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Stability of a mixed type cubic–quartic functional equation in non-Archimedean spaces

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ABSTRACT

In this paper, we prove the Hyers–Ulam–Rassias stability of the mixed type cubic–quartic functional equation

$$f(x+2y)+f(x-2y)=4(f(x+y)+f(x-y))-24f(y)-6f(x)+3f(2y)$$

in non-Archimedean normed spaces.

1. Introduction

A classical question in the theory of functional equations is that "when is it true that a function which approximately satisfies a functional equation $\mathcal{E}$ must be somehow close to an exact solution of $\mathcal{E}$". Such a problem was formulated by Ulam [1] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [2]. It gave rise to the *stability theory* for functional equations. Subsequently, various approaches to the problem have been introduced by several authors. There are cases in which each 'approximate function' is actually a 'true function'. In such cases, we call the equation $\mathcal{E}$ *superstable*.

In 1978, Rassias [3] formulated and proved the following theorem, which implies Hyers' Theorem as a special case. Suppose that E and F are real normed spaces and that F is a complete normed space, $f : E \to F$ is a function such that for each fixed $x \in E$ the function $t \longmapsto f(tx)$ is continuous on $\mathbb{R}$. If there exist $\epsilon > 0$ and $p \in [0, 1)$ such that

$$\|f(x+y)-f(x)-f(y)\| \leq \epsilon(\|x\|^p+\|y\|^p) \tag{1.1}$$

for all $x, y \in E$, then there exists a unique linear function $T : E \to F$ such that

$$\|f(x)-T(x)\| \leq \frac{\epsilon\|x\|^p}{(1-2^{p-1})}$$

for all $x \in E$. In 1991, Gajda [4] answered the question for $p > 1$, which was raised by Th. M. Rassias. This new concept is known as the Hyers–Ulam–Rassias stability of functional equations. The terminology, Hyers–Ulam–Rassias stability, is originated from these historical backgrounds. The terminology can also be applied to the case of other functional equations. For the history and various aspects of this theory we refer the reader to monographs [4–10].

Jun and Kim [11] introduced the following functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.2}$$

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and they established the general solution and the Hyers–Ulam–Rassias stability for the functional equation (1.2). The function $f(x) = x^3$ satisfies the functional equation (1.2), which is thus called a cubic functional equation and every solution of the cubic functional equation is said to be a cubic function.

Park and Bae [12] introduced the following quartic functional equation

$$f(x+2y) + f(x-2y) = 4[f(x+y) + f(x-y)] + 24f(y) - 6f(x) \tag{1.3}$$

(see also [13,14]). It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.3), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

For more detailed definitions of such terminologies, we can refer to [15–18].

By a non-Archimedean field we mean a field K equipped with a function (valuation) $|.|$ from K into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r + s| \le \max\{|r|, |s|\}$ for all $r, s \in K$. Clearly $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$.

Definition 1.1. Let X be a vector space over a scalar field K with a non-Archimedean non-trivial valuation $|.|$. A function $\|.\| : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r|\|x\|$ for all $r \in K, x \in X$;
(iii) the strong triangle inequality (ultrametric); namely,

$$\|x + y\| \le \max\{\|x\|, \|y\|\} \quad (x, y \in X).$$

Then $(X, \|.\|)$ is called a non-Archimedean normed space.

Due to the fact that

$$\|x_n - x_m\| \le \max\{\|x_{j+1} - x_j\| : m \le j \le n-1\} \quad (n > m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space, we mean one in which every Cauchy sequence is convergent.

Moslehian and Rassias [19] proved the Hyers–Ulam–Rassias stability of the Cauchy functional equation and the quadratic functional equation in non-Archimedean normed space.

Eshaghi Gordji and Bavand Savadkouhi [13] proved the Hyers–Ulam–Rassias stability of cubic and quartic functional equations in non-Archimedean normed space.

