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An extension of the Lyndon–Schützenberger result to pseudoperiodic words ☆,☆☆

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ABSTRACT

One of the particularities of information encoded as DNA strands is that a string u contains basically the same information as its Watson–Crick complement, denoted here as $\theta(u)$. Thus, any expression consisting of repetitions of u and $\theta(u)$ can be considered in some sense periodic. In this paper, we give a generalization of Lyndon and Schützenberger's classical result about equations of the form $u^l = v^n w^m$, to cases where both sides involve repetitions of words as well as their complements. Our main results show that, for such extended equations, if $l \geqslant 5$, $n, m \geqslant 3$, then all three words involved can be expressed in terms of a common word t and its complement $\theta(t)$. Moreover, if $l \geqslant 5$, then $n = m = 3$ is an optimal bound. These results are established based on a complete characterization of all possible overlaps between two expressions that involve only some word u and its complement $\theta(u)$, which is also obtained in this paper.

1. Introduction

Periodicity and primitiveness of words are fundamental properties in combinatorics on words and formal language theory. Their wide-ranging applications include pattern-matching algorithms (see, e.g., [1,2]) and data-compression algorithms (see, e.g., [3]). Sometimes motivated by their applications, these classical notions have been modified or generalized in various ways. A representative example is the "weak periodicity" of [4] whereby a word is called *weakly periodic* if it consists of repetitions of words with the same Parikh vector. This type of period was also called *abelian period* in [5]. Carpi and de Luca extended the notion of periodic words into that of periodic-like words according to the extendability of factors of a word [6]. Czeizler et al. have proposed the notion of *pseudo-primitiveness* (and pseudoperiodicity) of words in [7], motivated by the properties of information encoded as DNA strands.

DNA stores genetic information primarily in its single-stranded form as an oriented chain made up of four kinds of nucleotides: adenine ($\mathtt{A}$), cytosine ($\mathtt{C}$), guanine ($\mathtt{G}$), and thymine ($\mathtt{T}$). Thus, a single-stranded DNA can be viewed as a word over the four-letter alphabet $\{\mathtt{A}, \mathtt{C}, \mathtt{G}, \mathtt{T}\}$. Due to the Watson–Crick complementarity of DNA strands, whereby $\mathtt{A}$ is complementary

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to T, and C is complementary to G, single-stranded DNA molecules interact with each other. Indeed, two Watson–Crick complementary DNA single strands with opposite orientation will bind to each other by weak hydrogen bonds between their individual bases and form the well-known DNA double helix structure. In the process of DNA replication, a DNA double strand is separated into its two constituent single strands, each of which serves as a template for the enzyme called DNA polymerase. Starting from one end of a DNA single strand, DNA polymerase has the ability to build up, one nucleotide at a time, a new DNA strand that is perfectly complementary to the template, resulting in two copies of the DNA double strand. Thus, two DNA strands which are Watson–Crick complementary to each other can be considered "equivalent" in terms of the information they encode.

The fact that one can consider a DNA strand and its Watson–Crick complement "equivalent" led to natural and theoretically interesting extensions of various notions in combinatorics of words and formal language theory such as pseudo-palindrome [8], pseudo-commutativity [9], as well as hairpin-free and bond-free languages (e.g., [10–12]). Watson–Crick complementarity has been modeled mathematically by an antimorphic involution θ, i.e., a function that is an antimorphism ($\theta(uv) = \theta(v)\theta(u)$ for any words u, v), and an involution (θ^2 is the identity function). The aforementioned new concepts and notions are based on extending the notion of identity between words to that of "equivalence" between words u and $\theta(u)$, in the sense that an occurrence of $\theta(u)$ will be treated as another occurrence of u, albeit disguised by the application of θ.

In [7], a word w is said to be *θ-primitive* if we cannot find any word x that is strictly shorter than w such that w can be written as a combination of x and $\theta(x)$. For instance, if θ is the Watson–Crick complementarity then ATCG is θ-primitive, whereas TCGA is not because $\mathtt{TCGA} = \mathtt{TC}\theta(\mathtt{TC})$. The periodicity theorem of Fine and Wilf – one of the fundamental results on periodicity of words, see, e.g., [13,14] – was also extended as follows "For given words u and v, how long does a common prefix of a word in $\{u, \theta(u)\}^+$ and a word in $\{v, \theta(v)\}^+$ have to be, in order to imply that $u, v \in \{t, \theta(t)\}^+$ for some word t?".

In this paper, we continue the theoretical study of θ-primitive words by extending another central periodicity result, due to Lyndon and Schützenberger [15]. The original result states that, if the concatenation of two periodic words v^n and w^m can be expressed in terms of a third period u, i.e., $u^\ell = v^n w^m$, for some $\ell, m, n \geqslant 2$, then all three words u, v, and w can be expressed in terms of a common word t, i.e., $u, v, w \in \{t\}^+$ (see also [16] and Chapter 5 from [14] for some of its shorter proofs and [17,18] for some other generalizations). Replacing identity of words by the weaker notion of "equivalence" between words u and $\theta(u)$, for a given antimorphic involution θ, we define an extended Lyndon and Schützenberger equation as

$$u_1 \cdots u_\ell = v_1 \cdots v_n w_1 \cdots w_m, \tag{1}$$

where $u_1, \ldots, u_\ell \in \{u, \theta(u)\}$, $v_1, \ldots, v_n \in \{v, \theta(v)\}$, and $w_1, \ldots, w_m \in \{w, \theta(w)\}$ with $\ell, n, m \geqslant 2$. For this extended Lyndon and Schützenberger equation we ask the following question: "What conditions on ℓ, n, m imply that all three words u, v, w can be written as a combination of a word and its image under θ, i.e., $u, v, w \in \{t, \theta(t)\}^+$ for some word t?"

This paper gives a partial answer to the question that whenever $\ell \geqslant 5$, $n, m \geqslant 3$, Eq. (1) implies $u, v, w \in \{t, \theta(t)\}^+$ for some word t (Theorem 27), and that once either n or m becomes 2, we can construct u, v, w which satisfy Eq. (1), but such a word t does not exist (Examples 1 and 2). Therefore, for any $\ell \geqslant 5$, $n = m = 3$ is an optimal bound. In the case when $\ell = 3$ or $\ell = 4$, the problem of finding optimal bounds remains open, though Examples 1 and 2 work even in these cases. Our proofs are not generalizations of the methods used in the classical case, since one of the main properties used therein, i.e., the fact that the conjugate of a primitive word is still primitive, does not hold for θ-primitiveness any more.

Prior to the proof of the positive result, we characterize all non-trivial overlaps between two expressions $\alpha(v, \theta(v))$, $\beta(v, \theta(v)) \in \{v, \theta(v)\}^+$ for a θ-primitive word v. Formally speaking, we show that the equality $\alpha(v, \theta(v)) \cdot x = y \cdot \beta(v, \theta(v))$ with x and y shorter than v is possible, and we provide all possible representations of the involved words v, x, y (Theorem 14). Note that this result is in contrast to the classical case (where the two expressions involve only a word v, but not its image under θ).

The paper is organized as follows. In Section 2, we fix our terminology and recall some known results. In Section 3, we provide the characterization of all possible overlaps of the form $\alpha(v, \theta(v)) \cdot x = y \cdot \beta(v, \theta(v))$ with $\alpha(v, \theta(v)), \beta(v, \theta(v)) \in \{v, \theta(v)\}^+$ and x, y shorter than v. Finally, in Section 4 we provide our extension of Lyndon and Schützenberger's result.

2. Preliminaries

Here we introduce notions and notation used in the following sections. For details of each, readers are referred to [13,14].

Let Σ be a finite alphabet. We denote by Σ^* the set of all finite words over Σ, by λ the empty word, and by Σ^+ the set of all nonempty finite words. The catenation of two words $u, v \in \Sigma^*$ is denoted by either uv or $u \cdot v$. The *length* of a word $w \in \Sigma^*$ is denoted by $|w|$. We say that u is a *factor* (a *prefix*, a *suffix*) of v if $v = t_1 u t_2$ (resp. $v = ut_2$, $v = t_1 u$) for some $t_1, t_2 \in \Sigma^*$. We denote by $\mathrm{Pref}(v)$ (resp. $\mathrm{Suff}(v)$) the set of all prefixes (resp. suffixes) of the word v. We say that two words u and v overlap if $ux = yv$ for some $x, y \in \Sigma^*$ with $|x| < |v|$. An integer $p \geqslant 1$ is a *period* of a word $w = a_1 \ldots a_n$, with $a_i \in \Sigma$ for all $1 \leqslant i \leqslant n$, if $a_i = a_{i+p}$ for all $1 \leqslant i \leqslant n - p$.

A word $w \in \Sigma^+$ is called *primitive* if it cannot be written as a power of another word; that is, if $w = u^n$, then $n = 1$ and $w = u$. For a word $w \in \Sigma^+$, the shortest $u \in \Sigma^+$ such that $w = u^n$ for some $n \geqslant 1$ is called the *primitive root* of the word

w and is denoted by $\rho(w)$. It is well-known that two words u, v commute, i.e., $uv = vu$ if and only if u, v have the same primitive root. This is rephrased as the following proposition.

Proposition 1. *Let $u \in \Sigma^+$ be a primitive word. If $u^2 = xuy$, then either $x = \lambda$ or $y = \lambda$.*

A mapping $\theta : \Sigma^* \to \Sigma^*$ is called an *antimorphism* if for any words $u, v \in \Sigma^*$, $\theta(uv) = \theta(v)\theta(u)$. A mapping $\theta : \Sigma^* \to \Sigma^*$ is called an *involution* if θ^2 is the identity. As mentioned in Section 1, an antimorphic involution is a mathematical formalization and extension of the Watson–Crick complementarity. Throughout this paper we will assume that θ is an antimorphic involution on a given alphabet Σ. A word $w \in \Sigma^*$ is called a *θ-palindrome* if $w = \theta(w)$ (see [8,19]). A word is called a *pseudo-palindrome* if it is a θ-palindrome for some antimorphic involution θ.

The notions of periodic and primitive words were extended in [7] in the following way. A word $w \in \Sigma^+$ is θ-periodic if $w = w_1 \dots w_k$ for some $k \geqslant 2$ and words $t, w_1, \dots, w_k \in \Sigma^+$ such that $w_i \in \{t, \theta(t)\}$ for all $1 \leqslant i \leqslant k$. Following [8], in less precise terms, a word which is θ-periodic with respect to some antimorphic involution θ is also called *pseudoperiodic*. The word t in the definition of a θ-periodic word w is called a θ-period of w. We call a word $w \in \Sigma^+$ *θ-primitive* if it is not θ-periodic. The set of θ-primitive words is strictly included in the set of primitive ones, see [7]; for instance, if we take $a \neq b$ and $\theta(a) = b, \theta(b) = a$, then the word ab is primitive, but not θ-primitive. We define *the θ-primitive root of w*, denoted by $\rho_\theta(w)$, as the shortest word t such that $w \in t\{t, \theta(t)\}^*$. Note that if w is θ-primitive, then $\rho_\theta(w) = w$.

The Fine and Wilf theorem, originally formulated for sequences of real numbers in [20], illustrates another fundamental periodicity property in its form for words [13,14]. It states that for two words $u, v \in \Sigma^*$, if a power of u and a power of v have a common prefix of length at least $|u| + |v| - \gcd(|u|, |v|)$, then u and v are powers of a common word, where $\gcd(n_1, n_2)$ denotes the greatest common divisor of two integers n_1, n_2. Moreover, the bound $|u| + |v| - \gcd(|u|, |v|)$ is optimal.

This theorem was extended in [7] for the case when instead of powers of two words u and v, we look at expressions over $\{u, \theta(u)\}$ and $\{v, \theta(v)\}$, respectively. The extended theorem consists of the following two variants.

