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Determinant preserving maps on matrix algebras[☆]

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Abstract

Let M_n be the algebra of all $n \times n$ complex matrices. If $\phi : M_n \to M_n$ is a surjective mapping satisfying $\det(A + \lambda B) = \det(\phi(A) + \lambda\phi(B))$, $A, B \in M_n$, $\lambda \in \mathbb{C}$, then either ϕ is of the form $\phi(A) = MAN$, $A \in M_n$, or ϕ is of the form $\phi(A) = MA^{\mathrm{t}}N$, $A \in M_n$, where $M, N \in M_n$ are nonsingular matrices with $\det(MN) = 1$. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction and statement of the result

The study of linear operators on algebras that leave invariant certain functions, subsets or relations is now commonly referred to as the study of linear preserver problems. The first result on linear preservers is due to Frobenius [1] who studied linear maps on matrix algebras preserving the determinant. Let M_n be the algebra of all complex $n \times n$ matrices. If $A \in M_n$, then A^{t} denotes its transpose. Frobenius proved that if $\phi : M_n \to M_n$ is a bijective linear mapping satisfying $\det \phi(A) = \det A$, $A \in M_n$, then either $\phi(A) = MAN$, $A \in M_n$, or $\phi(A) = MA^{\mathrm{t}}N$, $A \in M_n$, for some $M, N \in M_n$ with $\det(MN) = 1$. One of the most known results in the theory of linear preservers is the Gleason–Kahane–Żelazko theorem [2,3] stating

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that every unital linear invertibility preserving functional f defined on a unital commutative complex Banach algebra $\mathscr{A}$ is multiplicative. For unital linear functionals the assumption of preserving invertibility is easily seen to be equivalent to the condition $f(x) \in \sigma(x)$, $x \in \mathscr{A}$. Here, $\sigma(x)$ denotes the spectrum of x. An interesting improvement of this result has been obtained by Kowalski and Slodkowski [4] who replaced the two assumptions in the Gleason–Kahane–Żelazko theorem, that is, the assumption of linearity and the assumption of preserving invertibility, by a single weaker assumption and showed that under this weaker condition the same conclusion holds true. More precisely, they proved that every functional f (no linearity of f is assumed) defined on a unital commutative complex Banach algebra $\mathscr{A}$ satisfying $f(0) = 0$ and

$$f(x) - f(y) \in \sigma(x - y), \quad x, y \in \mathscr{A}, \tag{1}$$

is linear and multiplicative. Obviously, every linear functional f on $\mathscr{A}$ with the property that $f(x) \in \sigma(x)$, $x \in \mathscr{A}$, satisfies (1). It is the aim of this note to improve the classical result of Frobenius in a similar way.

Theorem 1.1. *Let $\phi : M_n \to M_n$ be a surjective mapping satisfying*

$$\det(A + \lambda B) = \det(\phi(A) + \lambda\phi(B)), \quad A, B \in M_n, \ \lambda \in \mathbb{C}. \tag{2}$$

Then there exist $M, N \in M_n$ with $\det(MN) = 1$ such that either

$$\phi(A) = MAN, \quad A \in M_n,$$

or

$$\phi(A) = MA^{\mathrm{t}}N, \quad A \in M_n.$$

2. Proof

We need two simple lemmas.

Lemma 2.1. *Let $A, B \in M_n$ be matrices such that $\det(A + X) = \det(B + X)$ for every $X \in M_n$. Then $A = B$.*

Proof. If we denote $Y = A + X$ and $C = B - A$, then $\det Y = \det(C + Y)$ for every $Y \in M_n$. Denote $\operatorname{rank} C = r$. Then there exists Y_0 of rank $n - r$ such that $C + Y_0$ is invertible. Hence, $\det Y_0 \neq 0$, or equivalently, $r = 0$. It follows that $C = 0$, as desired. □

Lemma 2.2. *Let $A \in M_n$ be a matrix of rank k and $X \in M_n$ a nonsingular matrix. Then $X + \lambda A$ is singular for at most k different values of λ.*

Proof. Define $p(\lambda) = \det(X + \lambda A)$ and note that this is a nonzero polynomial since $p(0) \neq 0$. Moreover, since A is of rank k, it is equivalent to $I_k \oplus 0_{n-k}$, the diagonal matrix having first k diagonal entries equal to one and all others equal to zero. Hence, the degree of p is at most k. The desired conclusion follows trivially. □

Proof of Theorem 1.1. Suppose that $\phi(A) = \phi(B)$ for some $A, B \in M_n$. Then for every $X \in M_n$, we have $\det(A + X) = \det(\phi(A) + \phi(X)) = \det(B + X)$, and so, by Lemma 2.1, $A = B$. Hence, ϕ is bijective.

