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Modular functors in homotopy quantum field theory and tortile structures

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Abstract

We study a version of a modular functor for Turaev's homotopy quantum field theories using 2-categories of surfaces. We define the homotopy surface 2-category of a space X and define an \mathcal{S}_X -structure to be a monoidal 2-functor from this to the 2-category of idempotent-complete additive k -linear categories. We initiate the study of the algebraic structure arising from these functors. In particular we show that a unitary \mathcal{S}_X -structure gives rise to a lax tortile π -category when the background space is an Eilenberg–MacLane space $X = K(\pi, 1)$, and to a tortile category with lax $\pi_2 X$ -action when the background space is simply connected.

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0. Introduction

The motivation for this paper was to study a version of a modular functor for Turaev's homotopy quantum field theories. A homotopy quantum field theory, as defined by Turaev in [13], is a variant of a topological quantum field theory in which manifolds come equipped with a map to some auxiliary space X . The definition can be formulated in terms of representations of categories of cobordisms in X [2,8] and in 1 + 1 dimensions classification theorems in terms of generalised Frobenius algebras are possible [13,2]. From a geometrical point of view a 1 + 1-dimensional homotopy

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quantum field theory is something like a vector bundle on the free loop space of X with a generalised flat connection, giving “parallel transport” across surfaces. For a more precise statement see [3]. In topological quantum field theory Tillmann [10,11] has pioneered the use of 2-categories in the study of modular functors. The relevant 2-category is one whose objects are circles, whose morphisms are surfaces and whose 2-morphisms are (path components of) diffeomorphisms of surfaces, and a modular functor is viewed as a certain 2-representation of this 2-category. Tillmann’s 2-category is also an approximation to Segal’s category of Riemann surfaces appearing in the definition of conformal field theory, in the sense that the morphism spaces consisting of Riemann surfaces are replaced by categories whose classifying spaces have the same rational homotopy type. In this paper, we generalise Tillmann’s work to study structure arising from representations of a 2-category of surfaces where a background space is incorporated. For a space X , we call such a representation an \mathcal{S}_X -structure. An \mathcal{S}_X -structure should be viewed as a version of a modular functor for homotopy quantum field theories (a homotopy modular functor in Turaev’s terminology). Alternatively, an \mathcal{S}_X -structure may be viewed as an approximation to the conformal version of a $1+1$ homotopy quantum field theory, where surfaces are given a Riemann structure (this is a homotopy version of the category introduced by Segal [9] in the context of elliptic cohomology). We restrict ourselves to examining the genus zero part and show how the additive categories arising have a rich structure inherited from the underlying geometry.

In more detail the paper contains the following. In Section 1, we start by constructing a 2-category \mathcal{S}_X whose objects are circles mapped into X , whose morphisms are surfaces (with boundary) mapped to X (with deformation up to homotopy), and whose 2-morphisms are path components of diffeomorphisms between the surfaces, commuting up to homotopy with the maps to X . This model is based on the construction of Tillmann [10] in the background free case. We also introduce two operations on the 2-category, reflection and rotation, which play a central role in the geometric arguments later in the paper. In Section 2, we define an \mathcal{S}_X -structure as a 2-functor from \mathcal{S}_X to the 2-category of idempotent complete additive categories over an algebraically closed field k and define unitary \mathcal{S}_X -structures using the reflection and rotation operations.

Following topological quantum field theory, we study the algebraic structure inherited by the categories associated by the 2-functor to maps of the circle into X . In the case $X = K(\pi, 1)$, an \mathcal{S}_X -structure gives rise to a π -graded category, assembled from the categories associated to the homotopy classes of the circle mapped into X . In Section 3, we display a graded version of a balanced structure on this category. For a general space X , we show how the graded component associated to the circle collapsed to the basepoint inherits a compatible $\pi_2 X$ -action. With no extra assumptions the category inherits a weak version of duality in both the above cases as defined by Bakalov and Kirillov [1]. Also by a result of Tillmann [11] the above categories are semi-simple Artinian. Note this relies on assuming idempotent completeness of the resulting category.

In Section 4, we restrict our attention to unitary \mathcal{S}_X -structures (these can be viewed as a version of unitary homotopy modular functors): We show how this leads to a rigid duality. Namely, for $X = K(n, 1)$ we obtain a graded version of a tortile

structure. We call the resulting structure a lax tortile π -category, to emphasise that the crossed structure and the duality interact in a looser way than that required by Turaev [14]. For general X , we show how the duality interacts with the $\pi_2 X$ -action.

We gather together the definitions of appropriate variants of balanced and tortile categories in Appendix A and Appendix B recalls the basics of additive categories over k (or k -additive categories) and Tillmann’s construction of the weak duality.

1. The homotopy surface 2-category of a space

We begin with a few recollections about 2-categories mainly to establish terminology. Recall that a 2-category \mathcal{B} is essentially a category in which the morphism sets are categories and composition $\mathcal{B}(A, B) \times \mathcal{B}(B, C) \rightarrow \mathcal{B}(A, C)$ is functorial. The morphisms of the morphism categories are called 2-morphisms. We shall denote objects by A, B, C, \dots , 1-morphisms by f, g, h, \dots and 2-morphisms by $\alpha, \beta, \gamma, \dots$. By $\mathcal{B}_{A, B}(f, g)$ we mean the set of 2-morphisms between 1-morphisms $f, g \in \mathcal{B}(A, B)$. 2-morphisms have two kinds of compositions \circ_1 and \circ_2 called vertical and horizontal composition:

$$\begin{aligned} \circ_1 : \mathcal{B}_{A, B}(f, g) \times \mathcal{B}_{A, B}(g, h) &\rightarrow \mathcal{B}_{A, B}(f, h), \\ \circ_2 : \mathcal{B}_{A, B}(f_1, g_1) \times \mathcal{B}_{B, C}(f_2, g_2) &\rightarrow \mathcal{B}_{A, C}(f_1 f_2, g_1 g_2), \end{aligned}$$

\mathcal{B} comes equipped with associativity and identity 2-isomorphisms a_{fgh} , l_f and r_f for 1-morphisms f, g, h :

$$\begin{aligned} a_{fgh} : (fg)h &\rightarrow f(gh), \\ l_f : 1_A f &\rightarrow f, \\ r_f : f 1_B &\rightarrow f, \end{aligned}$$

where f is in $\mathcal{B}(A, B)$, and satisfying the associativity pentagons and identity triangles. A strict 2-category is one in which all associativity and identity 2-isomorphisms are identities. If $G, H : \mathcal{D} \rightarrow \mathcal{E}$ are 2-functors between strict 2-categories \mathcal{D} and \mathcal{E} then a pseudo-2-natural transformation $G \rightarrow H$ is a collection of 1-morphisms $N_U : GU \rightarrow HU$ and 2-isomorphisms

$$\begin{array}{ccc} GU & \xrightarrow{N_U} & HU \\ Gf \downarrow & \Downarrow N_f & \downarrow Hf \\ GV & \xrightarrow{N_V} & HV \end{array}$$

for objects $U, V \in \mathcal{D}$ and morphism $f : U \rightarrow V$. These must satisfy

$$N(id_U) = id_U, \quad N(fg) = N(f)N(g)$$

and

$$\begin{array}{ccc}
 \begin{array}{ccc}
 GU & \xrightarrow{N_U} & HU \\
 \text{\scriptsize Gh} \curvearrowleft & & \text{\scriptsize Gf} \xleftarrow{N_f} \text{\scriptsize Hf} \\
 \text{\scriptsize $G\gamma$} & & \\
 \curvearrowright & & \\
 GV & \xrightarrow{N_V} & HV
 \end{array}
 & = &
 \begin{array}{ccc}
 GU & \xrightarrow{N_U} & HU \\
 \text{\scriptsize Gh} \downarrow & & \text{\scriptsize Hh} \curvearrowleft \\
 \text{\scriptsize Nh} & & \text{\scriptsize $H\gamma$} \\
 \text{\scriptsize Hf} \curvearrowright & & \\
 GV & \xrightarrow{N_V} & HV
 \end{array}
 \end{array}$$

for morphisms $f, h: U \rightarrow V$ and 2-morphism $\gamma: f \rightarrow h$.

A *strict monoidal 2-category* is a 2-category \mathcal{B} with a strict 2-functor $\otimes: \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$ which is strictly associative and has a strict left and right identity element 1. The monoidal structure is *semi-strict* if \otimes is allowed to have non-trivial 2-isomorphisms $\otimes_{f_1, g_1}: (f_1 f_2) \otimes (g_1 g_2) \rightarrow (f_1 \otimes g_1)(f_2 \otimes g_2)$ (with \otimes being strict elsewhere; \otimes is no longer a 2-functor).

A *monoidal 2-functor* $G: \mathcal{D} \rightarrow \mathcal{E}$ between semi-strict monoidal strict 2-categories comes equipped with isomorphisms $M_{U, V}^G: G(U \otimes V) \rightarrow GU \otimes GV$, for objects $U, V \in \mathcal{D}$; these must satisfy

$$\begin{array}{ccc}
 G(U \otimes V) & \xrightarrow{M_{U, V}^G} & GU \otimes GV \\
 G(f \otimes g) \downarrow & & \downarrow Gf \otimes Gg \\
 G(U' \otimes V') & \xrightarrow{M_{U', V'}^G} & GU' \otimes GV'
 \end{array}$$

for morphisms $f: U \rightarrow U'$ and $g: V \rightarrow V'$ in \mathcal{D} . We also require that via the isomorphisms $M^G, G(\gamma \otimes \delta) = G(\gamma) \otimes G(\delta)$ for 2-morphisms γ, δ (to be more precise the equality resembles that of 2-morphisms given in the definition of 2-natural transformation above, but with the squares now the identity 2-morphisms of the previous diagram).

A pseudo-2-natural transformation N between monoidal 2-functors G and H is said to be *monoidal* if the following diagram commutes:

$$\begin{array}{ccc}
 G(U \otimes V) & \xrightarrow{N_{U \otimes V}} & H(U \otimes V) \\
 M_{U, V}^G \downarrow & & \downarrow M_{U, V}^H \\
 GU \otimes GV & \xrightarrow{N_U \otimes N_V} & HU \otimes HV
 \end{array}$$

We also require that for morphisms f and $g, N_{f \otimes g} = N_f \otimes N_g$ under the isomorphisms M^G and M^H (again we omit the obvious pasting diagram). Notice the definitions just given clearly admit laxer and stricter versions; for further details on 2-categories we refer to [5,7].

The 2-categories central to this paper are ones, roughly speaking, whose objects are collections of loops in a space X , whose morphisms are 2-manifolds (with boundary) in X , and whose 2-morphisms are diffeomorphisms of 2-manifolds in X . In practice

great care is needed with the definition and all models have their own advantages and deficiencies. As in topological quantum field theory it is extremely useful to be able to make geometric arguments using surfaces and we will extensively use surface diagram manipulation. For this we need a fairly explicit hold on surfaces and we generalise the 2-category of Tillmann [10] in which all surfaces are embedded in \mathbb{R}^3 .