Theorem 2 ([7]). *Let $u, v \in \Sigma^+$ be two distinct words with $|u| > |v|$. If there exist two expressions $\alpha(u, \theta(u)) \in u\{u, \theta(u)\}^*$ and $\beta(v, \theta(v)) \in v\{v, \theta(v)\}^*$ having a common prefix of length at least $2|u| + |v| - \gcd(|u|, |v|)$, then $\rho_\theta(u) = \rho_\theta(v)$. Moreover, the bound $2|u| + |v| - \gcd(|u|, |v|)$ is optimal.*

Theorem 3 ([7]). *For $u, v \in \Sigma^+$, if a word is in both $u\{u, \theta(u)\}^*$ and $v\{v, \theta(v)\}^*$, then $\rho_\theta(u) = \rho_\theta(v)$.*

In the following, we present several results on word equations which involve the antimorphic involution θ.

Lemma 4 ([7]). *For $u, v \in \Sigma^*$, if $uv = \theta(uv)$ and $vu = \theta(vu)$, then $u, v \in \{t, \theta(t)\}^*$ for some word $t \in \Sigma^+$.*

When considering word equations that involve the antimorphic involution like those in the previous lemmas, one often encounters the θ-commutativity of words. For two words u, v, u *θ-commutes* with v if $uv = \theta(v)u$ [9]. This is a special case of the *conjugacy* of words $xz = zy$. The solution to this conjugacy equation is well-known: $x = (pq)^j$, $y = (qp)^j$, and $z = (pq)^ip$ for some $i \geqslant 0, j \geqslant 1$ and $p, q \in \Sigma^*$ such that pq is primitive (we can assume $j = 1$ if we give up the primitivity of pq). Thus, the solution to the θ-commutativity of words is characterized as follows:

Proposition 5 ([9]). *For words $u, v \in \Sigma^+$ and an antimorphic involution θ, if $uv = \theta(v)u$ holds, then $u = (rt)^ir$ and $v = (tr)^j$ for some $i \geqslant 0, j \geqslant 1$, and θ-palindromes $r, t \in \Sigma^*$ such that rt is primitive.*

Using the characterizations of commutativity and θ-commutativity, we prove several results of use.

Lemma 6. *Let $x, y, z \in \Sigma^+$ with $x = \theta(x)$ and $y = \theta(y)$. If $xz = zy$ holds, then $x, y, z \in \{t, \theta(t)\}^*$ for some $t \in \Sigma^+$.*

Proof. As mentioned above, the conjugacy implies that $x = pq$, $y = qp$, and $z = (pq)^jp$ for some $p, q \in \Sigma^*$ and $j \geqslant 0$. Since $x = \theta(x)$ and $y = \theta(y)$, we have $pq = \theta(pq)$ and $qp = \theta(qp)$. Then, Lemma 4 implies that there exists a word $t \in \Sigma^+$ such that $p, q \in \{t, \theta(t)\}^*$. □

Lemma 7. *Let $x, y, z \in \Sigma^+$ with $x = \theta(x)$ and $z = \theta(z)$, and let v be a primitive word. If $v = xz = zy$ holds, then $x = r(tr)^{i+j}$, $y = (tr)^jr(tr)^i$, and $z = r(tr)^i$ for some $i \geqslant 0, j \geqslant 1$, and two non-empty θ-palindromes r, t such that rt is primitive.*

Proof. Let $v = xz = zy$. This conjugacy equation can be solved as $x = pq$, $y = qp$, and $z = (pq)^kp$ for some words p, q and $k \geqslant 0$. We claim that neither p nor q is empty. Indeed, if $q = \lambda$, then $v = p^{k+2}$, but this contradicts the primitivity of v. If $p = \lambda$, then we reach the same contradiction when $k \geqslant 1$; and if $k = 0$, then z would be empty, which is against the non-emptiness assumption on z. Thus, $p \neq \lambda$ and $q \neq \lambda$.

The assumption $z = \theta(z)$ implies $p = \theta(p)$, which derives $pq = \theta(q)p$ from $x = \theta(x)$. Moreover, k must be 0 because otherwise $z = \theta(z)$ implies $q = \theta(q)$, and this turns the above θ-commutativity equation into the commutativity equation $pq = qp$. Since both p and q are non-empty, this equation implies that p and q are powers of the same word, and hence,

v would not be primitive. As seen in Proposition 5, the solution to the θ-commutativity $pq = \theta(q)p$ is characterized as $p = r(tr)^i$ and $q = (tr)^j$ for some $i \geqslant 0, j \geqslant 1$, and θ-palindromes r, t such that rt is primitive. Thus, $x = r(tr)^{i+j}$, $y = (tr)^j r(tr)^i$, and $z = p = r(tr)^i$. $\square$

Lemma 8. *Let $y \in \Sigma^+$ be a nonempty θ-palindrome and $z \in \Sigma^+$ be a nonempty word with $|y| \leqslant |z|$. If $z\theta(z)$ is a prefix of $y\theta(z)yz$, then $\rho(y) = \rho(z)$.*

Proof. The conclusion is trivial if $|y| = |z|$ so that let us assume that $|y| < |z|$. Due to $z\theta(z) \in \mathrm{Pref}(y\theta(z)yz)$, it is clear that $y \in \mathrm{Pref}(z)$, that is, $y \in \mathrm{Suff}(\theta(z))$. Combining this with the prefix relation gives the θ-commutativity

$$zy = y\theta(z). \tag{2}$$

Using this, we can rewrite the prefix relation as $z\theta(z) \in \mathrm{Pref}(zyyz)$, that is, $\theta(z) \in \mathrm{Pref}(yyz)$. Due to this and $|y| < |z|$, we can let $\theta(z) = yz_1$ for some $z_1 \in \mathrm{Pref}(yz)$.

If $\frac{1}{2}|z| \leqslant |y| < |z|$, then $|z_1| \leqslant \frac{1}{2}|z|$, and hence, $z_1 \in \mathrm{Pref}(yz)$ means $z_1 \in \mathrm{Pref}(y)$. This actually means that $z_1 \in \mathrm{Pref}(z)$ because $y \in \mathrm{Pref}(z)$ and the prefix relation is transitive. Thus, $\theta(z_1) \in \mathrm{Suff}(\theta(z))$. Combining this with $\theta(z) = yz_1$, we have $z_1 = \theta(z_1)$. Now substituting $\theta(z) = yz_1$ into Eq. (2) gives $z_1y^2 = y^2z_1$, and hence, $\rho(z_1) = \rho(y)$, which means $\rho(y) = \rho(z)$.

If $|y| < \frac{1}{2}|z|$, then $z_1 \in \mathrm{Pref}(yz)$ means $y \in \mathrm{Pref}(z_1)$ so that let $z_1 = yz_2$ for some z_2. We can easily see that $z_2 \in \mathrm{Pref}(z)$. In addition, $z_2 \in \mathrm{Suff}(z_1) \subseteq \mathrm{Suff}(\theta(z))$ holds, and hence, $\theta(z_2) \in \mathrm{Pref}(z)$. As a result, $z_2 = \theta(z_2)$. Hence, $\theta(z) = y^2z_2$ and this derives $z_2y^3 = y^3z_2$ from Eq. (2). The commutativity implies $\rho(z_2) = \rho(y)$, which means $\rho(y) = \rho(z)$. $\square$

3. Overlaps between θ-primitive words

As mentioned in Proposition 1, a primitive word v cannot occur nontrivially inside v^2. Thus, two expressions v^i and v^j, with $i, j \geqslant 1$, cannot overlap nontrivially on a sequence longer than $|v|$. A natural question is whether we can have some nontrivial overlaps between two expressions $\alpha(v, \theta(v)), \beta(v, \theta(v)) \in \{v, \theta(v)\}^+$ when $v \in \Sigma^+$ is a θ-primitive word. In this section, we completely characterize all such nontrivial overlaps, and, moreover, in each case we also give the set of all solutions of the corresponding equation.

We begin our analysis by giving two lemmas and a proposition of use.

Lemma 9 ([7]). *Let $v \in \Sigma^+$ be a θ-primitive word. Then, $\theta(v)vx = yv\theta(v)$ for some words $x, y \in \Sigma^*$ with $|x|, |y| < |v|$, if and only if $v = \theta(v)$ and $x = y = \lambda$. Similarly, $v\theta(v)v = xv^2y$ for some $x, y \in \Sigma^*$ if and only if $v = \theta(v)$ and either $x = \lambda$ or $y = \lambda$.*

Lemma 10. *Let v be a θ-primitive word and let $v_1, v_2 \in \{v, \theta(v)\}$. For nonempty words x, y with $|x| = |y| < |v|$, the equation $v\theta(v)x = yv_1v_2$ implies $v_1 = v$.*

Proof. If $v_1 = \theta(v)$, then, in light of Proposition 1 and Lemma 9, v_2 can be neither v nor $\theta(v)$. $\square$

Proposition 11. *Let $v \in \Sigma^+$ be a θ-primitive word. Neither $v\theta(v)$ nor $\theta(v)v$ can be a proper factor of any word in $\{v, \theta(v)\}^3$.*

Proof. Let $v_1, v_2, v_3 \in \{v, \theta(v)\}$, and let $v_1v_2v_3 = xv\theta(v)y$ for some $x, y \in \Sigma^+$. If we apply Lemma 10 to the overlap between v_1v_2 and $v\theta(v)$, then we obtain that $v_2 = \theta(v)$. If we do so to the overlap between v_2v_3 and $v\theta(v)$, then $v_2 = v$. Thus, $v = \theta(v)$ must hold, but then v^2 would be a proper factor of v^3. However, this contradicts the assumption that v is primitive and thus also θ-primitive. $\square$

Next, we provide two intermediate results on the nontrivial overlaps between $\alpha(v, \theta(v))$ and $\beta(v, \theta(v))$ of the form $\alpha(v, \theta(v)) \cdot x = y \cdot \beta(v, \theta(v))$.

Theorem 12. *Let $v \in \Sigma^+$ be a θ-primitive word, $m \geqslant 1$, and $v_1, v_2, \ldots, v_m \in \{v, \theta(v)\}$. For $\beta(v, \theta(v)) \in \{v, \theta(v)\}^m$, if $v_1v_2\cdots v_mx = y\beta(v, \theta(v))$ for some x, y with $0 < |x| = |y| < |v|$, then there do not exist two indices $1 \leqslant i, j < m$ such that $v_i = v_{i+1} = v$ and $v_j = v_{j+1} = \theta(v)$.*

Proof. Suppose that such indices i, j existed. Note that m has to be at least 2 if $v = \theta(v)$, or has to be at least 4 if $v \neq \theta(v)$.

If $v = \theta(v)$, then we have $v^mx = yv^m$ with $m \geqslant 2$, but this obviously contradicts Proposition 1 due to $0 < |x| = |y| < |v|$ and the primitivity of v.

Let us consider the other case $v \neq \theta(v)$. As mentioned above, $m \geqslant 4$ in this case. Since θ is an involution, we can assume that $i \leqslant j$. This implies the existence of an index $2 \leqslant k \leqslant m-2$ such that $v_kv_{k+1} = v\theta(v)$. However, if so, then v could not be θ-primitive in light of Proposition 11 because v_kv_{k+1} is a proper factor of β. $\square$

Table 1
Characterization of possible proper overlaps of the form $\alpha(v, \theta(v)) \cdot x = y \cdot \beta(v, \theta(v))$. For the second and third equations, p and q are nonempty words. For the last three equations, $i \geqslant 0, j \geqslant 1, r, t \in \Sigma^+$ such that $r = \theta(r)$, $t = \theta(t)$, and rt is primitive. Note that the 4th and 5th equations are the same up to the antimorphic involution θ.

