(0) = y_n(T),
\end{cases}
\end{align*}
\]

where the maps $A_n, B_n, f_n$ were introduced in Section 5, the derivative $y'_n$ of an element $y_n \in X_n$ is considered in the sense of $D^*(S; H_n)$.

Let $W_{per} := \{ y \in W \mid y(0) = y(T) \}$, let us introduce the map

\[ L : D(L) = W_{per} \subset X \to X^* \]

by $Ly = y'$ for each $y \in W_{per}$.

From the main resolvability theorem it follows the next

Corollary 3. Let $A : X_1 \to C_v(X_1^*)$ and $B : X_2 \to C_v(X_2^*)$ be multi-valued maps such that

1. $A$ is $\lambda_0$-pseudomonotone on $W_1$ and it satisfies Condition (II);
2. $B$ is $\lambda_0$-pseudomonotone on $W_2$ and it satisfies Condition (II);
3. the sum $C = A + B : X \rightrightarrows X^*$ is finite-dimensionally locally bounded and weakly $\pm$-coercive.

Furthermore, let $\{h_j\}_{j \geq 1} \subset V$ be a complete system of vectors in $V_1, V_2, H$ such that the triple $((h_j)_{j \geq 1} : V_i; H)$ for $i = 1, 2$ satisfies Condition $(\gamma)$.

Then for each $f \in X^*$ the set

\[ K_{H_{per}}^f := \{ y \in W \mid y \text{ is the solution of (46), obtained by the FG method} \} \]

is nonempty and the representation

\[ K_{H_{per}}^f = \bigcap_{n \geq 1} \left[ \bigcup_{m \geq n} K_{m_{per}}^f \right]_{X^w} \]

is true, where for each $n \geq 1$,

\[ K_{n_{per}}^f = \{ y_n \in W_n \mid y_n \text{ is the solution of (47)} \}. \]

Moreover, if the operator $A + B : X \rightrightarrows X^*$ is $-\pm$-coercive, then $K_{H_{per}}^f$ is weakly compact in $X$ and in $W$.

Proof. At first let us prove the maximal monotonicity of $L$ on $W_{per}$. For $v \in X, w \in X^*$ such that for each $u \in W_{per} \langle w - Lu, v - u \rangle \geq 0$ is true, let us prove that $v \in W_{per}$ and $v' = w$. By the analogy with the proof of Corollary 2, we obtain $v' = w \in X^*$. Now we prove $v(0) = v(T)$. If we use [22, Theorem VI.1.17] with $u(t) \equiv v(T) \in W_{per}$, we obtain that

\[ 0 \leq \langle v' - Lu, v - u \rangle = \langle v' - u', v - u \rangle \]


\[ = \frac{1}{2} \left( \|v(T) - v(T)\|_H^2 - \|v(0) - v(T)\|_H^2 \right) \]

and then \(v(0) = v(T)\).

In order to prove the given statement, it is sufficient to show that \(L\) satisfies Conditions \(L_1-L_3\). Condition \(L_1\) follows from Proposition 7. Condition \(L_2\) follows from [22, Lemma VI.1.5] and from the fact that the set \(C^1(S; H_n)\) is dense in \(L_{P_{0}}(S; H_n) = X_n\). Condition \(L_3\) follows from [22, Lemma VI.1.7] with \(V = H = H_n\) and \(X = X_n\). \(\square\)

Remark 12. In the last Corollary we may get free from Condition \((\gamma)\) in the way, introduced in Remark 10. The proof is similar.

7.3. Example

Let us consider a bounded domain \(\Omega \subset \mathbb{R}^n\) with sufficiently smooth boundary \(\partial \Omega\), \(S = [0, T]\), \(Q = \Omega \times (0; T), \Gamma_T = \partial \Omega \times (0; T)\). Let, for \(i = 1, 2, m_i \in \mathbb{N}, N_1^i\) (respectively \(N_2^i\)) is the number of the derivatives with respect to the variable \(x\) of order \(\leq m_i - 1\) (respectively \(m_i\)) and \(\{A^i_\alpha(x, t, \xi)\}_{|\alpha| \leq m_i}\) be a family of real functions defined in \(Q \times \mathbb{R}^{N_1^i} \times \mathbb{R}^{N_2^i}\). Let

\[
D^k u = \{D^\beta u, |\beta| = k\} \quad \text{be the derivatives in} \ x,
\]

\[
\delta_i u = \{u, Du, \ldots, D^{m_i-1}u\},
\]

\[
A^i_\alpha(x, t, \delta_i u, D^{m_i}v) : x, t \rightarrow A^i_\alpha(x, t, \delta_i u(x, t), D^{m_i}v(x, t)).
\]