Let S_m denote m circles of radius $\frac{1}{4}$ centred at $(1,0), (2,0), \dots, (m,0)$ in \mathbb{R}^2 , each with basepoint $(n - 1/4, 0)$, $n = 1, \dots, m$; let S_0 denote the empty set. By *surface* we shall mean a smooth cobordism in \mathbb{R}^3 with boundary circles lying on the planes $z = 0$ and t for some $t > 0$. The circles are oriented counterclockwise when viewed from $z \gg 0$. We require that in some neighbourhood of each boundary component, the surfaces be straight cylinders of radius $\frac{1}{4}$ and further for simplicity that projection onto the z -coordinate is a Morse function. Notice that such surfaces are canonically oriented by choosing inward pointing normals. The boundary components in the plane $z = 0$ are called *inputs* and those on the plane $z = t$ are called *outputs*. Two surfaces Σ_1 and Σ_2 can be glued together by shifting Σ_2 vertically by t_1 (the height of Σ_1) and gluing along the boundary circles; the result is again smooth since the collars are straight cylinders.

Definition 1.1. Let X be a based topological space. Define a 2-category $\mathcal{S}_{D,X}$ as follows:

- Objects: based continuous functions $s : S_m \rightarrow X$, for $m \in \mathbb{N}$ (the m base-points of S_m sent to the basepoint of X).
- 1-Morphisms: continuous functions $g : \Sigma \rightarrow X$ where Σ is a surface (as above). The source and target are $g|_{z=0}$ and $g|_{z=t}$, respectively. On the straight boundary collars g must factor through the projection to the boundary.
- 2-Morphisms: orientation preserving diffeomorphisms $T : \Sigma_1 \rightarrow \Sigma_2$ that fix boundary collars pointwise and such that the following diagram commutes up to homotopy relative to the boundary (the homotopy is constant on the boundary).

$$\begin{array}{ccc}
 \Sigma_1 & \xrightarrow{T} & \Sigma_2 \\
 & \searrow g_1 & \swarrow g_2 \\
 & X &
 \end{array}$$

Composition of 1-morphisms is defined by gluing surfaces and taking the induced map to X . Similarly vertical composition \circ_1 of 2-morphisms is composition of diffeomorphisms and horizontal composition \circ_2 is union of diffeomorphisms induced by the gluing of surfaces. Where there is no ambiguity we will write s for the object $s : S_m \rightarrow X$ and g for the morphism $g : \Sigma \rightarrow X$.

Strictly speaking this is not a 2-category as there are no identity morphisms. This will be remedied in what follows where we introduce limited isotopy of surfaces needed in order to have a well-defined monoidal product given by disjoint union. Again we appeal to [10] and use Tillmann’s RS-moves which we now recall. Let Σ be a surface. By choosing $0 = t_0 < t_1 < \dots < t_{k-1} < t_k = t$ cut up Σ into slices $\Sigma_i \subset \mathbb{R}^2 \times [t_{i-1}, t_i]$ for

$i=1, \dots, k$, with components Σ_i^j for $j=1, \dots, j_i$. Define a *rescaling* to be a collection of functions $R_{i,j}(x, y, z) = (x, y, r_{i,j}(z))$ where $r_{i,j}: [t_{i-1}, t_i] \rightarrow [t_{i-1}, \tilde{t}_i]$ is a smooth function with derivative 1 in a neighbourhood of the boundary of $[t_{i-1}, t_i]$. Here \tilde{t}_i is independent of j , and the surfaces $\Sigma_{i+1}, \dots, \Sigma_k$ are shifted vertically by the appropriate amount. Define a *shift* to be a collection of functions $S_{i,j}(x, y, z) = (x, y, z) + (f_{i,j}^1(z), f_{i,j}^2(z), 0)$ where $f_{i,j}^l: [t_{i-1}, t_i] \rightarrow \mathbb{R}$ are smooth functions vanishing on some neighbourhood of the boundary of $[t_{i-1}, t_i]$. A 2-morphism T in $\mathcal{S}_{D,X}$ which is defined by a finite number of rescalings and shifts is called an *RS 2-morphism*.

We now define the *homotopy surface 2-category* \mathcal{S}_X by taking successive quotients of $\mathcal{S}_{D,X}$ on the 1 and 2-morphisms. Define an equivalence relation \equiv_2 on the set of 2-morphisms $\mathcal{S}_{D,X}(g_1, g_2)$ by setting $T_1 \equiv_2 T_2$ if and only if they are in the same connected component of $\text{Diffeo}^+(\Sigma_1, \Sigma_2; \partial)$. Next define an equivalence relation \equiv_1 on the 1-morphisms by setting $g \equiv_1 g'$ if and only if g and g' are isomorphic via an RS 2-morphism.

Definition 1.2. The *homotopy surface 2-category* \mathcal{S}_X of a space X is defined as the 2-category obtained by taking successive quotients of $\mathcal{S}_{D,X}$, first by \equiv_2 on the 2-morphisms sets, followed by \equiv_1 on the 1-morphism sets.

The quotient by \equiv_2 is straightforward to define. We briefly explain how to define the 2-category obtained under the quotient by \equiv_1 (details can also be found in Tillmann [10]).

Let \mathcal{B} be a 2-category and \equiv_1 an equivalence relation on the 1-morphism sets. Define the quotient $\bar{\mathcal{B}}$ of \mathcal{B} by \equiv_1 as follows. The objects of $\bar{\mathcal{B}}$ are those of \mathcal{B} . Pick representatives f_0 of all equivalence classes of 1-morphisms of \mathcal{B} , together with isomorphisms $\phi_f: f \rightarrow f_0$ from every 1-morphisms to its representative. Take as 1-morphisms of $\bar{\mathcal{B}}$ the representatives f_0 , and as 2-morphisms the sets $\mathcal{B}(f_0, g_0)$ between representatives. Define composition of 1-morphisms by $f_0 \circ g_0 = (f_0 g_0)_0$. Use the isomorphisms ϕ to obtain the associativity and identity isomorphisms of $\bar{\mathcal{B}}$ from those of \mathcal{B} (see Tillmann [10] for details). Vertical composition of 2-morphisms is as in \mathcal{B} ; horizontal composition of 2-morphisms is defined using the ϕ s in the obvious way. It is now easy to check that $\bar{\mathcal{B}}$ has the structure of a 2-category.

We return to the surface 2-category. The identity 1-morphism in $\mathcal{S}_X(s, s)$ can be taken to be a collection of straight cylinders mapping to X via $s \circ p$ where p is projection onto $\mathbb{R}^2 \times 0$. Also, notice that by taking X to be a one point space the resultant category is Tillmann's. We can also form a category from the above 2-category by identifying all 2-isomorphic 1-morphisms and this is a model for the homotopy surface category of a space defined in [2].

We can now define a monoidal structure \mathcal{S}_X using disjoint union. Define a 2-functor $\sqcup: \mathcal{S}_X \times \mathcal{S}_X \rightarrow \mathcal{S}_X$ on objects $s: S_m \rightarrow X$ and $s': S_n \rightarrow X$ to be the obvious map $s \sqcup s': S_{m+n} \rightarrow X$. On 1-morphisms $g: \Sigma \rightarrow X$ and $g': \Sigma' \rightarrow X$ let $\Sigma \cup \Sigma'$ be the surface obtained by rescaling the height of Σ' to that of Σ and shifting the result by a diffeomorphism $(\dot{x}, y, z) \mapsto (x + f(z), y, z)$ where $f: [0, t] \rightarrow \mathbb{R}$ is constantly k in a neighbourhood of 0 and constantly l in a neighbourhood of t and such that result is disjoint from Σ . Then $g \sqcup g'$ is defined by the induced maps. Finally, on 2-morphisms

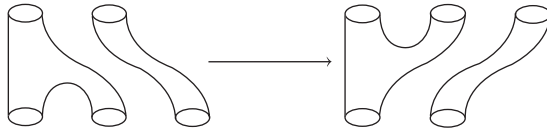
we take the disjoint union of diffeomorphisms. The analysis in [10] proves that \mathcal{S}_X is a semi-strict monoidal strict 2-category.

The arguments in this paper rest crucially on the fact that surfaces in \mathbb{R}^3 can be manipulated by a number of geometric operations we now define. Given $s: S_1 \rightarrow X$ let s^{-1} be the map s precomposed with reflection in the line $x=1$. For $s=s_1 \sqcup \dots \sqcup s_n: S_n \rightarrow X$ let $s^{-1} = s_n^{-1} \sqcup \dots \sqcup s_1^{-1}$.

Reflection: There is a contravariant monoidal equivalence of 2-categories

$$\hat{}: \mathcal{S}_X \rightarrow \mathcal{S}_X$$

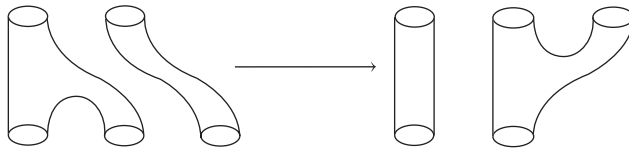
defined on objects by $s \mapsto s$ and on a morphism $g: \Sigma \rightarrow X$ by reflecting Σ in the plane $z=0$ and then translating the result. The map to X is the induced one and we denote the result by \hat{g} . Note that reflecting in the plane $z=0$ leaves the object circles and their maps to X unchanged. A 2-morphism will induce a 2-morphism on reflected 1-morphisms.



Rotation: \mathcal{S}_X has another identification with its opposite by rotating surfaces by 180° . Define a contravariant 2-functor

$$\tilde{}: \mathcal{S}_X \rightarrow \mathcal{S}_X$$

on objects by $s \mapsto s^{-1}$ and on a morphism $g: \Sigma \rightarrow X$ by rotating Σ by 180° around the y -axis and then adjusting by using a diffeomorphism $(x, y, z) \mapsto (x + f(z), y, z)$ where $f: [0, t_1] \rightarrow \mathbb{R}$ is as in the definition of monoidal product. The map to X is the induced one. Denote the result by \tilde{g} and again take induced 2-morphisms. Note that this is “anti-monoidal” rather than monoidal.



The composite

$$\mathcal{S}_X \xrightarrow{\tilde{}} \mathcal{S}_X \xrightarrow{\hat{}} \mathcal{S}_X$$

is given on objects by $s \mapsto s^{-1}$ and on a morphism $g: \Sigma \rightarrow X$ by reflecting in the plane $x=0$ and then adjusting similarly to above.

We make use of the reflection and rotation 2-functors in defining a unitary \mathcal{S}_X -structure in Section 2. They will then be repeatedly used in Section 4, where we examine unitary \mathcal{S}_X -structures and display the associated tortile structure.

We end this section with a remark on another possible version of the 2-category of surfaces defined above. In [12], Tillmann defines a 2-category of surfaces by considering a small generating set of surfaces, and then taking all those obtained by repeated gluing and disjoint union. This approach can be generalised to include a background space as in the context of this paper. This model is manifestly simpler to define than \mathcal{S}_X . However, the arguments involving reflection and rotation rely on all surfaces in the category being embedded in \mathbb{R}^3 , so that this alternative approach leads to a less natural construction and more cumbersome proofs.

2. \mathcal{S}_X -Structures

Now for the main definition of the paper. Let k be an algebraically closed field and let $\widehat{k\text{-Add}}$ be the 2-category of idempotent complete k -additive categories with monoidal structure given by the tensor product (see Appendix B for details).

Definition 2.1. Let X be a based space. An \mathcal{S}_X -structure is a monoidal 2-functor of strict 2-categories $F: \mathcal{S}_X \rightarrow \widehat{k\text{-Add}}$ such that the crossed cylinders are mapped to the functor changing components.