Equation	Solution
$v^k x = y\theta(v)^k, k \geqslant 1$	$v = yp, x = \theta(y), p = \theta(p)$, and whenever $k \geqslant 2, y = \theta(y)$
$vx = yv$	$v = (pq)^{i+1}p, x = qp, y = pq$, with $i \geqslant 0$
$v\theta(v)x = yv\theta(v)$,	$v = (pq)^{i+1}p, x = \theta(pq), y = pq$, with $i \geqslant 0, qp = \theta(qp)$
$v^{k+1}x = y\theta(v)^k v, k \geqslant 1$	$v = r(tr)^{i+j}r(tr)^i, x = (tr)^j r(tr)^i, y = r(tr)^{i+j}$
$v\theta(v)^k x = yv^{k+1}, k \geqslant 1$	$v = (rt)^i r(rt)^{j+i} r, y = (rt)^i r(rt)^j, x = (rt)^{j+i} r$
$v\theta(v)^k x = yv^k\theta(v), k \geqslant 2$	$v = (rt)^i r(rt)^{j+i} r, y = (rt)^i r(rt)^j, x = (tr)^j r(tr)^i$

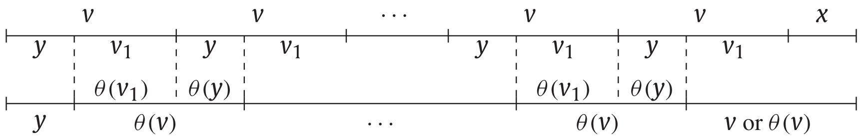

Fig. 1. The case when $\alpha(v, \theta(v)) = v^k$.

Theorem 13. *For a θ-primitive word $v \in \Sigma^+$, let $\alpha(v, \theta(v)), \beta(v, \theta(v)) \in \{v, \theta(v)\}^+$ such that $\alpha(v, \theta(v)) \cdot x = y \cdot \beta(v, \theta(v))$ for some $x, y \in \Sigma^+$ with $|x|, |y| < |v|$. Then, for any $i \geqslant 1$, neither $v\theta(v)^i v$ nor $\theta(v)v^i\theta(v)$ can occur either in $\alpha(v, \theta(v))$ or in $\beta(v, \theta(v))$.*

Proof. Suppose that $v\theta(v)^i v$ occurs in $\alpha(v, \theta(v))$ for some $i \geqslant 1$. We assumed that $x, y \in \Sigma^+$ and $|x|, |y| < |v|$ so that the factor $v\theta(v)^i v$ contains as a proper factor $\gamma(v, \theta(v)) \in \{v, \theta(v)\}^{i+1}$, i.e., there exist some $p, q \in \Sigma^+$ such that $v\theta(v)^i v = p\gamma(v, \theta(v))q$. Due to Lemma 9 and $\theta(v)$ being primitive, $\gamma(v, \theta(v)) = v^{i+1}$. Now we have $v\theta(v)^i v = pv^{i+1}q$ and hence $v\theta(v)v = pv^2q$. However, this contradicts Lemma 9. The other cases can be proved similarly. $\square$

As an immediate consequence of the previous two theorems, for a given θ-primitive word v, if $\alpha(v, \theta(v)) \cdot x = y \cdot \beta(v, \theta(v))$ with $x, y \in \Sigma^+$, $|x|, |y| < |v|$, then $\alpha(v, \theta(v))$ and $\beta(v, \theta(v))$ can be only of the following types v^k, $v^k\theta(v)$, $v\theta(v)^k$, $\theta(v)^k$, $\theta(v)^k v$, or $\theta(v)v^k$ for some $k \geqslant 1$. The next result refines this characterization further.

Theorem 14. *Let $v \in \Sigma^+$ be a θ-primitive word. Then, the only possible proper overlaps of the form $\alpha(v, \theta(v)) \cdot x = y \cdot \beta(v, \theta(v))$ with $\alpha(v, \theta(v)), \beta(v, \theta(v)) \in \{v, \theta(v)\}^+$, $x, y \in \Sigma^+$ and $|x|, |y| < |v|$ are given in Table 1 (modulo a substitution of v by $\theta(v)$) together with the characterization of their sets of solutions.*

Proof. Since θ is an involution, we can assume without loss of generality that α starts with v. Then the equation $\alpha x = y\beta$ enables us to let $v = yv_1$ for some non-empty word $v_1 \in \Sigma^+$. Due to the previous observation we know that $\alpha \in \{v^k, v^k\theta(v), v\theta(v)^k \mid k \geqslant 1\}$.

Case 1. First we consider the case when $\alpha = v^k$ for some $k \geqslant 1$. Since v is θ-primitive, $v^k x = y\beta$, and $|y|, |x| < |v|$, the border between any two consecutive v's falls inside a $\theta(v)$, see Fig. 1; otherwise v would occur inside v^2 which would contradict its primitivity. Thus, $\beta \in \{\theta(v)^k, \theta(v)^{k-1}v\}$.

The first subcase we investigate is when $\beta = \theta(v)^k$. Then, we immediately obtain $v_1 = \theta(v_1)$ and in addition, if $k \geqslant 2$, then $y = \theta(y)$. Moreover, if we look at the end of the two sides of the equation $v^k x = y\theta(v)^k$, we also obtain that $x = \theta(y)$. Thus, a proper overlap of the form $v^k x = y\theta(v)^k$ with v being θ-primitive is possible, and, moreover, the set of all solutions of this equation is characterized by the following formulas: $v = yv_1$ and $x = \theta(y)$, where $v_1 = \theta(v_1)$ and $y = \theta(y)$ whenever $k \geqslant 2$.

The second subcase is when $\beta = \theta(v)^{k-1}v$. If $k = 1$, then we have $vx = yv$, and this conjugacy equation is solved as $v = (pq)^{i+1}p$, $x = qp$, and $y = pq$ for some words p, q and an integer $i \geqslant 0$. Note that both p and q must be non-empty because $q = \lambda$ implies $v = p^{i+2}$, which contradicts the primitivity of v; the emptiness of p with $i \geqslant 1$ lead us to the same contradiction. If $p = \lambda$ and $i = 0$, then $v = x = q$ and contradicts the length condition $|x| < |v|$. When $k \geqslant 2$, substituting $v = yv_1$ into $v^k x = y\theta(v)^{k-1}v$ results in $yv_1y(v_1y)^{k-2}v_1x = y(\theta(v_1)\theta(y))\theta(v)^{k-2}v$, which implies that $v_1 = \theta(v_1)$, $y = \theta(y)$, and $v = v_1x$. Now we have $v = yv_1 = v_1x$, and to this equation we can apply Lemma 7 to obtain the characterization $y = r(tr)^{i+j}$, $v_1 = r(tr)^i$, and $x = (tr)^j r(tr)^i$ for some $i \geqslant 0, j \geqslant 1$, and two non-empty θ-palindromes r, t such that rt is primitive. Based on this characterization, $v = r(tr)^{i+j}r(tr)^i$.

Case 2. Suppose now that $\alpha = v^k\theta(v)$ for some $k \geqslant 1$. If $k \geqslant 2$, then β has to start with $\theta(v)^{k-1}$ because otherwise it would contradict the primitivity of v. This $\theta(v)^{k-1}$ has to be followed by v in light of Lemma 10. However, then, $v\theta(v)$ would overlap with $\theta(v)v$ with the overlap properly longer than v. This contradicts Lemma 9.

Thus, k has to be 1. Lemma 10 implies that β is either v^2 or $v\theta(v)$. Firstly, we consider the case when $\beta = v^2$, that is, we have $v\theta(v)x = yv^2$ (see Fig. 2 left). Note that for any $x, y \in \Sigma^+$ with $|x|, |y| < |v|$, $v\theta(v)x = yv^2$ holds if and only

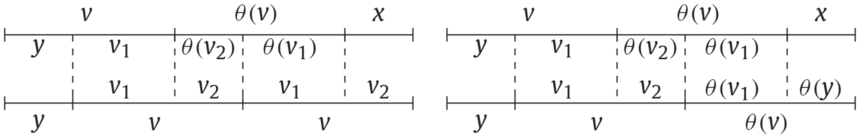

Fig. 2. The equations: $v\theta(v)x = yv^2$ and $v\theta(v)x = yv\theta(v)$.

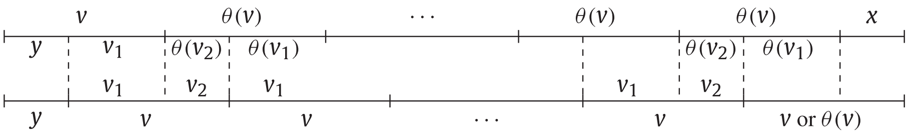

Fig. 3. The case when $\alpha(v, \theta(v)) = v\theta(v)^k$ and $\beta(v, \theta(v))$ starts with v.

if $v\theta(v)^k x = yv^{k+1}$ holds for any $k \geqslant 1$. Furthermore, the latter equation is the same as the equation $v^{k+1}x' = y'\theta(v)^k v$, which was considered in Case 1, up to the antimorphic involution θ. Using the result obtained in Case 1, we have that the set of all solutions of $v\theta(v)^k x = yv^{k+1}$ is characterized by the formulae $v = (rt)^i r(rt)^{j+i} r$, $x = (rt)^{j+i} r$, and $y = (rt)^i r(rt)^j$.

The remaining case for $\alpha = v\theta(v)$ is when $\beta = v\theta(v)$, see Fig. 2 right. Then, we can write $v = yv_1 = v_1v_2$ and we obtain immediately $x = \theta(y)$ and $v_2 = \theta(v_2)$. Thus, a proper overlap of the form $v\theta(v)x = yv\theta(v)$, with v being θ-primitive, is possible. Furthermore, the set of all solutions of this equation is characterized by the following formulas: $v = (pq)^{i+1}p$, $y = pq$, $x = \theta(pq)$ for some $i \geqslant 0$ and $p, q \in \Sigma^*$ such that $qp = \theta(qp)$. We can easily check that p, q have to be non-empty as done previously.

Case 3. Finally we consider the case when $\alpha = v\theta(v)^k$ for some $k \geqslant 2$; the case when $k = 1$ was already considered in Case 2. Since $\theta(v)$ is primitive, the border between any two $\theta(v)$'s falls inside v. According to Lemma 10, β has to begin with v, or more strongly, $\beta \in v^k\{v, \theta(v)\}$, see Fig. 3. Since the equation $v\theta(v)^k x = yv^{k+1}$ has been already characterized in Case 2, it suffices to consider the case when $\beta = v^k\theta(v)$, that is, we have the equation $v\theta(v)^k x = yv^k\theta(v)$. This equation implies that we can let $v = v_1v_2$ for some non-empty word $v_2 \in \Sigma^+$ as illustrated in Fig. 3. By substituting $v = yv_1 = v_1v_2$ into this equation, we can easily obtain $v_1 = \theta(v_1)$, $v_2 = \theta(v_2)$, and $x = \theta(y)$. Now Lemma 7 is applicable to $\theta(v) = v_2v_1 = v_1\theta(y)$ and result in the characterization that $x = \theta(y) = (tr)^j(rt)^i r$, $v_1 = (rt)^i r$, and $v_2 = (rt)^{i+j} r$ for some $i \geqslant 0, j \geqslant 1$, and two nonempty θ-palindromes r, t such that rt is primitive. $\square$

4. An extension of Lyndon and Schützenberger's result

As an application of the obtained characterization of non-trivial overlaps, now we consider the extended Lyndon–Schützenberger equation. Let us recall first the original result by Lyndon and Schützenberger [15].

Theorem 15. *If words u, v, w satisfy the relation $u^\ell = v^n w^m$ for some positive integers $\ell, n, m \geqslant 2$, then they are all powers of a common word, i.e., there exists a word t such that $u, v, w \in \{t\}^*$.*

Let us extend the equation as follows: for $u, v, w \in \Sigma^+$ and $\ell, n, m \geqslant 2$,

$$u_1 \cdots u_\ell = v_1 \cdots v_n w_1 \cdots w_m, \tag{3}$$

where $u_1, \ldots, u_\ell \in \{u, \theta(u)\}$, $v_1, \ldots, v_n \in \{v, \theta(v)\}$, and $w_1, \ldots, w_m \in \{w, \theta(w)\}$. We call Eq. (3) the *extended Lyndon–Schützenberger equation* (abbreviated as exLS equation).