In this paper, we establish the Hyers–Ulam–Rassias stability of the following functional equation

$$f(x+2y) + f(x-2y) = 4(f(x+y) + f(x-y)) - 24f(y) - 6f(x) + 3f(2y) \tag{1.4}$$

in non-Archimedean normed space. It is easy to show that the function $f(x) = x^3 + x^4$ satisfies the functional equation (1.4), which is called a mixed type cubic–quartic functional equation. For more detailed definitions of mixed type functional equations, we can refer to [14,20].

2. Main results

Throughout this section, we assume that G is an additive group and X is a complete non-Archimedean normed space. Now before taking up the main subject, for a given $f : G \to X$, we define the difference operator

$$Df(x, y) = f(x+2y) + f(x-2y) - 4[f(x+y) + f(x-y)] + 24f(y) + 6f(x) - 3f(2y)$$

for all $x, y \in G$. we consider the following function inequality:

$$\|Df(x, y)\| \le \varphi(x, y)$$

for an upper bound $\varphi : G \times G \to [0, \infty)$.

Theorem 2.1. *Let $\varphi : G \times G \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2|^{3n}} = 0 \tag{2.1}$$

for all $x, y \in G$ and let for each $x \in G$ the limit

$$\lim_{n\to\infty} \max\left\{ \frac{\varphi(0, 2^j x)}{|2|^{3j}} : 0 \le j < n \right\}, \tag{2.2}$$

denoted by $\tilde{\varphi}_C(x)$, exist. Suppose that $f : G \to X$ is an odd function satisfying

$$\|Df(x, y)\| \le \varphi(x, y) \tag{2.3}$$

for all $x, y \in G$. Then there exists a cubic function $C : G \to X$ such that

$$\|C(x) - f(x)\| \leq \frac{1}{|3 \cdot 2^3|}\tilde{\varphi}_C(x) \tag{2.4}$$

for all $x \in G$, and if, in addition,

$$\lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\varphi(0, 2^j x)}{|2|^{3j}} : i \leq j < n+i\right\} = 0$$

then C is the unique cubic function satisfying (2.4).

Proof. Setting $x = 0$ in (2.3), we get

$$\|3f(2y) - 24f(y)\| \leq \varphi(0, y) \tag{2.5}$$

for all $y \in G$. If we replace y in (2.5) by x and divide both sides of (2.5) by 24, then we get

$$\left\|\frac{f(2x)}{2^3} - f(x)\right\| \leq \frac{1}{|3 \cdot 2^3|}\varphi(0, x) \tag{2.6}$$

for all $x \in G$. Replacing x by $2^{n-1}x$ in (2.6), we get

$$\left\|\frac{1}{2^{3n}}f(2^n x) - \frac{1}{2^{3(n-1)}}f(2^{n-1}x)\right\| \leq \frac{\varphi(0, 2^{n-1}x)}{|3 \cdot 2^{3n}|} \tag{2.7}$$

for all $x \in G$. It follows from (2.7) and (2.1) that the sequence $\{\frac{f(2^n x)}{2^{3n}}\}$ is Cauchy. Since X is complete, we conclude that $\{\frac{f(2^n x)}{2^{3n}}\}$ is convergent. Set $C(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^{3n}}$.
Using induction one can show that

$$\left\|\frac{f(2^n x)}{2^{3n}} - f(x)\right\| \leq \frac{1}{|3 \cdot 2^3|} \max\left\{\frac{\varphi(0, 2^i x)}{|2|^{3i}} : 0 \leq i < n\right\} \tag{2.8}$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking n to approach infinity in (2.8) and using (2.2) one obtains (2.4). By (2.1) and (2.3), we get

$$\|DC(x, y)\| = \lim_{n\to\infty} \frac{1}{|2^{3n}|}\|f(2^n x, 2^n y)\| \leq \lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2|^{3n}} = 0$$

for all $x, y \in G$. Therefore the function $C : G \to X$ satisfies (1.4). To prove the uniqueness property of C, let C' be another cubic function satisfying (2.4). Then

