Deciding nonconstructibility of 3-balls with spanning edges and interior vertices

Satoshi Kamei

Department of Media and Information Systems, Teikyo University of Science and Technology, 2525 Yatsuzawa Uenohara-shi, Yamanashi 409-0193, Japan

Received 20 September 2005; received in revised form 28 February 2007; accepted 15 March 2007
Available online 21 March 2007

Abstract

Constructibility is a combinatorial property of simplicial complexes. In general, it requires a great deal of time to decide whether a simplicial complex is constructible or not. In this paper, we consider sufficient conditions for nonconstructibility of simplicial 3-balls to investigate efficient algorithms for the decision problem.

© 2007 Elsevier B.V. All rights reserved.

Keywords: Simplicial complex; Shellability; Constructibility

1. Introduction

Shellability is a fundamental and important concept for the study of the combinatorics of simplicial complexes. Since the proof of the Upper Bound Conjecture for convex polytopes, due to McMullen [10], many researchers have studied this concept in many fields of combinatorics. Constructibility can be viewed as a relaxation of shellability. Thus each shellable simplicial complex is constructible, but the converse is not necessarily true. Constructibility appears in different combinatorial contexts in [2,3,11]. As in the case of shellability, it can be shown that every constructible pseudomanifold is either a simplicial ball or a simplicial sphere. For the converse, many examples of nonconstructible simplicial balls and spheres were constructed in dimensions greater than 2 [4,5,7–9].

In general, there are no efficient algorithms known to decide whether a simplicial complex is constructible or not. However, for some restricted classes of simplicial complexes, there are tools available to determine constructibility. In [4,7], relations between constructibility of a simplicial 3-ball and the bridge index of a knot which consists of edges of the simplicial 3-ball are described. In [5], Hachimori presented an efficient algorithm to decide constructibility for simplicial 3-balls with less than three interior vertices. In this paper, we consider constructibility of simplicial 3-balls with more than two interior vertices.

In the followings, an edge of a simplicial 3-ball $B$ is called spanning if the edge is not contained in $\partial B$ but both end vertices of the edge belong to $\partial B$. A spanning edge $e$ is called strict if there is no vertex $v$ of $B$ such that both end vertices of $e$ are contained in a single component of the intersection $\partial B \cap \text{Star}_B v$. We notice that Star$_B v$ means the
closed star neighborhood of \( v \) in \( B \). A simplicial 3-ball \( B \) is reduced if each 2-face \( T \) of \( B \) which is not contained in \( \partial B \) has at least two edges which are contained in \( \text{int} \ B \).

The main theorem in this paper is the following.

**Theorem 3.1.** Let \( B \) be a reduced simplicial 3-ball which has interior vertices and spanning edges. If there are no edges whose end vertices are both interior with respect to \( B \) and all spanning edges are strict, then \( B \) is not constructible.

We choose \( B'_1 \) and \( B'_2 \) so that for the first time in the deconstruction of \( B \) a spanning edge appears in the disk \( B'_1 \cap B'_2 \). We show that either this edge is not strict, or there is an edge in \( B'_1 \cap B'_2 \) whose end vertices are interior in \( B \), or \( B'_1 \cap B'_2 \) contains a 2-simplex with at least two edges in \( \partial B \). All cases contradict one of the hypothesis of Theorem 3.1, hence, \( B \) was not constructible.

In Section 2, we prepare some basic notations. In Section 3, we prove the main theorem. In Section 4, we present an example which satisfies the conditions of the main theorem.

### 2. Notations

In this section, we prepare some notations. First we review terminology of simplicial complexes.

A simplicial complex \( C \) is a finite set of closed simplices \( \sigma \) in some Euclidean space such that (1) if \( \sigma \in C \), all the faces of \( \sigma \) (including the empty set) are contained in \( C \), and (2) if \( \sigma, \sigma' \in C \), then \( \sigma \cap \sigma' \) is a face of both \( \sigma \) and \( \sigma' \).

