Weakly submodular rank functions, supermatroids, and the flat lattice of a distributive supermatroid

Marcel Wild

Department of Mathematics, University of Stellenbosch, Matieland 7602, South Africa

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In Memoriam Gian-Carlo Rota

Abstract

Distributive supermatroids generalize matroids to partially ordered sets. Completing earlier work of Barnabei, Nicoletti and Pezzoli we characterize the lattice of flats of a distributive supermatroid. For the prominent special case of a polymatroid the description of the flat lattice is particularly simple. Large portions of the proofs reduce to properties of weakly submodular rank functions. The latter are also investigated for their own sake, and some new results on general supermatroids are derived.

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1. Introduction

The correspondence between closure operators on a lattice $\mathcal{D}$ and $\wedge$-subsemilattices of $\mathcal{D}$ is well known. In Section 2 we review from [18] the correspondence between $\wedge$-subsemilattices and certain weakly submodular rank functions $\mathcal{D} \to \mathbb{N}$, and indicate how the applicability of weakly submodular rank functions (WSRFs) extends beyond the present article. In Section 3, starting with matroids, we introduce the more general class of Faigle-WSRFs and then the intermediate class of distributive supermatroids (DSMs). As is the case for matroids, each Faigle-WSRF admits a “simple” Faigle-WSRF with an isomorphic flat lattice. The corresponding fact for a DSM is less obvious and is established in Section 4. This is the basis for Section 5, where the flat lattice of a DSM is characterized. Clearly, the flat lattices of DSMs are more general than the geometric lattices linked to matroids, but more specific than the arbitrary upper semimodular lattices linked to Faigle-WSRFs. For DSMs both sides of the characterization are difficult: to (i) determine what kind of lattices arise, and (ii) to argue that any abstract such lattice stems from a suitable DSM.

In Section 6 we explain in quite a bit of detail how DSMs also fit the framework of selectors, greedoids, and of course (general) supermatroids. Some novelties concerning the latter, i.e. (15), (16), (19), will be established along the way. Sections 2 and 6 are likely the ones of broadest interest. This justifies the order of terms in the title, despite the fact that characterizing the flat lattice of a DSM constitutes the article’s lion share. Without further mention, all sets and structures in this article are finite.

E-mail address: mwild@sun.ac.za.

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2. The equivalence of \( \wedge \)-subsemilattices and weak submodularity

We assume a basic familiarity with lattice theory. Since submodular functions and semimodular lattices occur frequently, the book of Stern [13] is particularly recommended as a reference for terms not fully defined and for additional background. Besides [13], we will also refer to [18], which in fact was triggered by previous drafts of the present article. Eventually enough results accumulated that held in a framework more general than DSMs, and which were sieved into [18]. Section 2 summarizes the key findings of [18], and adds Theorem 2 which was “missed” in [18].

Recall that a function \( \text{cl}: D \to D \) is a closure operator if it is monotone, idempotent, and extensive in the sense that \( A \leq \text{cl}(A) \) for all \( A \in D \). If \( A = \text{cl}(A) \), then \( A \) is closed. A \( \wedge \)-subsemilattice is a subset \( L \) of \( D \) such that \( X \wedge Y \in L \) for all \( X, Y \in L \). Here we also postulate \( 1_D \in L \). If \( D = \mathcal{P}(E) \) happens to be a Boolean lattice, i.e. the powerset of \( E \), then the pair \( (E, \text{cl}) \) is often called a closure space. Generally, each closure operator \( \text{cl}: D \to D \) yields the \( \wedge \)-subsemilattice \( L = L[\text{cl}] \) defined by

\[
\mathcal{L} := \{ \text{cl}(A) : A \in D \}.
\]

On its own the set \( L \), partially ordered by \( \leq \), is a lattice with meet \( \triangle \) and join \( \lor \) given by

\[
X \triangle Y = X \wedge Y \quad \text{and} \quad X \lor Y = \text{cl}(X \lor Y).
\]

Conversely, each \( \wedge \)-subsemilattice \( \mathcal{L} \) of \( D \) yields a closure operator \( \text{cl} = \text{cl}[\mathcal{L}] \) on \( D \) defined by

\[
\text{cl}(A) := \bigwedge \{ X \in \mathcal{L} : X \supseteq A \}.
\]

These processes are mutually inverse in the sense that \( \mathcal{L}[\text{cl}][\mathcal{L}] = \mathcal{L} \) and \( \text{cl}[\mathcal{L}][\text{cl}] = \text{cl} \). All of that is well known.

Apparently novel is the following correspondence between \( \wedge \)-subsemilattices \( \mathcal{L} \subseteq D \) and “weakly submodular” rank functions on \( D \). Some definitions beforehand. A map is a monotone function \( f: D \to \mathbb{N} \) which satisfies \( f(0) = 0 \).

The map \( f \) is:

1. \( f \) modular, if \( f(A \lor B) + f(A \land B) = f(A) + f(B) \);
2. \( f \) submodular, if \( f(A \lor B) - f(B) \leq f(A) - f(A \land B) \);
3. \( f \) weakly submodular, if \( f(A) = f(A \land B) \) implies \( f(A \lor B) = f(B) \);
4. \( f \) locally submodular, if it follows from \( A \succ A \land B \prec B \) and \( f(A) = f(A \land B) = f(B) \) that \( f(A \lor B) = f(A \land B) \).

The stated identities and inequalities are supposed to hold for all \( A, B \in D \). Obviously

\[
\text{locally submodular} \implies \text{weakly submodular} \implies \text{submodular} \implies \text{modular}.
\]

A rank function is a map which satisfies \( r(A) \leq |A| \) for all \( A \in D \). Here \( |A| \) is the natural rank (or height) of \( A \), i.e. the length \( n \) of a longest chain \( 0 < A_1 < A_2 < \cdots < A_n = A \). All unit increase maps \( r \), i.e. \( A < B \implies r(B) \leq r(A) + 1 \), are easily seen to be rank functions. Here \( \prec \) denotes the covering relation. We do not assume that \( D \) is graded.

Given a map \( f: D \to \mathbb{N} \), associate with it the set \( \mathcal{L}[f] \) of all \( f \)-maximal elements \( X \in D \), thus

\[
\mathcal{L}[f] := \{ X \in D : (\forall A \in D) (A \succ X \implies f(A) > f(X)) \}.
\]

Given a \( \wedge \)-subsemilattice \( \mathcal{L} \) of \( D \) or, equivalently, a closure operator \( \text{cl}: D \to D \), define a rank function \( r = r[\mathcal{L}] = r[\text{cl}] \) from \( D \) to \( \mathbb{N} \) by

\[
r(A) := \| \text{cl}(A) \|,
\]

where \( \| \| \) gives the height within \( \mathcal{L} \). Again, we do not assume that \( D \) or \( \mathcal{L} \) are graded. Given a \( \wedge \)-subsemilattice \( \mathcal{L} \) of \( D \) one checks that \( \mathcal{L}[r[\mathcal{L}]] = \mathcal{L} \). More interesting is the question: given a map \( f: D \to \mathbb{N} \), when is \( \mathcal{L}[f] \) a \( \wedge \)-subsemilattice?

Theorem 1 (Wild [18, Theorem 1]). Let \( D \) be any lattice.

(a) If the map \( f: D \to \mathbb{N} \) is weakly submodular, then \( \mathcal{L}[f] \) is a \( \wedge \)-subsemilattice of \( D \).

(b) If \( \mathcal{L} \) is a \( \wedge \)-subsemilattice of \( D \), then \( r[\mathcal{L}] \) is a weakly submodular rank function on \( D \).
In short, provided that \( f \) is weakly submodular, we may confuse the terms “\( f \)-maximal” and “closed”. Notice that the \( f \) in part (a) need not be a rank function. We mention that for \( D \) a Boolean lattice, part (a) has essentially been shown in [2] (using a condition equivalent to (R3)\(^{-} \)). What we call a map \( f : D \to \mathbb{N} \) is referred to as weighted lattice \( D_f \) in [15], but we shall henceforth follow [15, p. 128] to speak of flats rather than \( f \)-maximal elements. Besides [2], Proposition 3.1 in [15] is another attempt to characterize when the collection of flats is closed under infima. It is flawed in that a “respectful closure-operator” makes its appearance, instead of an intrinsic condition on \( f \), i.e. (R3)\(^{-} \).

By the way, our intrinsic closure operator \( \text{cl} \) coupled to a weakly submodular map \( f \) via \( L[f] \) features in a handy definition that relaxes “unit increase”. Namely, \( f \) is right [18, p. 365] if \( f(A) = \| \text{cl}(A) \| \) for all \( A \in D \). Here \( \| \) is the height function of \( L[f] \).

When speaking of a WSRF we shall have in mind a triple \( (D, r, L) \) consisting of a lattice \( D \), a weakly submodular rank function \( r : D \to \mathbb{N} \), and its associated (Theorem 1) lattice \( L = L[r] \). Depending on which aspect is more prominent in a particular situation, we may sometimes write \((D, r)\) or \((D, L)\) instead of \((D, r, L)\). Call \((D, r, L)\) simple if the join irreducibles of \( L \) and \( D \) coincide.

The “geometry” of all sorts of closure spaces has often been linked to the associated flat lattices. Suffice it to mention:

(i) projective geometry and complemented modular lattices (generalizable, on the one hand, to geometric lattices and matroids [8]; on the other hand, to modular lattices and their bases of lines [6]);

(ii) topology and distributive lattices;

(iii) convex geometry and locally lower distributive lattices [13, 7.3].

