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Generalized quadrangles and regularity

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Abstract

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a generalized quadrangle of order (s, t), $s, t > 1$, and assume that $\mathcal{S}$ has a regular point X. In this paper we survey some basic results on such generalized quadrangles (GQs) as well as the known examples. We also study a general representation of such GQs using the net associated with the regular point X and specialise the representation to the case where X is abelian centre of symmetry, as in all of the known examples.

MSC: 51E12; 51E14

Keywords: Generalized quadrangle; Regular point; Net; Centre of symmetry

1. Introduction and definitions

A (finite) *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (non-empty) sets of objects called *points* and *lines*, respectively, and for which $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $1 + t$ lines ($t \geqslant 1$) and two distinct points are incident with at most one line.
(ii) Each line is incident with $1 + s$ points ($s \geqslant 1$) and two distinct lines are incident with at most one point.

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(iii) If X is a point and ℓ is a line not incident with X, then there is a unique pair $(Y, m) \in \mathscr{P} \times \mathscr{B}$ for which $X \mathrm{I} m \mathrm{I} Y \mathrm{I} \ell$.

For a comprehensive introduction to GQs see [26]. The integers s and t are the *parameters* of the GQ and $\mathscr{S}$ is said to have *order* (s, t). If $s = t$, then $\mathscr{S}$ is said to have order s. If $\mathscr{S}$ has order (s, t), then it follows that $|\mathscr{P}| = (s+1)(st+1)$ and $|\mathscr{B}| = (t+1)(st+1)$ [26, 1.2.1]. If $\mathscr{S} = (\mathscr{P}, \mathscr{B}, \mathrm{I})$ is a GQ of order (s, t), then the incidence structure $\mathscr{S}^* = (\mathscr{B}, \mathscr{P}, \mathrm{I})$ is a GQ of order (t, s) called the *dual* of $\mathscr{S}$. By a result of Higman [12,13] (a short proof is found in [5]), if $s, t > 1$, then $s \leqslant t^2$ and $t \leqslant s^2$.

The classical GQs (with $s, t > 1$) arise as the classical rank 2 polar spaces and are: $Q(4, q)$, from the parabolic quadric in $\mathrm{PG}(4, q)$, and, $W(q)$, from the symplectic polarity in $\mathrm{PG}(3, q)$, GQs of order q; $Q(5, q)$, from the non-singular elliptic quadric in $\mathrm{PG}(5, q)$, a GQ of order (q, q^2); $H(3, q^2)$, from the Hermitian variety in $\mathrm{PG}(3, q^2)$, a GQ of order (q^2, q); and $H(4, q^2)$, from the Hermitian variety in $\mathrm{PG}(4, q^2)$, a GQ of order (q^2, q^3). Note that $Q(4, q) \cong W(q)^*$ and $Q(4, q) \cong W(q)$ if and only if q is even, while $Q(5, q) \cong H(3, q^2)^*$ (see [26, Chapter 3]).

A (finite) *net* is an incidence structure $\mathscr{N} = (\mathscr{P}, \mathscr{B}, \mathrm{I})$ in which $\mathscr{P}$ and $\mathscr{B}$ are disjoint (non-empty) sets of objects called *points* and *lines*, respectively, and for which $\mathrm{I} \subseteq (\mathscr{P} \times \mathscr{B}) \cup (\mathscr{B} \times \mathscr{P})$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $1 + t$ lines $(t \geqslant 1)$ and two distinct points are incident with at most one line.
(ii) Each line is incident with $1 + s$ points $(s \geqslant 1)$ and two distinct lines are incident with at most one point.
(iii) If X is a point and ℓ is a line not incident with X, then there is a unique line m such that $X \mathrm{I} m$ and there is no point incident with both ℓ and m (that is, ℓ and m are non-concurrent).

The *order* of $\mathscr{N}$ is $s + 1$, while the *degree* of $\mathscr{N}$ is $t + 1$. The incidence structure $\mathscr{N}^* = (\mathscr{B}, \mathscr{P}, \mathrm{I})$ is the *dual* of $\mathscr{N}$. Two non-concurrent lines of $\mathscr{N}$ are said to be *parallel* and parallelism is an equivalence relation on $\mathscr{B}$. Straightforward counts show that $|\mathscr{P}| = (s+1)^2$, $|\mathscr{B}| = (s+1)(t+1)$, the number of parallel classes is $t + 1$ and the number of lines in a parallel class is $s + 1$.

Important examples of nets are the affine planes of order s, which are nets of order s and degree $s + 1$. Another important example is the dual net H_q^n, $n > 2$, which is constructed as follows: the points of H_q^n are the points of $\mathrm{PG}(n, q)$ not in a given subspace $\mathrm{PG}(n-2, q) \subset \mathrm{PG}(n, q)$, the lines of H_q^n are the lines of $\mathrm{PG}(n, q)$ which have no point in common with $\mathrm{PG}(n-2, q)$, the incidence in H_q^n is the natural one. The axiom of Veblen for a dual net is the following: If $\ell_1 \mathrm{I} X \mathrm{I} \ell_2$, $\ell_1 \neq \ell_2$, $m_1 \not\mathrm{I}\, X \not\mathrm{I}\, m_2$, and if ℓ_i is concurrent with m_j for all $i, j \in \{1, 2\}$, then m_1 is concurrent with m_2. In fact, the axiom of Veblen characterizes the H_q^n in the following way.

Theorem 1 (*Thas and De Clerck [34]*). *Let $\mathscr{N}^*$ be a dual net with $s + 1$ points on any line and $t + 1$ lines through any point, where $t + 1 > s$. If $\mathscr{N}^*$ satisfies the axiom of Veblen, then $\mathscr{N}^* \cong H_q^n$ with $n > 2$ (hence $s = q$ and $t + 1 = q^{n-1}$).*

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, t). Given two (not necessarily distinct) points X, X' of $\mathcal{S}$, we write $X \sim X'$ and say that X and X' are *collinear*, provided that there is some line ℓ for which $X \,\mathrm{I}\, \ell \,\mathrm{I}\, X'$; hence $X \not\sim X'$ means that X and X' are not collinear. Dually, for $\ell, \ell' \in \mathcal{B}$, we write $\ell \sim \ell'$ or $\ell \not\sim \ell'$ according to whether ℓ and ℓ' are *concurrent* or nonconcurrent. For $X \in \mathcal{P}$ put $X^{\perp} = \{X' \in \mathcal{P} : X \sim X'\}$, and note that $X \in X^{\perp}$. For $X \neq X'$, the set $\{X, X'\}^{\perp}$ is defined to be $X^{\perp} \cap^{\perp}_{X'}$. Hence $|\{X, X'\}^{\perp}| = s + 1$ or $t + 1$ according as $X \sim X'$ or $X \not\sim X'$. More generally, if $A \subset \mathcal{P}$, $A^{\perp}$ is defined to be $\bigcap_{X \in A} X^{\perp}$. For $X \neq X'$, the set $\{X, X'\}^{\perp\perp}$ is defined to be $\{U \in \mathcal{P} : U \in Z^{\perp}$ for all $Z \in X^{\perp} \cap X'^{\perp}\}$. We have $|\{X, X'\}^{\perp\perp}| = s + 1$ or $|\{X, X'\}^{\perp\perp}| \leqslant t + 1$ according as $X \sim X'$ or $X \not\sim X'$.

If $X \sim X'$, $X \neq X'$, or if $X \not\sim X'$ and $|\{X, X'\}^{\perp\perp}| = t + 1$, where $X, X' \in \mathcal{P}$, we say the pair $\{X, X'\}$ is *regular*. The point X is *regular* provided $\{X, X'\}$ is regular for all $X' \in \mathcal{P}$, $X \neq X'$. Regularity for lines is defined dually. Note that if X is a regular point and $Y, Z \in X^{\perp}$, $Y \neq Z$, then $\{Y, Z\}$ is regular.

2. Basic results on regularity

In this section we review some fundamental results concerning regularity in GQs. Many more are to be found in Chapter 1 of [26]. Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, t).

Theorem 2 (*1.3.6 of Payne and Thas [26]*). *If $\mathcal{S}$ has a regular pair of points $\{X, Y\}$, then either $s = 1$ or $s \geqslant t$.*

Theorem 3 (*Thas [28]*). *If $s \neq 1 \neq t$ and $\mathcal{S}$ has a regular point X and a regular line ℓ with $X \,\mathrm{I}\!\!\!/\, \ell$, then $s = t$ is even.*

A point X of $\mathcal{S}$ is *coregular* if each line incident with X is regular.

Theorem 4 (*Payne and Thas [25]*). *If $\mathcal{S}$ has a coregular point X and t is odd, then $|\{X, Y\}^{\perp\perp}| = 2$ for all $Y \notin X^{\perp}$.*

Theorem 5 (*Payne [20], Payne and Thas [25]*). *If $\mathcal{S}$ has a coregular point X and $s = t \neq 1$, then X is regular if and only if s is even.*

Perhaps the most important structural result concerning regularity in GQs is the following theorem giving a construction of a (dual) net from a regular point of a GQ.

Theorem 6 (*1.3.1 of Payne and Thas [26]*). *Let $\mathcal{S}$ have a regular point X. Then the incidence structure with pointset $X^{\perp}\backslash\{X\}$, with lineset the set with elements $\{Y, Z\}^{\perp\perp}$, where $Y, Z \in X^{\perp}\backslash\{X\}$, $Y \not\sim Z$, and with the natural incidence, is the dual of a net of order s and degree $t + 1$. If in particular $s = t > 1$, then there arises a dual affine plane of order s. Moreover, in this case the incidence structure π_X with pointset $X^{\perp}$, with lineset the set with elements $\{Y, Z\}^{\perp\perp}$, where $Y, Z \in X^{\perp}\backslash\{X\}$, and with the natural incidence, is a projective plane of order s.*

The net arising as in Theorem 6 from a regular point X of a GQ $\mathcal{S}$ will be referred to as the net *associated* with X.