The zero-dimensional simplices in \( C \) are the vertices, the one-dimensional simplices are the edges of \( C \). The inclusion-maximal faces are called facets. The dimension of \( C \) is the largest dimension of a facet. A simplicial \( d \)-complex is short for a \( d \)-dimensional simplicial complex. If all facets of \( C \) have the same dimension, then \( C \) is called pure. In particular, the simplicial complex which only has the empty set as a face is a pure complex of dimension \(-1\), with a single facet.

The union of all simplices of \( C \) is called the underlying space of \( C \) and denote by \(|C|\). If \(|C|\) is homeomorphic to a manifold \( M \), then \( C \) is a triangulation of \( M \). If \( C \) is a triangulation of a \( d \)-ball or a \( d \)-sphere, then \( C \) is simply called a simplicial \( d \)-ball or a simplicial \( d \)-sphere, respectively. For any triangulation \( C \) of a manifold, the boundary simplicial complex \( \partial C \) is the collection of all simplices of \( C \) which lie on the boundary of the manifold.

The interior \( \text{int} \ C \) is the set \( C \setminus \partial C \). A \( d \)-dimensional pure simplicial complex is strongly connected if for any two of its facets \( F \) and \( F' \), there is a sequence of facets \( F = F_1, F_2, \ldots, F_k = F' \) such that \( F_i \cap F_{i+1} \) is a face of dimension \( d - 1 \), for \( 1 \leq i \leq k - 1 \).

If a \( d \)-dimensional pure simplicial complex is strongly connected and each \((d - 1)\)-dimensional face belongs to at most two facets, then it is called a pseudomanifold. Every triangulation of a connected manifold is a pseudomanifold.

For a simplicial complex \( C \) and a face \( \sigma \), the star neighborhood \( \text{Star}_C \sigma \) is the subcomplex of \( C \) which is formed by all faces of facets of \( C \) containing \( \sigma \). The link \( \text{Link}_C \sigma \) is the subcomplex of \( \text{Star}_C \sigma \) which is formed by all faces \( \sigma' \) such that the intersection of \( \sigma' \) and \( \sigma \) is empty. For a simplex \( \sigma \) and a vertex \( v \notin \sigma \), the join \( v * \sigma \) is a simplex whose vertices are those of \( \sigma \) plus the extra vertex \( v \). For a simplicial complex \( C \) and a vertex \( v \notin C \), the join \( v * C \) is defined as \( v * C = \{ v * \tau | \tau \in C \} \cup C \).

Now we recall the definition of constructibility of simplicial complexes. Constructibility is a relaxation of shellability; see [12,13] for the definition of shellability.

**Definition.** A pure \( d \)-dimensional simplicial complex \( C \) is constructible if

1. \( C \) is a simplex, or
2. there exist \( d \)-dimensional constructible subcomplexes \( C_1 \) and \( C_2 \) such that \( C = C_1 \cup C_2 \) and that \( C_1 \cap C_2 \) is a \((d - 1)\)-dimensional constructible complex.

As in the case of shellable pseudomanifolds, a constructible pseudomanifold is a simplicial ball or a simplicial sphere. But the converse is not necessarily true. In general, one needs to check all possible ways to divide a pseudomanifold for deciding whether it is constructible or not at least. We do not know of a more efficient procedure.

The operations discussed in the following proposition were introduced by Hachimori.
Proposition 2.1 (Hachimori [6, Proposition 3]). Let $C$ be a simplicial 3-ball and $T$ be a 2-face contained in $\text{int } C$.

1. Consider the case where all edges of $T$ are contained in $\partial C$. Divide $C$ into two simplicial 3-balls $C_1$ and $C_2$ by $T$. Then $C$ is constructible if and only if $C_1$ and $C_2$ are constructible.

2. Consider the case where some 2-face $T$ is contained in $\text{int } C$ and precisely two of its edges are contained in $\partial C$. Let $e$ be the remaining edge of $T$ contained in $\text{int } C$. We split $T$ into $T'$ and $T''$ such that the intersection of $T'$ and $T''$ is $e$. Denote the resulting simplicial 3-ball by $C'$. Then $C$ is constructible if and only if $C'$ is constructible.