$$\begin{aligned}\|C(x) - C'(x)\| &= \lim_{i\to\infty} |2|^{-3i}\|C(2^i x) - C'(2^i x)\| \\ &\leq \lim_{i\to\infty} |2|^{-3i} \max\{\|C(2^i x) - f(2^i x)\|, \|f(2^i x) - C'(2^i x)\|\} \\ &\leq \frac{1}{|3 \cdot 2^3|} \lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\varphi(0, 2^j x)}{|2|^{3j}} : i \leq j < n+i\right\}\end{aligned}$$

for all $x \in G$. If

$$\lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\varphi(0, 2^j x)}{|2|^{3j}} : i \leq j < n+i\right\} = 0,$$

then $C = C'$, and the proof is complete. □

Theorem 2.2. *Let $\varphi : G \times G \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2|^{4n}} = 0 \tag{2.9}$$

for all $x, y \in G$ and let for each $x \in G$ the limit

$$\lim_{n\to\infty} \max\left\{\frac{\varphi(0, 2^j x)}{|2|^{4j}} : 0 \leq j < n\right\}, \tag{2.10}$$

denoted by $\tilde{\varphi}_Q(x)$, exist. Suppose that $f : G \to X$ is an even function satisfying

$$\|Df(x, y)\| \leq \varphi(x, y) \tag{2.11}$$

for all $x, y \in G$. Then there exists a quartic function $Q : G \to X$ such that

$$\|Q(x) - f(x)\| \leq \frac{1}{|2^4|}\tilde{\varphi}_Q(x) \tag{2.12}$$

for all $x \in G$, and if, in addition,

$$\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(0, 2^j x)}{|2|^{4j}} : i \leq j < n+i\right\} = 0$$

then Q is the unique quartic function satisfying (2.12).

Proof. Setting $x = 0$ in (2.11), we get

$$\|f(2y) - 16f(y)\| \leq \varphi(0, y) \tag{2.13}$$

for all $y \in G$. If we replace y in (2.13) by x and divide both sides of (2.13) by 16, then we get

$$\left\|\frac{f(2x)}{2^4} - f(x)\right\| \leq \frac{1}{|2^4|}\varphi(0, x) \tag{2.14}$$

for all $x \in G$. Replacing x by $2^{n-1}x$ in (2.14) to obtain

$$\left\|\frac{1}{2^{4n}}f(2^n x) - \frac{1}{2^{4(n-1)}}f(2^{n-1}x)\right\| \leq \frac{\varphi(0, 2^{n-1}x)}{|2^{4n}|} \tag{2.15}$$

for all $x \in G$. From (2.15) and (2.9) it follows that the sequence $\{\frac{f(2^n x)}{2^{4n}}\}$ is Cauchy. Since X is complete, we conclude that $\{\frac{f(2^n x)}{2^{4n}}\}$ is convergent. Set $Q(x) := \lim_{n\to\infty}\frac{f(2^n x)}{2^{4n}}$.
By using induction one can show that

$$\left\|\frac{f(2^n x)}{2^{4n}} - f(x)\right\| \leq \frac{1}{|2^4|}\max\left\{\frac{\varphi(0, 2^i x)}{|2|^{4i}} : 0 \leq i < n\right\} \tag{2.16}$$

for all $n \in \mathbb{N}$ and all $x \in G$. Letting $n \to \infty$ in (2.16) and using (2.10) one can obtain (2.12). By (2.9) and (2.11), we get

$$\|DQ(x, y)\| = \lim_{n\to\infty}\frac{1}{|2^{4n}|}\|f(2^n x, 2^n y)\| \leq \lim_{n\to\infty}\frac{\varphi(2^n x, 2^n y)}{|2|^{4n}} = 0$$

for all $x, y \in G$. Therefore the function $Q : G \to X$ satisfies (1.4). Let now

$$\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(0, 2^j x)}{|2|^{4j}} : i \leq j < n+i\right\} = 0$$