Thus, to each collection of loops $s: S_m \rightarrow X$ we assign an additive category; to each surface $g: \Sigma \rightarrow X$ a functor of additive categories and to each diffeomorphism T of surfaces we assign a natural transformation of functors. This assignment is monoidal taking disjoint union to tensor product.

We make a few remarks on how the above definition relates to other notions in homotopy quantum field theory. Let us first consider the background free case, or topological quantum field theory. As was the point of view in [11], \mathcal{S} -structures have the flavour of a modular functor in dimension 2, since they essentially consist of some kind of representations of the mapping class groups. To be more precise, the standard notion of a modular functor appears in an \mathcal{S} -structure from those surfaces with an empty target 1-manifold (see [11] for details). On the other hand \mathcal{S} -structures fit nicely into the picture of extended topological quantum field theories in dimension 3. This becomes clear if one considers the definition given in [6], where an extended TQFT is defined as a representation of the double category of circles, surfaces between these, and relative 3-cobordisms (a 2-category is a special case of a double category). From this point of view \mathcal{S} should be viewed as a sub-2-category of the latter (we avoid being precise here). One could repeat the above remarks for the more general \mathcal{S}_X -structures, comparing them to homotopy modular functors and extended homotopy quantum field theories in dimension 3.

By recalling that there is a contravariant functor $(-)^{\vee}: \widehat{k\text{-Add}} \rightarrow \widehat{k\text{-Add}}$ taking an additive category \mathcal{A} to its dual $\mathcal{A}^{\vee} = \widehat{k\text{-Add}}(\mathcal{A}, \hat{k})$, the dual of an \mathcal{S}_X -structure is defined as follows.

Definition 2.2. The dual of $F: \mathcal{S}_X \rightarrow \widehat{k\text{-Add}}$, denoted F^{\vee} is defined as the composite

$$\mathcal{S}_X \xrightarrow{\widehat{\quad}} \mathcal{S}_X \xrightarrow{F} \widehat{k\text{-Add}} \xrightarrow{(-)^{\vee}} \widehat{k\text{-Add}}.$$

Later we will restrict ourselves to theories which are unitary in a way we now make precise. Given a finitely generated k -additive category \mathcal{A} i.e. one whose morphism vector spaces are finitely generated, one can define a functor $\text{hom} : \mathcal{A} \rightarrow \mathcal{A}^\vee$ given by $Y \mapsto \mathcal{A}(Y, -)$. Suppose F is an \mathcal{S}_X -structure taking values among k -additive categories that are finitely generated (in fact this is always the case) then there is a family of maps

$$\{\text{hom}_s : F(s) \rightarrow F^\vee(s)\}_{s \in \text{Ob}(\mathcal{S}_X)}.$$

Definition 2.3. An \mathcal{S}_X -structure F is (*lax*) *unitary* if the above family provide a monoidal pseudo-2-natural transformation N between F and F^\vee , and cylinders $g : I \rightarrow X$ satisfy $N(g) = F(g)^{-1}$.

Two important consequences of this definition are the following:

S-I. If $g : \Sigma \rightarrow X$ is a morphism then $F(\widehat{g})$ is right adjoint to $F(g)$ i.e. there are natural isomorphisms

$$\mathcal{B}(F(g)(U), V) \cong \mathcal{A}(U, F(\widehat{g})(V)),$$

where $F(g) : \mathcal{A} \rightarrow \mathcal{B}$. For the collapsed cylinders this isomorphism is the identity.

S-II. If $T : g_1 \rightarrow g_2$ is a 2-morphism then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{B}(F(g_1)U, V) & \xrightarrow{\cong} & \mathcal{A}(U, F(\widehat{g}_1)V) \\ F(T)_U \uparrow & & \downarrow F(\widehat{T})_V \\ \mathcal{B}(F(g_2)U, V) & \xrightarrow{\cong} & \mathcal{A}(U, F(\widehat{g}_2)V) \end{array}$$

where $\widehat{T} : \widehat{g}_1 \rightarrow \widehat{g}_2$ is obtained from T under reflection and the vertical arrows are given by pre- and post-composition.

3. Balanced categories from \mathcal{S}_X -structures

Let X be a based space. An \mathcal{S}_X -structure $F : \mathcal{S}_X \rightarrow \widehat{k\text{-Add}}$ determines a collection of categories $\{\mathcal{A}_\alpha\}_{\alpha \in \pi}$ indexed by the group $\pi = \pi_1 X$ as follows. A loop α in X determines a map $s_\alpha : S_1 \rightarrow X$, taking $(1, -1/4)$ as basepoint of S_1 . For each element of $\pi = \pi_1(X)$ choose a representative loop α and set

$$\mathcal{A}_\alpha = F(s_\alpha) \quad \mathcal{A} = \bigsqcup_{\alpha \in \pi} \mathcal{A}_\alpha.$$

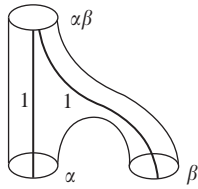
The following proposition refers to the definitions of balanced categories in Appendix A.

Theorem 3.1. (a) *Let π be a discrete group and let X be a based Eilenberg-MacLane space $K(\pi, 1)$. The k -additive category \mathcal{A} associated to an \mathcal{S}_X -structure is a balanced π -category.*

- (b) For any space X , the subcategory \mathcal{A}_1 is a balanced category with $\pi_2 X$ -action.
- (c) The categories above are semi-simple Artinian categories.

The proof of this proposition will take up the rest of this section.

Firstly consider part (a) in which $X = K(\pi, 1)$. To define a monoidal structure on \mathcal{A} let $\alpha, \beta \in \pi$ and pick a pair of pants surface P with two inputs and one output and let $p_{\alpha, \beta}: P \rightarrow X$ be a map inducing the maps indicated below.



Here the label indicates the map on the given line or boundary component and any two choices of $p_{\alpha, \beta}$ give the same 1-morphism in \mathcal{S}_X (note this is no longer true if $\pi_2 X \neq 0$). Define functors

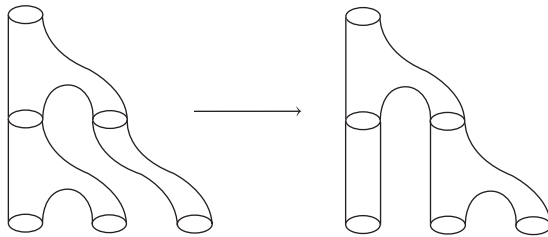
$$*_{\alpha, \beta} = F(p_{\alpha, \beta}): \mathcal{A}_\alpha \otimes \mathcal{A}_\beta \rightarrow \mathcal{A}_{\alpha\beta},$$

$$* = \bigsqcup_{\alpha, \beta \in \pi} *_{\alpha, \beta} : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}.$$

Now choose a disc D with one input only and let $d: D \rightarrow X$ be the collapse map to a basepoint of X . Noting that $F(d): \hat{k} \rightarrow \mathcal{A}_1$ define

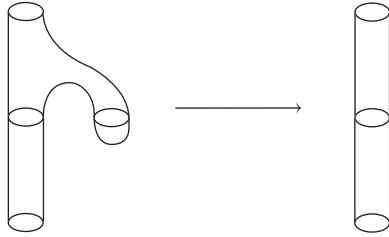
$$\mathbf{1} = F(d)(k) \in \mathcal{A}_1.$$

It can now be seen that $(\mathcal{A}, *, \mathbf{1})$ is a π -graded monoidal category by considering the obvious diffeomorphism



which gives a natural isomorphism $a: (- * -) * - \xrightarrow{\bullet} - * (- * -)$ i.e. a collection of isomorphisms $a_{U, V, W}$ as required. The isomorphism r_U is obtained from a

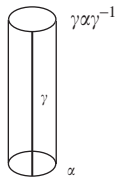
diffeomorphism



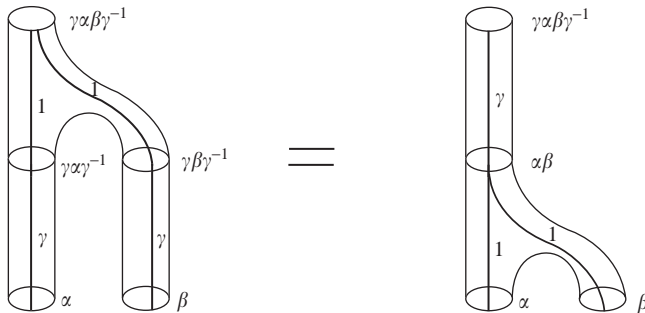
and similarly for l_U . Finally, commutativity of the associativity pentagons and identity triangles holds since by definition the compositions differ by at most an isotopy.

Next we show how to obtain structure leading to a balanced π -category.

Crossing: Let $\alpha, \gamma \in \pi$ and pick a cylinder I and a map $i_{\alpha, \gamma}: I \rightarrow X$ as shown below:



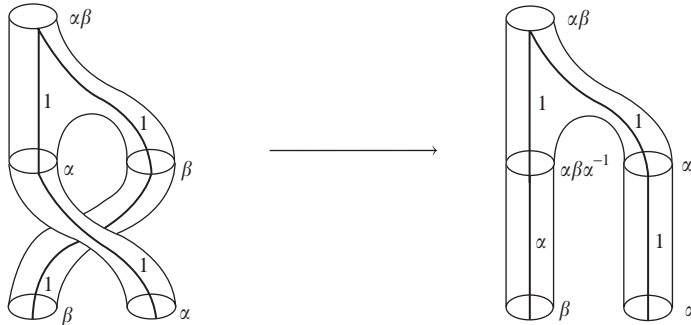
Define $\varphi(\gamma)_\alpha := F(i_{\alpha, \gamma}): \mathcal{A}_\alpha \rightarrow \mathcal{A}_{\gamma\alpha\gamma^{-1}}$ and assemble these into a map $\varphi: \pi \rightarrow \text{Aut}(\mathcal{A})$. Note that $\varphi(\gamma)$ is invertible with inverse $\varphi(\gamma^{-1})$ and also that $\varphi(\gamma)$ respects $*$ as required, by the equality below.



Furthermore, φ is a group homomorphism by construction, and the monoidal unit and other structure is preserved.

Braiding: Recall that the monoidal structure comes from $*_{\alpha, \beta} = F(p_{\alpha, \beta})$ for choices of cobordisms $p_{\alpha, \beta}: P \rightarrow X$. Let T be an untwisting diffeomorphism as

pictured



Letting $\tau_{\alpha, \beta} : J \rightarrow X$ be the crossed cylinders in the picture above and recalling that $F(\tau_{\alpha, \beta})$ is the functor changing components it follows that for each α and β there is a 2-morphism

$$\begin{array}{ccc}
 P \circ J & \xrightarrow{T} & P \circ (I \sqcup I) \\
 P_{\alpha, \beta} \circ \tau_{\alpha, \beta} \searrow & & \swarrow P_{\alpha\beta\alpha^{-1}, \alpha} \circ (i_{\beta, \alpha} \sqcup i_{\alpha, 1}) \\
 & X &
 \end{array}$$

This induces a natural isomorphism (of functors $\mathcal{A}_\beta \otimes \mathcal{A}_\alpha \rightarrow \mathcal{A}_{\alpha\beta}$)

$$s_{\alpha, \beta} : *_{\alpha, \beta} \circ \text{twist}_{\beta, \alpha} \xrightarrow{\bullet} *_{\alpha\beta\alpha^{-1}, \alpha} \circ (\varphi(\alpha)_\beta \otimes id).$$

Thus, for each $U \in \mathcal{A}_\alpha$ and $V \in \mathcal{A}_\beta$ we get isomorphisms

$$s_{U, V} : U * V \rightarrow \varphi(\alpha)V * U.$$

Since φ is a homomorphism into the group of $*$ preserving automorphisms of \mathcal{A} , the braiding result in [11] now implies condition (1) of Definition A.3. Condition (2) follows immediately from the naturality of s .