While a particular structure of interest often features additional concepts (a set of “points”, implicational bases [17], partial alphabets [3], independent sets, hyperplanes, etc.), the derived WSRF is stripped of all the fuzz and may showcase properties shared by a variety of structures. This is e.g. demonstrated in Theorem 2 and Section 6.

Let \( M(E) \) be a matroid on a set \( E \) and \( T \subseteq E \) any subset (the precise definition of “matroid” will be given in Section 3). According to [8, p. 118] the flats of the submatroid \( M(E - T) \) correspond bijectively to the flats \( X \) of \( M(E) \) satisfying \( \text{cl}(X - T) = X \). In order to raise the matter to the WSRF level \((D, r, L)\), we define the WSRF induced by \( C \in D \) as \(((0, C], r', L')\), where \([0, C]\) is the interval \( \{A \in D | 0 \leq A \leq C\} \) and \( r' \) is the (weakly submodular) restriction of \( r \) to \([0, C]\), and \( L':= L[r'] \). In the matroid example above, \( C = E - T \).

**Theorem 2.** With notation as above, the following holds:

(a) \( L'' = \{X \land C | X \in L', \text{ and} \}

(b) \( L'' \simeq L_0 := \{X \in L' | \text{cl}(X \land C) = X \} \), where \( L_0 \) is a \( \lor \)-subsemilattice of \( L' \).

**Proof.** As to (a), taking any \( X \in L' \), we need to show that \( X \land C \) is \( r' \)-maximal within \([0, C]\). Assuming the contrary, there would be \( A \in [0, C] \) with \( X \land A \subsetneq X \land C \) and \( r'(X \land C) = r'(A) \). Since \( X \land A = X \land C \), the interval \([X \land C, A]\) transposes up to \([X, X \lor A]\), and so \( r(X \lor A) = r(X) \) by the weak submodularity of \( r \). But this contradicts the \( r \)-maximality of \( X \). Conversely, let \( X' \in L' \). Put \( X := \text{cl}(X') \). Thus \( X \in L' \). Let \( X' \subseteq X \land C \subseteq X \), and \( r(X') = r(X \land C) \) forces \( r'(X') = r'(X \land C) \). Because \( X' \subsetneq X \land C \) is impossible by the \( r' \)-maximality of \( X' \), we must have \( X' = X \land C \).

As to (b), consider the function \( f(X) := X \land C \) from the poset \( L_0 \) to the lattice \( L' \). To see that \( f \) is onto, take any element \( X \land C \) from \( L'' \) (see (a)). Generally \( X \not\in L_0 \), but putting \( Y := \text{cl}(X \land C) \) it follows from \( X \land C \subseteq Y \subseteq X \) that \( Y \land C = X \land C \), and hence that \( Y = \text{cl}(Y \land C) \subseteq L_0 \) achieves \( f(Y) = X \land C \). In order to verify that \( L_0 \) is in fact a lattice and \( f \) a lattice isomorphism, it now suffices to show that \( X \subseteq Y \iff f(X) \subseteq f(Y) \) for all \( X, Y \in L_0 \). To see that the lattice \( L_0 \) is actually a \( \lor \)-subsemilattice of \( L' \), we pick any \( X, Y \in L_0 \) and tempt to show that \( X \lor Y = \text{cl}(X \lor Y) \) is an element of \( L_0 \). From \( X \land C \subseteq (X \lor Y) \land C \) follows that \( X = \text{cl}(X \land C) \subseteq \text{cl}((X \lor Y) \land C) \).

Motivated by applications to characteristic polynomials of weighted lattices, Whittle [15] focuses on the contraction and deletion of a weighted lattice \( D_f \) by an element \( x \) of \( D_f \), rather than on “weighted sublattices” akin to Theorem 2. It would be interesting to examine contraction and deletion for WSRFs.
It is well known [13, Corollary 1.9.10] that for any lattice \( \mathcal{D} \) with natural rank \( | | \) one has

\[
| | \text{ is submodular } \iff \mathcal{D} \text{ is upper semimodular.} \tag{1}
\]

Once more, this can be raised to the WSRF level. Namely, if \( \mathcal{L} \subseteq \mathcal{D} \) is any \( \land \)-subsemilattice, then by [18, Theorem 3]

\[
r[\mathcal{L}] \text{ is submodular } \iff \mathcal{L} \text{ is upper semimodular.} \tag{2}
\]

By duality (1) yields

\[
| | \text{ is modular } \iff \mathcal{D} \text{ is modular.}
\]

Obviously (2) cannot be dualized because \( \land \)-subsemilattices become \( \lor \)-subsemilattices under dualization. Thus characterizing modular \( \land \)-subsemilattices \( \mathcal{L} \subseteq \mathcal{D} \) in terms of \( r[\mathcal{L}]: \mathcal{D} \to \mathbb{N} \) remains an open problem. Similarly, characterizing various types of matroids by their rank functions is brand new territory. In the present article the task will rather be the converse: \( r \) is given and \( \mathcal{L}[r] \) is sought.

3. Faigle-WSRFs and DSMs

A matroid on a set \( E \) can be defined in various equivalent cryptomorphic ways [8]. For instance, it can be conceived as a WSRF \( (\mathcal{B}, r) \) such that:

\[(R1)^B \quad \mathcal{B} \text{ is a Boolean lattice with a rank function } r: \mathcal{B} \to \mathbb{N};\]

\[(R2) \quad r \text{ is unit increase;}\]

\[(R3) \quad r \text{ is submodular.}\]

Equivalently, a matroid is a pair \( (E, \mathcal{F}) \) where \( \mathcal{F} \subseteq \mathcal{B}(E) \) is a nonempty family of independent sets satisfying

\[(I1) \quad \text{For all } I \in \mathcal{F} \text{ and all } K \in \mathcal{B}(E) \text{ with } K \subseteq I \text{ one has } K \in \mathcal{F};\]

\[(I2) \quad |I| = |K| \text{ for all maximal } I, K \in [\phi, S] \cap \mathcal{F} \text{ and all } S \in \mathcal{B}(E).\]

We may use \( \preceq \) for comparable members of a lattice, even when these members are sets and \( \leq \) amounts to \( \subseteq \). Further note that \( X \prec Y \) means \( (X \preceq Y \text{ but } X \neq Y) \), and that \( [\phi, S] \) is the interval sublattice \( \{T: \phi \preceq T \preceq S\} \) of \( \mathcal{B}(E) \). It is well known how to get \( \mathcal{F} \) from \( r \) and conversely. We note that \( (I2) \) is equivalent to the augmentation property

\[(I2)' \quad \text{Whenever } I, K \in \mathcal{F} \text{ and } |I| < |K|, \text{ there is some } a \in K - I \text{ with } I \cup \{a\} \in \mathcal{F}.
\]

Gian-Carlo Rota has asked [11] for the most natural way to generalize matroids on sets \( E \) to “matroids on posets” \( (E, \preceq) \). There is no universal answer. In terms of the rank function the likely answer is: switch from \( \mathcal{B}(E) \) to the distributive lattice \( \mathcal{D}(E, \preceq) \) of all order ideals \( A \) of \( (E, \preceq) \) and postulate the same properties for the rank function. This was done by Faigle; see Section 6 for a discussion in the framework of greedoids.

Thus we speak of a Faigle-WSRF \( (\mathcal{D}, r) \) if:

\[(R1)^D \quad \mathcal{D} \text{ is a distributive lattice with a rank function } r: \mathcal{D} \to \mathbb{N},\]

\[(R2) \quad r \text{ is unit increase;}\]

\[(R3) \quad r \text{ is submodular.}\]

In particular, when \( \mathcal{D} = \mathcal{B} \) is Boolean, we are back to matroids. In general (Birkhoff’s Theorem), \( \mathcal{D} \) is isomorphic to \( \mathcal{D}(E, \preceq) \) if \( E \) is taken as the poset \( J(\mathcal{D}) \) of join irreducibles of \( \mathcal{D} \). Conversely, the join irreducibles of any lattice of type \( \mathcal{D}(E, \preceq) \) are exactly the principal order ideals \( J(a) := \{b \in E: b \preceq a\} \) of \( (E, \preceq) \) where \( a \) runs through \( E \). We shall freely switch back and forth between \( \mathcal{D}(E, \preceq) (= \text{ concrete distributive lattice}) \) and \( \mathcal{D} (= \text{ abstract distributive lattice}) \).

**Example 1.** Consider the 18-element distributive lattice \( \mathcal{D} = \{A, B, \ldots, U\} \). The numbers at the letters define a rank function \( r: \mathcal{D} \to \mathbb{N} \). For instance, \( B \prec C \) and \( r(B) = 1 < 2 = r(C) \) in accordance with (R2). Ditto
Fig. 1.

\[ r(P \lor C) - r(P) = 3 - 3 = 0 \] is smaller than \[ r(C) - r(P \land C) = 2 - 1 = 1, \] in accordance with (R3). In fact, one can show that \((\mathcal{D}, r)\) is a Faigle-WSRF (Fig. 1).

The join irreducibles of \(\mathcal{D}\) are \(B, C, D, H, N\) (displayed bold), endowed with the order induced by \(\mathcal{D}\). If \(E := \{b, c, d, h, n\}\) is an isomorphic “abstract” poset, then \(\mathcal{D} \simeq \mathcal{D}(E, \leq)\). For instance, \(P \in \mathcal{D}\) becomes the order ideal \(\{b, h, n\} \in \mathcal{D}(E, \leq)\) because the join irreducibles below \(P\) are \(B, H, N\).

