On acyclic colorings of planar graphs

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Abstract

The conjecture of B. Grünbaum on existing of admissible vertex coloring of every planar graph with 5 colors, in which every bichromatic subgraph is acyclic, is proved and some corollaries of this result are discussed in the present paper.

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1. Introduction and statement of the result

In 1973 Grünbaum has published a large paper [5] on graph colorings, in which various restrictions were given to the type of all 2- and 3-chromatic subgraphs. The main attention in this paper was attached to the planar graphs.

Definition 1. An admissible coloring of a graph is called acyclic (in narrow sense), if every bichromatic subgraph, induced by this coloring, is a forest (acyclic graph).

The acyclic coloring of a graph should obviously be considered only for loopless graphs without multiple edges, which is assumed below.

The first example of a planar graph, which is not acyclically 4-colorable, has been constructed by Grünbaum [5]. Afterwards Wegner has constructed [12] a planar graph, which possess a cycle in every 2-chromatic subgraph in every admissible 4-coloring.

Definition 2. Graph G is called k-degenerated, if each subgraph H of G contains a vertex, which induced degree is less than k, i.e.

\[ W(G) = \max_{G' \subseteq G} \min_{v \in V(G')} s_{G'}(v) + 1 \leq k, \]

where W(G) is known as Vizing-Wilf’s number.

In particular, a graph is 1-degenerated, iff it contains no edges, and is 2-degenerated, iff it is a forest.

Kostochka and Melnikov have shown [8] (answering Grünbaum’s question), that graphs, acyclically not colorable with 4 colors, can be found even among 3-degenerated bipartite planar graphs.
The main conjecture of Grünbaum in [5] was: *every planar graph is acyclically 5-colorable*. The main result in [5] was

**Theorem 1.** *Every planar graph is acyclically 9-colorable.*

Short after Mitchem [10] sharpened this bound to 8. In a year, Kostochka reduced this bound to 6. After that, Albertson and Berman [1] proved the result for 7 colors.

As it was pointed out by Grünbaum, the positive solution of his problem absorbs some results from [3,4,6,9,11], in which coverings of planar graphs by forests and edgeless graphs are investigated.

Considering his main conjecture as being “rather hard”, Grünbaum suggested a few relaxed versions of it. Under acyclic colorings in a broad sense I mean the class of admissible colorings in which all \( r \)-chromatic subgraphs \( r \geq 2 \), belong to a certain graph type \( G_r \). Below the further examples of acyclic colorings, suggested by Grünbaum, are discussed.

**Definition 3.** A graph is called **outerplanar**, if it can be imbedded into the plane in such a way, that all it's vertices are incident to some single face.

Certainly, each acyclic graph is outerplanar, but convers is not true. A conjecture was proposed, that every planar graph can be admissibly 5-colored in such a way, that all it's 2-chromatic subgraphs are outerplanar.

**Definition 4.** An admissible coloring with \( k_1 + \cdots + k_s \) colors is called **partially acyclic** \( (k_1, \ldots, k_s) \)-coloring, \( s \geq 1, \ k_i \geq 1, \) for \( 1 \leq i \leq s \), if for every \( i \), the subgraph, colored with \( k_i \) colors from \( i \)th subset, is colored acyclically.

A conjecture was made, that every planar graph has a partial acyclic \((4, 1)\)-coloring.

**Definition 5.** The **fan chromatic number** \( b(G) \) of the graph \( G \) is the least number of colors in admissible coloring, in which every connected component of each 2-chromatic subgraph is a fan (a tree of diameter 2).

Grünbaum proved [5], that for every planar graph \( G \), \( b(G) \leq 2304 \). If the main conjecture is true, it follows, that \( b(G) \leq 80 \), but this bound also seems fail to be sharp.

The conjecture of Grünbaum looks strong once more in the following relation. Replace the requirement of all 2-chromatic subgraphs being acyclic in the definition of acyclic coloring by another requirement of the same sort: all 3-chromatic subgraphs should be outerplanar. Then, as it was shown by Kostochka and Melnikov [8] (answering Grünbaum’s question), for arbitrary \( k \geq 3 \) there exist planar graphs (moreover, bipartite, 3-degenerated, and with \( 2k - 1 \) vertices), which fail to possess the cited \( k \)-coloring.

Nevertheless, the Grünbaum’s conjecture has been proved correct. The result of the present paper is

**Theorem 2.** *Every planar graph is acyclically 5-colorable.*

It involves the truthfulness of other cited statements.

2. Proof of theorem 2

Let \( G \) be a minimal on the number of vertices (obviously, \( V(G) \geq 7 \)) counterexample to the theorem, and it’s imbedding into the plane is fixed, which can be assumed to be a triangulation.

2.1. Basic properties of \( G \)

It is an easy task to prove the following

**Lemma 1.** There are no separating 3-cycles in \( G \).
Corollary 1.1. G does not contain vertices of degree 3.

Lemma 2. There are no 4-neighbours in G.