Condition (3) is obtained as follows. By post-composing the two functors above with $\varphi(\gamma)$ we get two functors $\mathcal{A}_\beta \otimes \mathcal{A}_\alpha \rightarrow \mathcal{A}_{\gamma\alpha\beta\gamma^{-1}}$. The natural transformation s induces a natural transformation of these two functors which for objects $U \in \mathcal{A}_\alpha$ and $V \in \mathcal{A}_\beta$ is given by $\varphi(\gamma)(s_{U, V})$. On the other hand, by pre-composing by $\varphi(\gamma)$ we again get an induced natural transformation which is given by $s_{\varphi(\gamma)U, \varphi(\gamma)V}$ for objects $U \in \mathcal{A}_\alpha$ and $V \in \mathcal{A}_\beta$. However, the diffeomorphisms inducing the natural transformation above are the same as 2-morphisms in \mathcal{S}_X so the induced natural transformations are the same i.e. $\varphi(\gamma)(s_{U, V}) = s_{\varphi(\gamma)U, \varphi(\gamma)V}$, proving condition (3).

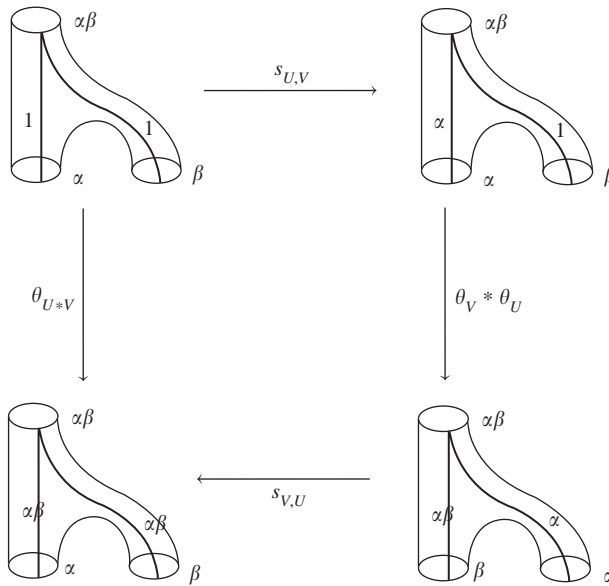
Twist: The Dehn twist of a cylinder $D : I \rightarrow I$ gives for each α, β a 2-morphism

$$\begin{array}{ccc}
 I & \xrightarrow{D} & I \\
 i_{\alpha, \beta} \searrow & & \swarrow i_{\alpha, \beta\alpha} \\
 & X &
 \end{array}$$

which induces a natural isomorphism $\varphi(\beta)_\alpha \xrightarrow{\bullet} \varphi(\beta\alpha)_\alpha$, which taking $\beta = 1$ gives a natural isomorphism

$$\theta_\alpha : \varphi(1)_\alpha \xrightarrow{\bullet} \varphi(\alpha)_\alpha$$

that is, for each $U \in \mathcal{A}_\alpha$ there is an isomorphism $\theta_U : U \rightarrow \varphi(\alpha)U$. Conditions (1) and (2) of Definition A.4 are satisfied by the equivalent mapping class group identities of [11] and the homotopy identity displayed below (where we represent a half twist by swapping the labels of the incoming boundary components).



Condition (3) follows immediately from the naturality of s and condition (4) follows by pre- and post-composing θ with $\varphi(\gamma)$, observing that the underlying geometry is the same in both cases, so the induced natural transformations agree. This implies $\theta_{\varphi(\gamma)U} = \varphi(\gamma)(\theta_U)$.

This completes the proof of part (a) of Theorem 3.1.

For part (b) let X be any space. We claim that \mathcal{A}_1 is a balanced category with $\pi_2 X$ -action (see Appendix A for definitions). Let $c : S_1 \rightarrow X$ be the collapse map and suppose this is the choice made to represent $1 \in \pi_1$ i.e. $\mathcal{A}_1 = F(c)$. By restricting to degenerate loops we obtain a monoidal product $*$, monoidal unit $\mathbf{1}$, braiding s and twist θ in a similar way to that in part (a) and $(\mathcal{A}_1, *, \mathbf{1}, s, \theta)$ is a balanced category (note that to obtain the braiding we must take $p_{1,1} = id$). The only genuinely new structure in part (b) is the $\pi_2 X$ -action. Let I be a cylinder and let $I \rightarrow X$ be a map as indicated

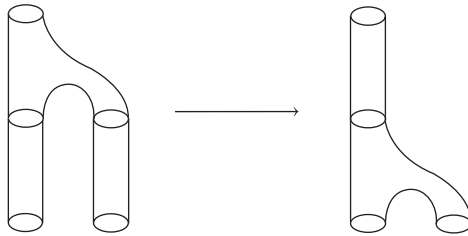
below:



Such a map determines an element of $\pi_2(X)$ and any two maps giving the same element are homotopic and hence 2-isomorphic. For each $g \in \pi_2(X)$ choose $i_g : I \rightarrow X$ and define $\rho : \pi_2(X) \rightarrow \text{Aut}(\mathcal{A})$ by

$$\rho(g) = F(i_g).$$

The composition of morphisms i_g corresponds to addition in $\pi_2 X$ and hence ρ is a group homomorphism. There is an RS-equivalence



and so $F(p \circ (i_g \cup i_1)) = F(i_g \circ p)$ for $g \in H_2(X)$. It follows that for $U, V \in \mathcal{A}_1$

$$(\rho(g)U) * V = \rho(g)(U * V) \quad \text{and} \quad \rho(g)f * h = \rho(g)(f * h)$$

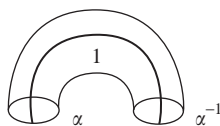
and similarly

$$U * \rho(g)V = \rho(g)(U * V) \quad \text{and} \quad \rho(g)(f * h) = f * \rho(g)h.$$

The proof of conditions (5) and (6) in definition A.7 are similar to the proofs of A.3(3) and A.4(4) in part (a) of Theorem 3.1. Conditions (3) and (4) can also be shown using similar arguments.

This completes the proof of part (b) of Theorem 3.1.

To prove part (c) of Theorem 3.1 we prove \mathcal{A} is semi-simple Artinian for the case $X = K(\pi, 1)$. A similar argument shows \mathcal{A}_1 is semi-simple Artinian for a general space X . By the work of Tillmann (see Appendix B) we must produce a non-degenerate form $\langle -, - \rangle : \mathcal{A}_\alpha \otimes \mathcal{A}_{\alpha^{-1}} \rightarrow \hat{k}$ for each $\alpha \in \pi$. Let C be a cylinder with two inputs and for $\alpha \in \pi$ choose a map $c_\alpha : C \rightarrow X$ as indicated below:



Now define

$$\langle -, - \rangle_\alpha := F(c_\alpha) : \mathcal{A}_\alpha \otimes \mathcal{A}_{\alpha^{-1}} \rightarrow \hat{k}$$

and

$$\Delta_\alpha := F(\tilde{c}_\alpha) : \hat{k} \rightarrow \mathcal{A}_\alpha \otimes \mathcal{A}_{\alpha^{-1}}$$

and write the image of the canonical element $k \in \hat{k}$ as

$$E_\alpha := \Delta_\alpha(k) = \sum_{i=1}^{n_\alpha} P_\alpha^i \otimes Q_{\alpha^{-1}}^i,$$

where $P_\alpha^i \in \mathcal{A}_\alpha$, $Q_{\alpha^{-1}}^i \in \mathcal{A}_{\alpha^{-1}}$.

Standard topological quantum field theory manipulations show that $\langle -, - \rangle_\alpha$ and E_α provide a non-degenerate form and it follows from Appendix B that \mathcal{A}_α is semi-simple Artinian proving part (c) of Theorem 3.1.

4. Tortile categories from unitary \mathcal{S}_X -structures

In this section, we consider under what conditions we can guarantee the category \mathcal{A} has (rigid) duals. In particular we show that an \mathcal{S}_X -structure that is unitary does have duals. Balanced categories with duals are known as tortile categories (ribbon categories) and we give appropriate variants of these in Appendix A. Referring to these definitions we prove the following theorem.

Theorem 4.1. (a) *Let π be a discrete group and let X be a based Eilenberg–MacLane space $K(\pi, 1)$. The k -additive category \mathcal{A} associated to a unitary \mathcal{S}_X -structure, is a semi-simple Artinian lax tortile π -category.*

(b) *For any space X , the subcategory \mathcal{A}_1 is a semi-simple Artinian tortile category with lax $\pi_2 X$ -action.*

Until further notice let $X = K(\pi, 1)$ and let $F : \mathcal{S}_X \rightarrow \widehat{k\text{-Add}}$ be a unitary \mathcal{S}_X -structure. The key to getting duality in the category \mathcal{A} is to use Tillmann’s involutions $(-)^* : \mathcal{A}_\alpha \rightarrow \mathcal{A}_{\alpha^{-1}}$ which arise from the non-degenerate forms $\langle -, - \rangle_\alpha$ of the previous section. Non-degeneracy is courtesy of functors

$$\mathcal{I} : \mathcal{A}_{\alpha^{-1}} \rightarrow \mathcal{A}_\alpha^\vee, \quad Y \mapsto \langle Y, - \rangle,$$

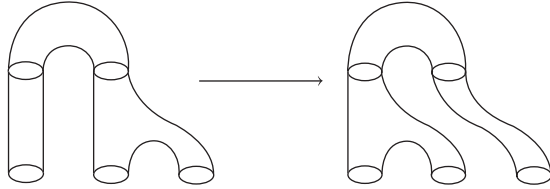
$$\mathcal{J} : \mathcal{A}_\alpha^\vee \rightarrow \mathcal{A}_{\alpha^{-1}}, \quad H \mapsto \sum_{i=1}^n H(P_\alpha^i) \otimes Q_{\alpha^{-1}}^i$$

for which there are natural transformations $N : \text{id} \cong \mathcal{I}\mathcal{J}$ and $M : \text{id} \cong \mathcal{J}\mathcal{I}$.

