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Differential Geometry and its Applications 26 (2008) 339–345
DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS
www.elsevier.com/locate/difgeo

Generalizations of the Liouville theorem

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Received 7 July 2005; received in revised form 27 January 2007

Available online 21 December 2007

Communicated by D. Krupka

Abstract

The purpose of this paper is to generalize the Liouville theorem for functions which are defined on the complete Riemannian manifolds. Then, we apply it to the isometric immersions between complete Riemannian manifolds in order to obtain an estimate for the size of the image of immersions in terms of the supremum of the length of their mean curvature vector in a quite general setting. The proofs are based on the Calabi's generalization of maximum principle for functions which are not necessarily differentiable.

MSC: primary 53C40; secondary 53C20

Keywords: Generalized Liouville theorem; Generalized maximum principle; Laplacian; Mean curvature; Isometric immersion

Introduction

The classical Liouville theorem asserts that harmonic and bounded functions on Euclidean spaces are constant. In this paper, we extend the Liouville theorem to functions which are defined on a complete (non-compact) Riemannian manifold whose Ricci curvature is bounded from below. Moreover, we relax the assumptions harmonicity and boundedness of functions. We prove the following Liouville type theorem (Theorem 2.1, see also Remarks 2.2, 2.3, 2.4 and Theorem 2.9):

- Let N be an n-dimensional complete (non-compact) Riemannian manifold whose Ricci curvature is bounded from below. Let $w : N \to \mathbf{R}$ be a $\mathcal{C}^2$-function such that $\Delta w \geqslant 1$. Then, we have

$$\limsup_{r_N(x)\to\infty} \frac{w(x)}{r_N(x)} > 0,$$

where r_N denotes the distance function on N, $r_N(y) := d_N(y, p)$. In particular, w is unbounded.

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[1] The author was partially supported by IPM Grant 5/80/1140.

0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2007.11.026

The above theorem can be interpreted as a generalization of maximum principle (on unbounded domains), compare with [3] and [11]. See also [12]. Then, this Liouville type theorem is applied, similar to [4] and [9], in order to sharpen and generalize the following theorem which is due to Jorge and Xavier [8]:

- Let $f : M^n \to \overline{M}^{n+k}$ be an isometric immersion between complete Riemannian manifolds. Suppose that the scalar curvature of M is bounded from below by some constant and the sectional curvature of $\overline{M}$ is bounded from above by some constant K. Denote the supremum of the length of mean curvature vector of f by $H_0 < \infty$. Suppose that the image of f is inside the closed normal ball $B(p, R) \subset \overline{M}$; and $R < \frac{\pi}{2\sqrt{K}}$, if $K > 0$. Then, we have

$$R \geqslant \begin{cases} \frac{1}{\sqrt{K}} \tan^{-1}\left(\frac{\sqrt{K}}{H_0}\right) & \text{if } K > 0, \\ \frac{1}{H_0} & \text{if } K = 0, \\ \frac{1}{\sqrt{-K}} \tanh^{-1}\left(\frac{\sqrt{-K}}{H_0}\right) & \text{if } K < 0. \end{cases}$$

We extend the above theorem of Jorge and Xavier (see also [7]) for quite general manifolds $\overline{M}$. In fact, in our result (Theorem 3.1), we do not need to compare manifold $\overline{M}$ with a space form.

The proof of above Liouville type theorem is based on the Calabi's generalization of maximum principle for continuous functions which are not necessarily differentiable (see [2] or [5]).

1. Preliminaries

In this section, we recall some of the basic definitions in order to state the Calabi's generalization of maximum principle.

Let N be a Riemannian manifold (of class $\mathcal{C}^3$) and let $\langle\cdot,\cdot\rangle$ denote the Riemannian metric on N. We denote the associated covariant derivative of N by D. For $p \in N$, we denote the distance from p to y by $r_N(y) := d_N(y, p)$. The function $r_N(y)$ is smooth on $N \backslash (\{p\} \cup C_p)$, where C_p denotes the cut locus of p. Also, we denote the Hessian of $r_N(y)$ by $\operatorname{Hess}(r_N)(v, w) := \langle D_v^{\nabla r_N}, w\rangle$, for all vectors v and w in the tangent bundle of N. We denote the closed ball with center $q \in N$ and radius $R > 0$ by $B(q, R)$ and the Laplacian (on N) by $\Delta = \Delta_N$.