Denote by $S_i$ the set of $i$-cycles in $G$, where $i \leq 6$, which possess no chords and contain at least 2 vertices in their interiors, moreover, if $i = 6$, we require, that the interior differ from a pair of adjacent 4- and 6-vertices. If $S_4 \neq \emptyset$, then let $C_4$ be an element of $S_4$, which contains the least number of vertices in it’s interior; $I_4 = \text{Int}(C_4)$, and $\bar{I}_4 = G \setminus \text{Out}(C_4)$.

In the opposite case let $C_4$ be the bound of the infinite 3-face. Further, denote by $C$ the cycle $C_4$, if there are no elements of $S_5 \cup S_6$, enclosed in $\bar{I}_4$, otherwise the element of $S_5 \cup S_6$ with the least interior among those enclosed in $\bar{I}_4$. Afterwards we put $I = \text{Int}(C)$, $\bar{I} = G \setminus \text{Out}(C)$.

If $H$ is a graph, then $H_i$ denote the set of $i$-vertices of $H$, and $H_m = H^4 \cup H^5$. Let $W(v)$ be a neighbourhood of a vertex $v$, i.e. a subgraph, induced by those vertices, adjacent to $v$.

Lemma 3. If $x, y \in I_4^m$, then $x, y$ are not adjacent.

Proof. In view of Lemma 2, we are to consider the two cases.

Case 1: $s(x) = 5$, $s(y) = 4$ (see Fig. 1).

Delete $x, y$ from $G$ and identify $c, e$. By Lemma 1, there are no loops in the obtained graph $G'$. Suppose, multiple edges have appeared. Then in $G$ there were chains of the length 2 and of $cle$ type, where $l \neq \{x, y, d\}$. Among 4-cycles of the type $[xcle]$, we take 4-cycle $[xcl_0e]$ with the least interior; by Lemma 1, it has no chords and contains at least 2 vertices in it’s interior. On the other hand, $[xcl_0e] \subseteq \bar{I}_4$, and $|\text{Int}([xcl_0e])| > |I_4|$, which is a contradiction to the definition of $C_4$.

Remark 1. In all succeeding lemmas the similar argument shows, that while identifying the vertices at the distance 2, loops and multiple edges can not appear.

By the definition of $G$, $G'$ can be acyclically colored with five colors $1, 2, 3, 4, 5$. The color of the vertex $v$ in the acyclic coloring we denote by $a(v)$; an ordered set with the elements $x_1, \ldots, x_n$ by $(x_1, \ldots, x_n)$, and let

$$a(x_1, \ldots, x_n) = (a(x_1), \ldots, a(x_n)), \quad \text{and} \quad a[x_1, \ldots, x_n] = \{a(x_1), \ldots, a(x_n)\}.$$  

Without loss of generality, we may assume, that

$$a(a, b, c * e) = (3, 2, 1), \quad \text{and} \quad a(d) \in \{2, 4\}.$$  

The coloring of $G'$ induces the partial coloring of $G$. It is easily seen, that we obtain acyclic coloring of $G$ by putting $a(x, y) = (5, 3)$.

Case 2 $s(x) = s(y) = 5$ (see Fig. 2).

Delete $x, y$ and identify $c, f$. Let we obtained $a(a, b, c * f) = (3, 2, 1)$.

1. $a(d, e) \notin 5$: we put $a(x, y) = (4, 5)$.
2. $a(d, e) = (4, 5)$: if there is no bichromatic 3, 4-chain between $a, d$, then we put $a(x, y) = (4, 3)$, else there is no 2, 5-chain between $b, e$, and we put $a(x, y) = (5, 2)$. 

Fig. 1.
Lemma 4. A vertex $v \in I^6_4$ cannot be adjacent to:

(a) $x, y \in I^m_4$, if $s(x) = 4$;
(b) $x, y \in I^5_4$, if $d_{w(v)}(x, y) = 2$, or $y \in I^m_4$ and $x \in I^5_4$, if $d_{w(v)}(x, y) = 2$, and $W(x) \cap I^4_4 \neq \emptyset$.

Proof. Case a: Subcase a1: $s(x) = s(y) = 4$ (see Fig. 3a and 3b).

Delete $x, v, y$ and insert an edge $ac$. Identify $c, e$ in the first case, and $f, e$ in the second one. Let $a(a, c, f) = (1, 2, 3)$. Put $a(v) \in \{4, 5\} \setminus a(d)$; $x$ we color with 3 only if forcedly, i.e. when $|a\{a, b, c, v\}| = 4$; afterwards $y$ is colored in an arbitrary admissible way, i.e. differently from the colors of all neighbour vertices.

Subcase a2: $s(x) = 4, s(y) = 5$ (see Fig. 4a and 4b).

Delete $x, y, v$, then in the first case identify $d, b, g$ and in the second $b, d, a, g$. Let $a(b, a) = (1, 2)$. If $a(c) = 2$, we put $a(v) \in \{3, 4, 5\} \cap a\{e, f\}$, then $a(y) \in \{3, 4, 5\}$. Let $a(c) = 3$; if $a\{e, f\} \neq \{4, 5\}$, we demand $a(v) \in \{4, 5\} \setminus a\{e, f\}$, then $a(x) \in \{4, 5\}$, otherwise we make $a(x, y, v) = (5, 4, 3)$.