The implications of this result for regularity in GQs will be investigated in great detail in the rest of the paper.

Also important for the rest of the paper is the concept of a centre of symmetry. Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, t) and X a point of $\mathcal{S}$. A *symmetry* of $\mathcal{S}$ about X is a collineation of $\mathcal{S}$ fixing $X^{\perp}$ elementwise. Any non-trivial symmetry about X fixes no point of $\mathcal{P}\backslash X^{\perp}$. The symmetries about X form a group whose order divides t (see [26, Chapter 8]). If the symmetry group about X has maximal order t, then X is called a *centre of symmetry*. Dually we have the concept of an *axis of symmetry*. If X is a centre of symmetry with symmetry group G and $P \in \mathcal{P}\backslash X^{\perp}$, then $\{X, P\}^{\perp\perp} = \{X\} \cup \{g(P) : g \in G\}$ and $\{X, P\}$ is a regular pair. Consequently, we have the following result:

Lemma 7. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, t) and X a centre of symmetry of $\mathcal{S}$. Then X is a regular point of $\mathcal{S}$.*

3. Examples of GQs and regularity

In this section we list the known examples of GQs with regularity.

The classical GQ $W(q)$ of order q has all points regular. If q is even, then all lines of $W(q)$ are regular, otherwise $W(q)$ has no regular lines (recall that for q even $W(q)$ is self dual). Since $Q(4, q)$ is the dual of $W(q)$ it has all lines regular, all points regular if q is even and no regular points otherwise. The classical GQ $Q(5, q)$ of order (q, q^2) has all lines regular (and since $t > s$ may have no regular points) while its dual $H(3, q^2)$ has all points regular. These results may be found in [26, 3.3.1].

Now we introduce our first examples of non-classical GQs and their regularity properties. These examples are due to Tits, first appearing in [8] (see also [26]). For these GQs we will need the following two definitions:

An *oval* of $\mathrm{PG}(2, q)$ is a set of $q + 1$ points, no three collinear. Lines of $\mathrm{PG}(2, q)$ meet an oval in $0, 1$ or 2 points and are called external lines, tangent lines and secant lines, respectively. If q is even, then the $q + 1$ tangent lines to an oval are concurrent in a point called the *nucleus* of the oval. See [15] for details and references on ovals.

An *ovoid* of $\mathrm{PG}(3, q)$, $q > 2$, is a set of $q^2 + 1$ points, no three collinear. An ovoid of $\mathrm{PG}(3, 2)$ is a set of five points no four of which are coplanar. Lines of $\mathrm{PG}(3, q)$ are either external, tangent or secant to the ovoid. The tangents at the point of the ovoid form a plane called the *tangent* plane at that point. Any plane of $\mathrm{PG}(3, q)$ not a tangent plane meets the ovoid in an oval and is called a *secant* plane. See [14] for details and references on ovoids.

Let $n = 2$ (respectively, $n = 3$) and let $\mathcal{O}$ be an oval (respectively, ovoid) of $\mathrm{PG}(n, q)$. Further, let $\mathrm{PG}(n, q)$ be embedded as a hyperplane in $\mathrm{PG}(n + 1, q)$. We now give the definition of the GQ $T_n(\mathcal{O})$ of order (q, q^{n-1}). Define points as (i) the points of $\mathrm{PG}(n + 1, q)\backslash \mathrm{PG}(n, q)$ (the *affine* points), (ii) the hyperplanes H of $\mathrm{PG}(n + 1, q)$ for which $|H \cap \mathcal{O}| = 1$, and (iii) one new symbol (∞). Lines are defined as (a) the lines of $\mathrm{PG}(n + 1, q)$ which are not contained in $\mathrm{PG}(n, q)$ and meet $\mathcal{O}$, and (b) the points of $\mathcal{O}$.

Incidence is that inherited from $\mathrm{PG}(n+1, q)$ plus each line of type (b) is incident with the point (∞).

The GQ $T_2(\mathscr{O})$ is isomorphic to $Q(4, q)$ if and only if $\mathscr{O}$ is a conic and is non-classical otherwise (see [26]). The GQ $T_3(\mathscr{O})$ is isomorphic to $Q(5, q)$ if and only if $\mathscr{O}$ is an elliptic quadric ovoid and is non-classical otherwise. Each line of $T_n(\mathscr{O})$ of type (b) is regular and if q is even, then the point (∞) of $T_2(\mathscr{O})$ is regular (see [26]). An oval $\mathscr{O}$ of $\mathrm{PG}(2, q)$, q even, is a *translation oval* if there exists a tangent ℓ to $\mathscr{O}$ and a group of q elations fixing $\mathscr{O}$ each element of which has axis ℓ. The line ℓ is called an *axis* of O. It was proved by Payne in [19] that every translation oval of $\mathrm{PG}(2, q)$ is projectively equivalent to an oval of the form $\{(1, t, t^\sigma) : t \in \mathrm{GF}(q)\} \cup \{(0, 0, 1)\}$ where σ is a generator of $\mathrm{Aut}(\mathrm{GF}(q))$. If $\mathscr{O}$ is a translation oval with axis ℓ, then any point of type (ii) of $T_2(\mathscr{O})$ that is a plane of $\mathrm{PG}(3, q)$ meeting $\mathrm{PG}(2, q)$ in ℓ is a regular point and consequently $P = \ell \cap \mathscr{O}$ is a coregular line of $T_2(\mathscr{O})$.

We now present a construction method for GQs introduced by Kantor [16].

Let G be a finite group of order s^2t, $1 < s, t$, together with a family $J = \{A_i : 0 \leqslant i \leqslant t\}$ of $1+t$ subgroups of G, each of order s. Assume that for each $A_i \in J$, there exists a subgroup A_i^* of G, order st, containing A_i. Put $J^* = \{A_i^* : 0 \leqslant i \leqslant t\}$ and define as follows a point-line incidence geometry $\mathscr{S} = (\mathscr{P}, \mathscr{B}, \mathrm{I}) = \mathscr{S}(G, J)$.

Points are of three kinds: (i) the elements of G; (ii) the right cosets A_i^*g, $A_i^* \in J^*$, $g \in G$; (iii) a symbol (∞).

Lines are of two kinds: (a) the right cosets A_ig, $A_i \in J$, $g \in G$; (b) the symbols $[A_i]$, $A_i \in J$.

A point g of type (i) is incident with each line A_ig, $A_i \in J$; a point A_i^*g of type (ii) is incident with $[A_i]$ and with each line A_ih contained in A_i^*g; the point (∞) is incident with each line $[A_i]$ of type (b).

Then Kantor [16] proved that the following holds: $\mathscr{S}(G, J)$ is a GQ of order (s, t) provided

$$K_1 : A_iA_j \cap A_k = \{1\}, \quad \text{for } i, j, k \text{ distinct, and}$$

$$K_2 : A_i^* \cap A_j = \{1\}, \quad \text{for } i \neq j.$$

If the conditions K_1 and K_2 are satisfied, then A_i^* is uniquely defined by A_i. Suppose K_1 and K_2 are satisfied. For any $h \in G$ let us define θ_h by $g^{\theta_h} = gh$, $(A_ig)^{\theta_h} = A_igh$, $(A_i^*g)^{\theta_h} = A_i^*gh$, $[A_i]^{\theta_h} = [A_i]$, $(\infty)^{\theta_h} = (\infty)$, with $g \in G$, $A_i \in J$ and $A_i^* \in J^*$. Then θ_h is an automorphism of $\mathscr{S}(G, J)$ which fixes the point (∞) and all lines of type (b). If $G' = \{\theta_h : h \in G\}$, then clearly $G' \cong G$ and G' acts regularly on the points of type (i). The centre $Z(G')$ of G' induces the group of symmetries of $\mathscr{S}(G, J)$ about (∞). Hence by Lemma 7 if $|Z(G')| = t$, then (∞) is a regular point.

If K_1 and K_2 are satisfied, then J is called a 4-*gonal family* or *Kantor family* for G.

Now put $G = \{(\alpha, c, \beta) : \alpha, \beta \in \mathrm{GF}(q)^2, c \in \mathrm{GF}(q)\}$. Define a binary operation $(\alpha, c, \beta)(\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta\alpha'^T, \beta + \beta')$. This makes G into a group whose centre is $Z(G) = \{(0, c, 0) : c \in \mathrm{GF}(q)\}$.

Let $\mathscr{C} = \{A_u : u \in \mathrm{GF}(q)\}$ be a set of q distinct upper triangular 2×2 matrices over $\mathrm{GF}(q)$. Then $\mathscr{C}$ is called a *q-clan* provided $A_u - A_r$ is anisotropic whenever $u \neq r$, that is

$\alpha(A_u - A_r)\alpha^T = 0$ has only the trivial solution $\alpha = (0,0)$. For $A_u \in \mathscr{C}$, put $K_u = A_u + A_u^T$. Let

$$A_u = \begin{pmatrix} x_u & y_u \\ 0 & z_u \end{pmatrix}, \quad x_u, y_u, z_u, u \in \mathrm{GF}(q).$$

For q odd, $\mathscr{C}$ is a q-clan if and only if

$$-\det(K_u - K_r) = (y_u - y_r)^2 - 4(x_u - x_r)(z_u - z_r) \tag{1}$$

is a nonsquare of $\mathrm{GF}(q)$ whenever $r, u \in \mathrm{GF}(q)$, $r \neq u$. For q even, $\mathscr{C}$ is a q-clan if and only if

$$y_u \neq y_r \quad \text{and} \quad \mathrm{Tr}((x_u + x_r)(z_u + z_r)(y_u + y_r)^{-2}) = 1, \tag{2}$$

whenever $r, u \in \mathrm{GF}(q)$, $r \neq u$, and Tr is the trace map from $\mathrm{GF}(q)$ to $\mathrm{GF}(2)$.