Definition 1.1. (See [2] and [5].) Let N be a Riemannian manifold and let $\eta : N \to \mathbf{R}$ be a continuous function. An *upper barrier* (*support function*) for η at the point x_0, is a $\mathcal{C}^2$-function, $\tilde{\eta}$, defined in some neighborhood of x_0 such that $\tilde{\eta} \geqslant \eta$ and $\eta(x_0) = \tilde{\eta}(x_0)$.

We say that $\Delta\eta(x_0) \leqslant a$ (or $\Delta(-\eta)(x_0) \geqslant -a$) in the *barrier sense* (*sense of support functions*), if for all $\epsilon > 0$, there is an upper barrier $\eta_{x_0,\epsilon}$ for η at x_0 such that $\Delta\eta_{x_0,\epsilon}(x_0) \leqslant a + \epsilon$.

Lemma 1.2. *Let N be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below by some constant b. Suppose that $r_N(y) := d_N(y, p)$ denotes the distance function on N. Let $\Psi : [0, \infty[\to \mathbf{R}$ be a $\mathcal{C}^2$-function such that $\Psi' \geqslant 0$. Then, we have*

$$\Delta_N(\Psi \circ r_N) \leqslant \Delta_b(\Psi \circ r_b)|_{r_b = r_N},$$

in the sense of barrier, where Δ_N and Δ_b denote the Laplacian respectively, on N and the space form of constant curvature b. In particular, when $\Psi(s) \equiv s$, we have

$$\Delta_N(r_N) \leqslant \Delta_b(r_b) = (n-1)m_b(r_b)|_{r_b = r_N},$$

where $m_b(r)$ is defined as the following:

$$m_b(r) := \begin{cases} \sqrt{b}\cot(r\sqrt{b}) & \text{if } b > 0, \\ \frac{1}{r} & \text{if } b = 0, \\ \sqrt{-b}\coth(r\sqrt{-b}) & \text{if } b < 0. \end{cases}$$

Proof. See [2, Prop. 7.7]. □

Lemma 1.3 *(Maximum principle). Let N be a connected complete Riemannian manifold and let $\eta : N \to \mathbf{R}$ be a continuous function. Suppose that $\Delta\eta \geqslant 0$, in the barrier sense. Then, η attains no (local) maximum unless it is constant.*

Proof. See [2, Thm. 7.8] or [5, 12.6]. □

2. The Liouville type theorem

We start this section with the following generalization of Liouville theorem for functions which are defined on a complete Riemannian manifold whose Ricci curvature is bounded from below. Compare with [4] and [9].

Theorem 2.1. *Let N be an n-dimensional complete (non-compact) Riemannian manifold whose Ricci curvature is bounded from below by some constant b. Let $w : N \to \mathbf{R}$ be a $\mathcal{C}^2$-function such that $\Delta w \geqslant 1$. Then, we have*

$$\limsup_{r_N(x)\to\infty} \frac{w(x)}{r_N(x)} \geqslant C,$$

where r_N denotes the distance function on N, $r_N(y) := d_N(y, p)$, and C is a positive constant which depends on b and n. In particular, w is unbounded.