In light of Theorem 15, we ask the question of under what conditions on ℓ, n, m, the exLS equation implies that $u, v, w \in \{t, \theta(t)\}^+$ for some word $t \in \Sigma^+$. If such t exists, we say that the triple (ℓ, n, m) *imposes θ-periodicity on* u, v, w, (or shortly, *imposes θ-periodicity*). Furthermore, we say that the triple (ℓ, n, m) *imposes θ-periodicity* if it imposes θ-periodicity on all u, v, w. Note that, if (ℓ, n, m) imposes θ-periodicity, then so does (ℓ, m, n), and vice versa. Indeed, assume that $u_1 \cdots u_\ell = v_1 \cdots v_m w_1 \cdots w_n$ holds. Applying θ to both of its sides, we obtain $\theta(u_\ell) \cdots \theta(u_1) = \theta(w_n) \cdots \theta(w_1)\theta(v_m) \cdots \theta(v_1)$, and if (ℓ, n, m) is assumed to impose θ-periodicity, then this equation gives $u, v, w \in \{t, \theta(t)\}^+$ for some $t \in \Sigma^+$. Note also that the fact that a certain triple (ℓ, n, m) imposes θ-periodicity does not imply that (ℓ', n', m') imposes θ-periodicity for $\ell' > \ell$ or $n' > n$ or $m' > m$.

The results of this section are summarized in Table 2. Overall, combining all the results from this section we obtain that $\ell \geqslant 5, n \geqslant 3, m \geqslant 3$ imposes θ-periodicity on u, v, and w (Theorem 27). In contrast, for $\ell \geqslant 3$, once either $n = 2$ or $m = 2$, (ℓ, n, m) does not always impose θ-periodicity, see Examples 1 and 2. Therefore, when $\ell \geqslant 5$, $(\ell, 3, 3)$ is the optimal bound. In the case when $\ell = 2$, $\ell = 3$, or $\ell = 4$, the problem of finding optimal bounds is still open.

Example 1. Let $\Sigma = \{a, b\}$ and $\theta : \Sigma^* \to \Sigma^*$ be the mirror image defined as $\theta(a) = a$, $\theta(b) = b$, and $\theta(w_1 \ldots w_n) = w_n \ldots w_1$, where $w_i \in \{a, b\}$ for all $1 \leqslant i \leqslant n$. Take now $u = a^k b^2 a^{2k}$, $v = \theta(u)^l a^{2k} b^2 = (a^{2k} b^2 a^k)^l a^{2k} b^2$, and $w = a^2$, for

Table 2
Result summary for the extended Lyndon–Schützenberger equation. As mentioned in the main text, if (ℓ, n, m) imposes θ-periodicity, then so does (ℓ, m, n). Hence, Theorems 25 and 26 prove that $(5, \geqslant 4, 4)$ and $(5, \geqslant 3, 3)$ also impose θ-periodicity, respectively.

ℓ	n	m	θ-periodicity	
$\geqslant 6$	$\geqslant 3$	$\geqslant 3$	Yes	Theorem 16
5	$\geqslant 5$	$\geqslant 5$	Yes	Theorem 17
5	4	$\geqslant 4$	Yes	Theorem 25
5	3	$\geqslant 3$	Yes	Theorem 26
$\geqslant 3$	2	$\geqslant 1$	No	Examples 1 and 2

some $k, l \geqslant 1$. Then, although $\theta(u)^{l+1}u^{l+1} = v^2w^k$, there is no word $t \in \Sigma^+$ with $u, v, w \in \{t, \theta(t)\}^+$, i.e., for any $k, l \geqslant 1$, the triple of numerical parameters $(2l + 2, 2, k)$ is not enough to impose θ-periodicity.

Example 2. Consider again $\Sigma = \{a, b\}$ and $\theta : \Sigma^* \to \Sigma^*$ be the mirror image defined in the previous example and take $u = b^2(aba)^k$, $v = u^lb = (b^2(aba)^k)^lb$, and $w = aba$ for some $k, l \geqslant 1$. Then, although $u^{2l+1} = v\theta(v)w^k$, there is no word $t \in \Sigma^+$ with $u, v, w \in \{t, \theta(t)\}^+$, i.e., for any $k, l \geqslant 1$, $(2l + 1, 2, k)$ is not enough to impose θ-periodicity.

In the rest of this section, we handle the cases when (ℓ, n, m) imposes θ-periodicity. Among them, we firstly consider some cases where enough amount of repetition is available for us to apply the extended Fine and Wilf's theorem (Theorem 2). The next two results analyze the cases when we have triples (ℓ, n, m) with $\ell \geqslant 6$ and $n, m \geqslant 3$ and, respectively $(5, n, m)$ with $n, m \geqslant 5$.

Theorem 16. *Let $u, v, w \in \Sigma^+$, $n, m \geqslant 3$, $\ell \geqslant 6$, $u_i \in \{u, \theta(u)\}$ for $1 \leqslant i \leqslant \ell$, $v_j \in \{v, \theta(v)\}$ for $1 \leqslant j \leqslant n$, and $w_k \in \{w, \theta(w)\}$ for $1 \leqslant k \leqslant m$. If $u_1 \dots u_\ell = v_1 \dots v_n\, w_1 \dots w_m$, then there exists a word $t \in \Sigma^+$ such that $u, v, w \in \{t, \theta(t)\}^+$.*

Proof. Let us suppose that $|v_1 \dots v_n| \geqslant |w_1 \dots w_m|$; the other case is symmetric and can be solved similarly. Then, $|v_1 \dots v_n| \geqslant \frac{1}{2}|u_1 \dots u_l| \geqslant 3|u|$, since $\ell \geqslant 6$. Since $n \geqslant 3$, this means that $u_1 \dots u_\ell$ and $v_1 \dots v_n$ share a common prefix of length larger than both $3|u|$ and $3|v|$. Thus, we can apply Theorem 2 to obtain that $u, v \in \{t, \theta(t)\}^+$ for some θ-primitive word $t \in \Sigma^+$. Moreover, since $u_1 \dots u_\ell = v_1 \dots v_n\, w_1 \dots w_m$, this implies $w_1 \dots w_m \in \{t, \theta(t)\}^*$. Since t is θ-primitive, Theorem 3 implies that also $w \in \{t, \theta(t)\}^+$. $\square$

This proof clarifies one important point: in order to prove that (ℓ, n, m) imposes θ-periodicity, it suffices to prove that two of u, v, w are in $\{t, \theta(t)\}^+$ for some t.

Theorem 17. *Let $u, v, w \in \Sigma^+$, $n, m \geqslant 5$, $u_i \in \{u, \theta(u)\}$ for $1 \leqslant i \leqslant 5$, $v_j \in \{v, \theta(v)\}$ for $1 \leqslant j \leqslant n$, and $w_k \in \{w, \theta(w)\}$ for $1 \leqslant k \leqslant m$. If $u_1u_2u_3u_4u_5 = v_1 \dots v_n\, w_1 \dots w_m$, then there exists a word $t \in \Sigma^+$ such that $u, v, w \in \{t, \theta(t)\}^+$.*

Proof. Since $u_1u_2u_3u_4u_5 = v_1 \dots v_n\, w_1 \dots w_m$ and $n, m \geqslant 5$, we immediately obtain that $|u| > |v|$ and $|u| > |w|$. Assume now that $n|v| \geqslant m|w|$; the other case is symmetric. Thus, $n|v| \geqslant 2|u| + \frac{1}{2}|u|$ and we take $n|v| = 2|u| + l$ for some $l \geqslant \frac{1}{2}|u|$.

We claim now that $l \geqslant |v|$. If $l \geqslant |u|$, then we are done since we already know that $|u| > |v|$. So, let $\frac{1}{2}|u| \leqslant l < |u|$. If $n \geqslant 6$, then $n|v| = 2|u| + l < 3|u|$ and thus $|v| < \frac{1}{2}|u| \leqslant l$. Thus, the only case remaining now is when $n = 5$. Then, $5|v| = 2|u| + l \geqslant 2|u| + \frac{1}{2}|u|$, which implies $|v| \geqslant \frac{1}{2}|u|$. But then we have that $4|v| \geqslant 2|u|$ while $5|v| = 2|u| + l$. Hence, also in this case we obtain $|v| \leqslant l$.

Thus, $u_1u_2u_3u_4u_5$ and $v_1 \dots v_n$ have a common prefix of length $n|v| = 2|u| + l \geqslant 2|u| + |v|$. This means, due to Theorem 2, that there exists a θ-primitive word $t \in \Sigma^+$ such that $u, v \in \{t, \theta(t)\}^+$. As mentioned previously, now we can also say that $w \in \{t, \theta(t)\}^+$. $\square$

The triple $(5, n, m)$ also turns out to impose θ-periodicity for any $n \geqslant 4$ and $m \geqslant 7$.

Theorem 18. *Let $u, v, w \in \Sigma^+$, $n \geqslant 4$, $m \geqslant 7$, $u_i \in \{u, \theta(u)\}$ for $1 \leqslant i \leqslant 5$, $v_j \in \{v, \theta(v)\}$ for $1 \leqslant j \leqslant n$, and $w_k \in \{w, \theta(w)\}$ for $1 \leqslant k \leqslant m$. If $u_1u_2u_3u_4u_5 = v_1 \dots v_n\, w_1 \dots w_m$, then there exists a word $t \in \Sigma^+$ such that $u, v, w \in \{t, \theta(t)\}^+$.*

Proof. Unless the border between v_n and w_1 falls inside u_3, Theorem 2 concludes the existence of such t. So, assume that the border falls inside u_3. Even under this assumption, if the border between u_2 and u_3 falls inside some v_i except v_n, then Theorem 2 leads us to the same conclusion. Otherwise, we have that $(n-1)|v| < 2|u|$, which means $|v| < \frac{2}{n-1}|u| \leqslant \frac{2}{3}|u|$. Similarly, if the border between u_3 and u_4 does not fall inside w_1, we reach the existence of such t; otherwise $|w| < \frac{2}{m-1}|u| \leqslant \frac{1}{3}|u|$. Under the condition that v_n and w_1 straddle these respective borders, the equation cannot hold because v and w are too short. $\square$

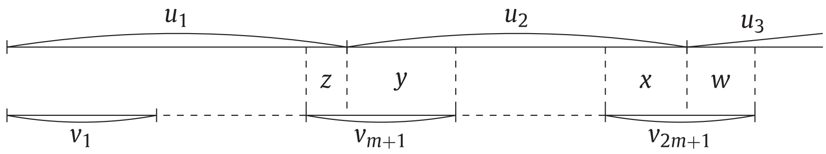

Fig. 4. The basic problem setting of Proposition 20.

We already know from Example 2 that for any $m \geqslant 1$, the triple $(5, 2, m)$ is not enough to impose θ-periodicity. So, we investigate next what would be the optimal bound for the extension of the Lyndon and Schützenberger result when the first parameter is 5. Note that, without loss of generality, we can assume $n \leqslant m$. Then, due to Theorem 17, all we have to investigate are the cases $(5, 3, m)$ for $m \geqslant 3$ and $(5, 4, m)$ for $m \geqslant 4$. The next intermediate lemma will be useful in the analysis of these cases.

Lemma 19. *Let $u \in \Sigma^+$ such that $u = xy$ and $y \in \mathrm{Pref}(u)$ for some θ-palindrome words $x, y \in \Sigma^+$. If $|y| \geqslant |x|$, then $\rho(x) = \rho(y) = \rho(u)$.*

Proof. We have $u = xy = yz$ for some $z \in \Sigma^+$ of the same length as x. The length condition implies that $x \in \mathrm{Pref}(y)$. Since $x = \theta(x)$ and $y = \theta(y)$, this means that $x \in \mathrm{Suff}(y)$ and hence $z = x$. So we have $u = xy = yx$, and hence x, y, and u share their primitive root. □

Unlike in the case of the original Lyndon–Schützenberger equation, the investigation of our extension entails the consideration of an enormous amount of cases since for each variable u_i, v_j, w_k we have two possible values. However, in almost all cases, it is enough to consider the common prefix between $u_1 \ldots u_\ell$ and $v_1 \ldots v_n$ or the common suffix between $u_1 \ldots u_\ell$ and $w_1 \ldots w_m$ to prove that either the equation imposes θ-periodicity or the equation cannot hold.