Now we can define a family of subgroups of G by $A(u) = \{(\alpha, \alpha A_u \alpha^T, \alpha K_u) \in G : \alpha \in \mathrm{GF}(q)^2\}$, $u \in \mathrm{GF}(q)$, and $A(\infty) = \{(0, 0, \beta) : \beta \in \mathrm{GF}(q)^2\}$.

With G, $A(u)$, $A^*(u)$ and J as above, the following theorem is a combination of results of Payne and Kantor.

Theorem 8 (*Payne [21,22] and Kantor [17]*). *The set J is a 4-gonal family for G if and only if $\mathscr{C}$ is a q-clan.*

In [29] Thas showed that (1) and (2) are exactly the conditions for the planes $x_u X_0 + z_u X_1 + y_u X_2 + X_3 = 0$, $u \in \mathrm{GF}(q)$, of $\mathrm{PG}(3,q)$ to define a flock $\mathscr{F}$ of the quadratic cone K with equation $X_0 X_1 = X_2^2$. (A *flock* is a set of planes of $\mathrm{PG}(3,q)$ that partition the non-vertex points of a quadratic cone.) In [32] Thas provides an elegant geometrical description of the flock GQ, which we denote $\mathscr{S}(G, \mathscr{C})$ or $\mathscr{S}(\mathscr{F})$, directly from the flock.

The flock GQ $\mathscr{S}(G, \mathscr{C})$ has order (q^2, q) and its centre $Z(G) = \{(0, c, 0) \in G : c \in \mathrm{GF}(q)\}$ has order q and so we have the following result:

Theorem 9. *The flock GQ $\mathscr{S}(G, \mathscr{C})$ has regular point (∞).*

There are many examples of GQs of order (q^2, q), for both q odd and even, constructed through this method (see [24] for example).

Next we will introduce a GQ construction that is a generalisation of the construction $T_n(\mathscr{O})$ of Tits. Let $\mathscr{S} = (\mathscr{P}, \mathscr{B}, \mathrm{I})$ be a GQ of order (s, t), $s \neq 1$, $t \neq 1$. A collineation θ of $\mathscr{S}$ is an *elation* about the point P if θ is the identity or if θ fixes all lines incident with P and fixes no point of $\mathscr{P} \backslash P^\perp$. If there is a group G of elations about P acting regularly on $\mathscr{P} \backslash P^\perp$, then we say that $\mathscr{S}$ is an *elation generalized quadrangle* (EGQ) with *elation group* G and *base point* P. Briefly we say that $(\mathscr{S}^{(P)}, G)$ or $\mathscr{S}^{(P)}$ is an EGQ. Note that if J is a 4-gonal family for a group G, then $\mathscr{S}(G, J)$ is an EGQ with elation group G and base point (∞). If for an EGQ the elation group G is abelian, then we say that the EGQ $(\mathscr{S}^{(P)}, G)$ is a *translation generalized quadrangle* (TGQ) and G is the *translation group*.

In $\mathrm{PG}(2n+m-1, q)$ consider a set $\mathscr{E}(n, m, q)$ of $q^m + 1$ $(n-1)$-dimensional subspaces, every three of which generate a $\mathrm{PG}(3n-1, q)$ and such that each element E of $\mathscr{E}(n, m, q)$ is contained in an $(n+m-1)$-dimensional subspace T_E having no point in common with any element of $E(n,m,q)\backslash\{E\}$. It is straightforward to check that T_E is uniquely determined for any element E of $\mathscr{E}(n,m,q)$. The space T_E is called the *tangent space* of $\mathscr{E}(n,m,q)$ at E. For $n=m=1$ such a set $\mathscr{E}(1,1,q)$ is an oval in $\mathrm{PG}(2,q)$ and more generally for $m=n$ such a set $\mathscr{E}(n,n,q)$ is a *pseudo-oval* of $\mathrm{PG}(3n-1,q)$. For $m=2n=2$ such a set $\mathscr{E}(1,2,q)$ is an ovoid of $\mathrm{PG}(3,q)$ and more generally for $m=2n$ such a set $\mathscr{E}(n,2n,q)$ is a *pseudo-ovoid*. In general we call such sets $\mathscr{E}(n,m,q)$ *eggs*.

Now embed $\mathrm{PG}(2n+m-1,q)$ in a $\mathrm{PG}(2n+m,q)$ and construct a point-line geometry $T(n,m,q)$ as follows:

Points are of three types: (i) the points of $\mathrm{PG}(2n+m,q)\backslash\mathrm{PG}(2n+m-1,q)$, called the *affine* points; (ii) the $(n+m)$-dimensional subspaces of $\mathrm{PG}(2n+m,q)$ which intersect $\mathrm{PG}(2n+m-1,q)$ in a tangent space of $\mathscr{E}(n,m,q)$; (iii) the symbol (∞).

Lines are of two types: (a) the n-dimensional subspaces of $\mathrm{PG}(2n+m,q)$ which intersect $\mathrm{PG}(2n+m-1,q)$ in an element of $\mathscr{E}(n,m,q)$; (b) the elements of $\mathscr{E}(n,m,q)$.

Incidence in $T(n,m,q)$ is inherited from $\mathrm{PG}(2n+m,q)$ plus each line of type (b) is incident with the point (∞).

Theorem 10 (*8.7.1 of Payne and Thas [26]*). *The incidence geometry $T(n,m,q)$ is a* TGQ *of order (q^n,q^m) with base point (∞). Conversely, every* TGQ *is isomorphic to a $T(n,m,q)$. It follows that the theory of* TGQs *is equivalent to the theory of the sets $\mathscr{E}(n,m,q)$.*

In the case where $n=m=1$ and $\mathscr{E}(1,1,q)$ is an oval $\mathscr{O}$ of $\mathrm{PG}(2,q)$, the GQ $T(1,1,q)$ is the Tits GQ $T_2(\mathscr{O})$. When $m=2n=2$ and $\mathscr{E}(1,2,q)$ is an ovoid Ω of $\mathrm{PG}(3,q)$, the GQ $T(1,2,q)$ is the Tits GQ $T_3(\Omega)$.

For more details on TGQ and EGQ see Chapter 8 of [26]. In the $m=2n$ case the $T(n,m,q)$ construction has led to non classical TGQ of order (q^n,q^{2n}) with q odd (see [24], for example).

From the above model for TGQ the following theorem is straightforward.

Theorem 11. *Each line of the* TGQ *$T(n,m,q)$ incident with (∞) is regular.*

Proof. Let ℓ be a line of $T(n,m,q)$ incident with (∞). Then ℓ is an element of the egg $\mathscr{E}(n,m,q)$, a $(n-1)$-dimensional subspace of $\mathrm{PG}(2n+m-1,q)$. The group of elations of $\mathrm{PG}(2n+m,q)$ with axis $\mathrm{PG}(2n+m-1,q)$ and centre a point in ℓ induces a symmetry group of order q^n of $T(n,m,q)$ about ℓ. Hence ℓ is regular.

Note that this result implies that $n\leqslant m$. In fact by [26, 8.7.2] we know that $n=m$ or $n(a+1)=ma$, with a odd, and that if q is even, then $n=m$ or $m=2n$. □

When $n=m$ by applying Theorem 5 (see also [27]) we have the following result:

Theorem 12. *The* TGQ *$T(n,n,q)$ has regular point (∞) if and only if q is even.*

Note that this result implies that when q is even any egg $\mathscr{E}(n,n,q)$ has the property that the tangent spaces of the egg intersect pairwise in a fixed $(n-1)$-dimensional subspace of $\mathrm{PG}(3n-1,q)$.

4. Property (G) and GQs with a regular point

Let $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ be a GQ of order (s,s^2), $s\neq 1$. By [1] we have that $|\{X,Y,Z\}^\perp|=s+1$ for any triple $\{X,Y,Z\}$ of pairwise noncollinear points. We say that $\{X,Y,Z\}$ is 3-*regular* provided $|\{X,Y,Z\}^{\perp\perp}|=s+1$. Now let X_1,Y_1 be distinct collinear points. We say that the pair $\{X_1,Y_1\}$ has *Property* (G), or that $\mathcal{S}$ has *Property* (G) *at* $\{X_1,Y_1\}$, if every triple $\{X_1,X_2,X_3\}$ of points, with X_1,X_2,X_3 pairwise noncollinear and $Y_1\in\{X_1,X_2,X_3\}^\perp$, is 3-regular. The GQ $\mathcal{S}$ has *Property* (G) *at the line* ℓ, or the line ℓ has *Property* (G), if each pair of points $\{X,Y\}$, $X\neq Y$ and $X\,\mathrm{I}\,\ell\,\mathrm{I}\,Y$, has Property (G). If (X,ℓ) is a flag, that is, if $X\,\mathrm{I}\,\ell$, then we say that $\mathcal{S}$ has *Property (G) at* (X,ℓ), or that (X,ℓ) has *Property* (G), if every pair $\{X,Y\}$, $X\neq Y$, and $Y\,\mathrm{I}\,\ell$, has Property (G).