Proof. Without loss of generality, we can assume that $w(p) = 0$ and $b \leqslant -1$. Suppose that $\Psi : [0, \infty[\to \mathbf{R}$ is a non-decreasing $\mathcal{C}^2$-function such that Ψ is zero on $[0, 4\sqrt{-b}]$ and $\Psi(s) = (3\sqrt{-b})s$, if $s \geqslant 8\sqrt{-b}$. Then, by Lemma 1.2, there is a positive constant L, which depends on b and n, such that

$$\Delta_N(\Psi \circ r_N) \leqslant \Delta_b(\Psi \circ r_b)|_{r_b=r_N} \leqslant L,$$

in the barrier sense, where L is a constant which depends on b and n. Therefore, there is a constant K, which depends on b and n, such that

$$\Delta_N(\Psi \circ r_N) \leqslant \Delta_N(Kw),$$

in the barrier sense. Now, by the maximum principle (Lemma 1.3) applied to the function $u := Kw - \Psi \circ r_N$ on the ball $B(p, t)$, for $t \geqslant 8\sqrt{-b}$, we get

$$0 = u(p) \leqslant \max_{B(p,t)} u = \max_{\partial B(p,t)} u.$$

Then, we see that, for any $t \geqslant 8\sqrt{-b}$, there exists $y \in \partial B(p, t)$ such that $0 \leqslant u(y)$ or equivalently $3\sqrt{-b}\, r_N(y) \leqslant Kw(y)$. This completes the proof of theorem. □

Remark 2.2. In Theorem 2.1, we can relax the assumption $\Delta w \geqslant 1$ as the following:

- $\Delta w(y) \geqslant 0$, for all $y \in N$ and $\Delta w(y) \geqslant 1$, for all y outside a compact (bounded) subset of N.

Moreover, by choosing a different (non-decreasing) $\mathcal{C}^2$-function Ψ (as in the proof of theorem), we can obtain a different type of Liouville theorem.

Remark 2.3. In Theorem 2.1, when $b = 0$, we can relax the assumption $\Delta w \geqslant 1$ as the following:

- $\Delta w(y) \geqslant 0$, for all $y \in N$ and $\Delta w(y) \geqslant \frac{1}{r_N(y)}$, for all y outside a compact (bounded) subset of N.

Remark 2.4. In Theorem 2.1, when $b = 0$, we can sharpen the conclusion of theorem as the following (by replacing r_N^2 instead of r_N in the proof of theorem):

- $\limsup_{r_N(x)\to\infty} \frac{w(x)}{r_N^2(x)} \geqslant C$.

Question 2.5. Is there any best choice for the function Ψ in the proof of Theorem 2.1? Compare with Remarks 2.2 and 2.4.

Next, we relax the assumptions of Theorem 2.1 for a 2-dimensional manifold N with non-negative curvature.

Theorem 2.6. *Let N be a 2-dimensional complete (non-compact) Riemannian manifold with non-negative Ricci curvature. Let $w : N \to \mathbf{R}$ be a non-constant $\mathcal{C}^2$-function such that $\Delta w \geqslant 0$. Then, we have*

$$\limsup_{r_N(x)\to\infty} \frac{w(x)}{\log(r_N(x))} > 0,$$

where r_N denotes the distance function on N, $r_N(y) := d_N(y, p)$. In particular, w is unbounded.

Proof. Let notations be as in Lemma 1.2 with $n = 2$, $b = 0$ and $\Psi(s) = \beta \log(s)$ where β is positive constant. Since that $n = 2$ and $b = 0$, it is easy to check that

$$\Delta_b(\log \circ r_b) = 0,$$

on $\mathbf{R}^2 \backslash \{0\}$. By Lemma 1.2, we have

$$\Delta_N(\Psi \circ r_N) \leqslant 0 \leqslant \Delta_N w,$$

in the barrier sense on $N\backslash\{p\}$. Put $\Omega_T := B(p, T)\backslash B(p, 1)$, for $T > 1$. Then, by Lemma 1.3, we can choose $\beta > 0$ small enough (depending on w) such that

$$\max_{\Omega_T} \phi = \max_{\partial B(p,T)} \phi,$$

where $\phi := w - \beta \log \circ r_N$. This completes the proof of theorem. □

Remark 2.7. Compare Theorem 2.6 with [6, Thm. 7.3]. Also, note that Theorem 3.1 is not necessarily correct for a complete manifold whose dimension is greater than 2. For example, consider the following function:

$$h(x, y, z) := -(1 + x^2 + y^2 + z^2)^{-1/2}.$$

It is easy to check that h is a smooth function on $\mathbf{R}^3$ and $\Delta h \geqslant 0$. But, h is a bounded and non-constant function.