and let Q' be another quartic function satisfying (2.12). Then

$$\begin{aligned}\|Q(x) - Q'(x)\| &= \lim_{i\to\infty}|2|^{-4i}\|Q(2^i x) - Q'(2^i x)\|\\ &\leq \lim_{i\to\infty}|2|^{-4i}\max\{\|Q(2^i x) - f(2^i x)\|, \|f(2^i x) - Q'(2^i x)\|\}\\ &\leq \frac{1}{|2^4|}\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(0, 2^j x)}{|2|^{4j}} : i \leq j < n+i\right\}\\ &= 0\end{aligned}$$

for all $x \in G$. Therefore $Q = Q'$. This completes the proof of the uniqueness of Q. □

Theorem 2.3. *Let $\varphi : G \times G \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty}\frac{\varphi(2^n x, 2^n y)}{|2|^{3n}} = \lim_{n\to\infty}\frac{\varphi(2^n x, 2^n y)}{|2|^{4n}} = 0 \tag{2.17}$$

for all $x, y \in G$ and let for each $x \in G$ the limit

$$\lim_{n\to\infty}\max\left\{\frac{\varphi(0, 2^j x)}{|2|^{3j}} : 0 \leq j < n\right\}, \tag{2.18}$$

denoted by $\tilde{\varphi}_C(x)$, and

$$\lim_{n\to\infty}\max\left\{\frac{\varphi(0, 2^j x)}{|2|^{4j}} : 0 \leq j < n\right\}, \tag{2.19}$$

denoted by $\tilde{\varphi}_Q(x)$*, exist. Suppose that* $f : G \to X$ *is a function satisfying*

$$\|Df(x,y)\| \leq \varphi(x,y) \tag{2.20}$$

for all $x, y \in G$*. Then there exist a cubic function* $C : X \to Y$ *and a quartic function* $Q : G \to X$ *such that*

$$\|f(x) - C(x) - Q(x)\| \leq \frac{1}{|2^4|} \max\left\{\frac{1}{|3|}\max\{\tilde{\varphi}_C(x), \tilde{\varphi}_C(-x)\}, \frac{1}{|2|}\max\{\tilde{\varphi}_Q(x), \tilde{\varphi}_Q(-x)\}\right\} \tag{2.21}$$

for all $x \in G$*, and if, in addition,*

$$\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(0,2^j x)}{|2|^{3j}} : i \leq j < n+i\right\} = \lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(0,2^j x)}{|2|^{4j}} : i \leq j < n+i\right\} = 0$$

then C *is the unique cubic function and* Q *is the unique quartic function.*

Proof. Let $f_o(x) = \frac{1}{2}[f(x) - f(-x)]$ for all $x \in G$. Then $f_o(0) = 0, f_o(-x) = -f_o(x)$, and

$$\|Df_o(x,y)\| \leq \frac{1}{|2|}\max\{\varphi(x,y), \varphi(-x,-y)\}$$

for all $x, y \in G$. From Theorem 2.1, it follows that there exists a unique cubic function $C : G \to X$ satisfying

$$\|f_o(x) - C(x)\| \leq \frac{1}{|3 \cdot 2^4|}\max\{\tilde{\varphi}_C(x), \tilde{\varphi}_C(-x)\} \tag{2.22}$$

for all $x \in G$.
Let $f_e(x) = \frac{1}{2}[f(x) + f(-x)]$ for all $x \in G$. Then $f_e(0) = 0, f_e(-x) = f_e(x)$, and

$$\|Df_e(x,y)\| \leq \frac{1}{|2|}\max\{\varphi(x,y), \varphi(-x,-y)\}$$

for all $x, y \in G$. From Theorem 2.2, it follows that there exists a unique quartic function $Q : G \to X$ satisfying

$$\|f_e(x) - Q(x)\| \leq \frac{1}{|2^5|}\max\{\tilde{\varphi}_Q(x), \tilde{\varphi}_Q(-x)\} \tag{2.23}$$

for all $x \in G$.
Hence, (2.21) follows from (2.22) and (2.23). □

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