**Example 2.** For instance, the element \(F \in \mathcal{D}\) in Example 1 is not \(r\)-maximal because \(G > F\) but \(r(G) = r(F)\). On the other hand, \(I \in \mathcal{D}\) is \(r\)-maximal because \(r(I) = 2\) and its upper covers \(V, L, P\) all have rank 3. Because of (R3) and Theorem 1, the subset \(\mathcal{L} \subseteq \mathcal{D}\) of \(r\)-maximal elements (shaded) must be a \(\land\)-subsemilattice; for instance \(I, G \in \mathcal{L} \Rightarrow B = I \land G \in \mathcal{L}\). The arrows indicate the action of \(\text{cl}\), e.g. \(\text{cl}(P) = U\), and \(\text{cl}(N) = N\) (For graphical reasons the other loops have been omitted.) (Fig. 2).

A natural generalization of the exchange axiom for matroids \((\mathcal{B}, \text{cl})\) to arbitrary lattices \(\mathcal{D}\) was proposed in [18]. Namely, given a closure operator \(\text{cl}: \mathcal{D} \to \mathcal{D}\), we demand that for all \(A \in \mathcal{D}\) and all \(a, b \in J(\mathcal{D})\):

If \(A \prec A \lor a\) and \(b \leq \text{cl}(A \lor a)\) and \(b \notin \text{cl}(A)\), then \(a \leq \text{cl}(A \lor b)\). \hspace{1cm} (3')

For matroids \((\mathcal{B}, \text{cl})\) the above reads:

If \(a \notin \text{cl}(A)\) and \(b \in \text{cl}(A \cup \{a\})\) and \(b \notin \text{cl}(A)\), then \(a \in \text{cl}(A \cup \{b\})\).
Fig. 2.

It is easy to see that (3′) is equivalent to the more handy fact that for all $A, B \in \mathcal{D}$:

$$\text{If } A \prec B \text{ in } \mathcal{D}, \text{ then } \operatorname{cl}(A) = \operatorname{cl}(B) \text{ or } \operatorname{cl}(A) \prec \operatorname{cl}(B) \text{ in } \mathcal{L}. \tag{3}$$

Returning to Faigle-WSRFs $(\mathcal{D}, r)$, an equivalent axiomatization is this:

A Faigle-WSRF is an ordered pair $(\mathcal{D}, \operatorname{cl})$ where $\mathcal{D}$ is a distributive lattice endowed with a closure operator $\operatorname{cl}: \mathcal{D} \to \mathcal{D}$ that satisfies the exchange axiom (3). \tag{4}

Indeed, from $(R2)$, $(R3)$ and [18, Theorem 2] it follows that $\operatorname{cl}=\operatorname{cl}[r]$ satisfies (3). Conversely, if $\operatorname{cl}: \mathcal{D} \to \mathcal{D}$ satisfies (3), then $r = r[\operatorname{cl}]$ satisfies $(R1)^D$, $(R2)$, $(R3)$ by [18, Theorem 5(a)].

For instance, since the closure operator $\operatorname{cl}$ in Example 2 stems from the Faigle-WSRF in Example 1, it must satisfy the exchange axiom. Thus, $C \prec V$ in $\mathcal{D}$ should force $\operatorname{cl}(C) \prec \operatorname{cl}(V)$ in $\mathcal{L}$. Indeed $G \prec U$ in $\mathcal{L}$ does hold. Note that in $\mathcal{D}$ we do not have $G \prec U$ since $G < M < U$.

If one takes “independence” instead of “rank function” (or “closure operator”) as the fundamental notion to be generalized from sets to posets, then the natural generalization of $(I1)$, $(I2)$ is this. Consider a family $\mathcal{F}$ of order ideals of the poset $(E, \leq)$ which satisfies (put $\mathcal{D} = \mathcal{D}(E, \leq)$):

$$\begin{align*}
(I1)^{SM} & \quad \text{For all } I \in \mathcal{F} \text{ and all } K \in \mathcal{D} \text{ with } K \leq I, \text{ one has } K \in \mathcal{F}. \\
(I2)^{SM} & \quad |I| = |K| \text{ for all maximal } I, K \in [\phi, S] \cap \mathcal{F} \text{ and all } S \in \mathcal{D}.
\end{align*}$$

Again it is easy to see that $(I2)^{SM}$ is equivalent to $(I2)'$. The order ideals $I \in \mathcal{F}$ are called independent and the structure $(E, \leq, \mathcal{F})$ is known as a DSM. The name will be clear in Section 6 where we deal with general supermatroids (SM). By $(I2)^{SM}$ all inclusion-maximal members of $\mathcal{F}$, called bases, have the same cardinality. When $(E, \leq)$ is trivial, then $(E, \leq, \mathcal{F}) = (E, \mathcal{F})$ is just a matroid.
Example 3. Consider this poset \((E, \leq)\) along with this family \(\mathcal{F}\) of order ideals (Fig 3).

As to \((I_1)_{SM}\), let \(I = \{d, h, n\} \in \mathcal{F}\). The subset \{d, n\} is no order ideal. But all subsets of \(I\) that happen to be members of \(\mathcal{D}(E, \leq)\), such as \{h, n\}, are in \(\mathcal{F}\).

As to \((I_2)\)', let \(I = \{b, d\}\) and \(K = \{b, c, h\}\). We have \(|I| < |K|\); and picking \(h \in K - I\) yields the order ideal \(I \cup \{h\} = \{b, d, h\}\) in \(\mathcal{F}\). As to \(loop \in E\), it is rather insignificant and dealt with in Lemma 4.

Lemma 3.

(a) Let \((E, \leq, \mathcal{F})\) be a DSM. Put \(\mathcal{D} := \mathcal{D}(E, \leq)\) and define \(r = r[\mathcal{F}]\) by

\[
    r(A) := \max\{|I| : I \in \mathcal{F} \text{ and } I \subseteq A\} \quad \text{(also called } \mathcal{F}\text{-rank).}
\]

Then \(r\) is a rank function on \(\mathcal{D}\) that satisfies (R2) and the following greedy chain property\(^1\):

\[(R4)\] For all \(A, B \in \mathcal{D}\) with \(A < B\) and \(r(A) < r(B)\) there is a chain \(A < A_1 < \cdots < A_k = B\) and a \(k \geq 1\) such that \(r(A) < r(A_1) < \cdots < r(A_k) = r(A_{k+1}) = \cdots = r(A_n)\).

(b) Let \(\mathcal{D} = \mathcal{D}(E, \leq)\) be a distributive lattice and \(r : \mathcal{D} \to \mathbb{N}\) a rank function that satisfies (R2) and (R4). Let \(\mathcal{F} = \mathcal{F}[r]\) be the family of all independent elements \(A \in \mathcal{D}\), where by definition

\[
    I \text{ is independent if } r(I) = |I| \quad \text{(also called } r\text{-independent).}
\]

Then \((E, \leq, \mathcal{F})\) is a DSM.

(c) The constructions in (a) and (b) are mutually inverse in the sense that

\[
    r[\mathcal{F}[r]] = r \quad \text{and} \quad \mathcal{F}[r[\mathcal{F}]] = \mathcal{F}.
\]

For later reference we record:

\[(R1)^{UM} \land (R2) \land (R3)^{--} \Rightarrow (R3)\] \quad [19, p. 94], \quad (7)
\[(R1)^{UD} \land (R4) \Rightarrow (R3)^{--}\] \quad [18, (12)], \quad (8)
\[(R1)^{UD} \land (R2) \land (R4) \Rightarrow (R3)\] \quad (combining (7) and (8)). \quad (9)

\(^1\) The name is adopted from [1, p. 101]. It appeals to greedoids, a type of combinatorial structure to be dealt with in Section 6.
Here UM, respectively UD, signify that the rank function $r$ is defined on a upper semimodular, respectively upper distributive, lattice.

As to Lemma 3(c), we invite the reader to verify $F[r[F]] = F$ and $r[F[r]] \leq r$, which boil down to a pleasant juggling of the definitions. The other parts of Lemma 3 will be established along the way in Section 6 as special cases of various generalizations. It follows from Lemma 3 and (9) that each DSM, alias $(\mathcal{D}, r)$, is a Faigle-WSRF. The converse fails. Fig. 4 displays the smallest instance of a Faigle-WSRF $(\mathcal{D}, r)$ which is not a DSM. Obviously $(R^1)^D$, $(R2)$, $(R3)$ are satisfied but $(R4)$ fails:

**Example 4.** Consider the DSM $(E, \leq, \mathcal{F})$ from Example 3. If we define $r: \mathcal{D}(E, \leq) \rightarrow \mathbb{N}$ as in (5), then e.g. $r([b, c, d]) = 2$ since $[b, c]$ (also $[b, d]$) is a maximal member of $\mathcal{F}$ contained in $[b, c, d]$. In fact, $r$ as defined above coincides with the rank function considered in Example 1. By the above $r$ is submodular and satisfies the greedy chain property. For instance, $r(D) = 1 < 3 = r(U)$, and indeed, e.g. the chain $D < K < S < T < U$ does the job since the corresponding ranks are $1 < 2 < 3 = 3$. Observe that $D < F < G < M < U$ is not "greedily" increasing since the ranks are $1 < 2 = 2 < 3 = 3$.