Property (G) was introduced by Payne in [23] and is studied in detail in the series of papers [30–32] by Thas.

Theorem 13 (*Thas and Van Maldeghem [35]*). *Let $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ be a GQ of order (q^2,q), q even, satisfying Property (G) at some point X. Then X is regular in $\mathcal{S}$ and the dual net $\mathcal{N}^*_X$ defined by X is isomorphic to H^3_q.*

In the previous section we saw the construction of a GQ $\mathcal{S}(\mathcal{F})$ of order (q^2,q) from a flock $\mathcal{F}$ of a quadratic cone in $\mathrm{PG}(3,q)$. The following theorem characterizes those GQs whose regular point (∞) has associated dual net H^3_q.

Theorem 14 (*Thas and Van Maldeghem [35]*). *For any GQ $\mathcal{S}(\mathcal{F})$ of order (q^2,q) arising from a flock $\mathcal{F}$, the point (∞) is regular. If q is even, then the dual net $\mathcal{N}^*_{(\infty)}$ is isomorphic to H^3_q. If q is odd, then the dual net $\mathcal{N}^*_{(\infty)}$ is isomorphic to H^3_q if and only if $\mathcal{F}$ is a Kantor–Knuth flock.*

The definition of the Kantor–Knuth flocks is as follows. Let $\mathcal{K}$ be the quadratic cone with equation $X_0X_1=X_2^2$ of $\mathrm{PG}(3,q)$, q odd. Then the q planes π_t with equation $tX_0-mt^\sigma X_1+X_3=0$, $t\in\mathrm{GF}(q)$, m a given nonsquare of $\mathrm{GF}(q)$, and σ a given automorphism of $\mathrm{GF}(q)$. Note that if σ is the identity automorphism of $\mathrm{GF}(q)$, then the Kantor–Knuth flock is linear, that is the elements of the flock contain a common line, giving rise to the classical GQ $H(3,q^2)$.

5. Representing GQs with a regular point

In this section we introduce a general representation for a GQ of order (s,t) with a regular point. The ideas in this section follow on from the work of Löwe [18] (see also Ghinelli and Ott [9] and the theses of Brown [2] and De Bruyn [7]).

To begin we will introduce the idea of a cover of a graph and a cover of a geometry.

If Γ is a graph, then a *t-fold cover* of Γ is a pair $(\overline{\Gamma},p)$ where $\overline{\Gamma}$ is a graph and p is a map from the vertex set of $\overline{\Gamma}$ onto the vertex set of Γ such that

(1) for any vertex X of Γ the set $p^{-1}(X)$ consists of t pairwise non-adjacent vertices,
(2) for any edge $\{X,Y\}$ of Γ, $p^{-1}(\{X,Y\})$ consists of t disjoint edges, and

(3) for any non-edge $\{X, Y\}$ of Γ, $p^{-1}(\{X, Y\})$ is a graph with no edges.
If Γ is the point-graph of a point-line geometry $\mathcal{N} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ and $(\overline{\Gamma}, p)$ satisfies
(4) for any line ℓ of $\mathcal{N}$, if $\mathcal{P}_\ell = \{P \in \mathcal{P} : P \,\mathrm{I}\, \ell\}$, then $p^{-1}(\mathcal{P}_\ell)$ consists of t disjoint complete graphs,

then we can form a geometry with points the vertices of $\overline{\Gamma}$, and lines defined to be the set of complete graphs from (4).

Let $\mathcal{N} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a point-line geometry with point-graph Γ. A *t-fold cover* of $\mathcal{N}$ is a pair $(\overline{\mathcal{N}}, p)$ where $\overline{\mathcal{N}} = (\overline{\mathcal{P}}, \overline{\mathcal{B}}, \overline{\mathrm{I}})$ is a point-line geometry with point-graph $\overline{\Gamma}$ and $p : \overline{\mathcal{P}} \to \mathcal{P}$ such that: (i) $(\overline{\Gamma}, p)$ is a t-fold cover of Γ; and (ii) $(\overline{\Gamma}, p)$ satisfies (4) giving rise to the geometry $\overline{\mathcal{N}}$.

We will abuse notation and also consider p as a map from $\overline{\mathcal{B}}$ to $\mathcal{B}$ induced by the map from $\overline{\mathcal{P}}$ to $\mathcal{P}$.

Remark 15. The definition of t-fold cover $\overline{\mathcal{N}} = (\overline{\mathcal{P}}, \overline{\mathcal{B}}, \overline{\mathrm{I}})$ of a geometry $\mathcal{N} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ above is convenient for our purposes in this paper. Perhaps a more standard definition is that there exists a surjective map $p : \mathcal{P} \times \mathcal{B} \to \overline{\mathcal{P}} \times \overline{\mathcal{B}}$ whose restriction to any point row or line pencil induces an isomorphism between point rows and line pencils, respectively.

Theorem 16. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, t) with a regular point X and associated net $\mathcal{N}_X = (\mathcal{P}_X, \mathcal{B}_X, \mathrm{I}_X)$. Then $(\overline{\mathcal{N}_X}, p)$ with $\overline{\mathcal{N}_X} = (\overline{\mathcal{P}_X}, \overline{\mathcal{B}_X}, \overline{\mathrm{I}_X})$ where $\overline{\mathcal{P}_X} = \mathcal{P} \backslash X^\perp$, $\overline{\mathcal{B}_X} = \mathcal{B} \backslash \{\ell \in \mathcal{B} : \ell \,\mathrm{I}\, X\}$, and $\overline{\mathrm{I}_X}$ is induced by* I, *together with $p : \overline{\mathcal{P}_X} \to \mathcal{P}_X$ defined by $p : Y \mapsto \{X, Y\}^\perp$ for $Y \in \mathcal{P} \backslash X^\perp$, is a t-fold cover of $\mathcal{N}_X$.*

Proof. For the point $\{X, Y\}^\perp$ of $\mathcal{N}_X$ the set $p^{-1}(\{X, Y\}^\perp) = \{X, Y\}^{\perp\perp} \backslash \{X\}$ has size t and no two elements are collinear in $\mathcal{S}$ and hence also in $\overline{\mathcal{N}_X}$. Next, if $\{X, Y\}^\perp$ and $\{X, Y'\}^\perp$ are two collinear points of $\mathcal{N}_X$, then there is a unique point $Z \in X^\perp \backslash \{X\}$ contained in both $\{X, Y\}^\perp$ and $\{X, Y'\}^\perp$. Each of the t lines of $\mathcal{S}$ incident with Z, but not X, is incident with a point of $p^{-1}(\{X, Y\}^\perp)$ and a distinct point of $p^{-1}(\{X, Y'\}^\perp)$, forming the t disjoint pairs of collinear points required for condition (2) of a t-fold cover. Now, suppose that $\{X, Y\}^\perp$ and $\{X, Y'\}^\perp$ are two non-collinear points of $\mathcal{N}_X$. Then $\{X, Y\}^\perp$ and $\{X, Y'\}^\perp$ are disjoint sets. If there is a point Z of $\{X, Y\}^{\perp\perp}$ collinear with a point Z' of $\{X, Y'\}^{\perp\perp}$, then the line ZZ' must meet $X^\perp$ in a point of $\{X, Y\}^\perp \cap \{X, Y'\}^\perp$. Hence it follows that $p^{-1}(\{\{X, Y'\}^\perp, \{X, Y\}^{\perp\perp}\})$ is a set of $2t$ pairwise non-collinear points of $\overline{\mathcal{N}_X}$.

If $Y \in X^\perp \backslash \{X\}$ is a line of $\mathcal{N}_X$, then $p^{-1}(Y)$ consists of t disjoint complete graphs corresponding to the t lines of $\mathcal{S}$, not YX, incident with Y.

Note that the covering geometry $\overline{\mathcal{N}_X}$ is "triangle free" in the sense that if three distinct points are pairwise collinear, then they are incident with a common line. In fact a cover of this type is enough to allow a reconstruction of the GQ.

Theorem 17. *Let $\mathcal{N} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a net with order s and degree $t+1$ and $(\overline{\mathcal{N}} = (\overline{\mathcal{P}}, \overline{B}, \overline{\mathrm{I}}), p)$ a t-fold cover of $\mathcal{N}$. Suppose that if W, Y, Z are three distinct, pairwise collinear, points of $\overline{\mathcal{N}}$, then $p(W)$, $p(Y)$, $p(Z)$ are incident with a common line of $\mathcal{N}$.*

Let $\mathcal{S}(\overline{\mathcal{N}}, p)$ *be the following incidence structure*:

Points: (i) *Points of* $\overline{\mathcal{N}}$,
(ii) *Lines of* $\mathcal{N}$, *and*
(iii) *A symbol* (∞).

Lines: (a) *Lines of* $\overline{\mathcal{N}}$, *and*
(b) *Parallel classes of* $\mathcal{N}$.

The incidence of $\mathcal{S}(\overline{\mathcal{N}}, p)$ *is as follows*: (i)(a), *is inherited from* $\overline{\mathcal{N}}$; (ii)(a), *a line of* $\mathcal{N}$ *is incident with each line of* $\overline{\mathcal{N}}$ *covered by it*; (ii)(b), *a line of* $\mathcal{N}$ *is incident with the parallel class containing it*; *and* (iii)(b), (∞) *is incident with each line of type* (b).

Then $\mathcal{S}(\overline{\mathcal{N}}, p)$ *is a GQ of order* (s, t) *with regular point* (∞).