Note that for the $(5, 3, m)$ or $(5, 4, m)$ extensions of the Lyndon–Schützenberger equation, we only have to consider the case when the border between v_n and w_1 is inside u_3 because otherwise Theorem 2 immediately implies that $u, v, w \in \{t, \theta(t)\}^+$ for some word $t \in \Sigma^+$. In addition, even if the border is inside u_3, as long as $m|w| \geqslant 2|u| + |w|$, we reach the same conclusion. Moreover, we can assume that w is θ-primitive since otherwise we would just increase the value of the parameter m. These observations justify the assumptions which will be made in the following propositions.

Proposition 20. *Let $u, v \in \Sigma^+$ such that v is a θ-primitive word, $u_1, u_2, u_3 \in \{u, \theta(u)\}$, and $v_1, \ldots, v_{2m+1} \in \{v, \theta(v)\}$ for some $m \geqslant 1$. If $v_1 \ldots v_{2m+1}$ is a proper prefix of $u_1u_2u_3$ and $2m|v| < 2|u| < (2m+1)|v|$, then $u_2 \neq u_1$ and $v_1 = \cdots = v_{2m+1}$. Moreover, $v_1 = z\theta(z)p$ and $u_1u_2 = (z\theta(z)p)^{2m}z\theta(z)$ for some non-empty word z and non-empty θ-palindrome p.*

Proof. Since θ is an involution, we may assume without loss of generality that $u_1 = u$ and $v_1 = v$. Note that $|v| < |u|$ and, due to the length condition, the border between u_1 and u_2 falls inside v_{m+1} while the one between u_2 and u_3 falls inside v_{2m+1}. Hence, we let

$$u_1 = v_1 \cdots v_m z \tag{4}$$

$$u_2 = y v_{m+2} \cdots v_{2m} x \tag{5}$$

for some non-empty words $w, x, y, z \in \Sigma^+$ such that $v_{m+1} = zy$, $v_{2m+1} = xw$, and $w \in \mathrm{Pref}(u_3)$. As such,

$$u_1u_2 = v_1v_2 \cdots v_{2m}x. \tag{6}$$

These are illustrated in Fig. 4. Due to the assumption $2|u| < (2m+1)|v|$ and Eq. (4), we have $2|z| = 2|u| - 2m|v| = |x| < |v|$, i.e., $|y| = |v| - |z| > |z|$. Now, we have two cases depending on whether u_2 is equal to u_1.

Case 1. Let us consider the case when $u_2 \neq u_1$, i.e., $u_2 = \theta(u)$, first. In this case, u_1u_2 is the θ-palindrome $u\theta(u)$, and hence, Eq. (6) implies

$$u\theta(u) = v_1v_2 \cdots v_{2m-1}v_{2m}x = \theta(x)\theta(v_{2m}) \cdots \theta(v_2)\theta(v_1). \tag{7}$$

Applying Theorem 14 to this equation leaves two subcases to be considered: (a) $v_1 = \cdots = v_{2m} = v$, and (b) $v_1 = v$, $v_2 = \cdots = v_{2m} = \theta(v)$.

By catenating w to the right of both sides of Eq. (7), we can easily observe that $v_s v_{2m} v_{2m+1} = \theta(v_2)\theta(v_1)w$ for some $v_s \in \mathrm{Suff}(v_{2m-1})$. Substituting the values of $v_1, \ldots, v_{2m}$ being specified as (b) into this equation results in $v_s\theta(v)v_{2m+1} = v\theta(v)w$. Lemma 10, however, denies that this equation holds so that the subcase (b) is impossible. It is worth noting, in passing, that this impossibility of (b) does not rely on the parity of the index of v_k on which the border between u_2 and u_3 lies (in our current proof, k is assumed to be odd). This fact will free us from repeating the argument of this paragraph in the context of k being even, which will be the case in Proposition 21.

Let us consider (a) next. Applying Theorem 14 to Eq. (7) provides us with the characterization $x = \theta(x)$ and $v = xp$ for some θ-palindrome p. In turn, by replacing Eq. (6) with $v = xp$ and Eq. (4), we can obtain $u\theta(u) = (xp)^{2m}x = v^m z\theta(z)\theta(v)^m$, from which $x = z\theta(z)$. We claim that v_{2m+1} must be v. To verify this claim, suppose $v_{2m+1} = \theta(v)$, and we will prove that u_3 can be neither u nor $\theta(u)$. If $u_3 = u$, then $v_1 \cdots v_{2m}v_{2m+1} \in \mathrm{Pref}(u_1u_2u_3)$ implies $v_1 \cdots v_{2m}v_{2m+1} \in \mathrm{Pref}(\theta(x)\theta(v_{2m}) \cdots \theta(v_1)v)$ because of $v \in \mathrm{Pref}(u)$ and Eq. (7). This prefix relation means that $v_{2m}v_{2m+1} = v\theta(v)$ overlaps with $\theta(v_1)v_1 = \theta(v)v$ in a way contradicting Lemma 10. For the case $u_3 = \theta(u)$, note that $\theta(v) = px = pz\theta(z)$ and recall that $x = z\theta(z) \in \mathrm{Pref}(v_{2m+1})$. If $|z\theta(z)| \geqslant |p|$, then Lemma 19 implies that $\rho(z\theta(z)) = \rho(p)$, which contradicts the primitivity of v. To consider the case $|z\theta(z)| < |p|$, we need the fact that $u = v^m z$, i.e., $\theta(u) = \theta(z)(pz\theta(z))^m$. The fact that v_{2m+1} is a prefix of $xu_3 = z\theta(z)\theta(u)$ can be rewritten as $pz\theta(z) \in \mathrm{Pref}(z\theta(z)\theta(z)p)$. In fact, now we have $pz\theta(z)^2 = z\theta(z)^2p$ because this prefix condition and $|z\theta(z)| < |p|$ imply $\theta(z) \in \mathrm{Suff}(p)$. This commutativity brings $\rho(p) = \rho(z\theta(z)^2)$, and hence, $p, z \in \{t, \theta(t)\}^+$ for some word t. However, this contradicts the θ-primitivity of v.

In conclusion, if $u_1 \neq u_2$, then $v_1 = \cdots = v_{2m+1}$ must hold. Furthermore, $v_1 = v = z\theta(z)p$ and $u_1u_2 = (z\theta(z)p)^{2m}z\theta(z)$.

Case 2. Here we consider the other case $u_2 = u_1 = u$, and will see that we cannot avoid a contradiction, that is, the assumption $u_2 = u_1 = u$ is invalid. In this case, replacing the formulas given by Eqs. (4) and (5) within the equation $u_1 = u_2$ gives

$$v_1 \cdots v_m z = yv_{m+2} \cdots v_{2m}x, \tag{8}$$

and hence, we can let $v_1 = v = yz'$ for some $z' \in \mathrm{Pref}(v_{m+2})$. By catenating w to the right of both sides of Eq. (8), we obtain

$$v_1 \cdots v_m zw = yv_{m+2} \cdots v_{2m}v_{2m+1}. \tag{9}$$

Moreover, by catenating z to the left of both sides of this equation, we obtain

$$zv_1 \cdots v_m zw = v_{m+1}v_{m+2} \cdots v_{2m}v_{2m+1}. \tag{10}$$

This means that $v_1 \cdots v_m$ is a proper factor of $v_{m+1}v_{m+2} \cdots v_{2m}v_{2m+1}$. Thus, $v_1 = \cdots = v_m = v$ must hold due to Proposition 11, and hence, $v_{m+2} = \cdots = v_{2m} = \theta(v)$ must hold due to Proposition 1. Recall that $|x| = 2|z|$; hence substituting $v_m = v = yz'$ into Eq. (8) gives $x = z'z$. Furthermore, $v_{m+1}v_{2m+1} \neq v^2$; indeed, otherwise, Eq. (10) implies $zvzw = v^2$ (if $m = 1$) or $zv^mzw = v\theta(v)^{m-1}v$ (if $m \geqslant 2$), and hence, $zv^2w = v\theta(v)v$. The first contradictions Proposition 1 and the latter contradicts Lemma 9.

To begin, we consider the case when $v_{m+1} = v$. As just mentioned, in this case, v_{2m+1} must be $\theta(v)$. Then, substituting $v = yz'$ into Eq. (10) gives $z' = \theta(z')$, and if further $m \geqslant 2$, then $y = \theta(y)$. Hence, if $m \geqslant 2$, then these θ-palindromic properties, with $v = zy = yz'$ and $|y| > |z'|$, imply that $\rho(y) = \rho(z')$ in light of Lemma 19, which contradicts the primitivity of v. If $m = 1$, then we have $v_1 = v$, $v_2 = v_{m+1} = v$, and $v_3 = v_{2m+1} = \theta(v)$. Combining $v = yz'$, $y \in \mathrm{Suff}(v_{m+1})$, and $|y| > |z'|$ together, we have $z' \in \mathrm{Suff}(y)$. This means that $z'^2 \in \mathrm{Suff}(v)$, and hence, $z'^2 \in \mathrm{Pref}(v_{2m+1})$, which is actually equal to x because $x \in \mathrm{Pref}(v_{2m+1})$ and $|x| = 2|z'|$. Now Eq. (6) becomes $u^2 = v^2z'^2$, which implies $u, v, z' \in \{t\}^*$ for some word $t \in \Sigma^+$ due to Theorem 15. However, this contradicts the primitivity of v because $|v| > |z'|$.

Next, we consider the case when $v_{m+1} = \theta(v)$; i.e., $v_1 = \cdots = v_m = v$ and $v_{m+1} = \cdots = v_{2m} = \theta(v)$. Since we have let $v_{m+1} = zy$, in this case we have $\theta(v) = zy$, and this implies with $v = yz'$ that $y = \theta(y)$ and $z' = \theta(z)$. Thus, $x = \theta(z)z = z'\theta(z')$. Moreover, if either $m \geqslant 2$ or $v_{2m+1} = \theta(v)$, then $z' = \theta(z')$, and hence $z = \theta(z')$ is also a θ-palindrome. At any rate, we have to consider the following three subcases depending on the values of u_3 and v_{2m+1}.

The first subcase is when $u_3 = u$; then $u_1u_2 = u_2u_3$. From $v_1 \cdots v_mv_{m+1} \in \mathrm{Pref}(u_1u_2)$ and $yv_{2m+2} \cdots v_{2m}v_{2m+1} \in \mathrm{Pref}(u_2u_3)$, we have

$$y\theta(v)^{m-1}v_{2m+1} \in \mathrm{Pref}(v^m\theta(v)). \tag{11}$$

Due to Lemma 10, v_{2m+1} has to be $\theta(v)$. So, substituting $v = yz'$ into Eq. (11) gives $z' = \theta(z')$ and $y \in \mathrm{Pref}(z'y)$. From these and $|y| > |z'|$, $\rho(y) = \rho(z')$ follows due to Lemma 19, but this contradicts the primitivity of v.