As to Lemma 3(b), let us focus on the rank function $r$ defined in Example 1 and discuss $r$-independency of order ideals. For instance, the order ideal $V = [b, c, h]$ is $r$-independent because $r(V) = 3$ and its lower covers in $\mathcal{D}(E, \leq)$, namely $I = [b, h]$ and $C = [b, c]$, have rank 2. On the other hand, $M = [b, c, d, h]$ is $r$-dependent since for the lower cover $L = [b, d, h]$ we have $r(L) = r(M)$.

It follows from (2) that the flat lattice $\mathcal{L}$ of a Faigle-WSRF $(\mathcal{D}, r, \mathcal{L})$, considered on its own, is upper semimodular. Conversely, let $\mathcal{L}$ be any upper semimodular lattice. Then there is a Faigle-WSRF whose flat lattice is isomorphic to $\mathcal{L}$. To see this, put $(E, \leq) := (J(\mathcal{L}), \leq)$ and $J(X) := \{p \in E : p \leq X\}$ for all $X \in \mathcal{L}$. We shall identify $\mathcal{L}$ with the $\wedge$-subsemilattice $\{J(X) : X \in \mathcal{L}\}$ of $\mathcal{L}' := \mathcal{D}(E, \leq)$. Thus $J(\mathcal{D}') = J(\mathcal{L})$ by construction. Let us call $\mathcal{D}'$ the distributive hull of $\mathcal{L}$. If $r'$ is the rank function associated to $\mathcal{L}'$, then $r': \mathcal{D}' \rightarrow \mathbb{N}$ satisfies $(R1)^D$, $(R2)$, $(R3)$ by [18, Theorem 5(b)]. So $(\mathcal{D}', r', \mathcal{L}')$ is a Faigle-WSRF.

Recall that a WSRF $(\mathcal{D}, \mathcal{L})$ is simple if $J(\mathcal{D}) = J(\mathcal{L})$. By the above each Faigle-WSRF $(\mathcal{D}, \mathcal{L})$ admits a simple Faigle-WSRF $(\mathcal{D}', \mathcal{L}')$ with an isomorphic flat lattice. For matroids this is all well known.

**Example 5.** The Faigle-WSRF $(\mathcal{D}, \mathcal{L})$ in Example 2 is not simple because $C \in J(\mathcal{D})$ but $C \notin J(\mathcal{L})$. In order to "simplify" it we put (Fig. 5)
on its own feet. The upper semimodularity of \( \mathcal{L} \) is verified at once. Its join irreducibles are highlighted. Let us compute the distributive hull \( \mathcal{D}' \) of \( \mathcal{L} \) which yields the simple Faigle-WSRF \( (\mathcal{D}', \mathcal{L}) \) (Fig. 6):

4. Simple DSMs

Suppose in particular that the Faigle-WSRF \( (\mathcal{D}, r) \) is a DSM. Surprisingly it is not obvious (not to the author) whether the simple Faigle-WSRF \( (\mathcal{D}', r') \) is again a DSM! It will take a couple of lemmas to establish this fact. These
lemmas are based on [10], yet we enhance the notation and add some helpful pictures. Why do we strive for a simple DSM? Because then the closure operator becomes cl(X) = \bigvee J(X) which helps the characterization\(^3\) of the flat lattice in Section 5. To unclutter notation we shall henceforth write a for the join irreducible J(a) of D(E, \leq).

**Lemma 4.** Let (E, \leq, \mathcal{F}) be a DSM. Then the set \( S := \{a \in E \mid a \notin \mathcal{F} \} \) of “loops” is an order filter of (E, \leq), and (E - S, \leq, \mathcal{F}) is a loopless DSM with isomorphic flat lattice.

**Proof.** That \( S \) is an order filter of (E, \leq) is trivial in view of (I1)\(^{SM} \). Moreover \( A \subseteq E' := E - S \) for all \( A \in \mathcal{F} \), so (E', \leq, \mathcal{F}) is again a DSM, and trivially without loops. Let \( \mathcal{L}' \) and \( \mathcal{L} \) be the flat lattices of (E', \leq, \mathcal{F}) and (E, \leq, \mathcal{F}), respectively. By Theorem 2 (applied to \( C = E - S \)) the map \( f : \mathcal{L} \to \mathcal{L}' : X \mapsto X - S \) is well defined and onto. We claim that \( X \leq Y \iff f(X) \leq f(Y) \) for all \( X, Y \in \mathcal{L} \). The implication \( \Rightarrow \) being trivial, suppose \( X - S \subseteq Y - S \). If one had \( X \cap S \not\subseteq Y \cap S \), then \( Y \cup (X \cap S) = Y \cup (X \cap S) \cup (X - S) = Y \cup X \) would be an order ideal properly containing \( Y \). Obviously (why?) \( r(Y \cup (X \cap S)) = r(Y) \). This contradicts \( Y \in \mathcal{L} \). Hence \( X \cap S \subseteq Y \cap S \), which implies \( X = (X - S) \cup (X \cap S) \subseteq (Y - S) \cup (Y \cap S) = Y \). It follows that \( f \) is a lattice isomorphism.

**Lemma 5.** Let (E, \leq, \mathcal{F}) be a DSM without loops. If \( a \in E \) and \( Y < a \) in \( D(E, \leq) \), then \( Y \) is closed.

**Proof.** Consider first the special case \( a_* := a - \{a\} \), i.e. \( a_* \) is the unique lower cover of the join irreducible \( a \) of \( D := D(E, \leq) \). Because there are no loops and because of (I1)\(^{SM} \) one has \( r(a) = |a| > |a_*| = r(a_*) \). Assume \( a_* \) were not closed, i.e. not \( r \)-maximal. Then there is a \( b \in E, b \neq a \), with \( a_* < a_* \lor b \) in \( D \) and \( r(a_* \lor b) = r(a_*) \). Notice that \( a < a \lor b \) by the distributivity of \( D \). By submodularity (R3), \( r(a \lor b) = r(a) \), see Fig. 7. It follows that

\[
r(a \lor b) - r(b) = r(a) - r(b) = |a| - |b| = |a \lor b| - |b| - 1.
\]

Because the intervals \([a \land b, a]\) and \([b, a \lor b]\) are isomorphic, the unique lower cover of \( a \lor b \) within \([b, a \lor b]\) is \( a_* \lor b \). In view of (10) and (R4) one therefore has \( r(a_* \lor b) = r(a \lor b) \). But this contradicts \( r(a \lor b) > r(a_*)_\). Hence

\(^3\)Pezzoli and Ross attempted a characterization of the flat lattice in [10, Theorem 3.2.3] which was shown to be false in [16, p. 8]. Our approach in Section 5 will be along different lines, i.e. based on the rank function rather than the behaviour of the hyperplanes.
the lower cover \(a_s \prec a\) must be closed. Now let \(Y < a\) be arbitrary. Then \(Y \leq a_s\). Because \(Y \leq \text{cl}(Y) \leq a_s\), and all of them are independent by \((11)^{SM}\), it follows that \(Y = \text{cl}(Y)\). \(\Box\)

**Lemma 6.** For each DSM \((E, \leq, \mathcal{F})\) there is a DSM \((E', \leq, \mathcal{F}')\) with an isomorphic flat lattice and the following property: if \(a \in E'\) and \(Y \leq a\) in \(\mathcal{D}(E', \leq)\), then \(Y\) is closed.

**Proof.** By Lemma 4 we may assume that all join irreducibles \(a\) are independent. By Lemma 5 only maximal elements \(a\) of \((E, \leq)\) might give nonclosed \(a\). For such \(a\) we show in a moment that:

For all \(A \in \mathcal{D}(E, \leq)\) there is a \(B \in \mathcal{D}(E, \leq)\) with \(B \not\supset a\) and \(\text{cl}(B) = \text{cl}(A)\). \(11\)

Assuming \((11)\), put \(E' := E - \{a\}\) and \(\mathcal{F}' := \{I - \{a\}\}I \in \mathcal{F}\). Because of the maximality of \(a\) one has \(\mathcal{F}' \subseteq \mathcal{F}\). Having checked \((12)'\) (rather than \((12)^{SM}\)) we welcome a new DSM \((E', \leq, \mathcal{F}')\). Let \(\mathcal{L}\) and \(\mathcal{L}'\) be the flat lattices of \((E, \leq, \mathcal{F})\), respectively \((E', \leq, \mathcal{F}')\). Consider the map \(f: \mathcal{L} \to \mathcal{L}': X \mapsto X - \{a\}\) (so \(f(X) = X\) if \(a \notin X\)). By Theorem 2 (applied to \(C = E - \{a\}\)) \(f\) is well defined and onto. We claim that \(X \leq Y \iff f(X) \leq f(Y)\) for all \(X, Y \in \mathcal{L}\). The implication \(\Rightarrow\) being trivial, suppose one had \(f(X) \leq f(Y)\) but \(X \not\leq Y\). Then necessarily \(a \in X, a \notin Y, X - \{a\} \leq Y\). Thus \(X - \{a\} = X \cap Y\) is closed. Let \(A := X\). By \((11)\) there is an ideal \(B\) of \(E - \{a\}\) with \(\text{cl}(B) = \text{cl}(A)\). But \(B \leq X - \{a\}\) implies \(\text{cl}(B) \leq \text{cl}(X - \{a\}) = X - \{a\}\). This contradiction shows that \(X \leq Y\). It follows that \(f\) is a lattice isomorphism. Iteratively dropping nonclosed (maximal) join irreducibles one eventually arrives\(^4\) at a DSM with an isomorphic flat lattice and all join irreducibles closed. So Lemma 6 follows from Lemma 5.