Furthermore, if $\mathcal{S}$ *is a GQ of order* (s, t) *with regular point X, associated net* $\mathcal{N}_X$ *and t-fold cover* $(\overline{\mathcal{N}_X}, p)$ *of* $\mathcal{N}_X$, *then there exists an isomorphism from* $\mathcal{S}$ *to* $\mathcal{S}(\overline{\mathcal{N}_X}, p)$ *mapping X to* (∞).

Proof. It is straightforward to check that $\mathcal{S}(\overline{\mathcal{N}}, p)$ is a geometry satisfying the first two axioms of a GQ and most of the cases for the third axiom. We consider the two problematic cases of non-incident point-line pairs (Y, ℓ) of $\mathcal{S}(\overline{\mathcal{N}}, p)$.

Let Y be a point of $\overline{\mathcal{N}}$ and ℓ a line of $\overline{\mathcal{N}}$. If $p(Y)\ \mathrm{I}\ p(\ell)$, then Y is collinear in $\mathcal{S}(\overline{\mathcal{N}}, p)$ with the unique point $p(\ell)$ of ℓ . If $p(Y) \not\mathrm{I}\ p(\ell)$, then consider the t covers of the line $p(\ell)$ of $\mathcal{N}$ (including ℓ). For any such line of $\overline{\mathcal{N}}$ the point Y is collinear with at most one point on the line, since otherwise we have a triangle of $\overline{\mathcal{N}}$ not incident with a common line. In $\mathcal{N}$ the point $p(Y)$ is incident with one line parallel to $p(\ell)$ and with t meeting $p(\ell)$ and hence Y is collinear in $\overline{\mathcal{N}}$ with t points on a cover of $p(\ell)$, hence exactly one per line.

Next let Y be a line of $\mathcal{N}$ and ℓ a line of $\overline{\mathcal{N}}$. Since $Y \neq p(\ell)$ we have that either Y is in the same parallel class of $\mathcal{N}$ as $p(\ell)$ or not. In the first case Y is collinear in $\mathcal{S}(\overline{\mathcal{N}}, p)$ with the unique point $p(\ell)$. In the second case, as lines of $\mathcal{N}$, Y and $p(\ell)$ intersect in a unique point Q of $\mathcal{N}$. Exactly one of the covers $\overline{Q}$ of Q is incident with ℓ, which is the unique point of $\mathcal{S}(\overline{\mathcal{N}}, p)$ incident with ℓ and collinear in $\mathcal{S}(\overline{\mathcal{N}}, p)$ with Y. □

Remark 18. From this theorem we see that considering GQs of order (s, t) with a regular point is equivalent to considering t-fold covers of nets of order s and degree $t + 1$ with no non-collinear triangles. This motivates the following definition.

Definition 19. Let $\mathcal{N}$ be a net of order s and degree $t + 1$ and $(\overline{\mathcal{N}}, p)$ a t-fold cover of $\mathcal{N}$. Then $(\overline{\mathcal{N}}, p)$ (or just $\overline{\mathcal{N}}$ if p is understood) will be called a GQ-cover if for every triple of pairwise collinear points X, Y, Z of $\overline{\mathcal{N}}$ the triple of points $p(X), p(Y), p(Z)$ is incident with a common line in $\mathcal{N}$.

Note that it follows from the definition of a GQ-cover of a net of order s and degree $t + 1$ that $s \geqslant t$.

We now proceed to give a general description of t-fold covers of the point-graph of a net and calculate the conditions for such a cover to define a cover of the net and in particular a GQ-cover.

Let $\mathcal{N}$ be a net of order s and degree $t+1$ with point-graph Γ and let $(\overline{\Gamma}, p)$ be a t-fold cover of Γ. If P is a point of $\mathcal{N}$, then label the elements of $p^{-1}(P)$ arbitrarily such that $p^{-1}(P) = \{(P,1),(P,2),\dots,(P,t)\}$. For P and Q collinear points of $\mathcal{N}$ define a permutation ϕ_{PQ} of $\{1,2,\dots,t\}$ by $(P,i) \sim (Q, i^{\phi_{PQ}})$. It follows that $\phi_{QP}^{-1} = \phi_{PQ}$. Note that any set of permutations $\{\phi_{PQ} : P, Q \in \mathcal{P}, P \sim Q, P \neq Q\}$ of $\{1,2,\dots,t\}$ such that $\phi_{QP}^{-1} = \phi_{PQ}$ defines a t-fold cover of Γ as above.

Theorem 20. *Let $\mathcal{N} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a net of order s and degree $t+1$ equipped with a set of permutations $\{\phi_{PQ} : P, Q \in \mathcal{P}, P \sim Q, P \neq Q\}$ of $\{1,2,\dots,t\}$ such that $\phi_{QP}^{-1} = \phi_{PQ}$. The permutations define a cover of $\mathcal{N}$ if and only if $\phi_{PQ}\phi_{QR}\phi_{RP}$ is the identity when P, Q, R are collinear.*

Further, this cover of $\mathcal{N}$ is a GQ-cover if and only if $\phi_{PQ}\phi_{QR}\phi_{RP}$ is fixed point free when P, Q, R are non-collinear.

Proof. Let Γ be the point-graph of $\mathcal{N}$ and $\overline{\Gamma}$ the graph with vertices (P,i), $P \in \mathcal{P}$, $i \in \{1,\dots,t\}$ and $(P,i) \sim (Q,j)$ if and only if $P \sim Q$ and $j = i^{\phi_{PQ}}$. With $p : (P,i) \mapsto P$, the pair $(\overline{\Gamma}, p)$ defines a t-fold cover of Γ which extends to a t-fold cover of $\mathcal{N}$ if the preimage under p of the pointset of a line of $\mathcal{N}$ is a set of t disjoint complete graphs. Now $(P,i) \sim (Q, i^{\phi_{PQ}}) \sim (R, i^{\phi_{PQ}\phi_{QR}})$ and so $(\overline{\Gamma}, p)$ extends to a cover of $\mathcal{N}$ if and only if $(P,i) \sim (R, i^{\phi_{PQ}\phi_{QR}})$ for all collinear triples of points $\{P, Q, R\}$ and $i \in \{1,\dots,t\}$. That is, $\phi_{PQ}\phi_{QR}\phi_{RP} = 1$ if P, Q, R are collinear. In this case we call the covering geometry $\overline{\mathcal{N}}$. For $(\overline{\mathcal{N}}, p)$ to be a GQ-cover (P,i) must not be collinear to $(R, i^{\phi_{PQ}\phi_{QR}})$ if P, Q, R are not collinear, since otherwise we have a non-collinear triangle in $\overline{\mathcal{N}}$. Hence $\phi_{PQ}\phi_{QR}\phi_{RP}$ must be fixed point free. □

6. Some applications to subquadrangles of GQs with a regular point

In this section we look at a couple of applications to subquadrangles of the representation of GQs with a regular point via the cover of the associated net.

The following result has a different published proof to that given here, although equally as short.

Theorem 21 (*Thas [33]*). *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s,t), $s, t > 1$, with a regular point X and associated net $\mathcal{N}_X = (\mathcal{P}_X, \mathcal{B}_X, \mathrm{I}_X)$ of order s and degree $t+1$. If $\mathcal{N}_X$ has a proper subnet of degree $t+1$, then $\mathcal{S}$ has a proper subquadrangle $\mathcal{S}'$ of order (s', t) containing X as a regular point. Further, the proper subnet must be an affine plane $(s' = t)$ and also $s = t^2$.*

Proof. Let $(\overline{\mathcal{N}_X} = (\overline{\mathcal{P}_X}, \overline{\mathcal{B}_X}, \overline{\mathrm{I}_X}), p)$ be the t-fold cover of $\mathcal{N}_X$ defined by $\mathcal{S}$. Suppose that $\mathcal{N}'_X = (\mathcal{P}'_X, \mathcal{B}'_X, \mathrm{I}'_X)$ is a proper subnet of $\mathcal{N}_X$ of order $s' < s$ and degree $t+1$. Define the geometry $\overline{\mathcal{N}'_X}$ to be the proper subgeometry of $\overline{\mathcal{N}_X}$ defined on the pointset $p^{-1}(\mathcal{P}'_X)$, and p' the restriction of the map p to $p^{-1}(\mathcal{P}'_X)$. Then $(\overline{\mathcal{N}'_X}, p')$ is necessarily a GQ-cover of $\mathcal{N}'_X$. By the construction of Theorem 17 this yields a subquadrangle of order (s', t) of $\mathcal{S}$ with regular point X.

Now since the subquadrangle has order (s', t) and a regular point, it follows that $s' \geqslant t$. By [26, 2.2.2(ii)] it must be that $s' = t$, that is, $\mathcal{N}'_X$ is an affine plane, and $s = t^2$. □

Theorem 22 (*Brown and Thas [4]*). *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s^2, s) with regular point X such that the associated net is the dual of H_s^3 and $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ is a subquadrangle of $\mathcal{S}$ of order s not containing the point X. Then $\mathcal{S}'$ is isomorphic to the dual of $T_2(\mathcal{O})$ for some oval $\mathcal{O}$ of* $\mathrm{PG}(2, s)$.