The second subcase is when $u_3 = \theta(u)$ and $v_{2m+1} = v$. In this case, using $x = \theta(z)z$, we have $v = y\theta(z) = \theta(z)zw$. Since $|y| > |z|$, this means that $\theta(z) \in \mathrm{Pref}(y)$, i.e., $z \in \mathrm{Suff}(y)$. Note that $x = \theta(z)z \in \mathrm{Suff}(u_2)$ so that when $u_3 = \theta(u)$, $\theta(z)z \in \mathrm{Pref}(u_3)$, and by definition $w \in \mathrm{Pref}(u_3)$. Hence, if $|w| \leqslant 2|z|$, then $w \in \mathrm{Pref}(\theta(z)z)$, which implies $w \in \mathrm{Pref}(y)$ because $y\theta(z) = \theta(z)zw$ and $|w| = |y| - |z|$. Thus, $y = wz$, and hence, we have $v = wz\theta(z) = \theta(z)zw$. Since $z\theta(z)$ and $\theta(z)z$ are θ-palindromes, Lemma 6 implies that $w, z\theta(z) \in \{t, \theta(t)\}^+$, which contradicts the θ-primitivity of v. If $|w| > 2|z|$, then $\theta(z)z, w \in \mathrm{Pref}(u_3)$ enables us to let $w = \theta(z)zw'$ for some $w' \in \Sigma^+$. Since $w \in \mathrm{Pref}(u_3)$, $\theta(w')\theta(z)z \in \mathrm{Suff}(u_2) = \mathrm{Suff}(yv_{m+2} \cdots v_{2m}\theta(z)z)$. From this we can observe that $\theta(w') \in \mathrm{Suff}(v_{2m})$, that is, $w' \in \mathrm{Pref}(v)$. Actually, this means that $w' \in \mathrm{Pref}(y)$ because $|w'| = |w| - 2|z| = |y| - 3|z|$. Due to this length condition and $v = y\theta(z) = (\theta(z)z)^2w'$, $\theta(z)z\theta(z) \in \mathrm{Pref}(y)$, that is, $z\theta(z)z \in \mathrm{Suff}(y)$. Now we have $y = w'z\theta(z)z$ so that $v = w'(z\theta(z))^2 = (\theta(z)z)^2w'$, but this causes a contradiction using Lemma 6 as done for the case $|w| \leqslant 2|z|$.

The last subcase is when $u_3 = \theta(u)$ and $v_{2m+1} = \theta(v)$. As mentioned previously, in this case $z = \theta(z) = z' = \theta(z')$. Hence, we have $v = yz = z^2w$. Recall that $w \in \mathrm{Pref}(u_3)$ and $u_3 = \theta(u) = z(zy)^m$. Thus, $yz \in \mathrm{Pref}(z^2z(zy)^m)$. Applying Theorem 4 in [7] to this prefix relation gives $\rho(z) = \rho(y)$, but this contradicts the primitivity of v. □

Proposition 21. *Let $u, v \in \Sigma^+$ such that v is θ-primitive, $u_1, u_2, u_3 \in \{u, \theta(u)\}$, and $v_1, \ldots, v_{2m} \in \{v, \theta(v)\}$ for some $m \geqslant 2$. If $v_1 \cdots v_{2m} \in \mathrm{Pref}(u_1u_2u_3)$ and $(2m-1)|v| < 2|u| < 2m|v|$, then one of the following two statements is true:*

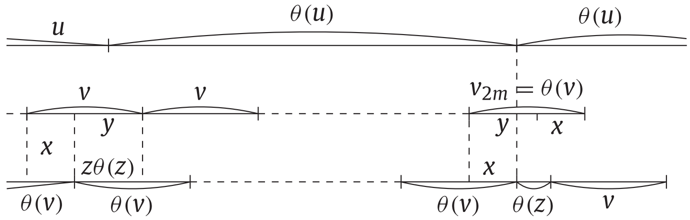

Fig. 5. When $u_3 = \theta(u)$, $v_{2m} = \theta(v) = xy$ overlaps with $y\theta(z)v$ because $\theta(z)v \in \mathrm{Pref}(\theta(u))$.

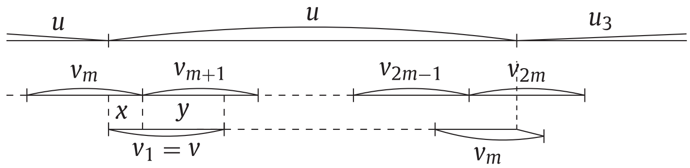

Fig. 6. If $u_2 = u$, we can regard that $v_1 \dots v_m$ overlaps with $v_m \dots v_{2m-1}$ not depending on the value of u_3.

1. *$u_1 \neq u_2$ and $v_1 = \cdots = v_{2m}$, with $v_1 = xz\theta(z)$ and $u_1u_2 = (xz\theta(z))^{2m-1}x$ for some $x, z \in \Sigma^+$ such that $x = \theta(x)$,*
2. *$u_1 = u_2$, $v_1 = \cdots = v_m$, and $v_{m+1} = \cdots = v_{2m} = \theta(v_1)$, with $u_1 = [r(tr)^i(rt)^{i+j}r]^{m-1}r(tr)^i(rt)^j$ and $v_1 = r(tr)^i(rt)^{i+j}r$ for some $i \geqslant 0$, $j \geqslant 1$, and $r, t \in \Sigma^*$ such that $r = \theta(r)$, $t = \theta(t)$, and rt is primitive.*

Proof. Just as in the proof of Proposition 20, we can assume without loss of generality that $u_1 = u$ and $v_1 = v$. Then, we analyze two cases depending on whether $u_2 = u_1$.

Case 1. Let us first look at the case when $u_2 \neq u_1$, i.e., $u_2 = \theta(u)$. Eq. (7) with $2m$ being replaced by $2m-1$, i.e.,

$$u\theta(u) = v_1v_2\cdots v_{2m-1}x = \theta(x)\theta(v_{2m-1})\cdots\theta(v_2)\theta(v_1) \tag{12}$$

holds for some $x \in \mathrm{Pref}(v_{2m})$ so that we have 2 subcases (a) $v_1 = \cdots = v_{2m-1} = v$ and (b) $v_1 = v$, $v_2 = \cdots = v_{2m-1} = \theta(v)$. As remarked in Case 1 of the proof of Proposition 20, (b) cannot hold. Moreover, the proof of Proposition 20 showed that under the condition of subcase (a), if $u_3 = u$, then v_{2m} has to be v. Hence, it suffices to prove that $v_{2m} = v$ under the condition of (a) with $u_3 = \theta(u)$. Recall that in this case we have $x = \theta(x)$ and $v = xy$ for some nonempty θ-palindrome y. Due to Eq. (12), $|u| = (m-1)|v| + |x| + \frac{1}{2}|y|$, and this means that the θ-palindrome y must be of even length; so one can let $y = z\theta(z)$ for some $z \in \Sigma^+$. Hence, $u = (xy)^{m-1}xz$. From this $\theta(z)x \in \mathrm{Pref}(u_3)$ and combining this with Eq. (12) gives

$$v_{2m} \in \mathrm{Pref}(x\theta(z)xy). \tag{13}$$

Let us suppose for now that $v_{2m} = \theta(v)$ (see Fig. 5). Since x was defined to be a prefix of v_{2m}, we have $x \in \mathrm{Pref}(\theta(v))$, and note that $\theta(v) = yx = z\theta(z)x$. If $|x| \geqslant |y| = 2|z|$, then Lemma 19 implies that $\rho(x) = \rho(y)$, which is a contradiction with θ-primitivity of v. If $|z| \leqslant |x| < 2|z|$, then Eq. (13), i.e., $z\theta(z)x \in \mathrm{Pref}(x\theta(z)xy)$, gives $z \in \mathrm{Pref}(x)$ and $z\theta(z)x \in \mathrm{Pref}(x\theta(z)x)$, which imply $z\theta(z)x = x\theta(z)z$. However, this equation implies $x, z \in \{t, \theta(t)\}^+$ for some $t \in \Sigma^+$ due to Lemma 6, contradicting the θ-primitivity of v. If $|y| < |z|$, then we can apply Lemma 8 to $z\theta(z) \in \mathrm{Pref}(x\theta(z)xz)$, which is from Eq. (13), to obtain $\rho(x) = \rho(z)$, the same contradiction as above.

Consequently, if $u_1 \neq u_2$, then we must have $v_1 = \cdots = v_{2m} = v$. Substituting $v = xz\theta(z)$ into Eq. (12) gives us $u_1 = u = (xz\theta(z))^{m-1}xz$.

Case 2. Let us look next at the case when $u_2 = u_1 = u$, illustrated in Fig. 6 and let $v = xy$ with $x \in \mathrm{Suff}(v_m)$ and $y \in \mathrm{Pref}(v_{m+1})$. Note that

$$ux = v_1v_2\cdots v_m \tag{14}$$

holds. Moreover, note that $|x| < |y|$ since $|x| = m|v| - |u|$ and $(2m-1)|v| < 2|u|$. Now, if we look at the overlap between $v_1\cdots v_m$ and $v_m\cdots v_{2m-1}$, then, due to Theorem 14, we get the following two subcases: (a) $v_1 = \cdots = v_{m-1} = v$ and $v_m = v_{m+1} = \cdots = v_{2m-1} = \theta(v)$; (b) $v_1 = \cdots = v_m = v$, $v_{m+1} = \cdots = v_{2m-1} = \theta(v)$.

First, let us consider the subcase (a). If $u_3 = u$, then $v_{m-1}v_m = v\theta(v)$ overlaps with $v_{2m-1}v_{2m} = \theta(v)v_{2m}$ and thus, due to Theorem 14, v_{2m} cannot be either v or $\theta(v)$. Otherwise, $u_3 = \theta(u)$ and note that $x = \theta(x)$ and $y = \theta(y)$ since $v_m = v_{m+1} = \theta(v)$. Then, since the overlapped part between v_{2m-1} and v_m is x, we obtain $x \in \mathrm{Pref}(\theta(v))$. Since $\theta(v) = yx$ and $|x| < |y|$, we have $x \in \mathrm{Pref}(y)$, i.e., $x \in \mathrm{Suff}(y)$. By the way, Eq. (14) can be rewritten in this case as $ux = (xy)^{m-1}yx$ so that $y \in \mathrm{Suff}(u)$. Due to the transitivity of suffix relation, $x \in \mathrm{Suff}(u)$, that is, $x \in \mathrm{Pref}(\theta(u))$. Since $u_3 = \theta(u)$ and $v_m = \theta(v) = yx$, we can say that $v_{m-1}v_m$ overlaps with $v_{2m-1}v_{2m}$, which results in the same conclusion as above. Thus, the subcase (a) is not possible.

For the subcase (b), we prove that $v_{2m} = \theta(v)$. Let us start our analysis by supposing that $v_{2m} = v$. First, since $v_m = v$ ends with x, let $v = zwx$ for some $z, w \in \Sigma^+$ with $|w| = |x|$. If $u_3 = \theta(u)$, since $v_{2m} = v = zwx$, we obtain that $w \in \mathrm{Pref}(u_3)$, i.e., $\theta(w) \in \mathrm{Suff}(u)$. But this means that $w = \theta(w)$, since the right end of the first u cuts $v_m = v = zwx$ after exactly $|zw|$

characters. Since the overlap between v_{2m-1} and v_m is x, we have $xz = zw$ with $x = \theta(x)$ and $w = \theta(w)$. Then Lemma 6 implies that $x, z, w \in \{t, \theta(t)\}^+$ for some $t \in \Sigma^+$, a contradiction with the θ-primitivity of $v = zwx$. If $u_3 = u$, note that $v_1 \cdots v_m = (xy)^m \in \mathrm{Pref}(xv_{m+1} \cdots v_{2m-1}v_{2m})$. Hence, y is a prefix of $v_{2m} = v$. With $|x| < |y|$ and $v = xy$, this prefix relation would lead us to the same contradiction due to Lemma 19.

In conclusion, for this case, i.e., when $u_2 = u_1$, we obtain that $v_1 = \cdots = v_m = v$ and $v_{m+1} = \cdots = v_{2m} = \theta(v)$. By applying Theorem 14 to the overlap between $v_1 \ldots v_m$ and $v_m \ldots v_{2m-1}$, we get the representations of u and v by two θ-palindromes r and t. □

From the presentations of u and v obtained in Propositions 20 and 21, the next corollary is obtained.

Corollary 22. *Let $u, v \in \Sigma^+$ such that v is θ-primitive, $u_1, u_2, u_3 \in \{u, \theta(u)\}$, and $v_1, \ldots, v_n \in \{v, \theta(v)\}$ for some $n \geqslant 3$. If $v_1 \cdots v_n = u_1u_2z$ for some $z \in \mathrm{Pref}(u_3)$ and $(n-1)|v| < 2|u|$, then z is a θ-palindrome.*

These propositions show that if we suppose v to be θ-primitive, then the values of u_1, u_2, u_4, and u_5 determine the values of $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ uniquely, modulo a substitution of v by $\theta(v)$, or of w by $\theta(w)$. Thus, they significantly decrease the number of cases to be considered. Furthermore, the value of u_3 may put an additional useful restriction on v or w.