The operations (1) and (2) are indicated in Fig. 1. The thick lines are the edges of $\partial C \cap T$.

From any simplicial 3-ball, we can obtain reduced 3-balls by the operations above. Thus it is essential to consider the constructibility of reduced 3-balls.

3. Main argument

In this section, we prove the main theorem. First we prepare some notation.

Definition. Let $B$ be a simplicial 3-ball. Let $D$ be a simplicial 2-ball which is a subcomplex of $B$. We assume that $D$ contains a spanning edge, that is, there is an edge of $D$ which is not contained in $\partial B$ and both end vertices of the edge belong to $\partial B$. We also assume that $\partial D$ contains no spanning edges. A spanning edge $e$ is outermost if there exists a connected one-dimensional subcomplex $P$ of $\partial D$ such that $P \cup e$ bounds a 2-ball $\Delta$ which is a subcomplex of $D$ and that $\Delta$ contains no spanning edges except $e$. The 2-ball $\Delta$ is called an outermost disk.

We stress here that the notation “spanning” is defined for an edge of a simplicial 3-ball. In this paper, an edge of a simplicial 2-ball is called spanning only if the simplicial 2-ball is a subcomplex of a simplicial 3-ball and the edge satisfies the conditions of spanning edge with respect to the simplicial 3-ball.

Now we prove the main theorem.

Theorem 3.1. Let $B$ be a reduced simplicial 3-ball which has interior vertices and spanning edges. If there are no edges whose end vertices are both interior with respect to $B$ and all spanning edges are strict, then $B$ is not constructible.

Proof. We assume that the reduced 3-ball $B$ is constructible. Let $B'$ be a constructible 3-ball which is a subcomplex of $B$ such that $B' = B'_1 \cup B'_2$ is part of the construction of the constructible 3-ball $B$. Then $B'_1 \cap B'_2$ is a 2-ball, and we call such a 2-ball a divide. We notice that the interior of any divide is contained in the interior of $B$ by constructibility. Thus no interior vertices of a divide belong to $\partial B$. In the following, an edge of a 2-ball which is not contained in the boundary of the 2-ball is called an interior edge. We use the notation “interior edge” only for an edge of a divide or an outermost disk of a divide. Note that any interior edge of a divide and an outermost disk of a divide is not contained in $\partial B$. Hence if both end vertices of an interior edge of a divide belong to $\partial B$, the edge satisfies the conditions of spanning edge.
We choose $B'_1$ and $B'_2$ so that for the first time of the deconstruction of $B$ there appears a spanning edge. Our choice is possible since the constructible ball $B$ can be completely decomposed into simplices. It follows that $\partial B'$ contains no spanning edges, since otherwise such an edge would have been contained in a divide occurring at an earlier stage of the deconstruction of $B$, thus contradicting our choice of $B'_1$ and $B'_2$. In particular, the boundary of the divide $B'_1 \cap B'_2$ contains no spanning edges. We choose an outermost spanning edge $e$, cutting off an outermost disk corresponding to $e$ on the divide $B'_1 \cap B'_2$. Let $P$ be $\partial A \setminus e$.

First we assume that all vertices of the outermost disk $A$ belong to $\partial B$. Then from the notice of the first paragraph, the interior of $A$ contains no vertices. All edges of $P$ are contained in $\partial B$, since otherwise there would be an edge which is not contained in $\partial B$ and both end vertices belong to $\partial B$. Thus it would be a spanning edge and it contradicts the assumption that $\partial B'$ contains no spanning edges. We assume that there is an interior edge of $A$. Notice that the interior edge is not contained in $\partial B$. Furthermore the interior edge connects two vertices of $\partial B$, since all vertices of $A$ belong to $\partial B$. Then the interior edge satisfies the conditions of spanning edge and it contradicts the assumption that $A$ is an outermost disk. Hence all edges of $A$ are contained in $\partial A$, then $A$ is a 2-face. Furthermore two edges of the 2-face are contained in $P$, thus contained in $\partial B$. It contradicts the hypothesis that $B$ is reduced.