Proof. Suppose that the net $\mathcal{N}_X \cong (H_s^3)^*$ is constructed from the line ℓ of $\mathrm{PG}(3, s)$; that is by taking as points the lines of $\mathrm{PG}(3, s)$ not meeting ℓ, as lines the points of $\mathrm{PG}(3, s)\backslash\ell$, and the incidence from $\mathrm{PG}(3, s)$. Then each plane π of $\mathrm{PG}(3, s)$ *not* containing ℓ gives rise to an affine plane subnet of $(H_s^3)^*$, which is the dual of π with the point $\pi \cap \ell$ and the lines on $\pi \cap \ell$ removed. By Theorem 21 we see that π gives rise to a subquadrangle of order s. The subquadrangle has points X, the points of π not on ℓ and the points of $\mathcal{P}\backslash X^{\perp}$ that are covers of a line of π not meeting ℓ. The lines of the subquadrangle are the lines of $\mathcal{S}$ incident with X and the lines of $\mathcal{S}$ not incident with X that are covers of a point of $\pi\backslash(\pi \cap \ell)$. This gives $s^3 + s^2$ distinct subquadrangles of order s containing X, the maximal number possible.

Now suppose that $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ is a subquadrangle of order s of $\mathcal{S}$, not containing X. The geometry $\mathcal{S}\backslash X^{\perp}$ is an s-fold cover of the net $(H_s^3)^*$ with covering map p taking the point $P \in \mathcal{S}\backslash X^{\perp}$ to the point $\{X, P\}^{\perp}$ of $\mathcal{N}_X \cong (H_s^3)^*$. The subquadrangle $\mathcal{S}'$ contains a unique line, m say, incident with X. The points of $\mathcal{S}$ incident with m, but distinct from X, form a parallel class of $(H_s^3)^*$ the elements of which are contained in a plane of $\mathrm{PG}(3, s)$ containing ℓ which we will denote by $p(m)$. The subquadrangle $\mathcal{S}'$ contains $s + 1$ points of m, which we denote by $\mathcal{O}$. In H_s^3 the set $\mathcal{O}$ is a set of $s + 1$ points on the plane $p(m)$, none of which is incident with ℓ. Consider a line n of $\mathcal{S}'$ not concurrent with m. Thus $p(n)$ is a point of $\mathrm{PG}(3, s)$ not on the plane $p(m)$. Further, since no two lines of $\mathcal{S}'$ may be incident with a common point of $X^{\perp}$ not on m, it follows that the covering map p gives a one-to-one correspondence between the s^3 lines of $\mathcal{S}'$not concurrent with m and the s^3 points of $\mathrm{PG}(3, s)\backslash p(m)$. Each point of $\mathcal{S}'$ not incident with m is collinear with a unique point of $\mathcal{O}$ and so under the map p is a line of $\mathrm{PG}(3, s)$ meeting $p(m)$ in a point of $\mathcal{O}$. Since no two lines of $\mathcal{S}'$ are concurrent in a point of $X^{\perp}$ not on m, it must also be the case that no two points of $\mathcal{S}'$, not incident with m, correspond under p to the same line of $\mathrm{PG}(3, s)$. Thus p gives a one-to-one correspondence between the set $\mathcal{P}'\backslash m$ and the lines of $\mathrm{PG}(3, s)$ not in $p(m)$ meeting $p(m)$ in a point of $\mathcal{O}$. It is now a straightforward exercise to verify that $\mathcal{O}$ is an oval and that $\mathcal{S}'$ is isomorphic to the dual of $T_2(\mathcal{O})$.

7. Representing GQs with an abelian centre of symmetry

Each known example of a regular point of a GQ is also a centre of symmetry with an (elementary) abelian symmetry group. We call such a point an *abelian centre of symmetry*. In this section we will introduce a representation of GQs with an abelian centre of symmetry.

Suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a GQ of order (s, t) with centre of symmetry X, (additive) abelian symmetry group A, associated net $\mathcal{N} = (\mathcal{P}_X, \mathcal{B}_X, \mathrm{I})$ and t-fold cover $(\overline{\mathcal{N}}, p)$. For each point P of $\mathscr{N}$ arbitrarily label some element of $p^{-1}(P)$ by $(P, 0)$, where 0 is the identity of A. For $\alpha \in A$, label the point $(P, 0)^{\alpha}$ of $\overline{\mathscr{N}}$ by (P, α). Hence for each $\alpha \in A$ the map

$$(P, \beta) \mapsto (P, \alpha + \beta) \quad \text{for } P \in \mathscr{P}_X \text{ and } \beta \in A$$

is the symmetry α of $\overline{\mathscr{N}}$. Consequently, we have that two points (P, α) and (Q, β) of $\overline{\mathscr{N}}$ are collinear if and only if $\beta - \alpha$ is a constant depending on P and Q. We define a function c from pairs of collinear points of $\mathscr{N}$ to A such that

$$(P, \alpha) \sim (Q, \beta) \iff c(P, Q) = \alpha - \beta \quad \text{for } P \sim Q,\ P \neq Q, \text{ and } c(P, P) = 0.$$

Note that the function c is alternating, that is, $c(P, Q) = -c(Q, P)$ for all $P, Q \in \mathscr{P}$, $P \sim Q$.

The function c will often be referred to as a *covering function*.

Remark 23. An alternative way to view the function c is as follows. Let Γ be the simplicial complex constructed from the point graph of $\mathscr{N}$. Then c is a 1-cochain on Γ mapping into the abelian group A. We will not say anything more concerning the cohomological aspects of this situation, except to mention that it motivates some of the considerations and terminology/notation that follows and that more details on this aspect of covering geometries may be found in [3].

Now for a general alternating function c mapping from pairs of collinear points of $\mathscr{N}$ to A we can define the collinearity

$$(P, \alpha) \sim (Q, \beta) \iff c(P, Q) = \alpha - \beta.$$

If the function c defines a t-fold cover of $\mathscr{N}$, then this geometry is denoted $\mathscr{N}^c$. Since all GQs with a centre of symmetry arise in this way we are interested which alternating functions define a cover of $\mathscr{N}$ which is also a GQ-cover.

Recall the general representation of GQs with a regular point discussed in Section 5. If instead of considering permutations of the set $\{1, 2, \ldots, t\}$ we consider permutations of the group A, then we have

$$\alpha^{\phi_{PQ}} = \alpha - c(P, Q) \quad \text{for } P, Q \in \mathscr{P}, P \sim Q \text{ and } \alpha \in A. \tag{3}$$

Now applying Theorem 20 we have the following result:

Theorem 24. *Let $\mathscr{N} = (\mathscr{P}, \mathscr{B}, \mathrm{I})$ be a net of order s and degree $t + 1$, A an (additive) abelian group of order t and c mapping from pairs of collinear points of $\mathscr{N}$ to A is an alternating function. Let $\overline{\Gamma}$ be the graph with vertex set $\overline{\mathscr{P}} = \{(X, \alpha) : X \in \mathscr{P}, \alpha \in A\}$, adjacency $(X, \alpha) \sim (Y, \beta) \Leftrightarrow c(X, Y) = \alpha - \beta$, and $p : \overline{\mathscr{P}} \to \mathscr{P}$ such that $(X, \alpha) \mapsto X$. Then $(\overline{\Gamma}, p)$ gives rise to a GQ-cover of $\mathscr{N}$ if and only if*

$$\delta c(X, Y, Z) = 0 \iff X, Y, Z \text{ are collinear},$$

where $\delta c(X, Y, Z) = c(X, Y) - c(X, Z) + c(Y, Z)$.

Proof. Using (3) we see that c defines a cover of $\mathcal{N}$ if $\delta c(X, Y, Z) = 0$ whenever X, Y, Z are collinear. In addition, c defines a GQ-cover if $\delta c(X, Y, Z) \neq 0$ whenever X, Y, Z are non-collinear.

7.1. *Covering functions for the known GQs of order s*

In this section we will consider the known examples of (abelian) centres of symmetry of GQs of order s and give the corresponding covering function c. We restrict our attention to GQs of order s since in the known cases the associated net is always the affine plane $AG(2, q)$, which is straightforward to represent and calculate with.

The GQ $W(q)$ given by the symplectic polarity with form $x_0y_1 - x_1y_0 + x_2y_3 - x_3y_2 = 0$ has the property that each point is an abelian centre of symmetry. If P is a point of $W(q)$ and π_P the polar plane of P under the symplectic polarity, then the affine plane associated with P as a regular point of $W(q)$ is $\pi = (\pi_P \backslash \{P\})^*$. The symmetries of $W(q)$ about the point P are induced by the elations of $PG(3, q)$ with centre P and axis π_P. Using coordinates now, if $X = (x_0, x_1, x_2, x_3)$, then π_X, the polar plane of X, has coordinates $[-x_1, x_0, -x_3, x_2]$. If P is the point $(0, 1, 0, 0)$ then $\pi_p = [1, 0, 0, 0]$ and π has pointset $\{[0, 1, x_1, x_2] \cap \pi_P : x_1, x_2 \in GF(q)\}$ and the covers of the point $[0, 1, x_1, x_2] \cap \pi_P$ are the elements of the set $\{(1, \alpha, x_2, -x_1) : \alpha \in GF(q)\}$. If we denote $[0, 1, x_1, x_2] \cap \pi_P$ by (x_1, x_2), then π assumes the canonical form of $AG(2, q)$. Further, we can identify the group of symmetries about P with the additive group of $GF(q)$ and denote $(1, \alpha, x_2, -x_1)$ by $((x_1, x_2), \alpha)$. Now $((x_1, x_2), \alpha) \sim ((y_1, y_2), \beta)$ if and only if $(1, \alpha, x_2, -x_1) \sim (1, \beta, y_2, -y_1)$, which is the case if and only if $\beta - \alpha - x_2y_1 + x_1y_2 = 0$, that is $\alpha - \beta = x_1y_2 - x_2y_1$. In other words, the covering function

$$c((x_1, x_2), (y_1, y_2)) = x_1y_2 - x_2y_1$$

for $AG(2, q)$ gives rise to the classical GQ $W(q)$.