**Proof of \((11)\).** We may assume that \(a \leq A\), for otherwise we may take \(B := A\). Because the maximal element \(a\) of \((E, \leq)\) yields a nonclosed \(a\), there is a \(b \in E\) with \(a \not\prec a \vee b\) and \(r(a \vee b) = r(a)\). By maximality \(b \not\supset a\). It follows that \(a_s \vee b \neq a \vee b\) (why?), whence \([a, a \vee b]\) and \([a_s, a_s \vee b]\) are distinct transposed intervals.

**First case:** \(b \not\leq A\). Then the configuration is depicted in Fig. 8. Note that \(A - \{a\}\) is an order ideal because of the maximality of \(a\). It may be that \([A - \{a\}, A]\) coincides with \([a_s \vee b, a \vee b]\). Since \(a_s\) is closed (Lemma 5), it follows from \(a_s \prec a_s \vee b\) and \((R2)\) that \(r(a_s \vee b) = r(a_s) + 1\), and so \(r(a_s \vee b) = r(a)\). In view of the latter, it follows from \(r(a \vee b) = r(a)\) that \(r(a \vee b) = r(a_s \vee b)\). This implies \(r(A) = r(A - \{a\})\) by submodularity. Hence \(\text{cl}(A) = \text{cl}(B)\) for \(B := A - \{a\}\).

**Second case:** \(b \not\supset A\). The configuration is shown in Fig. 9. It may happen that \([A - \{a\}, A \vee b]\) coincides with \([a_s, a \vee b]\). Again \(r(a \vee b) = r(a)\) implies \(r(a \vee b) = r(a_s \vee b)\). Now submodularity yields \(r(A \vee b) = r((A \vee b) - \{a\})\). But also \(r(A \vee b) = r(a)\) because of \(r(a \vee b) = r(a)\) and submodularity. Hence \(\text{cl}(A) = \text{cl}(B)\) for \(B := (A \vee b) - \{a\}\). \(\Box\)

\(^4\)Because \(f(X) = X\) or \(f(X) = X - \{a\}\) for all \(X \in \mathcal{L}\), one has in particular \(f(b) = b \in \mathcal{L}'\) for all closed \(b\) of \((E, \leq, \mathcal{F})\) (since \(a \notin b\)). Thus \(b\) remains closed in \((E', \leq, \mathcal{F}')\). Conversely, a nonclosed \(b\) of \((E, \leq, \mathcal{F})\) may well become closed within \((E', \leq, \mathcal{F}')\). Just think about nonsimple matroids.
5. The flat lattice of a DSM

Let \( \mathcal{L} \) be a lattice and \( E := J(\mathcal{L}) \) its poset of join irreducibles. Recall that \( p \in E \) is prime if

\[
(\forall X, Y \in \mathcal{L}) \quad X \vee Y \supseteq p \Rightarrow (X \supseteq p \text{ or } Y \supseteq p).
\]

For a nonprime \( p \in E \) call an ideal \( F \subseteq E \) (so \( F \in \mathcal{D}(E, \leq) \)) a primality failure for \( p \) if \( q \nsubseteq p \) for all \( q \in F \) but \( \sqrt{F \supseteq p} \) in \( \mathcal{L} \). Call a nonprime \( p \in E \) harmless if for all primality failures \( F \) of \( p \) and relevant upper covers \( q \) of \( p \) within \( (E, \leq) \) one has \( \sqrt{F \supseteq q} \). Here “relevant” means that \( q \) must have \( J(q) \subseteq F \cup J(p) \).

**Theorem 7.** A lattice \( \mathcal{L} \) is isomorphic to the flat lattice of a DSM iff \( \mathcal{L} \) is upper semimodular and all nonprime join irreducibles are harmless.

**Proof.** Suppose first \( \mathcal{L} \) is isomorphic to the flat lattice of a DSM \( (\mathcal{D}, r) \) with \( \mathcal{D} = \mathcal{D}(E, \leq) \). Then \( \mathcal{L} \) is upper semimodular as seen at the end of Section 3. By Lemma 6 we may assume that \( \mathcal{L} \subseteq \mathcal{D} \) is a \( \wedge \)-subsemilattice and \( a, a_\ast \in \mathcal{L} \) for all \( a \in E \). By [18, Theorem 6] one then has \( J(\mathcal{D}) = J(\mathcal{L}) \simeq E \), so \( (\mathcal{D}, r) \) is simple. Also by [18, Theorem 6] one has \( \mathrm{cl}(A) = \bigvee A \) where \( \bigvee \) denotes the join within \( \mathcal{L} \), and \( r(A) = \| \bigvee A \| \) for all \( A \in \mathcal{D} \). Here \( \| \| \) yields the height within \( \mathcal{L} \). Now, for any nonprime \( p \in J(\mathcal{L}) \) fix a primality failure \( F \in \mathcal{D} \) and an upper cover \( q \) of \( p \) within \( (E, \leq) \) such that \( q \subseteq F \cup p \). We have to show that \( q \leq \bigvee F \) in \( \mathcal{L} \). Put \( A := F \cup p_* \), \( B := A \cup \{p\} \), and \( C := A \cup \{p, q\} \). Clearly \( A, B \in \mathcal{D} \), but also \( C \in \mathcal{D} \) since \( q \subseteq F \cup p \). Obviously \( A \prec B \prec C \in \mathcal{D} \). Because \( A \cup \{q\} \) is not an order ideal, the set \( \{A, B, C\} \subseteq \mathcal{D} \) is a 3-element interval. From the assumption \( p \leq \bigvee F \) follows \( r(A) = r(B) \). Hence \( r(B) = r(C) \) by the greedy chain property (R4). But this forces \( q \leq \bigvee F \) in \( \mathcal{L} \).

Before we come to the converse, we mention from [18, (10)] that for every locally submodular map \( f \) the greedy chain property (R4) is equivalent to this very local greedy chain property:

\[(R4)^- \quad \text{For all 3-element intervals } [A, C] = \{A \prec B \prec C\} \text{ it follows from } f(A) < f(C) \text{ that } f(A) < f(B).\]

Now let \( \mathcal{L} \) conversely be an upper semimodular lattice with all nonprime \( p \in J(\mathcal{L}) =: E \) harmless. Let \( (\mathcal{D}, r, \mathcal{L}) \) be the associated simple Faigle-WSRF, where \( \mathcal{D} := \mathcal{D}(E, \leq) \) is the distributive hull of \( \mathcal{L} \) (see end of Section 3). Thus \( J(\mathcal{D}) = J(\mathcal{L}) \) and \( \mathrm{cl}(A) = \bigvee A \) for all \( A \in \mathcal{D} \). It remains to verify \((R4)^-\). Thus assume that \( B = F \cup \{p\}, C = F \cup \{p, q\} \) for some \( p, q \in E \), and that \( r(F) = r(B) \). The fact that \( \{F, B, C\} \) is a 3-element interval forces \( q \) to be an upper cover of \( p \) within \( (E, \leq) \), so \( q \subseteq F \cup p \). From \( r(F) = r(B) \) follows \( p \leq \bigvee F \), so \( F \) is a primality failure for \( p \). Since \( p \) is harmless by assumption, one must have \( q \leq \bigvee F \), which implies \( r(B) = r(C) \). \( \square \)
Observe that each distributive lattice \( \mathcal{L} \) satisfies the condition of Theorem 7 because all \( p \in J(\mathcal{L}) \) are prime. Ditto each geometric lattice \( \mathcal{L} \) satisfies it because there are no upper covers \( q \) of \( p \). These flat lattices correspond to the so-called poset greedoids (see Section 6), respectively, to matroids.

In order to characterize the flat lattices of polymatroids we need two lemmata. For \( X \in \mathcal{L} \) call a maximal element \( p \) of \( (J(X), \leq) \) special with respect to \( X \), if there is a \( q \in J(\mathcal{L}) \) with \( p \leq q = q \wedge X \). For instance, let \( \mathcal{L} \) be the lattice in Example 5. Then \( H \) is special with respect to \( I \) since \( H \leq N = N \wedge I \).

**Lemma 8.** Let \( \mathcal{L} \) be isomorphic to the flat lattice of a DSM. Then the following holds:

All elements special with respect to \( X \in \mathcal{L} \) are prime in the interval \([0, X]\). \((12)\)

**Proof.** As in the proof of Theorem 7, there is a simple DSM \( (\mathcal{D}, r) \), \( \mathcal{D} := D(E, \leq) \), with \( \mathcal{L}[r] \cong \mathcal{L} \). We may identify \( \mathcal{L} = \mathcal{L}[r] \subseteq \mathcal{D} \). Consider any \( X \in \mathcal{L} \) and a maximal element \( p \in X \subseteq E \) which admits a \( q \in E \) with \( p \leq q = q \wedge X \) in \( \mathcal{D} \). Observe that \( A := X - \{p\} \) and \( C := X \cup \{q\} \) are order ideals of \( (E, \leq) \). Because of \( p < q \) the set \( A \cup \{q\} \) is not an order ideal. Hence \( A < X < C \) constitutes a 3-element interval in \( \mathcal{D} \). Because \( r(X) = \|X\| < \|X \wedge q\| = r(C) \), it follows from (R4) that \( r(A) < r(X) \). But this means \( p \nleq \sqcup A \) in \( \mathcal{L} \), i.e. \( p \) is prime in the interval \([0, X]\) of \( \mathcal{L} \). \( \square \)

Let \( (E, \leq) \) be a poset such that \( p < q, p < t \), implies \( q \leq t \) or \( t \leq q \). Then \( (E, \leq) \) is a disjoint union of “downwards” trees. For convenience, call it a downwards forest.