Remark 2.8. In Theorems 2.1 and 2.6, we can relax this assumption that Ricci curvature N is bounded from below (on entire manifold) as the following:

- The Ricci curvature N is bounded from below outside a compact (bounded) subset of N.

Now, in Theorem 2.1, we relax the assumption of lower bound on Ricci curvature of N.

Theorem 2.9. *Let N be an n-dimensional complete (non-compact) Riemannian manifold. Suppose that R and A are two positive numbers and h is a positive, unbounded and non-decreasing $\mathcal{C}^2$-function on $[R, +\infty[$ such that*

$$h'' + nh' \exp\left(\frac{1}{2}h\right) \leqslant A. \tag{2.1}$$

Suppose that $w : N \to \mathbf{R}$ is a subharmonic $\mathcal{C}^2$-function such that $\Delta w \geqslant 1$ on $N\backslash B(p, R)$, for some $p \in N$. Suppose that the Ricci curvature of N is bounded from below by $-\exp(h \circ r_N)$ on $N\backslash B(p, R)$, where r_N denotes the distance function on N, $r_N(y) := d_N(y, p)$. Then, we have

$$\limsup_{r_N(x)\to\infty} \frac{w(x)}{h(r_N(x))} > 0.$$

In particular, w is unbounded.

Proof. It is easy to check that

$$\nabla(h \circ r_N) = h'(r_N)\nabla r_N,$$
$$\Delta(h \circ r_N) = h''(r_N)\|\nabla r_N\|^2 + h'(r_N)\Delta r_N,$$

in the barrier sense on $N \backslash B(p, R)$. By Lemma 1.2, we have (note that $\|\nabla r_N\| = 1$)

$$\Delta(h \circ r_N) \leqslant h''(r_N) + h'(r_N)(n-1)\exp\left(\frac{1}{2}h \circ r_N\right)\coth\left(r_N \cdot \exp\left(\frac{1}{2}h \circ r_N\right)\right) \leqslant A \leqslant A\Delta w,$$

in the barrier sense on $N \backslash B(p, T)$, for large enough $T > R$. Put $\Omega_T := B(p, T)\backslash B(p, R)$. Then, by Lemma 1.3 and choosing $\beta > 0$ small enough (depending on w), we have

$$\max_{\Omega_T} \phi = \max_{\partial B(p,T)} \phi,$$

where $\phi := Aw - \beta h \circ r_N$. This completes the proof of theorem. □

Remark 2.10. In Theorem 2.9, we can replace the condition (2.1) with the following weaker condition:

$$h'' + nh' \exp\left(\frac{1}{2}h\right) \leqslant Ah.$$

Then, the conclusion of Theorem 2.9 becomes as the following (calculate $\Delta(\log(h \circ r_N))$):

$$\limsup_{r_N(x)\to\infty} \frac{w(x)}{\log(h \circ r_N(x))} > 0.$$

Moreover, by choosing $\exp h(s) := Ks^2 \log^2 s$, for some constant K, we can recover the main results of [3].

Question 2.11. Is it possible to strengthen the conclusion of Theorem 2.9 (similar to Theorem 2.1) as the following:

$$\limsup_{r_N(x)\to\infty} \frac{w(x)}{h(r_N(x))} \geqslant C,$$

where C is a positive constant (which depends on A and n)?

Question 2.12. Is there any best choice for the function h, the lower bound of Ricci curvature in Theorem 2.9? Compare with Question 2.5. See example of [3, p. 365] and also [11, p. 205].

3. Application

In this section, by using Theorem 2.1, we generalize the result of Jorge and Xavier [8]. See also [7] and [3].