Moreover, let \(\psi : \mathbb{R} \rightarrow \mathbb{R}\) be some locally Lipschitz real function and its Clarke's generalized gradient \(\Phi = \partial_{\text{Cl}} \psi : \mathbb{R} \rightarrow \mathbb{R}\) satisfies the growth condition

\[
\exists C > 0: \quad \|\Phi(t)\|_+ \leq C(1 + |t|), \quad [\Phi(t), t]_+ \geq \frac{1}{C}(t^2 - 1) \quad \forall t \in \mathbb{R}. \quad (48)
\]

Let us consider the next problem with the Dirichlet boundary conditions:

\[
\frac{\partial y(x, t)}{\partial t} + \sum_{|\alpha| \leq m_1} (-1)^{|\alpha|} \partial^\alpha \left( A^1_\alpha(x, t, \delta_1 y, D^{m_1}y) \right)
\]

\[
+ \sum_{|\alpha| \leq m_2} (-1)^{|\alpha|} \partial^\alpha \left( A^2_\alpha(x, t, \delta_2 y, D^{m_2}y) \right) + \Phi(y(x, t)) \ni f(x, t) \quad \text{in} \ Q, \quad (49)
\]

\[
D^\alpha y(x, t) = 0 \quad \text{on} \ \Gamma_T \quad \text{as} \ |\alpha| \leq m_i - 1 \quad \text{and} \ i = 1, 2, \quad (50)
\]

and

\[
y(x, 0) = y(x, T) \quad \text{in} \ \Omega, \quad (51)
\]

or

\[
y(x, 0) = 0 \quad \text{in} \ \Omega. \quad (52)
\]
Let us assume $H = L_2(\Omega)$ and $V_l = W^{m_l,p_l}_0(\Omega)$ with $p_l > 1$ such that $V_l \subset H$ with continuous embedding. Consider the function $\varphi : L_2(S; H) \to \mathbb{R}$ defined by

$$\varphi(y) = \int_Q \psi(y(x,t)) \, dx \, dt \quad \forall y \in L_2(S; H).$$

Using growth condition (48) and Lebourg's mean value theorem, we note that the function $\varphi$ is well defined and Lipschitz continuous on bounded sets in $L_2(S; H)$. Thus it is locally Lipschitz. So the Clarke's generalized gradient $\partial_C \varphi : L_2(S; H) \Rightarrow L_2(S; H)$ is well defined. Moreover, the Aubin–Clarke theorem (see [13, p. 83]) ensures that, for each $y \in L_2(S; H)$ we have

$$p \in \partial_C \varphi(y) \quad \Rightarrow \quad p \in L_2(Q) \quad \text{with} \quad p(x,t) \in \partial_C \psi(y(x,t)) \quad \text{for a.e.} \quad (x,t) \in Q.$$

Under suitable conditions on the coefficients $A^i$, the given problems can be written in the following form:

$$y' + A_1(y) + A_2(y) + \partial_C \varphi(y) \ni f, \quad y(0) = y(T), \quad (53)$$

or, respectively,

$$y' + A_1(y) + A_2(y) + \partial_C \varphi(y) \ni f, \quad y(0) = \bar{y}. \quad (54)$$

where

$$f \in X^* = L_2(S; L_2(\Omega)) + L_{q_1}(S; W^{-m_1,q_1}(\Omega)) + L_{q_2}(S; W^{-m_2,q_2}(\Omega)).$$

$p^{-1}_i + q^{-1}_i = 1$.

Each element $y \in W$ that satisfies (53) (or (54)) is called a generalized solution of problem (49), (50), (51) (respectively (49), (50), (52)).

**The choice of basis.** As complete system of vectors $\{h_j\}_{j \geq 1} \subset W^{m_1,p_1}_0(\Omega) \cap W^{m_2,p_2}_0(\Omega)$ we may consider the spacial basis for $H^l_0(\Omega)$ with $l \in \mathbb{N}$ such that $H^l_0(\Omega) \subset W^{m_1,p_1}_0(\Omega)$ with continuous embedding ($i = 1,2$), or we may assume that there is a complete vector system $\{h_j\}_{j \geq 1} \subset W^{m_1,p_1}_0(\Omega) \cap W^{m_2,p_2}_0(\Omega)$ such that the triples

$$\{h_j\}_{j \geq 1}; \quad W^{m_i,p_i}_0(\Omega); \quad L_2(\Omega), \quad i = 1,2$$

satisfy Condition (γ).

For example, when $n = 1$ as $\{h_j\}_{j \geq 1}$ we may take the “special” basis for the pair $(H^{\max[m_1,m_2]}_0(\Omega); L_2(\Omega))$ with a suitable $\varepsilon > 0$ [53,40]. It is well known that the triple $(\{h_j\}_{j \geq 1}; \ L_p(\Omega); \ L_2(\Omega))$ satisfies Condition (γ) for $p > 1$. Then, using (for example) the results of [40,41], we obtain the necessary condition.