**Lemma 9.** Let \( \mathcal{L} \) be a semimodular lattice satisfying the necessary condition (12) on special elements. If \( (J(\mathcal{L}), \leq) \) is a downwards forest, then \( \mathcal{L} \) is isomorphic to the flat lattice of a DSM.

**Proof.** Putting \( (E, \leq) := (J(\mathcal{L}), \leq) \) let \( \mathcal{D} := D(E, \leq) \) be the distributive hull of \( \mathcal{L} \) and \( (\mathcal{D}, r) \) the simple Faigle-WSRF with, up to isomorphism, \( \mathcal{L} \subseteq \mathcal{D} \) as \( \wedge \)-subsemilattice. We have to verify (R4). Thus, assuming \( r(B) < r(C) \), let us check that \( r(A) < r(B) \). Let \( p, q \in E \) be such that \( B = A \cup \{p\} \) and \( C = A \cup \{p, q\} \). By assumption \( A \cup \{q\} \) is not an order ideal, whence \( p < q \) and \( q^* \leq B \) must take place. From \( r(B) < r(C) \) follows \( q^* \subseteq X := \sqcup B \). We claim that \( p^* \) is special with respect to \( X \). First, note that \( p^* \leq q^* = q \wedge X \). Suppose there was a \( p' \in E \) with \( p < p' \leq X \). Since \((E, \leq) \) is a downwards forest, it follows from \( p < q \) that either \( q \leq p' \) or \( p' < q \). The former case contradicts \( q \nleq X \). The second case implies \( p' \leq q^* \), contradicting the fact that \( p^* \) is a maximal element of the order ideal \( B \). Thus \( p^* \) is special with respect to \( X \). Putting \( A' := X - \{p\} \in \mathcal{D} \), it follows from (12) that \( p^* \nleq \sqcup A' \). A fortiori \( p^* \nleq \sqcup A \), whence \( r(A) < r(B) \). \( \square \)

A polymatroid is an ordered pair \((P, g)\) where \( P \) is a finite set and \( g: \mathcal{B}(P) \rightarrow \mathbb{N} \) is a submodular map \([7, \text{p. 18}]\). Note \((P, g)\) is a matroid iff \( g(\{x\}) \leq 1 \) for all \( x \in P \). A polymatroid \((P, g)\) may be viewed as a DSM as follows \([14, \text{p. 338}]\): for all \( x \in P \) let \( C_x \) be a chain of cardinality \( g(\{x\}) \) and define the poset \((E, \leq)\) as the disjoint union of all chains \( C_x \) \((x \in P)\). Call an ideal \( I \subseteq E \) independent if \( \sum_{x \in S} |I \cap C_x| \leq g(S) \) for all \( S \subseteq P \). Letting \( \mathcal{B} \) be the family of all independent ideals \( I \subseteq E \) it turns out that \((E, \leq, \mathcal{B})\) is a DSM. If \( g(\{x\}) \neq 0 \) for all \( x \in P \), then the DSM is simple, and so its flat lattice \( \mathcal{L} \) has a poset \((J(\mathcal{L}), \leq)\) of join irreducibles which is isomorphic to \((E, \leq)\).

**Theorem 10.** Let \( \mathcal{L} \) be an upper semimodular lattice such that \((J(\mathcal{L}), \leq)\) is a disjoint union of chains. Then \( \mathcal{L} \) is isomorphic to the flat lattice of a polymatroid iff for all \( X \in \mathcal{L} \):

\[
\text{The maximal elements of } J(X) \text{ which are nonmaximal in } J(\mathcal{L}), \text{ are prime in } [0, X]. \quad (13)
\]

**Proof.** Let \( X \) be any element of an upper semimodular \( \mathcal{L} \) with the property that \((J(\mathcal{L}), \leq)\) is a disjoint union of chains. We claim that the maximal elements of \((J(X), \leq)\), which are nonmaximal in \((J(\mathcal{L}), \leq)\), are exactly the special elements with respect to \( X \). Trivially (independent of the shape of \( J(\mathcal{L}) \)) each element \( p \) which is special with respect to \( X \) is maximal in \( J(X) \) and nonmaximal in \( J(\mathcal{L}) \). Conversely, let \( p \) be maximal in \((J(X), \leq)\) but nonmaximal in \((J(\mathcal{L}), \leq)\). Take \( q \) as the unique upper cover of \( p \) in \((J(\mathcal{L}), \leq)\). Since \((J(\mathcal{L}), \leq)\) is a union of chains, we have \( p = q_* = q \wedge X \). Whence \( p \) is special. By the above, the necessity of (13) follows from Lemma 8, and the sufficiency of (13) follows from Lemma 9. \( \square \)
A lattice \( \mathcal{L} \) is strong [13, 4.6] if \( X \vee p \geq p \) implies \( X \geq p \) for all \( p \in J(\mathcal{L}) \) and \( X \in \mathcal{L} \). For instance, all “generalized matroid lattices” in the sense of [13, p. 258] are strong semimodular lattices. They comprise all geometric, and all modular lattices. It follows at once from the definition of harmlessness and Theorem 7 that the nonprime elements in any polymatroid flat lattice \( \mathcal{L} \) constitute an order filter of \( (J(\mathcal{L}), \leq) = C_1 \cup \cdots \cup C_n \). One easily sees that in a strong semimodular polymatroid flat lattice \( \mathcal{L} \) the nonprime elements also constitute an order ideal of \( (J(\mathcal{L}), \leq) \). Thus the set of nonprime (respectively prime) elements is a union of some \( C_i \).

Let us mention that the “fundamental example” of a DSM in [1, p. 104] is actually a case of polymatroids which can be proven quickly [16, p. 7] with a matching result of Rado e.g. stated in [14, p. 98]. Polymatroids also feature prominently in [15]. Are there “real life” applications of DSMs beyond the polymatroid case?

6. Selectors, greedoids and supermatroids

We conclude this article by demonstrating how DSMs also nicely fit the framework of selectors, greedoids, and supermatroids, respectively.

We have seen that the DSMs are exactly those Faigle-WSRFs that satisfy the greedy chain property (R4). Thus, in view of (9), the DSMs are exactly the WSRFs satisfying (R1), (R2), (R4).

A selector is a WSRF \((A, r)\) such that:

\((R1)_{UD}\) \(A\) is a (locally) upper distributive lattice with a rank function \(r : A \to \mathbb{N}\);

\((R2)\) \(r\) is unit increase;

\((R4)\) \(r\) satisfies the greedy chain property.

These structures were introduced by Crapo in [3]. In order to avoid trivial cases we shall at times assume that \((A, r)\) is loopless, i.e. \(r(A) = 0 \Rightarrow A = 0\). In any case, it follows from (9) that \(r\) is submodular. Clearly a selector \((A, r)\) is a matroid iff \(A = \mathcal{B}\) is Boolean, and it is a DSM if \(A = \mathcal{D}\) is distributive. Furthermore, the so-called antimatroids can be considered as selectors \((A, r)\) where \(r(A) = |A|\) for all \(A \in A\). In line with general WSRFs, the flat lattice of a selector is defined as the \(\wedge\)-subsemilattice \(\mathcal{L} \subseteq A\) of \(r\)-maximal elements, and \((A, r)\) is simple if \(J(A) = J(\mathcal{L})\). By (2) flat lattices of selectors are always upper semimodular. Conversely, every upper semimodular lattice \(\mathcal{L}\) determines, up to isomorphism, a unique simple selector [3, p. 248]. Thus upper semimodular lattices are “universal” for both Faigle-WSRFs and selectors, but not for DSMs (Theorem 7).

Each distributive lattice is isomorphic to the lattice of order ideals of a poset. More generally, a lattice is upper distributive iff \(A\) is isomorphic to a set system \(A \subseteq \mathcal{B}(E)\) which is closed under unions and accessible in the sense that for all \(A \in A\), \(|A| = n\), there is a chain \(\emptyset \subset A_1 \subset A_2 \subset \cdots \subset A_n = A\) with \(A_i \in A\) and \(|A_i| = i\) \((1 \leq i \leq n)\). Thus, we speak of a “concrete” selector \((A, r)\) whenever \(A \subseteq \mathcal{B}(E)\). Such \((A, r)\) is loopless iff \(E \in A\). The sets in \(A\) are called partial alphabets [3]. Call \(I \in A\) independent (or: \(r\)-independent) if \(r(I) = |I|\). The family \(\mathcal{F} = \mathcal{F}[r]\) of all independent sets is a certain subset \(\mathcal{F} \subseteq \mathcal{B}(E)\) which enjoys these properties:

\((I1)\) \text{ for all } I \in \mathcal{F} \text{ there is some } a \in I \text{ with } I - \{a\} \in \mathcal{F};

\((I2)\) \(|I| = |K|\) for all maximal \(I, K \in \emptyset, S \cap \mathcal{F}\) and all \(S \in \mathcal{B}(E)\).

For instance, suppose our concrete selector happens to be a DSM \((\mathcal{D}, r)\) with \(\mathcal{D} \subseteq \mathcal{B}(E)\). Then the truth of \((I1)\) is particularly plausible: since all \(I \in \mathcal{F}\) are order ideals, the set \(I - \{a\}\) can be in \(\mathcal{F}\) only for few choices \(a \in I\), namely the \(\leq\)-maximal elements \(a\) of \(I\). That in fact all \(\leq\)-maximal elements do the job is guaranteed by \((I1)^{\text{SM}}\).