Next, we will see that ϕ is rank nonincreasing. Let $A \in M_n$ be of rank k and denote $\operatorname{rank}\phi(A) = l$. Then there exist nonsingular matrices $U, V \in M_n$ such that $U\phi(A)V = I_l \oplus 0_{n-l}$. By surjectivity we can find $B \in M_n$ with $U\phi(B)V = \operatorname{diag}(1, 2, \ldots, n)$. It follows that

$$\begin{aligned}\det(B + \lambda A) &= \det(\phi(B) + \lambda\phi(A)) \\ &= \frac{1}{\det(UV)} \det\big(\operatorname{diag}(1, 2, \ldots, n) + \lambda(I_l \oplus 0_{n-l})\big).\end{aligned}$$

By Lemma 2.2, the left side is equal to zero for at most k different values of λ, while the right side has l zeroes as a polynomial in λ. Hence, $l \leqslant k$ and so, $\operatorname{rank}\phi(A) \leqslant \operatorname{rank} A$. The inverse ϕ^{-1} is also rank nonincreasing. Therefore, $\operatorname{rank}\phi(A) = \operatorname{rank} A$ for every $A \in M_n$.

Clearly, $\det\phi(I) = \det I = 1$, and so, after replacing ϕ by $A \mapsto \phi(A)\phi(I)^{-1}$, if necessary, we may assume that $\phi(I) = I$. In particular, $\det(A - \lambda I) = \det(\phi(A) - \lambda I)$, $A \in M_n$, $\lambda \in \mathbb{C}$. Thus, A and $\phi(A)$ have the same characteristic polynomial for every $A \in M_n$.

Let us denote by $N \in M_n$ the matrix with 1's above the main diagonal and zeroes elsewhere, $N = E_{12} + E_{23} + \cdots + E_{n-1,n}$. Since ϕ preserves the characteristic polynomial and rank, $\phi(N)$ must be similar to N, and after composing ϕ by a suitable similarity transformation we may assume with no loss of generality that $\phi(N) = N$. Thus, for an arbitrary $X \in M_n$ we have $\det(X + \lambda N) = \det(\phi(X) + \lambda N)$, $\lambda \in \mathbb{C}$. Comparing the coefficients at λ^{n-1} we arrive at $e_n^{\mathrm{t}} X e_1 = e_n^{\mathrm{t}} \phi(X) e_1$.

Let u and v be any orthogonal unit vectors. We can find an orthonormal basis $x_1, \ldots, x_n$ of $\mathbb{C}^n$ such that $x_1 = u$ and $x_n = v$. Let U be the unitary matrix satisfying $Ue_i = x_i$, $i = 1, \ldots, n$. Denote $L = UNU^*$ and $M = \phi^{-1}(L)$. Then M is a nilpotent of nilindex n, and consequently, there is a nonsingular $S \in M_n$ such that $SMS^{-1} = N$. Define an auxiliary mapping $\psi : M_n \to M_n$ by

$$\psi(X) = U^*\phi(S^{-1}XS)U.$$

Clearly, ψ is a bijective mapping satisfying (2), $\psi(I) = I$, and $\psi(N) = N$. It follows that $e_n^{\mathrm{t}} X e_1 = e_n^{\mathrm{t}} \psi(X) e_1$ for every $X \in M_n$. Replacing $S^{-1}XS$ by A, a straightforward computation gives us

$$v^*\phi(A)u = e_n^{\mathrm{t}} SAS^{-1} e_1, \quad A \in M_n.$$

Thus, the mapping $A \mapsto v^*\phi(A)u$ is linear for every pair of orthogonal unit vectors u and v. Let A, B be any matrices and λ, μ any scalars. Denote by $Z = \phi(\lambda A + \mu B) - \lambda\phi(A) - \mu\phi(B)$. Then $v^*Zu = 0$ for every pair of orthonormal vectors u

and v, and hence, $Z = \gamma I$ for some scalar γ. But ϕ preserves the characteristic polynomial, and therefore, it preserves the trace. Hence, $Z = 0$. Thus, ϕ is linear. Using the result of Frobenius we complete the proof. $\square$

References

[1] G. Frobenius, Über die Darstellung der endlichen Gruppen durch lineare Substitutionen, Sitzungsber. Deutsch. Akad. Wiss. (1897) 994–1015.
[2] A.M. Gleason, A characterization of maximal ideals, J. Anal. Math. 19 (1967) 171–172.
[3] J.-P. Kahane, W. Żelazko, A characterization of maximal ideals in commutative Banach algebras, Stud. Math. 29 (1968) 339–343.
[4] S. Kowalski, Z. Slodkowski, A characterization of multiplicative linear functionals in Banach algebras, Stud. Math. 67 (1980) 215–223.