**The definition of operators $A$.** Let $A^i(x,t,\eta,\xi)$, defined in $Q \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, satisfy the conditions: for almost all $x,t \in Q$ the map $\eta,\xi \to A^i(x,t,\eta,\xi)$ is continuous on $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$;

for all $\eta,\xi$ the map $x,t \to A^i(x,t,\eta,\xi)$ is measurable on $Q$. \quad (55)

for all $u,v \in L^{p_i}(0,T; \ V_i) =: \mathcal{V}_i$, \quad $A^i(x,t,\delta u, D^{m} u) \in L^{\bar{p}_i}(Q)$. \quad (56)
Then for each $u \in V_i$ the map
\[
 w \to a_i(u, w) = \sum_{|\alpha| \leq m_i} A^i_\alpha(x, t, \delta_i u, D^{m_i} u) D^\alpha w \, dx \, dt,
\]
is continuous on $V_i$ and then
\[
 a_i(u, w) = \left\langle A_i(u), w \right\rangle.
\]
(57)

**Conditions on $A_i$.** Similarly to [53, Sections 2.2.5, 2.2.6, 3.2.1] we have
\[
 A_i(u) = A_i(u, u), \quad A_i(u, v) = A_{i1}(u, v) + A_{i2}(u),
\]
where
\[
 \left\langle A_{i1}(u, v), w \right\rangle = \sum_{|\alpha| = m_i} \int_Q A^i_{\alpha}(x, t, \delta_i u, D^{m_i} v) D^\alpha w \, dx \, dt,
\]
\[
 \left\langle A_{i2}(u), w \right\rangle = \sum_{|\alpha| \leq m_i - 1} \int_Q A^i_{\alpha}(x, t, \delta_i u, D^{m_i} u) D^\alpha w \, dx \, dt.
\]
We add the following conditions:
\[
 \left\langle A_{i1}(u, u), u - v \right\rangle - \left\langle A_{i1}(u, v), u - v \right\rangle \geq 0 \quad \forall u, v \in V_i;
\]
(58)
if $u_j \rightharpoonup u$ in $V_i$, $u_j' \rightharpoonup u'$ in $V^*_i$ and if \( A_{i1}(u_j, u_j) - A_{i1}(u_j, u) \to 0 \), then $A^i_{\alpha}(x, t, \delta u_j, D^{m_i} u_j) \to A^i_{\alpha}(x, t, \delta u, D^{m_i} u)$ in $L^{q_i}(Q)$;
(59)
"coercivity".
(60)

**Remark 13.** Similarly to [53, Theorem 2.2.8] the sufficient conditions to obtain (58), (59) are:
\[
 \sum_{|\alpha| = m_i} A^i_{\alpha}(x, t, \eta, \xi) \xi_\alpha \frac{1}{|\xi| + |\xi|^{p_i - 1}} \to +\infty \text{ as } |\xi| \to \infty
\]
for almost each $x, t \in Q$ and bounded $|\eta|$;
\[
 \sum_{|\alpha| = m_i} (A^i_{\alpha}(x, t, \eta, \xi) - A^i_{\alpha}(x, t, \eta, \xi^*)) (\xi_\alpha - \xi^*_\alpha) > 0 \quad \text{as } \xi \neq \xi^*
\]
for almost each $x, t \in Q$ and $\forall \eta$.

The next condition implies the coercivity:
\[
 \sum_{|\alpha| = m_i} A^i_{\alpha}(x, t, \eta, \xi) \xi_\alpha \geq c|\xi|^{p_i} \quad \text{for rather large } |\xi|.
\]
A sufficient condition to obtain (56) (see [53, p. 332]) is the following:
\[
 |A^i_{\alpha}(x, t, \eta, \xi)| \leq c[|\eta|^{p_i - 1} + |\xi|^{p_i - 1} + k(x, t)], \quad k \in L^{q_i}(Q).
\]
(61)
By analogy with the proof of [53, Theorem 3.2.1] and [53, Proposition 2.2.6] we obtain the next proposition.

**Proposition 8.** Let operator $A_i : V_i \to V_i^{**}$ ($i = 1, 2$), defined in (57), satisfy (55), (56), (58), (59) and (60). Then $A_i$ is pseudomonotone on $W_i$ (even on $W_{i,\sigma}$ in classical sense). Moreover it is bounded if (61) holds true.

Due to the last statement, to Corollary 3, to Corollary 2, to Remark 10 and to Remark 12, it follows that under above mentioned conditions for each $f \in X^*$ there exists a generalized solution of problem (49)–(51) (respectively of problem (49)–(52)) $y \in W$, turned out by the FG method and representation (7) for all these solutions holds.

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**References**