Here comes another parallel between DSMs and selectors. If \((E, \leq, \mathcal{F})\) is a loopless DSM, then each principal order ideal is in \(\mathcal{F}\), and so each order ideal of \((E, \leq)\) is a union of members of \(\mathcal{F}\), i.e. \(\bigcup \mathcal{F} = \mathcal{D}(E, \leq)\). The analogue, and more, for selectors reads [3, Theorem 4]: if \(\mathcal{F} = \mathcal{F}[r]\) arises as the family of \(r\)-independent sets of a concrete loopless selector \((A, r)\), then \(A\) and \(r\) can be retrieved from \(\mathcal{F}\) alias

\[
A = \bigcup \mathcal{F}, \text{ respectively, } r(A) = \max\{|I| : I \in \mathcal{F}, I \subseteq A\}.
\]

This covers, in particular, the part \(r[\mathcal{F}[r]] = r\) in Lemma 3(c).

---

5 The property of \(r\) that \(r = | \cdot |\) could be coined \((R2)^{\dagger}\) and, by the way, the property of \(r\) being “tight” that was mentioned after Theorem 1, could be coined \((R2)^{\dagger}\). For the sake of completeness, recall that we also considered \((R3)^{\dagger}\), \((R3)^{\dagger}\), \((R3)^{\dagger}\) and \((R4)^{\dagger}\), \((R4)^{\dagger}\).
What about structures \((E, \mathcal{F})\) where \(\mathcal{F} \subseteq \mathcal{B}(E)\) is any set system satisfying (11)\(^{-}\) and (12) (thus not necessarily arising from a selector)? Such a pair \((E, \mathcal{F})\) is called a greedoid. One can show that a greedoid \((E, \mathcal{F})\) yields a selector via (14) if and only if for all \(I_1, I_2, K \in \mathcal{F}\) it follows from \(I_1, I_2 \subseteq K\) that \(I_2 \cup I_2 \subseteq \mathcal{F}\). Such a \((E, \mathcal{F})\) is also called an interval greedoid. Antimatroids and DSMs are the special cases of interval greedoids where \(\mathcal{F}\) is closed under arbitrary unions, respectively intersections.\(^{6}\) A *poset greedoid* is a DSM \((E, \leq, \mathcal{F})\) with \(\mathcal{F} = \mathcal{D}(E, \leq)\). A greedoid is a poset greedoid if and only if it simultaneously is a DSM and antimatroid.

How do Faigle-WSRFs fit the picture? In [5] Ulrich Faigle defined a quasi-geometry as a poset \((E, \leq)\) endowed with a closure operator \(cl: \mathcal{B}(E) \to \mathcal{B}(E)\) such that for all \(a, b \in E\) and all subsets \(S \subseteq E\):

\[
\begin{align*}
(F1) & \quad a \leq b \Rightarrow a \in cl(\{b\}). \\
(F2) & \quad \text{If } b \not\in S \text{ but } b \in cl(S \cup \{a\}) \text{ and } (J(a) - \{a\}) \subseteq cl(S), \text{ then } a \in cl(S \cup \{b\}).
\end{align*}
\]

To cite from [3, p. 249]: axiom (F1) guarantees that only order ideals can be closed, while axiom (F2) guarantees that the closure operator has the exchange property with respect to covering pairs in the lattice of order ideals of \((E, \leq)\). Influenced by these remarks, and disliking the clumsy \((F2)\), the author decided to shrink \(\mathcal{B} = \mathcal{B}(E)\) to \(\mathcal{D} = \mathcal{D}(E, \leq)\) and thereby obtained in (4) the more appealing\(^7\) concept of a Faigle-WSRF \((\mathcal{D}, cl')\) (where \(cl'\) is the restriction of \(cl\) to \(\mathcal{D}\)). Note that one cannot retrieve the full quasi-geometry \((\mathcal{B}, cl)\) from \((\mathcal{D}, cl')\), but at least the poset \((E, \leq)\). Does it matter? Quasi-geometries are also stubborn beyond their definition. For instance, while each quasi-geometry \((\mathcal{B}, cl)\) with underlying poset \((E, \leq)\) gives rise, naturally enough, to a certain greedoid \((E, \mathcal{F})\), the intrinsic characterization (not mentioning \(\leq\)) of the arising type of greedoid \([7, \text{p. 110}]\) is only marginally nicer\(^8\) than \((F2)\). What’s more, now the “original” poset \((E, \leq)\) can no longer be retrieved from \((E, \mathcal{F})\). In other words, quasi-geometries on nonisomorphic posets may yield the same greedoid. We could continue frowning upon nonequivalent notions of independency, respectively rank, associated with quasi-geometries, but let us close on a positive note. If nothing else, Faigle’s quasi-geometries kindled to considerable extent the development of selectors [3, p. 234], greedoids [7, Chapter VIII], and, for that matter, WSRFs.

Let us now deal with supermatroids, introduced in [19]. A pair \((\mathcal{S}, \mathcal{F})\) is a supermatroid on \(\mathcal{S}\) if \(\mathcal{S} = (\mathcal{S}, \leq)\) is a poset with 0 and height function \(|\cdot|\), and \(\mathcal{F} \subseteq \mathcal{S}\) is a subset such that:

\[
\begin{align*}
(11)_{SM} & \quad \text{For all } I \in \mathcal{F}, \text{ if } K \in \mathcal{S} \text{ and } K \leq I, \text{ then } K \in \mathcal{F}; \\
(12)_{SM} & \quad |I| = |K| \text{ for all maximal } I, K \in [0, A] \cap \mathcal{F} \text{ and all } A \in \mathcal{S}.
\end{align*}
\]

The name derives from the fact that matroids \(M(E)\) can be considered as supermatroids on \(\mathcal{S} = \mathcal{B}(E)\) if \(\mathcal{F}\) is taken as the family of all independent subsets of \(E\). Ditto our friends the DSMs, are supermatroids on a distributive lattice \(\mathcal{S} = \mathcal{D}(E, \leq)\), which explains the abbreviation DSM. In general, \(\mathcal{F}\) is not a family of subsets of some set \(E\), yet one still refers to the members \(I \in \mathcal{S}\), that happen to be in \(\mathcal{F}\), as being independent. Clearly condition \((12)_{SM}\) does not make sense for general supermatroids. It is natural to call a supermatroid *loopless* if \(\vee \mathcal{F}\) is the (existing) top element of \((\mathcal{S}, \leq)\).

Lemma 3 states an optimal compatibility of the independence and rank function point of view of DSMs. How much of this extends to general supermatroids? Given a supermatroid \((\mathcal{S}, \mathcal{F})\), we restate (5) and define

\[
r = r[\mathcal{F}] \quad \text{by } r(A) := \max\{|I|: I \in \mathcal{F}, I \leq A\}.
\]

From \((11)_{SM}\) it is clear that \(r: \mathcal{S} \to \mathbb{N}\) is monotone, whence a rank function, and \((12)_{SM}\) says that \(r(A)\) equals the height of any maximal independent element below \(A\). In a WSRF \((\mathcal{D}, r, \mathcal{L})\) each of \(r\) and \(\mathcal{L}\) determines the other.

\(^{6}\) The family \(\mathcal{F}\) of a DSM \((E, \leq, \mathcal{F})\) is trivially closed under intersections because of \((11)_{SM}\) and because intersections of order ideals are order ideals. That conversely every greedoid \((E, \mathcal{F})\) with \(\mathcal{F}\) closed under intersections comes from a DSM \((E, \leq, \mathcal{F})\), was independently discovered by [7, p. 128] and by the author (mentioned in [11]).

\(^{7}\) Faigle-WSRFs also tie in nicely with supermatroids, see (20).

\(^{8}\) Arguably the most elegant subclass of the [7, p. 110] type of greedoids is constituted by the greedoids \((E, \mathcal{F})\) with \(\mathcal{F}\) closed under intersections, i.e. DSMs.
With supermatroids, only $F$ determines $r$. Given a rank function $r: \mathcal{S} \to \mathbb{N}$, define the family of $r$-independent elements by

$$F[r] := \{ I \in \mathcal{S} : r(I) = |I| \}.$$ 

**Question:** Which rank functions $r': \mathcal{S} \to \mathbb{N}$ are induced by some supermatroid $(\mathcal{S}, F)$ in the sense that $r' = r[F]$?

Notice that the only $F$ that could possibly do the job is $F := F[r']$. Indeed, from $r' = r[F]$ one gets $F[r'] = F[r[F]]$. As mentioned after Lemma 3, it is a trivial matter to check $F[r[F]] = F$. Thus, we are left with the following question.

**Question:** For what kind of lattices $\mathcal{S}$ and rank functions $r: \mathcal{S} \to \mathbb{N}$ does $F[r]$ yield a supermatroid $(\mathcal{S}, F[r])$?

Here is a sufficient condition:

If $(\mathcal{S}, \leq)$ is a graded lattice and $r: \mathcal{S} \to \mathbb{N}$ a rank function satisfying (R2) and (R4), then $(\mathcal{S}, F[r])$ is a supermatroid. Moreover, $r = r[F[r]]$. \hfill (15)

**Proof of (15).** To see $(11)^{SM}$, pick $I \in F[r]$ (so $r(I) = |I|$) and $K \in \mathcal{S}$ with $K < I$. By definition of rank function $r(K) \leq |K|$. Assuming $r(K) < |K|$, the hypothesis (R2) applied to any maximal chain $K < \cdots < I$ immediately yields the contradiction $r(I) < |I|$. Hence $r(K) = |K|$, and so $K \in F[r]$.