Now we consider the GQ $T_2(\mathcal{O})$. If q is odd, then $T_2(\mathcal{O}) \cong W(q)^*$ and has no regular points, so we will suppose that q is even. Let $\mathcal{O}$ be the oval $\{(1, t, f(t)) : t \in GF(q)\} \cup \{(0, 0, 1)\}$ of $PG(2, q)$ with nucleus $N = (0, 1, 0)$ and with $f(0) = 0$ and $f(1) = 1$. Embed $PG(2, q)$ in $PG(3, q)$ as the hyperplane $x_3 = 0$ and construct $T_2(\mathcal{O})$ in the usual way.

The point (∞) is regular and the associated affine plane is

$$\pi_{(\infty)} = \frac{PG(3, q)}{N} \Big\backslash \frac{PG(2, q)}{N} \cong AG(2, q)\ .$$

The symmetry group of $T_2(\mathcal{O})$ about (∞) is induced by the group of elations of $PG(3, q)$ with axis $PG(2, q)$ and centre N. Identifying the symmetry group with the additive group of $GF(q)$ and representing $\pi_{(\infty)}$ in canonical form we have that $((x_1, x_2), \alpha)$ represents the point $(x_1, \alpha, x_2, 1)$. It follows that $((x_1, x_2), \alpha) \sim ((y_1, y_2), \beta)$ if and only if the point $(x_1 + y_1, \alpha + \beta, x_2 + y_2)$ is a point of the oval $\mathcal{O}$. Hence the corresponding covering function of $AG(2, q)$ is given by

$$c((x_1, x_2), (y_1, y_2)) = \begin{cases} 0 & \text{if } x_1 = y_1, \\ (x_1 + y_1) f^{-1}\left(\dfrac{x_2 + y_2}{x_1 + y_1}\right) & \text{if } x_1 \neq y_1. \end{cases}$$

Each point P of $\mathcal{O}$ is an axis of symmetry of $T_2(\mathcal{O})$ with associated affine plane $(\mathrm{PG}(3,q)/P)^*\backslash(NP/P)^*$ and symmetry group induced by the elations of $\mathrm{PG}(3,q)$ with axis $\mathrm{PG}(2,q)$ and centre P. By similar considerations to above we can calculate the corresponding covering functions of $\mathrm{AG}(2,q)$. If $P=(0,0,1)$, then the covering function is

$$c((x_1,x_2),(y_1,y_2)) = \begin{cases} 0 & \text{if } x_2 = y_2, \\ \dfrac{x_1+y_1}{x_2+y_2}(f(x_2)+f(y_2)) & \text{if } x_2 \neq y_2. \end{cases}$$

If $P=(1,t,f(t))$, then the covering function is

$$c((x_1,x_2),(y_1,y_2)) = \begin{cases} 0 & \text{if } x_2 = y_2, \\ \dfrac{x_1+y_1}{x_2+y_2}(f_t^{-1}(x_2)+f_t^{-1}(y_2)) & \text{if } x_2 \neq y_2, \end{cases}$$

where $f_t(x)=(f^{-1}((x+t)^{q-2})+f^{-1}(t^{q-2}))/x$.

In the special case where $\mathcal{O}$ is a translation oval, that is, $f(t)=t^{\sigma}$, where σ is a generator of $\mathrm{Aut}(\mathrm{GF}(q))$, each plane π of $\mathrm{PG}(3,q)$, not the plane of $\mathcal{O}$, that meets $\mathrm{PG}(2,q)$ in the axis $x_0=0$ of $\mathcal{O}$ is a centre of symmetry of $T_2(\mathcal{O})$. By [26, 12.5.2] $T_2(\mathcal{O})$ is self-dual with a duality interchanging the line $(0,0,1)$ of $T_2(\mathcal{O})$ with the point (∞) and π with an element of $\mathcal{O}\backslash\{(0,0,1)\}$. Hence the covering function associated with π is one of those above.

In the classical case, when q is even, the formulae above give a number of different covering functions. These include $x_1y_2+x_2y_1$, $\sqrt{(x_1+y_1)(x_2+y_2)}$ and $(x_1+y_1)(x_2+y_2)$. In the next section we shall investigate when different covering functions give rise to the same GQ.

Remark 25. An interesting question is whether any such covering functions can be found for non-desarguesian affine planes. This would immediately yield a new GQ of order s.

7.2. Equivalence of covering functions and the group of a GQ fixing an abelian centre of symmetry

Let $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ and $\mathcal{S}'=(\mathcal{P}',\mathcal{B}',\mathrm{I}')$ be two GQs of order (s,t) with abelian centres of symmetry X and X', respectively. Suppose that $\mathcal{S}$ has symmetry group A about X, associated net $\mathcal{N}_X$ and covering function c, while $\mathcal{S}'$ has symmetry group A' about X', associated net $\mathcal{N}_{X'}$ and covering function c'. Then we are interested in determining under what conditions there is an isomorphism from $\mathcal{S}$ to $\mathcal{S}'$ mapping X to X'.

So let $i:\mathcal{S}\rightarrow\mathcal{S}'$ be an isomorphism such that $X^i=X'$. We first observe that i must induce an isomorphism from $\mathcal{N}_X$ to $\mathcal{N}_{X'}$, so we will assume that $\mathcal{N}=\mathcal{N}_{X'}=\mathcal{N}_X$ and that i induces a collineation T of $\mathcal{N}$. Also, since $A=i^{-1}\circ A'\circ i$ we will assume that $A'=A$ and that for $\alpha\in A$ we have that $i^{-1}\circ\alpha\circ i=\alpha^{\sigma}$ for some automorphism σ of A. Now any GQ is uniquely determined by the geometry remaining after removing a point, all lines incident with that point and all points incident with those lines. Hence the GQs $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic with an isomorphism mapping X to X' if and only if the covering geometries $\mathcal{N}^c$ and $\mathcal{N}^{c'}$ are isomorphic. Thus we now determine when $\mathcal{N}^c$ and $\mathcal{N}^{c'}$ are isomorphic.

First we consider a useful normalisation of c. Suppose that b is a function from the pointset of $\mathcal{N}$ to A and δb acts on pairs of points of $\mathcal{N}$ by $\delta b(P,Q)=b(P)-b(Q)$, then for any fixed element k of A we have that $\mathcal{N}^c$ is isomorphic to $\mathcal{N}^{c+\delta b}$ by the map $(P, \alpha) \mapsto (P, \alpha + b(P) + k)$.

Note that for a fixed function δb there are exactly t functions $d : \mathcal{P} \to A$ such that $\delta b(P, Q) = d(P) - d(Q)$, namely the t functions $b(P) + k$ for $k \in A$.

Lemma 26. *The geometries $\mathcal{N}^c$ and $\mathcal{N}^{c'}$ are isomorphic if and only if*

$$c'(P^T, Q^T) = c(P, Q)^\sigma + \delta b(P, Q) \quad \textit{for } P \textit{ and } Q \textit{ collinear points of } \mathcal{N},$$

where T is some collineation of $\mathcal{N}$, σ some automorphism of A and δb is an alternating function that may be written in the form $\delta b(P, Q) = b(P) - b(Q)$ for some map b from the points of $\mathcal{N}$ to A.

Further, for such a fixed T the automorphism σ and the function δb are unique. Also, the isomorphisms from $\mathcal{N}^c$ to $\mathcal{N}^{c'}$ that induce the collineation T on $\mathcal{N}$ are $(P, \alpha) \mapsto (P^T, \alpha^\sigma + b(P) + k)$ for $k \in A$.

Proof. Suppose that $i : \mathcal{N}^c \to \mathcal{N}^{c'}$ is an isomorphism such that i induces T on $\mathcal{N}$ and for $\alpha \in A$, $i^{-1} \circ \alpha \circ i = \alpha^\sigma$ for some fixed automorphism σ of A. Then i must act by $(P, \alpha)^i = (P^T, t_P(\alpha))$ for some permutation t_P of A. Now

$$(P, \alpha)^i = (P, 0)^{\alpha \circ i} = (P^T, t_P(0))^{i^{-1} \circ \alpha \circ i} = (P^T, t_P(0) + \alpha^\sigma).$$

Defining $b(P) = t_P(0)$ we have that $(P, \alpha)^i = (P^T, b(P) + \alpha^\sigma)$ and consequently for $P \sim Q$

$$(P, \alpha) \sim (Q, \beta) \iff (P^T, b(P) + \alpha^\sigma) \sim (Q^T, b(Q) + \beta^\sigma),$$

which is the case if and only if $c'(P^T, Q^T) = c(P, Q)^\sigma + \delta b(P, Q)$.

Conversely, if $c'(P^T, Q^T) = c(P, Q)^\sigma + \delta b(P, Q)$, then it is straightforward to check that $(P, \alpha)^i = (P^T, \alpha^\sigma + b(P))$ is an isomorphism from $\mathcal{N}^c$ to $\mathcal{N}^{c'}$.

Suppose that we have $\overline{\sigma} \in \mathrm{Aut}(A)$ and $\overline{b} : \mathcal{P} \to A$ such that $c'(P^T, Q^T) = c(P, Q)^\sigma + \delta b(P, Q) = c(P, Q)^{\overline{\sigma}} + \delta\overline{b}(P, Q)$. Now for P, Q, R pairwise collinear points of $\mathcal{N}$ we have

$$\begin{aligned} \delta c'(P^T, Q^T, R^T) &= c'(P^T, Q^T) - c'(P^T, R^T) + c'(Q^T, R^T) \\ &= (\delta c(P, Q, R))^\sigma - (\delta\overline{c}(P, Q, R))^{\overline{\sigma}} = 0. \end{aligned}$$

It follows from this that $\overline{\sigma} = \sigma$ and consequently $\delta b = \delta\overline{b}$. Hence σ and δb are unique for a given T. If b is fixed, then recall that the functions $d : \mathcal{P} \to A$ such that $\delta d = \delta b$ are exactly $d(P) = b(P) + k$, $k \in A$, so all possible isomorphisms from $\mathcal{N}^c$ to $\mathcal{N}^{c'}$ that induce T on $\mathcal{N}$ have the form $(P, \alpha) \mapsto (P^T, \alpha^\sigma + b(P) + k)$ for $k \in A$.