Notice that these forms satisfy a “Frobenius” condition, namely, for $U \in \mathcal{A}_\alpha$, $V \in \mathcal{A}_\beta$ and $W \in \mathcal{A}_\gamma$ with $\alpha\beta\gamma = 1$ there are natural isomorphisms

$$\langle U, V * W \rangle_\alpha \simeq \langle U * V, W \rangle_{\gamma^{-1}},$$

which arise from the diffeomorphism indicated below:



To define Tillmann’s involutions let $\text{hom} : \mathcal{A}_\alpha \rightarrow \mathcal{A}_\alpha^\vee$ be defined by $Y \rightarrow \mathcal{A}_\alpha(Y, -)$ and set $(-)^* = \mathcal{J} \circ \text{hom}$. These functors satisfy $(-)^{**} \cong \text{id}$ and $\mathcal{A}_\alpha(-, -) \simeq \langle -^*, - \rangle_{\alpha^{-1}}$ and both the forms and involutions extend to tensor products. In particular we have non-degenerate forms and involutions

$$\langle -, - \rangle_{\underline{\alpha}} : \mathcal{A}_{\underline{\alpha}} \otimes \mathcal{A}_{\underline{\alpha}^{-1}} \rightarrow \hat{k}, \quad (-)_{\underline{\alpha}}^* : \mathcal{A}_{\underline{\alpha}} \rightarrow \mathcal{A}_{\underline{\alpha}^{-1}},$$

where $\underline{\alpha} = (\alpha_1, \dots, \alpha_n), \underline{\alpha}^{-1} = (\alpha_n^{-1}, \dots, \alpha_1^{-1})$ and $\mathcal{A}_{\underline{\alpha}} = \mathcal{A}_{\alpha_1} \otimes \dots \otimes \mathcal{A}_{\alpha_n}$. We will write \mathcal{I} and \mathcal{J} for the associated maps $\mathcal{A}_{\underline{\alpha}} \rightarrow \mathcal{A}_{\underline{\alpha}^{-1}}^\vee$ and $\mathcal{A}_{\underline{\alpha}^{-1}}^\vee \rightarrow \mathcal{A}_{\underline{\alpha}}$. See Appendix B for further discussion.

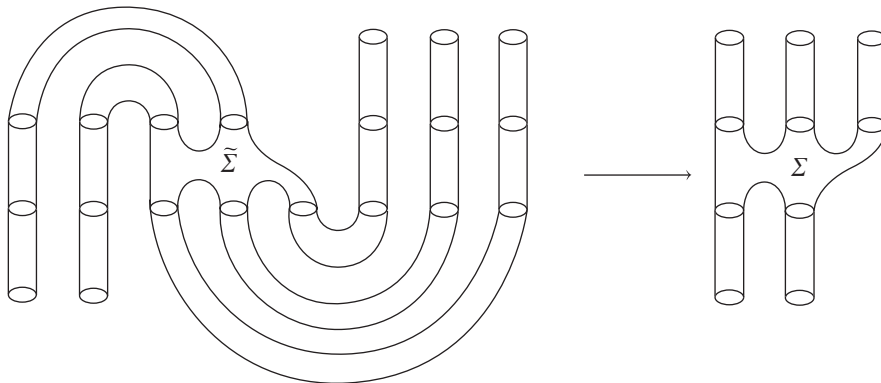
The functors \mathcal{J} have the following property.

Lemma 4.1. *Let $g : \Sigma \rightarrow X$ be a morphism in \mathcal{S}_X . Then there is a natural isomorphism*

$$t_{F(g)} : \mathcal{J} \circ F(\tilde{g})^\vee \xrightarrow{\bullet} F(g) \circ \mathcal{J}.$$

Proof. By using the diffeomorphism below we have natural isomorphisms

$$k : \mathcal{J} \circ F(\tilde{g})^\vee \circ \mathcal{J} \xrightarrow{\bullet} F(g).$$



Pre-composing with \mathcal{J} and using the equivalence $N : \text{id} \simeq \mathcal{J} \mathcal{J}$ gives the required natural isomorphisms. \square

For the crossing φ we have the following additional property:

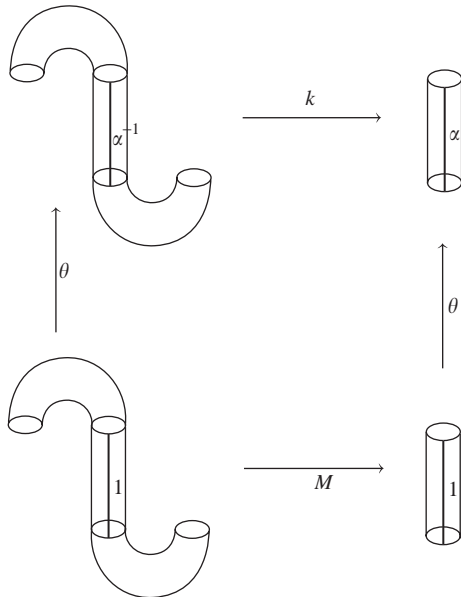
Lemma 4.2. *The following diagram commutes.*

$$\begin{array}{ccc}
 \mathcal{J} \circ \varphi(\alpha^{-1})^\vee & \xrightarrow{t_{\varphi(\alpha)}} & \varphi(\alpha) \circ \mathcal{J} \\
 \swarrow J\theta^\vee & & \searrow \theta_{\varphi(\alpha)}\mathcal{J} \\
 & \mathcal{J} &
 \end{array}$$

Proof. Consider the following diagram, where $H \in \mathcal{A}^\vee$.

$$\begin{array}{ccccc}
 J \circ \varphi(\alpha^{-1})^\vee(H) & \xrightarrow{J_{\varphi(\alpha^{-1})^\vee(N_H)}} & \mathcal{J} \circ \varphi(\alpha^{-1})^\vee \circ \mathcal{J} \circ \mathcal{J}(H) & \xrightarrow{k_{\varphi(\alpha), \mathcal{J}(H)}} & \varphi(\alpha) \circ \mathcal{J}(H) \\
 \mathcal{J}\theta_H^\vee \uparrow & & \mathcal{J}\theta_{\mathcal{J}(H)}^\vee \uparrow & & \downarrow \theta_{\varphi(\alpha), \mathcal{J}(H)} \\
 \mathcal{J}(H) & \xrightarrow{\mathcal{J}(N_H)} & \mathcal{J} \circ \mathcal{J} \circ \mathcal{J}(H) & \xrightarrow{M_{\mathcal{J}(H)}} & \mathcal{J}(H)
 \end{array}$$

The left square commutes by naturality of θ^\vee and the right square by the mapping class group identity given below:



The result now follows because the map along the top is $t_{\varphi(\alpha)}$ and the bottom map is the identity. \square

Now we investigate how the involution interacts with the monoidal structure and the crossing.

Proposition 4.1. *Let $U \in \mathcal{A}_\alpha$, $V \in \mathcal{A}_\beta$ and $\gamma \in \pi$. There are natural isomorphisms*

$$p_{U,V} : (U * V)^* \xrightarrow{\cong} V^* * U^*$$

and

$$c_{\gamma,U} : (\varphi(\gamma)U)^* \xrightarrow{\cong} \varphi(\gamma)U^*$$

and moreover the $p_{U,V}$ and $c_{\gamma,U}$ commute, that is, the following diagram commutes:

$$\begin{array}{ccccc} (\varphi(\gamma)U * \varphi(\gamma)V)^* & \xrightarrow{p} & (\varphi(\gamma)V)^* * (\varphi(\gamma)U)^* & \xrightarrow{c * c} & \varphi(\gamma)V^* * \varphi(\gamma)U^* \\ \downarrow = & & & & \downarrow = \\ (\varphi(\gamma)(U * V))^* & \xrightarrow{c} & \varphi(\gamma)(U * V)^* & \xrightarrow{\varphi(\gamma)(p)} & \varphi(\gamma)(V^* * U^*) \end{array}$$

Proof. Let $p : P \rightarrow X$ be the map giving the monoidal product $\mathcal{A}_\alpha \otimes \mathcal{A}_\beta \rightarrow \mathcal{A}_{\alpha\beta}$. Notice that \tilde{p} induces the opposite monoidal product $\mathcal{A}_{\alpha^{-1}} \otimes \mathcal{A}_{\beta^{-1}} \rightarrow \mathcal{A}_{\beta^{-1}\alpha^{-1}}$. By property S-I of unitary \mathcal{S}_X -structures we have natural isomorphisms

$$\mathcal{A}(U * V, -) \xrightarrow{\bullet} \mathcal{A} \otimes \mathcal{A}(U \otimes V, \Delta -)$$

i.e. isomorphisms

$$q_{U,V} : \text{hom} \circ F(p)(U \otimes V) \rightarrow F(\hat{p})^\vee \circ \text{hom}(U \otimes V).$$

Applying the functor \mathcal{J} to these and composing with $t : \mathcal{J} \circ F(\hat{p})^\vee \rightarrow F(\tilde{\hat{p}}) \circ \mathcal{J}$ we obtain

$$\mathcal{J} \circ \text{hom} \circ *(U \otimes V) \xrightarrow{J(q_{U,V})} \mathcal{J} \circ F(\hat{p})^\vee \circ \text{hom}(U \otimes V) \xrightarrow{t_{\text{hom}(U \otimes V)}} * \circ \mathcal{J} \circ \text{hom}(U \otimes V),$$

which define a collection of natural transformations

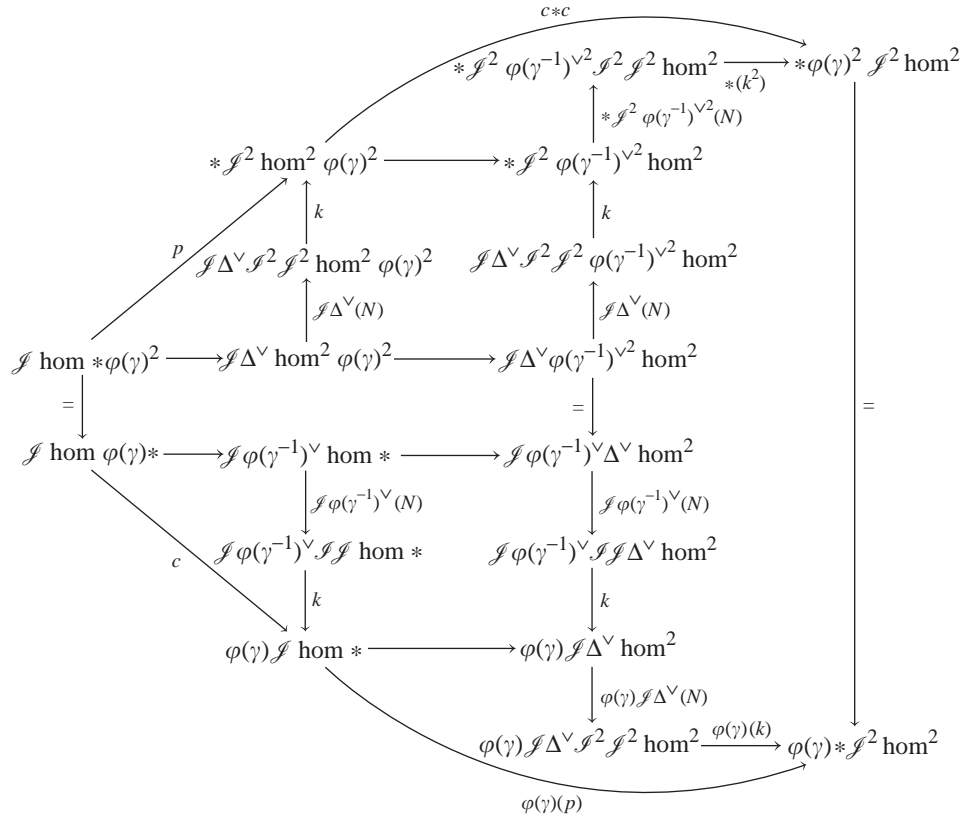
$$p_{U,V} : (U * V)^* \rightarrow V^* * U^*.$$

To construct $c_{\gamma,U}$ proceed as above but starting with the natural transformations

$$w_{\gamma,U} : \mathcal{A}(\varphi(\gamma)U, -) \xrightarrow{\bullet} \mathcal{A}(U, \varphi(\gamma^{-1})-)$$

again coming from property S-I of a unitary \mathcal{S}_X -structure.