As to $(12)^{SM}$, assume some $B \in \mathcal{S}$ admits maximal $I, K \in [0, B] \cap F[r]$ with $|I| < |K|$. For $B' = I \cup K$ we have $B' \leq B$ and $r(B') \geq r(K) = |K| > |I| = r(I)$. Hence (R4) guarantees some $A_1 \in \mathcal{S}$ with $I < A_1 \leq B'$ and $r(I) < r(A_1)$. From (R2) we get $r(A_1) = |I| + 1$. Since $\mathcal{S}$ is graded, $|I| + 1$ equals $|A_1|$, and so $r(A_1) = |A_1|$. Because $A_1 \in [0, B] \cap F[r]$, this contradicts the maximality of $I$.

As to $r[F[r]] = r$, in view of the trivial inequality $r[F[r]] \leq r$, it suffices to show that each $A \in \mathcal{S}$ admits an $I \in F[r]$ with $I \leq A$ and $r(I) = r(A)$. But this follows at once from (R2) and (R4). \hfill $\blacksquare$

A look at the proof shows that (R2) alone forces $F[r]$ to have $(11)^{SM}$, whether $\mathcal{S}$ is graded or not. However, easy counter examples show that $(\mathcal{S}, F[r])$ may lack $(12)^{SM}$ when the hypothesis “graded” in (15) is dropped. Condition (15) is far from being necessary; there are ranked lattices $(\mathcal{S}, r)$ that lack both (R2) and (R4), yet yield a supermatroid $(\mathcal{S}, F[r])$. Finally, note that Lemma 3(b) is covered by (15).

In the remainder of the article we take the supermatroid $(\mathcal{S}, F)$ for granted and derive, in (16)–(18), depending on $\mathcal{S}$, various properties of $r[F]$. Statement (19) is of a different type in that a property of $r[F]$ implies another property of $r[F]$.

Let $(\mathcal{S}, F)$ be a supermatroid on an upper semimodular lattice $\mathcal{S}$. Then $r := r[F]$ satisfies (R4)$^-$. \hfill (16)

**Proof of (16).** Let $[A, C] = \{ A < B < C \}$ be a 3-element interval of $\mathcal{S}$. We shall derive a contradiction from the hypothesis that $r(A) < r(C)$ but $r(A) = r(B)$. Pick $A' \in F$ with $A' \leq A$ and $r(A) = |A'|$. Because of $r(C) > |A'|$ there is some $C' \in F$ with $A' < C' \leq C$. Clearly $A \wedge C' = A'$. From $A \vee C' \in [A, C] = \{ A, B, C \}$ and $r(A \vee C') > r(A) = r(B)$ follows $A \vee C' = C$. Thus the 2-element interval $[A', C']$ transposes up to the 3-element interval $[A, C]$. This contradicts the upper semimodularity of $\mathcal{S}$. \hfill $\blacksquare$

If $(E, F)$ is a greedoid and $A, r$ are defined as in (14), then $A$ is again union-closed (trivial) and accessible [3, Theorem 1]. Deplorably [3, p. 272], $r: A \to \mathbb{N}$ can lack (R2), and even (R3)$^-$. It seems impossible to characterize greedoids $(E, F)$ in terms of rank functions $r: A \to \mathbb{N}$ on upper distributive lattices. Albeit they are characterized as pairs $(B, r)$ where $B$ is a Boolean lattice and $r$ satisfies (R3)$^-$, such a $(B, r)$ is neither a WSRF nor comparable with a selector $(A, r)$ since $B$ is stronger than $A$, but (R3)$^- \leq$ weaker than (R2) $\wedge$ (R4). However, greedoids nicely fit the framework of supermatroids, due to (I1)$^{SM}$. We first show:

Let $(\mathcal{S}, F)$ be a supermatroid on a upper distributive lattice $\mathcal{S}$.

Then $r := r[F]$ satisfies (R3)$^- \wedge$ and (R4). \hfill (17)

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9 An alternative definition of “$r$-independent” worthwhile investigating would be to postulate that $r(I') < r(I)$ for all $I' < I$ (this concept occurs in [4]).
Proof of (17). Recall that \( S \) can be taken as a set system \( S \subseteq \mathcal{B}(E) \) which is accessible and closed under unions. Hence also \( S \subseteq \mathcal{B}(E) \). From (11) SM follows (11)\(^-\). As to (12), assume for some \( S \in \mathcal{B}(E) \) we had maximal elements \( I, K \in [\emptyset, S] \cap \mathcal{F} \) with \( |I| < |K| \). Because \( S \) is union-closed, the element \( I \cup K \in S \) would give a contradiction with (12) SM. It follows that \( (E, \mathcal{F}) \) is a greedoid. As mentioned above, the corresponding rank function \( r' : \mathcal{B}(E) \to \mathbb{N} \) is locally submodular, and obviously \( (R3)^- \) is inherited by the restriction \( r : S \to \mathbb{N} \) of \( r' \). That \( r' \) satisfies (R4) is also well known, but the transfer to \( r \) is less obvious. In any case, from (15) we know that \( r \) enjoys \((R4)^-\), and according to [18, (10)] we have \((R4)^- \Leftrightarrow (R4)\) for all locally submodular \( r \). □

Inspection of the proof and the preceding remarks shows that a supermatroid \((S, F)\) on an upper distributive lattice \( S \) yields a greedoid \((E, F)\). Conversely, each greedoid \((E, F)\) yields the (loopless) supermatroid \((S, F)\) on the upper distributive lattice \( S := \cup F \).

We mention that supermatroids on lower distributive lattices are called cg-matroids, where cg alludes to convex geometry [12].

Let \((S, F)\) be a supermatroid on a modular lattice \( S \). Then \( r := r[F] \) satisfies (R2).

This has been shown in [19]. We refer to Pezzoli [9] for more about modular supermatroids. Since each distributive lattice is simultaneously upper distributive and modular, the properties (R2), (R4) of a DSM claimed in Lemma 3(a) follow from (17) and (18).

Let \( S \) be any lattice and let \((S, F)\) be a submodular supermatroid [4] in the sense that \( r := r[F] \) satisfies (R3). Then \( r \) also satisfies (R2).

Proof of (19). Consider any \( p \in J(S) \) with unique lower cover \( p_s \). If \( r(p) > r(p_s) \), then necessarily \( p \in F \), whence \( p_s \in F \) and

\[
r(p) = |p| = |p_s| + 1 = r(p_s) + 1
\]

(note that \(|p| = |p_s| + 1\) even for nongraded lattices \( S \)). In order to see that \( r \) is unit increase, take any covering \( A < B \) in \( S \). If \( p \in S \) is any minimal element with the property that \( p \leq B \) but \( p \not\leq A \), then \( p \in J(S) \), \( p \lor A = B \), \( p \land A = p_s \).

Since \( [p_s, p] \) transposes up to \([A, B] \), and \( r(p) \leq r(p_s) + 1 \) by the above, the submodularity (R3) of \( r \) implies that \( r(B) \leq r(A) + 1 \). □

Every submodular supermatroid \((S, F)\) induces a Faigle-WSRF \((S, r')\) where \( S \) is the distributive hull of \( S \) and \( r' \) is the canonical extension of \( r := r[F] \).

Proof of (20). By definition (see Section 3) the distributive hull of \( S \) is \( D := D(E, \leq) \) where \( E, \leq \) is the poset of join irreducibles of \( S \). The only sensible way to extend \( r : S \to \mathbb{N} \) to \( r' : D \to \mathbb{N} \) is by setting \( r'(A) := r(\text{cl}(A)) \), where \( \text{cl} \) is the closure operator associated to the \( \land \)-subsemilattice \( S \subseteq D \). By [18, (18)] the property (R3) of \( r \) is inherited by \( r' \). In order to see that \( r' \) satisfies (R2), pick any \( A < B \) in \( S \). There is a (now unique) \( p \in E \) such that \( B = A \cup \{p\} \). As in the proof of (19), the interval \([p_s, p]\) transposes up to \([A, B] \). From \( r' \) satisfying (R3) it follows again that \( r'(B) \leq r'(A) + 1 \). □

We mention that (20) is reminiscent of [4, Theorem 1] which is phrased in terms of quasi-geometries.

A supermatroid \((S, F)\) is strong [7, p. 108] if \( r[F] \) satisfies (R2) and \((R3)^-\). In particular, by (19), each submodular supermatroid is strong. The converse fails. Nevertheless, by (7), each strong supermatroid on a semimodular lattice is submodular. That is also stated in [7, pp. 108].

Many types of WSRFs \((S, r)\) are displayed in Fig. 10 according to various properties of \( r \) (plus two non-WSFRs that only satisfy \((R3)^-\)). The “e.g. from” in three of the boxes is to be interpreted as follows: say \((S, r)\) satisfies (R1), (R2), (R3). Then \( r \) need not bear a relation to a submodular supermatroid (try \( S := N_5 \) and the obvious \( r \)), yet conversely each strong supermatroid triggers, by (19), a WSRF satisfying (R1), (R2), (R3). The meaning of
“characterizes” is clear. The WSRFs in the shaded boxes are the ones that do not comprise (let alone characterize) a crisp subclass of supermatroids.

References