Theorem 3.1. *Let $f : M^n \to \overline{M}^{n+k}$ be an isometric $\mathcal{C}^2$-immersion between complete Riemannian manifolds. Suppose that the Ricci curvature of (non-compact) manifold M is bounded from below by some constant b. Denote the mean curvature vector of $f(M)$ in $\overline{M}$ by H. Suppose that the image of f is contained in $\overline{M}\backslash(C_p \cup \{p\})$, for some $p \in \overline{M}$, where C_p denotes the cut locus of p. Suppose that the Hessian of distance function on $\overline{M}$, $r(y) := d_{\overline{M}}(y, p)$, is bounded from below by $m(r) \geqslant 0$ on the tangent bundle of $\partial B(p, r)$, i.e.* $\mathrm{Hess}(r)(v, v) \geqslant m(r)\|v\|^2$ *for all vectors v in the tangent bundle of $\partial B(p, r)$. Moreover, suppose that $\Phi : [0, \infty[\to \mathbf{R}$ is a $\mathcal{C}^2$-function and $\Phi''(r) \geqslant \Phi'(r)m(r)$. Then, we have*

- *Either*

$$\limsup_{\rho(x)\to\infty} \frac{\Phi(r \circ f)(x)}{\rho(x)} \geqslant C,$$

where ρ denotes distance function on M, i.e. $\rho(x) := d_M(x, q)$, for some $q \in M$, and C is a positive constant which depends on b and n. In particular, the image of f, $f(M)$, is unbounded.

- *Or*

$$\inf_{x\in M}\big[\Phi'(r\circ f)(x)\big(m(r\circ f)(x)-\big\|H\big(f(x)\big)\big\|\big)\big]\leqslant 0.$$

Proof. By [13, Lemma 2.1] (see also [10, Lemma 2]), we have

$$\begin{aligned}\Delta_M\big(\Phi(r\circ f)\big)&\geqslant n\Phi'(r\circ f)\big[m(r\circ f)-\big\langle\nabla r, H\big(f(x)\big)\big\rangle\big]\\&\geqslant n\Phi'(r\circ f)\big[m(r\circ f)-\big\|H\big(f(x)\big)\big\|\big].\end{aligned}$$

Now, theorem follows from Theorem 2.1, applied to function $\Phi(r\circ f)$. □

Remark 3.2. We can improve Theorem 3.1 similar to Remarks 2.2, 2.3, 2.4 and Theorem 2.9. Moreover, when $n=2$, we can prove a Rigidity type theorem similar to [13, Cor. 2.7].

Corollary 3.3. *Suppose that the assumptions and notations are as in Theorem 3.1. Moreover, suppose that the sectional curvature of $\overline{M}$ is bounded from above by some constant K. Then, we can choose the function $m(r):=m_K(r)$, where $m_K(r)$ is defined as in Lemma 1.2, and function Φ can be chosen as the following*:

$$\Phi(r):=\begin{cases}\frac{1-\cos(r\sqrt{K})}{K} & \text{if } K>0,\\ \frac{r^2}{2} & \text{if } K=0,\\ \frac{1-\cosh(r\sqrt{-K})}{K} & \text{if } K<0.\end{cases}$$

In particular, this generalizes the result of Jorge and Xavier [8].

Proof. By the Hessian comparison theorem (see [14, p. 4]), we can choose $m(r):=m_K(r)$. It is easy to see that function Φ (as the above) satisfies the assumptions of Theorem 3.1. □

Remark 3.4. We can state and prove a theorem for maps between Riemannian manifolds (which are not necessarily isometric immersion) in terms of the energy density and torsion field, see [9, Thm. 2].

Remark 3.5. In Corollary 3.3 and Theorem 3.1, this condition that Ricci (scalar) curvature of M is bounded from below, cannot be omitted. Indeed, there exists an example, which is due to Calabi [1, 28.2.7], of a complete minimal ($H\equiv 0$) surface which is contained in a ball of $\mathbf{R}^4$.

Acknowledgement

The author would like to thank the Research Council of Sharif University of Technology and Institute for Studies in Theoretical Physics and Mathematics (IPM) for support.

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