Lemma 23. *Let $u, v \in \Sigma^+$ such that v is a θ-primitive word, $u_1, u_2, u_3 \in \{u, \theta(u)\}$, and $v_1, \ldots, v_n \in \{v, \theta(v)\}$ for some $n \geqslant 3$. If $v_1 \cdots v_n \in \mathrm{Pref}(u_1u_2u_3)$, $u_1 \neq u_2$, $u_1 = u_3$, and $(n-1)|v| < 2|u| < n|v|$, then $|v| < \frac{4}{2n-1}|u|$.*

Proof. Without loss of generality, we can assume that $u_1 = u$ and $v_1 = v$ because θ is an involution. Due to Propositions 20 and 21, $v_1 = \cdots = v_n = v = xy$ for some θ-palindromes x, y and $u\theta(u) = v^{n-1}x$. Since $v^n \in \mathrm{Pref}(u\theta(u)u)$, this equation implies that $y \in \mathrm{Pref}(u)$, but in fact this means that $y \in \mathrm{Pref}(v)$ because $v \in \mathrm{Pref}(u)$. Then Lemma 19 and the primitivity of v imply that $|y| < |x|$, which is equivalent to $|y| < \frac{1}{2}|v|$. This means that $|v| < \frac{4}{2n-1}|u|$ because $|y| = n|v| - 2|u|$. □

All we did so far in studying the extended Lyndon–Schützenberger equation $u_1 \ldots u_5 = v_1 \ldots v_n\, w_1 \ldots w_m$ was to consider either the common prefix of $v_1 \ldots v_n$ and $u_1 \ldots u_5$, or the common suffix of $w_1 \ldots w_m$ and $u_1 \ldots u_5$. Next, we combine them together and consider the whole equation. The following lemma proves to be useful for our considerations.

Lemma 24. *Let $u, v \in \Sigma^+$ such that v is a θ-primitive word, $u_1, u_2, u_3 \in \{u, \theta(u)\}$ and $v_1, \ldots, v_n \in \{v, \theta(v)\}$ for some $n \geqslant 3$. If $v_1 \cdots v_n = u_1u_2z$ for some $z \in \mathrm{Pref}(u_3)$, $u_1 = u_2$, and $(n-1)|v| < 2|u|$, then $v_1 = xyx$ and $z = x^2$ for some $x, y \in \Sigma^+$ such that $x = \theta(x)$ and $yx = \theta(yx)$.*

Proof. Just as before, we assume that $u_1 = u$ and $v_1 = v$. Propositions 20 and 21 imply that $n = 2m$ for some $m \geqslant 2$, $u = \{r(tr)^i(rt)^{i+j}r\}^{m-1}r(tr)^i(rt)^j$, and $v = r(tr)^i(rt)^{i+j}r$ for some $r, t \in \Sigma^*$ such that $r = \theta(r)$, $t = \theta(t)$, $i \geqslant 0$, and $j \geqslant 1$. By taking $x = r(tr)^i$ and $y = (rt)^j$, we complete the proof. □

Next, we prove that the triple $(5, 4, m)$ imposes θ-periodicity for any $m \geqslant 4$.

Theorem 25. *Let $u, v, w \in \Sigma^+$, $u_1, u_2, u_3, u_4, u_5 \in \{u, \theta(u)\}$, $v_1, v_2, v_3, v_4 \in \{v, \theta(v)\}$, and $w_1, \ldots, w_m \in \{w, \theta(w)\}$ for some $m \geqslant 4$. If these words satisfy $u_1u_2u_3u_4u_5 = v_1v_2v_3v_4\, w_1 \cdots w_m$, then u is not θ-primitive and $u, v, w \in \{t, \theta(t)\}^+$ for some $t \in \Sigma^+$.*

Proof. First note that we can assume that w is θ-primitive, since otherwise we would just increase the numerical parameter m. If u is not θ-primitive, that is, $u \in \{p, \theta(p)\}^k$ for some θ-primitive word $p \in \Sigma^+$ and $k \geqslant 2$, then the equation can be rewritten as $p_1p_2 \cdots p_{5k} = v_1v_2v_3v_4w_1 \ldots w_m$, where $p_i \in \{p, \theta(p)\}$ for $1 \leqslant i \leqslant 5k$. But then, due to Theorem 16, we obtain that $v, w \in \{p, \theta(p)\}^+$. Furthermore, we can assume that also v is θ-primitive. Indeed, if it is not, then $v \in \{q, \theta(q)\}^j$ for some θ-primitive word q and $j \geqslant 2$. Then, the equation becomes $u_1 \ldots u_5 = q_1 \ldots q_{4j}w_1w_2 \ldots w_m$, where $q_i \in \{q, \theta(q)\}$ for $1 \leqslant i \leqslant 4j$. But this implies that $u, w \in \{q, \theta(q)\}^+$ due to Theorem 18. Since u and w are assumed to be θ-primitive, $u, w \in \{q, \theta(q)\}$ and we have $5|q| < 4j|q| + m|q|$, which contradicts the fact that u, v, and w satisfy the equation $u_1 \ldots u_5 = q_1 \ldots q_{4j}w_1w_2 \ldots w_m$. Even when v is θ-primitive, if $m \geqslant 7$ then the same argument leads to the same contradiction.

Now we will show that if u, v, and w are θ-primitive, then the equation cannot hold for $m \leqslant 6$. Since θ is an involution, we can assume that $u_1 = u$, $v_1 = v$, and $w_1 = w$. Let us start by supposing that u, v, and w satisfy $u_1u_2u_3u_4u_5 = v_1v_2v_3v_4\, w_1 \cdots w_m$. Now, we have several cases depending on where the border between v_4 and w_1 is located. If it is left to or on the border between u_2 and u_3, then Theorem 2 implies that $u, w \in \{t, \theta(t)\}^+$ for some θ-primitive word $t \in \Sigma^+$, which further implies that also $v \in \{t, \theta(t)\}^+$. In fact, $u, v, w \in \{t, \theta(t)\}$ because they are θ-primitive. Then $|u_1 \ldots u_5| = 5|t|$, while $|v_1v_2v_3v_4w_1 \ldots w_m| = (4+m)|t|$ with $m \geqslant 4$, which is a contradiction. The case when the border between v_4 and w_1 is right to or on the border between u_3 and u_4 will lead the contradiction along the same argument.

Let us suppose that $|u_1u_2| < |v_1v_2v_3v_4| < |u_1u_2u_3|$. Note that under this supposition, $|v|, |w| < |u|$. If $m|w| \geqslant 2|u| + |w| - 1$, then $u_3u_4u_5$ and $w_1 \ldots w_m$ share a suffix long enough to impose the θ-periodicity onto u and w due to

Theorem 2. However, as explained before, this leads to a contradiction. This argument also applies to $u_1u_2u_3$ and $v_1v_2v_3v_4$. As a result, it is enough to consider the case when $3|v| < 2|u| < 4|v|$ and $(m-1)|w| < 2|u| < m|w|$.

We start our case analysis with the case when $u_3 \neq u_5$. Let

$$u_1u_2z = v_1v_2v_3v_4 \tag{15}$$

for some $z \in \text{Pref}(u_3)$. This means

$$u_3u_4u_5 = zw_1 \cdots w_m. \tag{16}$$

Actually, z is a θ-palindrome because of Corollary 22. From $z \in \text{Pref}(u_3)$, we can obtain $z \in \text{Suff}(u_5)$ due to this θ-palindromic property of z and $u_3 \neq u_5$. This means that either $w_m \in \text{Suff}(z)$ or $z \in \text{Suff}(w_m)$. In both cases, Eq. (16) implies that $u_3u_4u_5$ and $w_mw_1w_2 \cdots w_m$ share a common prefix of length at least $2|u| + |w| - 1$, but then Theorem 2 would lead us to the contradiction. This technique works also in the case when $u_1 \neq u_3$ by changing the roles of $v_1 \cdots v_4$ and $w_1 \cdots w_m$ in the above argument.

The cases to be investigated now are when $u_1 = u_3 = u_5$. Note that $|v| < \frac{2}{3}|u|$ because $3|v| < 2|u| < 4|v|$. If $u_2 \neq u_1$ or $u_4 \neq u_5$, then Lemma 23 implies that $|w| < \frac{4}{2m-1}|u|$. Then $5|u| - (4|v| + m|w|) > 0$ which contradicts the fact that u, v, and w satisfy the given equation.

Let us close our analysis by considering the case when $u_1 = u_2 = u_3 = u_4 = u_5 = u$. Applying Propositions 20 and 21, we have that $m = 2k$ for some $k \geqslant 2$, $w_1 = \cdots = w_k = w$, $w_{k+1} = \cdots = w_{2k} = \theta(w)$, $v_1 = v_2 = v$, and $v_3 = v_4 = \theta(v)$. Then, Lemma 24 implies that $u = xyxxy = (y'x'x')^{k-1}y'x'$, $v = xyx$, and $\theta(w) = x'y'x'$ for some $x, y, x', y' \in \Sigma^+$ with $x = \theta(x)$, $yx = \theta(yx)$, $x' = \theta(x')$, and $x'y' = \theta(x'y')$. Furthermore, $u^2x^2 = v_1v_2v_3v_4$ and $x'^2u^2 = w_1 \cdots w_m$ so that $u = x^2x'^2$. Note that $k \in \{2, 3\}$ since $4 \leqslant m \leqslant 6$.

When $k = 2$, i.e., $xyxxy = y'x'x'y'x'$, we have three subcases depending on the lengths of xy and $y'x'$. If $|xy| < |y'x'|$, then by looking at the two sides of the equality $xyxxy = y'x'x'y'x'$, we obtain $y'x' = xyz = \theta(z)xy$ and $x = zx'\theta(z)$ for some $z \in \Sigma^+$. Substituting $x = zx'\theta(z)$ into $xyz = \theta(z)xy$ we get $z = \theta(z)$, and hence, $y'x' = xyz = zxy$. Thus, $y'x', xy, z \in p^+$ for some primitive word p. Let $z = p^i$ and $y'x' = p^j$ for some $i, j \geqslant 1$. Then $y'x' = zxy$ and $x = zx'z$ imply that $p^j = p^{2i}x'p^iy$. Since p is primitive, p cannot be a proper factor of p^2 so that this equation implies that $x' \in p^+$. However, this contradicts the primitivity of $\theta(w) = x'y'x'$. The case when $|xy| > |y'x'|$ is symmetric to the previous case after reversing variables and sides of the equation. Finally, if $|xy| = |y'x'|$, then $x = x'$, which is a contradiction with the primitivity of u since $u = xxx'x'$.

When $k = 3$, i.e., $u = xyxxy = (y'x'x')^2y'x'$, we first note that $|xy| > |y'x'|$ and $|xyx| > |y'x'x'|$. If $|xy| \geqslant |y'x'x'|$, then, by the Fine and Wilf theorem, $\rho(xyx) = \rho(y'x'x')$. Since xyx is strictly longer than $y'x'x'$, this means that $v = xyx$ is not primitive, which is a contradiction. Otherwise, i.e., $|y'x'| < |xy| < |y'x'x'|$, let $xy = y'x'z$ for some $z \in \text{Pref}(x')$. Since $x' = \theta(x')$, the equation $xyxxy = (y'x'x')^2y'x'$ also implies that $xy = \theta(z)y'x'$. Moreover, since $xy = y'x'z = \theta(z)y'x'$ and $\theta(z) \in \text{Suff}(x')$, we obtain $z = \theta(z)$. Thus $xy, y'x', z \in \{q\}^+$ for some primitive word $q \in \Sigma^+$, which, just as above, contradicts the primitivity of $\theta(w)$.