To show c and p commute we claim the following diagram commutes where composition has been omitted and a superscript 2 refers to tensor product.



Let $m : P \rightarrow X$ and $i : I \rightarrow X$ induce the monoidal structure and crossing $\varphi(\gamma)$ on \mathcal{A} . The surface diagram for the crossing in Section 3 induces an equality $m \circ (i \sqcup i) = i \circ m$, thus the natural transformations in S-I has two decompositions, which result in the following commutative diagram:

$$\begin{array}{ccccc}
 \mathcal{A}(\varphi(\gamma)U * \varphi(\gamma)V, -) & \longrightarrow & \mathcal{A}^2(\varphi(\gamma)V \otimes \varphi(\gamma)U, \Delta-) & \longrightarrow & \mathcal{A}^2(V \otimes U, \varphi(\gamma^{-1})^2 \Delta-) \\
 \downarrow = & & & & \downarrow = \\
 \mathcal{A}(\varphi(\gamma)(U * V), -) & \longrightarrow & \mathcal{A}(U * V, \varphi(\gamma^{-1})-) & \longrightarrow & \mathcal{A}^2(V \otimes U, \Delta\varphi(\gamma^{-1})-)
 \end{array}$$

Applying \mathcal{J} shows that the central horizontal slice of the main diagram above commutes. The two large central rectangles commute by naturality, and the triangles and sections with curved arrow commute by the definitions of p and c . Finally the far

right part of the diagram follows from identities in the mapping class group, similar to those in Lemma 4.1. \square

After these preliminaries we now construct the duality. Notice that for $U \in \mathcal{A}_\alpha$, $V \in \mathcal{A}_\beta$ and $W \in \mathcal{A}_\gamma$ with $\alpha = \beta\gamma$ there are natural isomorphisms

$$\begin{aligned} \mathcal{A}_\alpha(U * W, V) &\simeq \langle (U * W)^*, V \rangle \xrightarrow{\langle p, id \rangle} \langle W^* * U^*, V \rangle \\ &\simeq \langle W^*, U^* * V \rangle \simeq \mathcal{A}_\gamma(W, U^* * V). \end{aligned}$$

Use these to define b_U and d_U by

$$\begin{aligned} \mathcal{A}_\alpha(U, U) &\simeq \mathcal{A}_1(\mathbf{1}, U^* * U) \quad id \mapsto b_U, \\ \mathcal{A}_{\alpha^{-1}}(U, U) &\simeq \mathcal{A}_1(U * U^*, \mathbf{1}) \quad id \mapsto d_U. \end{aligned}$$

Proposition 4.2. b_U and d_U provide \mathcal{A} with a right duality.

Proof. Since \mathcal{A} is a monoidal category it can be thought of as a 2-category with one object and it follows from the definition of adjoints in 2-categories that b_U and d_U give a right duality iff U is right adjoint to U^* in this category. It follows from the theory of adjoints in 2-categories (see, for example, [4, p. 158]) that U is left adjoint to U^* iff there are natural equivalences

$$\mathcal{A}_\alpha(U, V * W) \simeq \mathcal{A}_\gamma(V^* * U, W)$$

and

$$(U^* * V) * U \simeq U^* * (V * U).$$

The result now follows by the natural isomorphisms above and associativity. \square

We now prove that the $c_{\gamma, U}$'s defined above satisfy the definition of lax tortile π -category, but first we need one result about the crossing.

Lemma 4.3. *The following diagram commutes:*

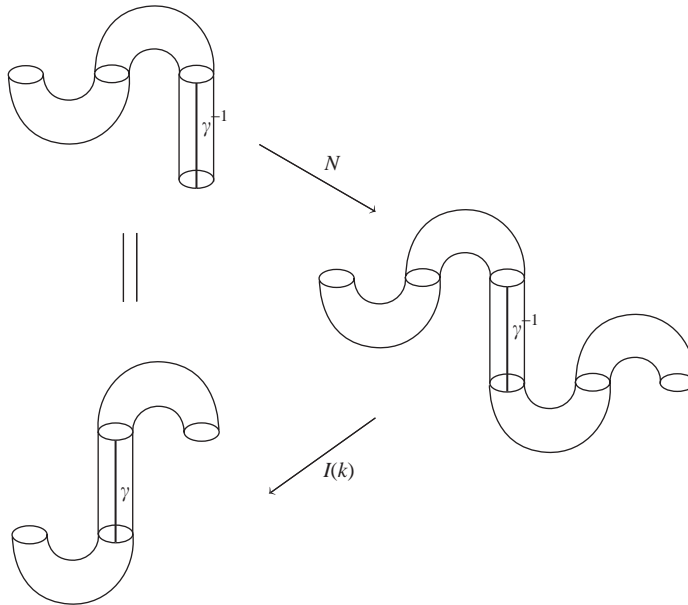
$$\begin{array}{ccc} \mathcal{A}(\varphi(\gamma)U, \varphi(\gamma)U) & \xleftarrow{\varphi(\gamma)} & \mathcal{A}(U, U) \\ \cong \downarrow & & \downarrow \cong \\ \langle (\varphi(\gamma)U^*, \varphi(\gamma)U) \rangle & \xrightarrow{\langle c_{\gamma, U}, id \rangle} \langle \varphi(\gamma)U^*, \varphi(\gamma)U \rangle & \xrightarrow{=} \langle U^*, U \rangle \end{array}$$

Proof. Recalling the definition of $w_{\gamma, U}$ from Proposition 4.1, since $\varphi(\gamma)$ is induced from a straight cylinder, $w_{\gamma, U}$ acts as the functor $\varphi(\gamma^{-1})$ by the definition of unitary

\mathcal{S}_X -structure. Now consider the diagram below:

$$\begin{array}{ccc}
 \text{hom} \circ \varphi(\gamma)(U) & \xrightarrow{N} & \mathcal{I} \circ \mathcal{J} \circ \text{hom} \circ \varphi(\gamma)(U) \\
 w_{\gamma,U} \downarrow & & \downarrow I \circ J(w_{\gamma,U}) \\
 \varphi(\gamma^{-1})^\vee \circ \text{hom}(U) & \xrightarrow{N} & \mathcal{I} \circ \mathcal{J} \circ \varphi(\gamma^{-1})^\vee \circ \text{hom}(U) \\
 \varphi(\gamma^{-1})^\vee(N) \downarrow & & \downarrow \mathcal{I} \circ \mathcal{J} \circ \varphi(\gamma^{-1})^\vee(N) \\
 \varphi(\gamma^{-1})^\vee \circ \mathcal{I} \circ \mathcal{J} \circ \text{hom}(U) & \xrightarrow{N} & \mathcal{I} \circ \mathcal{J} \circ \varphi(\gamma^{-1})^\vee \circ \mathcal{I} \circ \mathcal{J} \circ \text{hom}(U) \\
 = \downarrow & \swarrow \mathcal{I}(k) & \\
 \mathcal{I} \circ \varphi(\gamma) \circ \mathcal{J} \circ \text{hom}(U) & &
 \end{array}$$

To see that this diagram commutes: the two top sections follow immediately by naturality of N ; the bottom triangle is a consequence of the mapping class group identity below:



Now following the diagram around the right-hand path from the top left-hand corner gives the map $\mathcal{I}(c_{\gamma,U}) \circ N$ by definition of $c_{\gamma,U}$ and the left-hand side is the map $\varphi(\gamma^{-1})^\vee(N) \circ \varphi(\gamma^{-1})$, by the remark made above. Evaluating the diagram of functors on $\varphi(\gamma)U$ gives the desired result. \square

Proposition 4.3. \mathcal{A} is a lax tortile π -category.

Proof. Diagram (1) of Definition A.6 is simply naturality of the $c_{\alpha,U}$.

To prove diagram (2) of A.6 we claim the following diagram commutes:

$$\begin{array}{ccccc}
 J \circ \text{hom} \circ \varphi(\alpha)(U) & \xrightarrow{J(w_{\alpha,U})} & J \circ \varphi(\alpha^{-1})^\vee \circ \text{hom}(U) & \xrightarrow{t_{\varphi(\alpha), \text{hom}(U)}} & \varphi(\alpha) \circ \mathcal{J} \circ \text{hom}(U) \\
 \mathcal{J} \text{hom}(\theta_U) \downarrow & & \mathcal{J} \theta_{\text{hom}(U)}^\vee \uparrow & \swarrow & \\
 J \circ \text{hom}(U) & \xrightarrow{=} & J \circ \text{hom}(U) & \xleftarrow{\theta_{\varphi(\alpha) \circ \mathcal{J} \text{hom}(U)}} &
 \end{array}$$

The top line of this diagram is $c_{\alpha,U}$ by definition, the left map θ_U^* and the right map $\theta_{\varphi(\alpha)U^*}$.

By the consequence S-II of the unitary property the natural transformations $w_{\alpha,U}$ satisfy

$$\begin{array}{ccc}
 \mathcal{A}(\varphi(\alpha)U, -) & \xrightarrow{w_{\alpha,U}} & \mathcal{A}(U, \varphi(\alpha^{-1})-) \\
 \mathcal{A}(\theta_U, -) \downarrow & & \downarrow \mathcal{A}(U, \theta_{\varphi(\alpha^{-1})-}) \\
 \mathcal{A}(U, -) & \xrightarrow{id} & \mathcal{A}(U, -)
 \end{array}$$

This is obtained from S-II by taking the diffeomorphism T to be the inverse twist, which is mapped to the twist by reflection. The bottom map is the identity since the identity is sent to the identity in the definition of pseudo-2-natural transformation. Taking \mathcal{J} of this diagram proves the left square above commutes.

For the triangle on the right pre compose the diagram in Lemma 4.2 with hom .