Now we consider the special case in which $c = c'$, that is automorphisms of the geometry $\mathcal{N}^c$. If T is a collineation of $\mathcal{N}$ such that there exists a (necessarily unique) automorphism σ_T of A and function δb_T satisfying

$$c(P^T, Q^T) = c(P, Q)^{\sigma_T} + \delta b_T(P, Q) \quad \text{for all } P, Q \in \mathcal{P}, P \sim Q, P \neq Q,$$

then we say that T is *admitted* by c. The set of collineations of $\mathcal{N}$ admitted by c forms a group which we denote $\mathrm{Aut}_c(\mathcal{N})$. □

Lemma 27. *The full collineation group of the geometry* $\mathcal{N}^c$ *comprises the elements*

$$(P, \alpha) \mapsto (P^T, \alpha^{\sigma_T} + k + b_T(P)),$$

where $k \in A$, $T \in \mathrm{Aut}_c(\mathcal{N})$ *and* σ_T *and* b_T *are fixed such that* $c(P^T, Q^T) = c(P, Q)^{\sigma_T} + \delta b_T(P, Q)$.

Hence we have the corresponding result on the collineation group of a GQ with a centre of symmetry.

Theorem 28. *Let* $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ *be a GQ of order* (s, t) *with an abelian centre of symmetry* X *with symmetry group* A. *Suppose that* X *has associated net* $\mathcal{N}$ *and covering function* c. *Then the group of collineations of* $\mathcal{S}$ *fixing* X *is induced by the full collineation group of* $\mathcal{N}^c$ *which comprises the elements*

$$(P, \alpha) \mapsto (P^T, \alpha^{\sigma_T} + k + b_T(P)),$$

where $k \in A$, $T \in \mathrm{Aut}_c(\mathcal{N})$ *and* σ_T *and* b_T *are fixed such that* $c(P^T, Q^T) = c(P, Q)^{\sigma_T} + \delta b_T(P, Q)$.

Corollary 29. *Let* $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ *be a GQ of order* (s, t) *with an abelian centre of symmetry* X *with symmetry group* A. *Then* A *is normal in* $\mathrm{Aut}(\mathcal{S})_X$ *and* $\mathrm{Aut}(\mathcal{S})_X$ *is the semidirect product of* $\mathrm{Aut}_c(\mathcal{N})$ *with* A.

Any elation of $\mathcal{S}$ about X induces a collineation of $\mathcal{N}_X$ fixing each parallel class. If $\mathcal{S}$ is an EGQ with base point X and elation group G, then the group of $\mathcal{N}_X$ induced by G is transitive on points and is a translation group if and only if X is a centre of symmetry with symmetry group contained in G. In this case $\mathcal{N}_X$ is called a *skew translation generalized quadrangle* (see [23]).

In [33] Thas proves that if $\mathcal{S}$ is an EGQ with regular base point X and $\gcd(s-1, t) = 1$, then either X is a centre of symmetry with symmetry group contained in the elation group of $\mathcal{S}$, or $\mathcal{S}$ contains a proper subquadrangle of order t and consequently $s = t^2$.

If $\mathcal{S}$ is an EGQ with base point (∞), then the Kantor family contains a subgroup normal in the elation group if and only if the corresponding line incident with (∞) is an axis of symmetry with symmetry group contained in the elation group (see, for instance, [6,10,11] for studies of such Kantor families).

References

[1] R.C. Bose, S.S. Shrikhande, Geometric and pseudo-geometric graphs $(q^2+1, q+1, q)$, J. Geom. 2/1 (1973) 74–94.

[2] M.R. Brown, Generalized quadrangles and associated structures, Ph.D. Thesis, University of Adelaide, 1997.

[3] M.R. Brown, A characterisation of the generalized quadrangle $Q(5, q)$ using cohomology, J. Algebraic Combin. 15 (2) (2002) 107–126.

[4] M.R. Brown, J.A. Thas, Subquadrangles of order s of generalized quadrangles of order (s, s^2), Part I, J. Combin. Theory Ser. A 106 (1) (2004) 15–32.

[5] P.J. Cameron, Partial quadrangles, Quart. J. Math. Oxford (3) 25 (1974) 1–13.

[6] X. Chen, D. Frohardt, Normality in a Kantor family, J. Combin. Theory Ser. A 64 (1993) 130–136.
[7] B. De Bruyn, Characterization of near polygons, Ph.D. Thesis, Ghent University, 2000.
[8] P. Dembowski, Finite Geometries, Springer, Berlin, 1968.
[9] D. Ghinelli, U. Ott, Nets and generalized quadrangles, Geom. Dedicata 52 (1994) 1–14.
[10] D. Hachenberger, On finite elation generalized quadrangles with symmetries, J. London Math. Soc. (2) 53 (1994) 397–406.
[11] D. Hachenberger, Groups admitting a Kantor family and a factorized normal subgroup, Des. Codes Cryptogr. 8 (1996) 135–143.
[12] D. Higman, Partial geometries, generalized quadrangles and strongly regular graphs, in: A. Barlotti (Ed.), Atti Convegno di Geometria e sue Applicazioni, Perugia, 1971, pp. 263–293.
[13] D. Higman, Invariant relations, coherent configurations and generalized polygons, in: M. Hall Jr., J.H. van Lint (Eds.), Combinatorics, part 3, Mathematics Centre Tracts, vol. 57, 1974, pp. 27–43.
[14] J.W.P. Hirschfeld, Finite Projective Spaces of Three Dimensions, Oxford University Press, Oxford, 1985.
[15] J.W.P. Hirschfeld, Projective Geometries over Finite Fields, Oxford University Press, Oxford, 1998.
[16] W.M. Kantor, Generalized quadrangles associated with $G_2(q)$, J. Combin. Theory Ser. A 29 (1980) 212–219.
[17] W.M. Kantor, Some generalized quadrangles with parameters (q^2, q), Math. Z. 192 (1986) 46–50.
[18] S. Löwe, Über die Konstruktion von verallgemeinerten Vierecken mit einem regulären Punkt, Habilitationsschrift Technische Universität Braunschweig, 1992.
[19] S.E. Payne, A complete determination of translation ovals in finite Desarguesian planes, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 51 (1971) 328–331.
[20] S.E. Payne, Generalized quadrangles of even order, J. Algebra 31 (1974) 367–391.
[21] S.E. Payne, Generalized quadrangles as group coset geometries, Congr. Numer. 29 (1980) 717–734.
[22] S.E. Payne, A new infinite family of generalized quadrangles, Congr. Numer. 49 (1985) 115–128.
[23] S.E. Payne, An essay on skew translation generalized quadrangles, Geom. Dedicata 32 (1989) 93–118.
[24] S.E. Payne, Flocks and generalized quadrangles: an update, chapter 4 of 'Generalized Polygons', in: F. De Clerck et al. (Eds.), Proceedings of the Academy Contact Forum 'Generalized Polygons' 20 October, Palace of the Academies, Brussels, Belgium, 2001, pp. 295–303.
[25] S.E. Payne, J.A. Thas, Moufang conditions for finite generalized quadrangles, in: P.J. Cameron, J.W.P. Hirschfeld, D.R. Hughes (Eds.), Finite Geometries and Designs, Cambridge University Press, London Mathematical Society, Lecture Note Series, vol. 49, 1980, pp. 275–303.
[26] S.E. Payne, J.A. Thas, Finite Generalized Quadrangles, Pitman, London, 1984.
[27] J.A. Thas, The m-dimensional projective space $S_m(M_n(\mathrm{GF}(q)))$ over the total matrix algebra $M_n(\mathrm{GF}(q))$ of the $n \times n$ matrices with elements in the Galois field $\mathrm{GF}(q)$, Rend. Mat. (6) (4) (1971) 459–532.
[28] J.A. Thas, 4-gonal configurations, in: A. Barlotti (Ed.), Finite Geometric Structures and their Applications, Ed. Cremonese Roma, 1973, pp. 251–263.
[29] J.A. Thas, Generalized quadrangles and flocks of cones, European J. Combin. 8 (1987) 441–452.
[30] J.A. Thas, Generalized quadrangles of order (s, s^2), I, J. Combin. Theory Ser. A 67 (1994) 140–160.
[31] J.A. Thas, Generalized quadrangles of order (s, s^2), II, J. Combin. Theory Ser. A 79 (1997) 223–254.
[32] J.A. Thas, Generalized quadrangles of order (s, s^2), III, J. Combin. Theory Ser. A 87 (1999) 247–272.
[33] K. Thas, A theorem concerning nets arising from generalized quadrangles with a regular point, Des. Codes Cryptogr. 25 (3) (2002) 247–253.
[34] J.A. Thas, F. De Clerck, Partial geometries satisfying the axiom of Pasch, Simon Stevin 51 (1977) 123–137.
[35] J.A. Thas, H. Van Maldeghem, Generalized quadrangles and the axiom of Veblen, in: Geometry, Combinatorial Designs, and Related Structures, Cambridge University Press, Cambridge, 1997, pp. 241–253.