Next we assume that the outermost disk $A$ contains at least two vertices which are interior with respect to $B$. Denote two of them by $u_1$ and $u_2$. Then all vertices of the link $\text{Link}_{A}u_1 \ (\text{Link}_{A}u_2)$ belong to $\partial B$ from the hypothesis that there are no edges whose end vertices are both interior with respect to $B$. The link $\text{Link}_{A}u_1 \ (\text{Link}_{A}u_2)$ contains an interior edge of $A$ which separates $u_1$ and $u_2$ on $A$ (Fig. 2(a)–(c)). The end vertices of the interior edge belong to $\partial B$, since all vertices of $\text{Link}_{A}u_1 \ (\text{Link}_{A}u_2)$ belong to $\partial B$. Thus the interior edge satisfies the conditions of spanning edge. It contradicts the assumption that $A$ is an outermost disk.

We assume that the outermost disk $A$ contains exactly one vertex $v$ which is interior with respect to $B$. Assume that $v$ is not contained in $\partial A$. All vertices of $A$ except $v$ are contained in $P$, thus they belong to $\partial B$. If $P$ contains a vertex $v'$ which is not connected with $v$ by an edge of $A$, then the link $\text{Link}_{A}v$ contains an interior edge of $A$ which separates $v$ and $v'$ on $A$. The end vertices of the interior edge belong to $\partial B$. Thus the interior edge satisfies the conditions of spanning edge. It contradicts the assumption that $A$ is an outermost disk. Thus each vertex of $P$ is connected with $v$ by an edge of $A$; see Fig. 2(d). All edges of $P$ are contained in $\partial B$, since $\partial B'$ contains no spanning edges. Then the intersection $\partial B \cap \text{Star}_B v$ contains $P$. Hence the spanning edge $e$ is not strict and it contradicts the hypothesis.

Fig. 2.
Thus the interior edge satisfies the conditions of spanning edge. It contradicts the assumption that except the two edges which are incident to link \( \text{Link} \) \( P_1 \cap P_2 \) of the disk. Thus each vertex of \( 3\)-ball \( B \) contains an interior edge of the divide \( C_1 \cap C_2 \). Furthermore each vertex of the divide \( C_1 \cap C_2 \) is connected with \( v \) by an edge of the divide, since otherwise the link \( \text{Link}_{C_1 \cap C_2} \) \( v \) would contain an interior edge of the divide \( C_1 \cap C_2 \) which separates \( v \) and \( v' \) on \( A \). The end vertices of the interior edge belong to \( \partial B \). Thus the interior edge satisfies the conditions of spanning edge. It contradicts the assumption that \( A \) is an outermost disk. Thus each vertex of \( P \) except \( v \) is connected with \( v \) by an edge of \( A \); see Fig. 2(e). Notice that all edges of \( P \) except the two edges which are incident to \( v \) are contained in \( \partial B \), since \( \partial B' \) contains no spanning edges. Let \( C \) be a constructible 3-ball which is a subcomplex of \( B \) such that \( C = C_1 \cup C_2 \) is part of the construction of the constructible 3-ball \( B \) and that the divide \( C_1 \cap C_2 \) contains the vertex \( v \). Further we assume that \( C \) is decomposed into \( C_1 \) and \( C_2 \) before \( B' \) is decomposed into \( B'_1 \) and \( B'_2 \) in the deconstruction of \( B \). Then \( B'_1 \) is contained in \( C_1 \) or \( C_2 \). This assumption is possible since \( \partial B' \) contains the vertex \( v \). Thus \( \partial C_1 \) and \( \partial C_2 \) contain no spanning edges. The divide \( C_1 \cap C_2 \) contains less than two vertices which are interior with respect to \( B \), since otherwise the link \( \text{Link}_{C_1 \cap C_2} \) \( v \) would contain an interior edge of the divide \( C_1 \cap C_2 \) which separates \( v \) and the other interior vertex on the divide \( C_1 \cap C_2 \) and whose end vertices belong to \( \partial B \). Thus the divide \( C_1 \cap C_2 \) would contain a spanning edge and it contradicts the choice of \( C_1 \) and \( C_2 \). Therefore, the divide \( C_1 \cap C_2 \) contains exactly one vertex which is interior with respect to \( B \). Furthermore each vertex of the divide \( C_1 \cap C_2 \) except \( v \) is connected with \( v \) by an edge of the divide, since otherwise there would exist a spanning edge as in the previous discussions. There are two cases: Fig. 3(a) depicts the case that the vertex \( v \) is not contained in \( \partial C \), and Fig. 3(b) depicts the case that the vertex \( v \) is contained in \( \partial C \). In either case, the link \( \text{Link}_{C_1 \cap C_2} \) \( v \) is a 1-sphere such that all edges are contained in \( \partial B \), since all vertices of the link \( \text{Link}_{C_1 \cap C_2} \) \( v \) belong to \( \partial B \) and there are no spanning edges on \( \partial C_1 \) \( \partial C_2 \). With the notice that all edges of \( P \) except the two edges which are incident to \( v \) are contained in \( \partial B \), the intersection \( \partial B \cap \text{Star}_B \) \( v \) contains a path which connects the end vertices of \( e \) (Fig. 3(c)). Also in this case, the spanning edge \( e \) is not strict, and it contradicts the hypothesis.