Finally, diagram (3) for b_U in Definition A.6 is shown to commute by considering the following diagram:

$$\begin{array}{ccccc}
 \mathcal{A}(\varphi(\gamma)U, \varphi(\gamma)U) & \xleftarrow{\varphi(\gamma)} & \mathcal{A}(U, U) & & \\
 \cong \downarrow & & \downarrow \cong & & \\
 \langle (\varphi(\gamma)U)^*, \varphi(\gamma)U \rangle & \xrightarrow{\langle c_{\gamma,U}, id \rangle} & \langle \varphi(\gamma)U^*, \varphi(\gamma)U \rangle \xrightarrow{=} & \langle U^*, U \rangle & \\
 \cong \downarrow & & \downarrow \cong & & \\
 \langle (\varphi(\gamma)U^* \mathbf{1})^*, \varphi(\gamma)U \rangle \xrightarrow{=} & \langle (\varphi(\gamma)(U^* \mathbf{1}))^*, \varphi(\gamma)U \rangle \xrightarrow{\langle c_{\gamma,U^* \mathbf{1}}, id \rangle} & \langle \varphi(\gamma)(U^* \mathbf{1})^*, \varphi(\gamma)U \rangle \xrightarrow{=} & \langle (U^* \mathbf{1})^*, U \rangle & \\
 \langle p, id \rangle \downarrow & & \downarrow \langle \varphi(\gamma)p, id \rangle & & \downarrow \langle p, id \rangle \\
 \langle (\varphi(\gamma)\mathbf{1})^* (\varphi(\gamma)U)^*, \varphi(\gamma)U \rangle & \xrightarrow{\langle c_{\gamma, \mathbf{1}^* c_{\gamma,U}}, id \rangle} & \langle \varphi(\gamma)\mathbf{1}^* (\varphi(\gamma)U)^*, \varphi(\gamma)U \rangle \xrightarrow{=} & \langle \mathbf{1}^* U^*, U \rangle & \\
 \cong \downarrow & & \downarrow \cong & & \\
 \langle (\varphi(\gamma)\mathbf{1})^*, (\varphi(\gamma)U)^* (\varphi(\gamma)U) \rangle & \xrightarrow{\langle c_{\gamma, \mathbf{1}^* c_{\gamma,U}^*}, id \rangle} & \langle \varphi(\gamma)\mathbf{1}^*, (\varphi(\gamma)U)^* (\varphi(\gamma)U) \rangle \xrightarrow{=} & \langle \mathbf{1}^* U^* (\varphi(\gamma)U) \rangle & \\
 \cong \downarrow & & \downarrow \cong & & \\
 \mathcal{A}(\mathbf{1}, (\varphi(\gamma)U)^* (\varphi(\gamma)U)) & \xleftarrow{\mathcal{A}(id, c_{\gamma,U}^* id)} & \mathcal{A}(\varphi(\gamma)\mathbf{1}, (\varphi(\gamma)U)^* (\varphi(\gamma)U)) \xleftarrow{\varphi(\gamma)} & \mathcal{A}(\mathbf{1}, U^* (\varphi(\gamma)U)) &
 \end{array}$$

Notice that the left and right sides of the diagram define $b_{\varphi(\gamma)U}$ and b_U , respectively, as the image of the identity morphism. Thus, since $\varphi(\gamma)$ is a functor, the commutativity of this diagram proves (3) by mapping the identity in the top right-hand corner to the bottom left-hand corner via the two exterior paths.

The top and bottom parts of the diagram commute by Lemma 4.3 and naturality. The left part of the central section commutes by Lemma 4.1. The remaining sections of the diagram commute by naturality.

To complete the proof it remains to check diagram (3) for d_U . The arguments are essentially the same as those for the b_U case, so we simply give the corresponding diagram without further comment:

$$\begin{array}{ccccccc}
 \mathcal{A}((\varphi(\gamma)U)^*, (\varphi(\gamma)U)^*) & \xleftarrow{\mathcal{A}(c_{\gamma,U}, c_{\gamma,U}^{-1})} & \mathcal{A}(\varphi(\gamma)U^*, \varphi(\gamma)U^*) & \xleftarrow{\varphi(\gamma)} & \mathcal{A}(U^*, U^*) \\
 \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\
 \langle (\varphi(\gamma)U)^*, (\varphi(\gamma)U)^* \rangle & \xrightarrow{\langle c_{\gamma,U}^{-1}, c_{\gamma,U} \rangle} & \langle (\varphi(\gamma)U^*), \varphi(\gamma)U^* \rangle & \xrightarrow{\langle c_{\gamma,U}, id \rangle} & \langle \varphi(\gamma)U^{**}, \varphi(\gamma)U^* \rangle \xrightarrow{=} \langle U^{**}, U^* \rangle \\
 \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\
 \langle (\varphi(\gamma)U)^{**}, (\varphi(\gamma)U^* * \mathbf{1}) \rangle & \xrightarrow{\langle c_{\gamma,U}^{*-1}, c_{\gamma,U} * id \rangle} & \langle (\varphi(\gamma)U^*)^*, \varphi(\gamma)U^* * \mathbf{1} \rangle & \xrightarrow{\langle c_{\gamma,U}, id \rangle} & \langle \varphi(\gamma)U^{**}, \varphi(\gamma)U^* * \mathbf{1} \rangle \xrightarrow{=} \langle U^{**}, U^* * \mathbf{1} \rangle \\
 \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\
 \langle (\varphi(\gamma)U)^{**} * (\varphi(\gamma)U^*, \mathbf{1}) \rangle & \xrightarrow{\langle c_{\gamma,U}^{*-1} * id, id \rangle} & \langle (\varphi(\gamma)U^*)^* * (\varphi(\gamma)U^*, \mathbf{1}) \rangle & \xrightarrow{\langle c_{\gamma,U} * c_{\gamma,U}, id \rangle} & \langle \varphi(\gamma)U^{**} * \varphi(\gamma)U^*, \mathbf{1} \rangle \xrightarrow{=} \langle U^{**} * U^*, \mathbf{1} \rangle \\
 \langle p, id \rangle \uparrow & & \langle p, id \rangle \uparrow & & \langle \varphi(\gamma)p, id \rangle \uparrow \\
 \langle (\varphi(\gamma)U * (\varphi(\gamma)U^*)^*, \mathbf{1}) \rangle & \xrightarrow{\langle id * c_{\gamma,U}^{-1}, id \rangle} & \langle (\varphi(\gamma)U * \varphi(\gamma)U^*)^*, \mathbf{1} \rangle & \xrightarrow{\langle c_{\gamma,U} * U^*, id \rangle} & \langle \varphi(\gamma)(U * U^*)^*, \mathbf{1} \rangle \xrightarrow{=} \langle (U * U^*)^*, \mathbf{1} \rangle \\
 \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\
 \mathcal{A}(\varphi(\gamma)U * (\varphi(\gamma)U^*)^*, \mathbf{1}) & \xleftarrow{\mathcal{A}(id * c_{\gamma,U}, id)} & \mathcal{A}(\varphi(\gamma)U * \varphi(\gamma)U^*, \mathbf{1}) & \xleftarrow{\varphi(\gamma)} & \mathcal{A}(U * U^*, \mathbf{1})
 \end{array}$$

This finishes the proof. \square

Now let X be an arbitrary space and claim that \mathcal{A}_1 is a lax tortile category with $\pi_2(X)$ -action. Methods similar to those used for $X = K(\pi, 1)$ above, produce a duality on \mathcal{A}_1 . We need to relate this to the $\pi_2(X)$ -action.

Lemma 4.4. For all $U \in \mathcal{A}_1$ and $g \in \pi_2 X$ there are natural isomorphisms

$$h_{g,U}: (\rho(g)U)^* \rightarrow \rho(g^{-1})U^*.$$

Proof. The maps $i_g: I \rightarrow X$ giving $\pi_2(X)$ -action satisfy $\widehat{i}_g = i_{g^{-1}}$ so, by property S-1, the unitary property induces natural isomorphisms

$$v_{g,U}: \mathcal{A}(\rho(g)U, -) \xrightarrow{\bullet} \mathcal{A}(U, \rho(g^{-1})-).$$

Applying the functor \mathcal{J} and composing with the equivalence t of 4.1 we get

$$\mathcal{J} \circ \text{hom} \circ \rho(g)(U) \xrightarrow{J(v_{g,U})} \mathcal{J} \circ \rho(g^{-1})^\vee \circ \text{hom}(U) \xrightarrow{t_{\rho(g), \text{hom}(U)}} \rho(g^{-1}) \circ \mathcal{J} \circ \text{hom}(U)$$

giving the desired natural isomorphisms. \square

Proposition 4.4. \mathcal{A}_1 is a lax tortile category with $\pi_2(X)$ -action.

Proof. Diagram (1) of Definition A.8 is naturality of the $h_{g,U}$'s. To prove the diagrams in (2) for b_U and d_U , proceed as in the proof of Proposition 4.3 replacing $c_{\gamma,U}$ with $h_{g,U}$. \square

Combining Propositions 4.2, 4.3 and 4.4 with the results of Section 3 completes the proof of Theorem 4.1.

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Appendix A. Balanced and tortile structures

This appendix contains the variants of balanced and tortile categories that we need. Our lax tortile π -categories are lax versions of the ribbon π -categories defined by Turaev in [14].

Definition A.1. Let π be a discrete group. A monoidal category $(\mathcal{A}, *, 1)$ is said to be π -graded if it splits as a disjoint union of full subcategories $\mathcal{A} = \sqcup_{\alpha \in \pi} \mathcal{A}_\alpha$ such that objects belong to a unique grading, there are no morphisms between objects of different grading and the monoidal structure multiplies gradings (in the group π).

We denote the set of invertible functors on \mathcal{A} by $\text{Aut}(\mathcal{A})$. A functor is *monoidal* if the monoidal product, unit and all structure morphisms are preserved.

Definition A.2. A π -graded monoidal category is said to be *crossed* if it comes equipped with a group homomorphism $\varphi : \pi \rightarrow \text{Aut}(\mathcal{A})$ with $\varphi(\gamma) : \mathcal{A}_\beta \rightarrow \mathcal{A}_{\gamma\beta\gamma^{-1}}$ such that each $\varphi(\gamma)$ is a monoidal functor.

Definition A.3. A *braiding* on a crossed π -graded monoidal category \mathcal{A} is a collection of isomorphism $s_{U,V} : U * V \rightarrow \varphi(\alpha)V * U$ for objects U of \mathcal{A}_α , V of \mathcal{A}_β , such that

(1) For $U \in \mathcal{A}_\alpha, V \in \mathcal{A}_\beta$ and $W \in \mathcal{A}_\gamma$ the following two diagrams commute:

$$\begin{array}{ccc}
 U * V * W & \xrightarrow{s_{U*V,W}} & \varphi(\alpha\beta)W * U * V \\
 & \searrow^{id*s_{V,W}} & \nearrow^{s_{U,\varphi(\beta)W} * id} \\
 & U * \varphi(\beta)W * V &
 \end{array}$$

$$\begin{array}{ccc}
 U * V * W & \xrightarrow{s_{U,V*W}} & \varphi(\alpha)V * \varphi(\alpha)W * U \\
 & \searrow s_{U,V} * id & \nearrow id * s_{U,W} \\
 & & \varphi(\alpha)V * U * W
 \end{array}$$

(associativity morphisms have been suppressed; insert them and one gets the hexagons expected by the reader).

- (2) for $U, U' \in \mathcal{A}_\alpha$ and $V, V' \in \mathcal{A}_\beta$ and morphisms $f: U \rightarrow U'$ and $g: V \rightarrow V'$ the following diagram commutes

$$\begin{array}{ccc}
 U * V & \xrightarrow{f * g} & U' * V' \\
 s_{U,V} \downarrow & & \downarrow s_{U',V'} \\
 \varphi(\alpha)V * U & \xrightarrow{\varphi(\alpha)g * f} & \varphi(\alpha)V' * U'
 \end{array}$$

- (3) for $U \in \mathcal{A}_\alpha$ and $V \in \mathcal{A}_\beta$ and $\gamma \in \pi$,

$$s_{\varphi(\gamma)U, \varphi(\gamma)V} = \varphi(\gamma)(s_{U,V}).$$

Definition A.4. A *twist* on a braided crossed π -graded monoidal category \mathcal{A} is a collection of isomorphisms $\theta_U: U \rightarrow \varphi(\alpha)U$ for $U \in \mathcal{A}_\alpha$ such that:

- (1) $\theta_{\mathbf{1}} = id_{\mathbf{1}}$,
 (2) for $U \in \mathcal{A}_\alpha$ and $V \in \mathcal{A}_\beta$,

$$\theta_{U * V} = s_{\varphi(\alpha\beta)V, \varphi(\alpha)U} \circ (\theta_{\varphi(\alpha)V} * \theta_U) \circ s_{U,V}$$

- (3) for $U, V \in \mathcal{A}_\alpha$ and morphism $f: U \rightarrow V$ the following diagram commutes:

$$\begin{array}{ccc}
 U & \xrightarrow{f} & V \\
 \theta_U \downarrow & & \downarrow \theta_V \\
 \varphi(\alpha)U & \xrightarrow{\varphi(\alpha)f} & \varphi(\alpha)V
 \end{array}$$

- (4) for $\gamma \in \pi$ and $V \in \mathcal{A}$,

$$\theta_{\varphi(\gamma)V} = \varphi(\gamma)(\theta_V).$$

Definition A.5. A *balanced π -category* is a braided crossed π -graded monoidal category with twist.