In summary, we have proved the following:

1. the θ-primitivity of w can be assumed without loss of generality;
2. if either u or v is not θ-primitive, $u_1u_2u_3u_4u_5 = v_1v_2v_3v_4w_1 \cdots w_m$ implies $u, v, w \in \{t, \theta(t)\}^+$ for some $t \in \Sigma^+$;
3. under the assumption that u, v, w be θ-primitive, if $m \geqslant 7$, then we reach the same conclusion as above;
4. under the same assumption, if $m \leqslant 6$, the equation cannot hold.

Therefore, we can conclude that if the equation holds, then $u, v, w \in \{t, \theta(t)\}^+$ for some $t \in \Sigma^+$. $\square$

The next result shows that the triple $(5, 3, m)$ also imposes θ-periodicity for any $m \geqslant 3$.

Theorem 26. *Let $u, v, w \in \Sigma^+$, $u_1, u_2, u_3, u_4, u_5 \in \{u, \theta(u)\}$, $v_1, v_2, v_3 \in \{v, \theta(v)\}$, and $w_1, \ldots, w_m \in \{w, \theta(w)\}$ with $m \geqslant 3$. If these words verify the equation $u_1u_2u_3u_4u_5 = v_1v_2v_3\, w_1 \cdots w_m$, then u is not θ-primitive and $u, v, w \in \{t, \theta(t)\}^+$ for some $t \in \Sigma^+$.*

Proof. As in the proof of Theorem 25, we can assume that w is θ-primitive. Also if u is not θ-primitive, then, just as before, Theorem 16 results in $u, v, w \in \{t, \theta(t)\}^+$ for some $t \in \Sigma^+$. So let us assume that u is θ-primitive. Moreover, we can assume that v is θ-primitive. Indeed, if it is not, then $v \in \{p, \theta(p)\}^j$ for some θ-primitive word p and $j \geqslant 2$. Then the equation becomes $u_1u_2u_3u_4u_5 = p_1 \ldots p_{3j}w_1w_2 \ldots w_m$, where $p_i \in \{p, \theta(p)\}$ for $1 \leqslant i \leqslant 3j$. For the case $m \geqslant 5$ and the case $m = 4$, Theorems 17 and 25 lead us to the contradiction, respectively. If $m = 3$, we can change the roles of v and w, and reduce it to the case when v is θ-primitive. In the following, we assume that u, v, and w are θ-primitive and prove that the equation cannot hold.

Now, since θ is an involution, we can assume that $u_1 = u$, $v_1 = v$, and $w_1 = w$. As in the proof of Theorem 25, in all cases except when the border between v_3 and w_1 falls inside u_3, we get a contradiction. Furthermore, using the same arguments as in the previous proof, we can assume that $2|v| < 2|u| < 3|v|$ and $(m-1)|w| < 2|u| < m|w|$. Moreover, due to Proposition

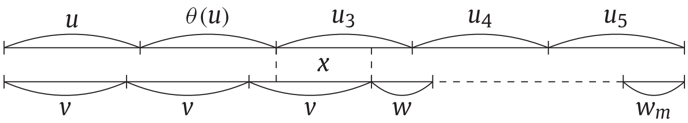

Fig. 7. $u_1u_2u_3u_4u_5 = v_1v_2v_3w_1 \cdots w_m$ for Theorem 26.

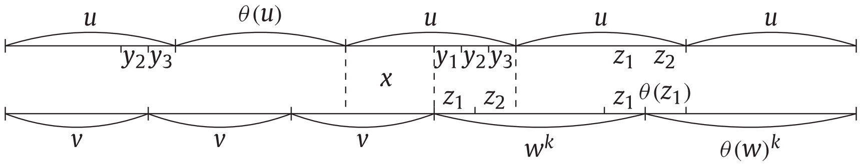

Fig. 8. The suffix of u_3 can be written in two ways as $y_1y_2y_3$ and z_1z_2.

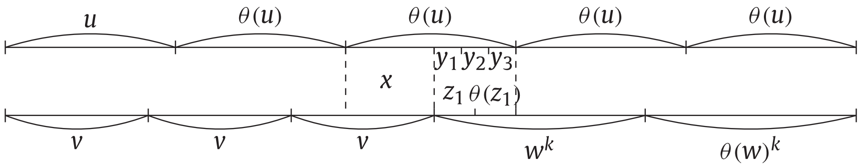

Fig. 9. The suffix of $u_3 = \theta(u)$ can be written in two ways as $y_1y_2y_3$ and $z_1\theta(z_1)$.

20, $u_2 = \theta(u)$ and $v_1 = v_2 = v_3 = v$, see Fig. 7. Then $u\theta(u)x = v^3$ for some $x \in \Sigma^+$, which satisfies $x = \theta(x)$ due to Corollary 22. Since $x \in \mathrm{Pref}(u_3)$, if $u_3 \neq u_5$, then $x \in \mathrm{Suff}(u_5)$ which implies that either $w_m \in \mathrm{Suff}(x)$ or $x \in \mathrm{Suff}(w_m)$. In both cases, we obtain that $u_3u_4u_5$ and $w_mw_1w_2 \cdots w_m$ share a common suffix of length at least $2|u|+|w|-1$. Hence, Theorem 2 implies that $u, w \in \{t, \theta(t)\}^+$ for some $t \in \Sigma^+$ and thus also $v \in \{t, \theta(t)\}^+$ which leads to the same contradiction as above. Otherwise, $u_3 = u_5$ and we have the following four cases left:

1. $u\theta(u)u\theta(u)u = vvvw_1 \cdots w_m$,
2. $u\theta(u)\theta(u)u\theta(u) = vvvw_1 \cdots w_m$,
3. $u\theta(u)uuu = vvvw_1 \cdots w_m$,
4. $u\theta(u)\theta(u)\theta(u)\theta(u) = vvvw_1 \cdots w_m$.

Let us start by considering the first equation. Since v is θ-primitive, using Lemma 23, we have $|v| < \frac{4}{5}|u|$ and $|w| < \frac{4}{2m-1}|u|$. However, then $5|u| - (3|v| + m|w|) > 5|u| - \frac{12}{5}|u| - \frac{4m}{2m-1}|u| = \frac{6m-13}{5(2m-1)}|u| > 0$ because $m \geqslant 3$. Hence, $5|u| > 3|v| + m|w|$ contradicting our supposition that the words u, v, and w satisfy the equation $u\theta(u)u\theta(u)u = vvvw_1 \cdots w_m$.

For the second equation, Propositions 20 and 21 imply that $w_1 = w_2 = \cdots = w_m = w$. Since $u\theta(u) = v^2v_p$ for some $v_p \in \mathrm{Pref}(v)$ and $u\theta(u)$ is a θ-palindrome, we have $u\theta(u) = \theta(v_p)\theta(v)^2$. Note that $\theta(v_p) \in \mathrm{Suff}(\theta(v))$. Also $u\theta(u) = w_sw^{m-1}$ for some $w_s \in \mathrm{Suff}(w)$. Since $m \geqslant 3$, the Fine and Wilf theorem implies that $\rho(\theta(v)) = \rho(w)$ and thus we obtain again the same contradiction as above.

Next we consider the third equation. Since $u_4 = u_5$, Propositions 20 and 21 imply that $m = 2k$ for some $k \geqslant 2$ and $w_1 = \cdots = w_k = w$ and $w_{k+1} = \cdots = w_{2k} = \theta(w)$. Let $w^k\theta(w)^k = z_1z_2u^2$ for some $z_1, z_2 \in \Sigma^+$ with $|z_1| = |z_2| = k|w| - |u|$, as illustrated in Fig. 8.

Then, $z_1z_2 \in \mathrm{Suff}(u)$, which due to length conditions means that $z_1 \in \mathrm{Suff}(w^k)$. Thus, $\theta(z_1) \in \mathrm{Pref}(\theta(w)^k)$ which implies immediately that $z_2 = \theta(z_1)$. Similarly, we can let $u\theta(u)u = v^3y_1y_2y_3$ for some $y_1, y_2, y_3 \in \Sigma^+$ with $|y_1| = |y_2| = |y_3| = |u| - |v|$. Then $y_1y_2y_3 = z_1\theta(z_1)$, which implies $y_3 = \theta(y_1)$ and $y_2 = \theta(y_2)$. Recall that $(2k-1)|w| < 2|u| < 2k|w|$ was assumed. So we have $|y_1y_2y_3| < |w|$ and $|w| < \frac{2}{2k-1}|u| \leqslant \frac{2}{3}|u|$. Thus, $|y_1y_2y_3| < \frac{2}{3}|u|$. This further implies that $|x| = |u| - |y_1y_2y_3| > |y_1|$. If we look at the second v, since $y_3 \in \mathrm{Suff}(u)$, using length arguments, we obtain that $y_3 \in \mathrm{Pref}(v)$, and hence $y_3 \in \mathrm{Pref}(u)$. Since $|y_3| < |x|$, this means that $y_3 \in \mathrm{Pref}(x)$ and hence $\theta(y_3) \in \mathrm{Suff}(x)$, which further implies $\theta(y_3) \in \mathrm{Suff}(v)$. Thus $y_2 = \theta(y_3)$ because $y_2 \in \mathrm{Suff}(v)$, which results in $y_1 = y_2 = y_3$ and, moreover, they are all θ-palindromes. Hence $y_1y_2 = \theta(y_2)\theta(y_1) = \theta(y_1y_2)$, which is a prefix of $\theta(v)$. This means that $u\theta(u)u$ and $v^3\theta(v)$ share a prefix of length at least $2|u| + |v| - 1$. Consequently $\rho_\theta(u) = \rho_\theta(v)$ which leads to the same contradiction as before.

Lastly, we consider the fourth equation, illustrated in Fig. 9. Just as in the case of the third equation, $y_3 = \theta(y_1)$ and $y_2 = \theta(y_2)$. Since $y_2y_3 \in \mathrm{Suff}(\theta(u))$, these equalities give $\theta(y_3)\theta(y_2) = y_1y_2 \in \mathrm{Pref}(u) \subseteq \mathrm{Pref}(v^2)$. Thus, we can see that $u\theta(u)^2$ and v^5 share their prefix of length at least $2|u| + |v|$. The rest is as same as for the third equation.

In conclusion, if u is θ-primitive, then, using length arguments, we always reach a contradiction. On the other hand, if u is not θ-primitive, then we proved that there exists a word $t \in \Sigma^+$ such that $u, v, w \in \{t, \theta(t)\}^+$. $\square$

Combining Theorems 16, 17, 25, and 26 and Examples 1 and 2 all together, now we conclude our analysis on the extended Lyndon–Schützenberger equation with the summarizing theorem.

Theorem 27. *For $u, v, w \in \Sigma^+$, let $u_1, \ldots, u_\ell \in \{u, \theta(u)\}$, $v_1, \ldots, v_n \in \{v, \theta(v)\}$, and $w_1, \ldots, w_m \in \{w, \theta(w)\}$. If $u_1 \ldots u_\ell = v_1 \ldots v_n w_1 \ldots w_m$ and $\ell \geqslant 5$, $n, m \geqslant 3$, then $u, v, w \in \{t, \theta(t)\}^+$ for some $t \in \Sigma^+$. Furthermore, $n = 3$ and $m = 3$ are optimal.*

5. Conclusion

This paper continues the investigation of an extended notion of primitiveness of words, based on replacing the identity between words by a weaker notion of "equivalence" between a word u and $\theta(u)$, where θ is a given antimorphic involution. Firstly, we completely characterize all non-trivial overlaps between two words in $\{v, \theta(v)\}^+$ of the form $\alpha(v, \theta(v)) \cdot x = y \cdot \beta(v, \theta(v))$. As an application of this characterization, we extend the Lyndon–Schützenberger equation to the equation $u_1 \cdots u_\ell = v_1 \cdots v_n w_1 \cdots w_m$, where $u_1, \ldots, u_\ell \in \{u, \theta(u)\}$, $v_1, \ldots, v_n \in \{v, \theta(v)\}$, and $w_1, \ldots, w_m \in \{w, \theta(w)\}$. The strongest result obtained states that for $\ell \geqslant 5$ and $n, m \geqslant 3$, $u, v, w \in \{t, \theta(t)\}^+$ for some word t, while once n or m become 2, the existence of such t is not guaranteed any more.

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