We obtained a contradiction in each case. Therefore, our assumption that \( B \) is constructible was wrong. This completes the proof of Theorem 3.1. \( \square \)

Next we observe the converse of Theorem 3.1.

**Observation 3.2.** Let \( B \) be a reduced simplicial 3-ball which has interior vertices. Assume that there are no edges whose end vertices are both interior with respect to \( B \). If \( B \) has no spanning edges then \( B \) is constructible.

**Proof.** Let \( v \) be a vertex which is interior with respect to \( B \). From the hypothesis, all vertices of \( \text{Link}_B \) \( v \) are contained in \( \partial B \). Since \( B \) has no spanning edges, all edges of \( \text{Link}_B \) \( v \) belong to \( \partial B \). Since \( B \) is reduced, the 2-face of \( \text{Link}_B \) \( v \) belong to \( \partial B \) as well. Thus the star neighborhood \( \text{Star}_B \) \( v \) coincides with the simplicial 2-ball \( B \). Since the 2-sphere \( \partial B \) is shellable, we can shell the 3-ball \( \text{Star}_B \) \( v \) = \( v \ast \partial B \) along the shelling of \( \partial B \). Shellable 3-balls are constructible so that the simplicial 3-ball \( B \) is constructible. \( \square \)
4. An example

In this section, we present an example which satisfies the conditions of Theorem 3.1.

Example 4.1. We consider an example which is constructed from the 3-ball known as “Bing’s house with two rooms” [1]. First, consider the 3-ball which is indicated in Fig. 4 and denote it by \( B \). The walls are made of one layer of cubes. There are two tunnels one of which connects the upper space and the lower floor and another of which connects the lower space and the upper floor. In [5], a triangulation of the 3-ball \( B \) with no interior vertices is presented. Furthermore it is shown that the simplicial 3-ball is reduced and nonconstructible. Here we give another triangulation which has many interior vertices. We triangulate the 2-skeleton of \( B \) as in [5]. Add an interior vertex in each cube. Consider a cone from the vertex over the triangulated boundary of each cube. Then we obtain a triangulation of \( B \). As in [5], the obtained simplicial 3-ball is reduced. Furthermore all spanning edges are strict. Thus the simplicial 3-ball satisfies the conditions of Theorem 3.1 and it is not constructible.

Acknowledgments

The author would like to thank Dr. Masahiro Hachimori for his helpful advice during the preparation of this manuscript. The author would like to express his gratitude to the anonymous referees for their thoughtful comments. The representations are improved tremendously by their comments.

References