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A Partition of $L(3, n)$ into Saturated Symmetric Chains

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For positive integers m and n let $L(m, n)$ denote the set of all m-tuples $(a_1, a_2, \ldots, a_m)$ of integers with $0 \leq a_1 \leq a_2 \leq \cdots \leq a_m \leq n$. The set $L(m, n)$ is partially ordered such that $(a_1, \ldots, a_m) \leq (b_1, \ldots, b_m)$ holds precisely when $a_i \leq b_i$ for $i = 1, 2, \ldots, m$. We prove that the partially ordered set $L(3, n)$ has a partition into saturated symmetric chains.

It is not out of the place to mention that D. E. Littlewood assumed that there is such a partition of $L(m, n)$ into symmetric chains for all $m, n \geq 1$ in his book *Theory of Group Characters*.

1. INTRODUCTION

For all positive integers m, n the partially ordered set $L(m, n)$ is defined as follows. The elements of $L(m, n)$ are all sequences $(a_1, \ldots, a_m)$ of integers with $0 \leq a_1 \leq a_2 \leq \cdots \leq a_m \leq n$. The partial order in $L(m, n)$ is defined by $(a_1, \ldots, a_m) \leq (b_1, \ldots, b_m)$ when $a_i \leq b_i$ for $i = 1, 2, \ldots, m$. It is easy to see that $L(m, n)$ is a distributive lattice of cardinality $\binom{m+n}{m}$.

It is also easy to see that $L(m, n)$ satisfies the Jordan–Dedekind chain condition: all maximal chains between two comparable elements have the same length. The length of a maximal chain between $(0, \ldots, 0)$ and $(a_1, \ldots, a_m)$ is $a_1 + a_2 + \cdots + a_m$. This is the *rank* of $(a_1, \ldots, a_m)$. A chain $x_i < x_{i+1} < \cdots < x_j$ in $L(m, n)$ is *symmetric* if $r(x_i) + r(x_j) = mn$, where $r(x_i)$ is the rank of x_i and mn is the rank of $(n, \ldots, n)$.

In [2, pp. 193–203] D. E. Littlewood evidently assumes that $L(m, n)$ has a decomposition into saturated symmetric chains when he relies on the 'method of chains' of Aitken [1]. It has been observed by R. Stanley in [5] that this method is not correct as stated by Aitken. A corrected version of Aitken's result appears in [4]. With the aid of the cohomology theory of projective varieties R. Stanley could prove that $L(m, n)$ has a weaker property S, which implies the *Sperner property* that the largest size of an antichain is equal to $\max\{p_i : 0 \leq i \leq mn\}$, where p_i is the number of elements of rank i. In fact $L(m, n)$ is only a special case of the vast class of partially ordered sets studied by R. Stanley in [5].

We will prove the following result.

THEOREM. *The partially ordered set $L(3, n)$ has a partition into saturated symmetric chains when $n \geq 0$.*

2. PROOF OF THE THEOREM

The theorem will be proved by induction over n with separate proofs for odd and even n.

CASE I. $n = 2k + 1$. Let S_n denote the subset of all (a_1, a_2, a_3) in $L(3, n)$ for which either $a_1 = 0$ or $a_3 = n$, or both. The remaining elements of $L(3, n)$ have $1 \leq a_1 \leq a_2 \leq a_3 \leq n - 1$ and form a p.o. set isomorphic to $L(3, n - 2)$. By the induction it is sufficient to give a partition of S_n into saturated symmetric chains. Let $C_i = (0, i, i), (0, i, i+1), \ldots, (0, i, n-i), (0, i+1, n-i), (0, i+2, n-i+1), \ldots, (0, 2i, n), (0, 2i+1, n), (1, 2i+1, n), (1, 2i+2, n), (2, 2i+2, n), \ldots, (n-2i-1, n, n), (n-2i, n, n)$, for $i = 0, 1, \ldots, k$. It is easy to see that each element in S_n belongs to just one chain C_i.

The induction starts with $L(3, 1)$: $(0, 0, 0), (0, 0, 1), (0, 1, 1), (1, 1, 1)$.

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CASE II. $n = 2k$. This case is more complicated (see Figure 1 for example).