When $\pi = \{1\}$ the usual notion of a balanced category is obtained. Now we introduce duality into balanced π -categories. A *right duality* assigns to an object $U \in \mathcal{A}_\alpha$ an object $U^* \in \mathcal{A}_{\alpha^{-1}}$ and morphisms

$$b_U: \mathbf{1} \rightarrow U^* * U \quad d_U: U * U^* \rightarrow \mathbf{1}$$

such that the following compositions are the identity:

$$U^* \simeq \mathbf{1} * U^* \xrightarrow{b_U * id} (U^* * U) * U^* \simeq U^* * (U * U^*) \xrightarrow{id * d_U} U^* * \mathbf{1} \simeq U^*,$$

$$U \simeq U * \mathbf{1} \xrightarrow{id * b_U} U * (U^* * U) \simeq (U * U^*) * U \xrightarrow{d_U * id} \mathbf{1} * U \simeq U.$$

The above duality is also called rigid duality, and U^* is a rigid dual, to distinguish it from weaker versions. By duality we shall always mean rigid duality.

Definition A.6. A *lax tortile π -category* is a balanced π -category with right duality and a collection of isomorphisms $c_{\gamma,U} : (\varphi(\gamma)U)^* \rightarrow \varphi(\gamma)U^*$ for $U \in \mathcal{A}$ and $\gamma \in \pi$ such that:

(1) For $U \in \mathcal{A}_\alpha$ and $f : U \rightarrow V$ the following diagram commutes:

$$\begin{array}{ccc} (\varphi(\gamma)U)^* & \xrightarrow{c_{\gamma,U}} & \varphi(\gamma)U^* \\ (\varphi(\gamma)f)^* \downarrow & & \downarrow \varphi(\gamma)f^* \\ (\varphi(\gamma)V)^* & \xrightarrow{c_{\gamma,V}} & \varphi(\gamma)V^* \end{array}$$

(2) For $U \in \mathcal{A}_\alpha$ the following diagram commutes:

$$\begin{array}{ccc} (\varphi(\alpha)U)^* & \xrightarrow{c_{\alpha,U}} & \varphi(\alpha)U^* \\ & \searrow (\theta_U)^* & \swarrow \theta_{\varphi(\alpha)U^*} \\ & U^* & \end{array}$$

(3) For $U \in \mathcal{A}_\alpha$ the following diagram commutes:

$$\begin{array}{ccccc} (\varphi(\gamma)U)^* * \varphi(\gamma)U & \xrightarrow{c_{\gamma,U} * id} & \varphi(\gamma)U^* * \varphi(\gamma)U \\ & \swarrow b_{\varphi(\gamma)U} & \searrow \varphi(\gamma)(b_U) \\ & \mathbf{1} & \end{array}$$

and similarly for d_U .

Note that a *strict* tortile π -category is one for which all $c_{\gamma,U}$'s are identities, and is the same thing as a ribbon crossed π -category (satisfying $\theta_1 = id_1$).

Now we introduce an ungraded version with an action of a group G .

Definition A.7. A *balanced category with G -action* is a balanced category $(\mathcal{A}, *, \mathbf{1}, s, \theta)$ together with a homomorphism $\rho : G \rightarrow \text{Aut}(\mathcal{A})$ such that:

- (1) $\rho(g)U * V = \rho(g)(U * V) = U * \rho(g)V$,
- (2) for $f : U \rightarrow U'$ and $h : V \rightarrow V'$ we have $\rho(g)f * h = \rho(g)(f * h) = f * \rho(g)h$,

- (3) $\rho(g)(a_{U,V,W}) = a_{\rho(g)U,V,W} = a_{U,\rho(g)V,W} = a_{U,V,\rho(g)W}$,
- (4) $\rho(g)(r_U) = r_{\rho(g)U}$ and $\rho(g)(l_U) = l_{\rho(g)U}$,
- (5) $s_{\rho(g)U,V} = \rho(g)(s_{U,V}) = s_{U,\rho(g)V}$,
- (6) $\theta_{\rho(g)U} = \rho(g)(\theta_U)$.

Note that ρ factors through the centre of $\text{Aut}(\mathcal{A})$ and on objects $\rho(g)$ is determined by $\rho(g)\mathbf{1}$ since $\rho(g)(U) = (\rho(g)\mathbf{1}) * U$. Incorporating duals we have the following definition.

Definition A.8. A *tortile category with lax G-action* is a balanced category with G-action with right duality and a collection of isomorphisms $h_{g,U} : (\rho(g)U)^* \rightarrow \rho(g^{-1})(U^*)$ such that the following diagrams commute:

(1) For $U, V \in \mathcal{A}$ and $f : U \rightarrow V$,

$$\begin{array}{ccc} (\rho(g)U)^* & \xrightarrow{h_U} & \rho(g^{-1})U^* \\ (\rho(g)f)^* \uparrow & & \uparrow \rho(g^{-1})f^* \\ (\rho(g)V)^* & \xrightarrow{h_V} & \rho(g^{-1})V^* \end{array}$$

(2)

$$\begin{array}{ccc} (\rho(g)U)^* * \rho(g)U & \xrightarrow{h_{g,U} * id} & U^* * U \\ & \swarrow b_{\rho(g)U} & \searrow b_U \\ & \mathbf{1} & \end{array}$$

and similarly for d_U .

Appendix B. k-Additive categories

Let k be an algebraically closed field. A *additive category over k*, or *k-additive category* is a category with the following properties:

- (1) morphism spaces are complex vector spaces and composition is bilinear,
- (2) there is a finite direct sum on objects,
- (3) there is a zero object 0 such that $\text{hom}(U, 0) = \text{hom}(0, U) = 0$ for all $U \in \mathcal{A}$.

k -additive functors between k -additive categories are additive functors acting linearly on the morphism spaces. The tensor product of two k -additive categories is defined in the usual way. For more details on additive categories we refer to [7].

The *idempotent completion* $\hat{\mathcal{A}}$ of a category \mathcal{A} has objects pairs (U, e) where $e : U \rightarrow U$ is an idempotent in \mathcal{A} and morphisms $f : (U, e) \rightarrow (U', e')$ where $f : U \rightarrow U'$ such that $fe = f = e'f$. The idempotent completion of an additive category over k remains an additive category over k . Any functor from \mathcal{A} to an idempotent complete

category \mathcal{B} can be completed to a functor on $\hat{\mathcal{A}}$; this construction yields a natural equivalence between the categories of such functors.

Definition B.1. Let $\widehat{k\text{-Add}}$ be the 2-category with objects idempotent complete additive small categories over k , 1-morphisms k -additive functors between these and 2-morphisms natural transformations.

The tensor product of k -additive categories induces a monoidal structure on $\widehat{k\text{-Add}}$: since k is algebraically closed the tensor product of idempotent complete k -additive categories remains idempotent complete. The category \hat{k} of finite dimensional vector spaces over k is the monoidal unit.

The set of k -additive functors between \mathcal{A} and \mathcal{B} is denoted $\widehat{k\text{-Add}}(\mathcal{A}, \mathcal{B})$ and this is again an idempotent k -additive category if both \mathcal{A} and \mathcal{B} are small, and \mathcal{B} is idempotent complete. If a k -additive category is small its *dual* can be defined as $\mathcal{A}^\vee = \widehat{k\text{-Add}}(\mathcal{A}, \hat{k})$.

An object X of an additive category is *simple* if $\mathcal{A}(X, X)$ is one dimensional and \mathcal{A} is *semi-simple* if every object is isomorphic to a finite sum of simple objects. \mathcal{A} is *Artinian* if there are finitely many isomorphism classes of simple objects.

We now discuss Tillmann's work on duality in idempotent complete k -additive categories. A *non-degenerate form* consists of a k -additive functor $\langle -, - \rangle : \mathcal{A} \otimes \mathcal{B} \rightarrow \hat{k}$ and an object $\sum_{i=1}^n P_i \otimes Q_i$ in $\mathcal{B} \otimes \mathcal{A}$ such that the functors

$$\mathcal{I} : \mathcal{A} \rightarrow \mathcal{B}^\vee \text{ by } Y \mapsto \langle Y, - \rangle,$$

$$\mathcal{J} : \mathcal{B}^\vee \rightarrow \mathcal{A} \text{ by } H \mapsto \sum_{i=1}^n H(P_i) \otimes Q_i$$

provide an equivalence of categories. Her methods prove that given a non-degenerate form and letting $X = \sum Q_i$ and $A = \mathcal{A}(X, X)$ then the following hold:

- (1) \mathcal{A} is equivalent to the category of finite-dimensional projective A -modules.
- (2) $\mathcal{A}(Y, Z)$ is a finitely generated vector space.

She also defines a contravariant functor $(-)^* : \mathcal{B} \rightarrow \mathcal{A}$ and similarly a contravariant functor $(-)^* : \mathcal{A} \rightarrow \mathcal{B}$ with the property that $(-)^* \circ (-)^*$ is naturally isomorphic to $id_{\mathcal{A}}$. To define these involutions consider the contravariant functor

$$\text{hom} : \mathcal{B} \rightarrow \mathcal{B}^\vee \quad Y \mapsto \mathcal{B}(Y, -),$$

which is defined since by the above $\mathcal{B}(Y, Z)$ is a finitely generated vector space and hence an object of \hat{k} . Set $(-)^* = \mathcal{J} \circ \text{hom}$. Tillmann proves that there is a natural equivalence $\mathcal{B}(-, -) \simeq \langle -, - \rangle$ and her methods also show that \mathcal{A} and \mathcal{B} are semi-simple Artinian categories.

If we are given a collection of non-degenerate forms $\langle -, - \rangle_{\alpha_i} : \mathcal{A}_{\alpha_i} \otimes \mathcal{B}_{\alpha_i} \rightarrow \hat{k}$ for $i = 1 \dots n$ we can construct a non-degenerate form

$$\langle -, - \rangle_{\underline{\alpha}} : \mathcal{A}_{\alpha_1} \otimes \dots \otimes \mathcal{A}_{\alpha_n} \otimes \mathcal{B}_{\alpha_n} \otimes \dots \otimes \mathcal{B}_{\alpha_1} \rightarrow \hat{k}$$

and similarly an involution

$$(-)^* : \mathcal{B}_{\alpha_n} \otimes \cdots \otimes \mathcal{B}_{\alpha_1} \rightarrow \mathcal{A}_{\alpha_1} \otimes \cdots \otimes \mathcal{A}_{\alpha_n},$$

which satisfies $\mathcal{B}_{\alpha_n} \otimes \cdots \otimes \mathcal{B}_{\alpha_1}(-, -) \simeq \langle (-)^*, - \rangle_{\underline{\alpha}}$.

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