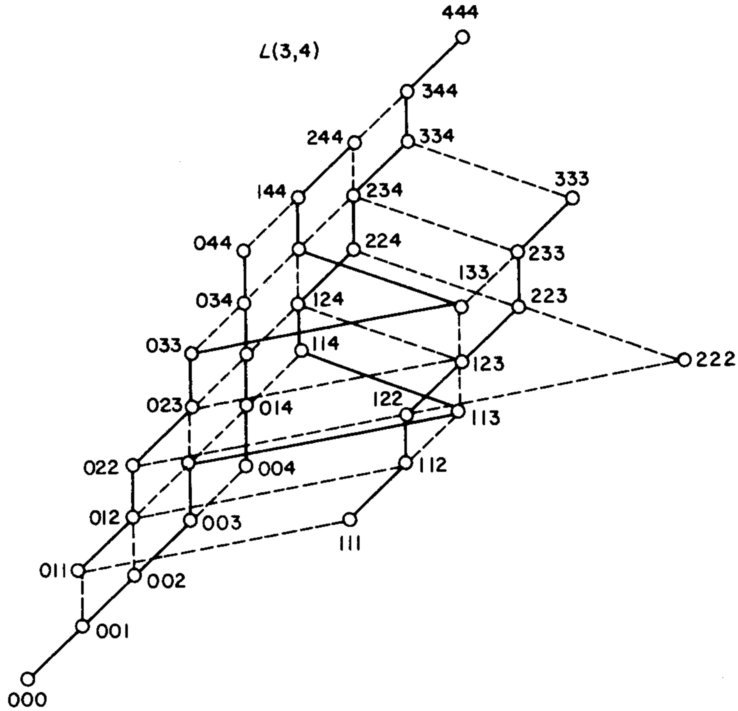

FIGURE 1

Let S_n, as before, denote the subset of all (a_1, a_2, a_3) with $a_1 = 0$ or $a_3 = n$, or both. We augment S_n by "exceptional" elements $(1, 2i+1, n-1)$ for $i = 0, 1, \ldots, k-1$. The augmented set S_n^+ is then partitioned into chains C and C_i $(i = 0, 1, \ldots, k-1)$, where $C = (0, 0, n)$, $(0, 1, n), \ldots, (0, n, n)$ and $C_i = (0, i, i)$, $(0, i, i+1), \ldots, (0, i, n-i-1)$, $(0, i+1, n-i-1)$, $(0, i+1, n-i)$, $(0, i+2, n-1)$, $(0, i+2, n-i+1), \ldots, (0, 2i+1, n-1)$, $(1, 2i+1, n-1)^*$, $(1, 2i+1, n)$, $(1, 2i+2, n)$, $(2, 2i+2, n)$, $(2, 2i+3, n), \ldots, (n-2i-1, n, n)$, $(n-2i, n, n)$. Each chain C_i contains an exceptional element $(1, 2i+1, n-1)$ from the set T_n, say, of all (a_1, a_2, a_3) in $L(3, n)$ with $a_1 = 1$ or $a_3 = n-1$, or both. T_n minus exceptional elements has a partition into saturated symmetric chains D_i, where $D_i = (1, i, i)$, $(1, i, i+1), \ldots, (1, i, n-i-1)$, $(1, i+1, n-i-1)$, $(1, i+1, n-i), \ldots, (1, 2i, n-1)$, $(2, 2i, n-1)$, $(2, 2i+1, n-1)$, $(3, 2i+1, n-1), \ldots, (n-2i, n-1, n-1)$, $(n-2i+1, n-1, n-1)$ for $i = 1, 2, \ldots, k-1$. Observe that the subset $S_n \cup T_n$ contains all elements (a_i, a_2, a_3) in $L(3, n)$ with $a_1 = 0$ or 1, or $a_3 = n-1$ or n. Therefore $L(3, n) - (S_n \cup T_n)$ is isomorphic to $L(3, n-4)$, and we may use induction. The induction starts by $L(3, 0)$ and $L(3, 2)$, which are easy to decompose into saturated symmetric chains.

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I have heard that Douglas West has found a symmetric chain decomposition of $L(4, n)$, but I have not seen his construction.

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