Case b: Subcase b1: $s(y) = 4$ (see Fig. 5a and 5b).
The case when $x'$ lies between $d$ and $e$ is equivalent to the second of considered ones. Delete $x'$, $x$, $v$, $y$, and insert an edge $eg$. In the first case identity $c$, $d$, and $b$, $e$, and in the second $b$, $c$, $e$. Let $a(b, a, g) = (1, 2, 3)$. Demand that $a(v) \in \{4, 5\} \setminus a(d), a(x) \neq a(x'')$, and if $a(v) = a(f)$, then $a(y) \in \{4, 5\}$.

**Subcase b2:** $s(y) = 5$ (see Fig. 6).

The proof is valid also for those cases, when $x'$ is situated between $b$ and $c$, or between $d$ and $e$, or when, $x'$ is absent at all. Actually: delete $x', x, y, v$, insert an edge of $W(x)$, and identify $b, e, h$. Let $a(e, f, g) = (1, 2, 3)$.

(1) $a\{c, d\} = \{2, 3\}$.
   (1.1) $a(a) = 2$, then we make $a(x, v, y) = (4, 3, 5)$.
   (1.2) $a(a) = 4$: there is no 2, 4-chain $a, f : a(x, v, y) = (5, 2, 4)$; there is no 3, 5-chains from $g$ to $\{c, d\}$:
      $a(x, v, y) = (5, 3, 5)$.

(2) $a\{c, d\} = \{2, 4\}$.
   (2.1) $a(a) \in \{2, 3\}: a(x, v, y) = (5, 4, 5)$;
   (2.2) $a(a) = 5$: $a(x, v, y) = (3, 4, 5)$.

(3) $a\{c, d\} = \{4, 5\}$, $a(a) = 2$: $a(x, v, y) = (3, 4, 5)$.

After all $x'$, if present, we color differently from $a(a), a(b), a(c)$, when $a(x) = a(x'')$, and admissibly otherwise.

**Corollary 4.1.** There is no vertex $c_2 \in C$, which is adjacent to exactly one vertex $c'_2 \in I$.

**Proof.** If in $[c_1c'_2c_3 \ldots c_{|C|}]$ (see Fig. 7) there is a chord, then it is inner one and looks as $c'_2c_i$, where $4 \leq i \leq |C|$. But 3-cycles, cutted by the chord, must by Lemma 1 have an empty interiors, and 4- and 5-cycles by the definition of $C$,
must contain exactly one vertex in their interiors, which contradicts to Lemmas 3 and 4. Let $[c_1c'_2c_3\cdots c_{|C|}]$ has no chords, but it’s interior consists of exactly one vertex, or of a pair of adjacent 4- and 6-vertices. But this is impossible by Lemmas 3 and 4, and the cycle $[c_1c'_2c_3\cdots c_{|C|}]$ is a contradiction to the definition of $C$.

From Lemma 2 and chordless properly of $C$ we obtain

**Corollary 4.2.** If $s(v) = 4$, then $v \notin C$.

**Lemma 5.** If $s(v) \geq 7$ and $v \in I_4$, then $|W(v) \cap I^m_4| \leq s(v) - 5$, with the only exception that $W(v) \cap I^4_4 = \emptyset$.

**Proof.** Case 1. $s(v) = 7$.

Subcase 1.1. $W(v) \cap I^5_4 = \emptyset$ (see Fig. 8).

Delete $x, y, z, v, a, c, e, f, g$. Let we obtained $a(e, d, e) = (1, 3, 2)$, demand only, that $a(v) \notin a\{b, f\}$, and $a\{x, z\} \cap \{1, 2\} = \emptyset$.

Subcase 1.2: $|W(v) \cap I^3_4| = 1$ (see Fig. 9a and 9b).

Delete $x, y, z, v, a, c, e, f, g$. Identify in the first case $a, c, e, h, e$; in the second $a, c, e$. Let $a\{a, c, e, h\} \cap \{3, 4, 5\} = \emptyset$, demand that $a(v) \notin a\{b, d\}$ and if it is obtained $a(v) \in a\{g, f\}$, then $a(z) \notin \{1, 2\}$.

Subcase 1.3. $|W(v) \cap I^5_4| = 2$ (see Fig. 10a and 10b).
Delete $x, y, z, v$. In the first case identify $a, d, g$, in the second $b, d, a$. Let $a\{a, b, d, g\} \cap \{3, 4, 5\} = \emptyset$. Among the colors 3, 4, 5 there is such $t$, which is represented at $\{c, e, f, h, i\}$ at most once; put $a(v) = t, a(x, y, v) \cap \{1, 2\} = \emptyset$ with the only exception that $W(y)$ or $W(z)$ already contains all the colors 3, 4, and 5.

Case 2: $s(v) = 8$.

Subcase 2.1: $W(v) \cap I^5_4 = \emptyset$ (see Fig. 11).

Delete $x, y, z, u, v$, insert an edge $bh$ and identify $b, d, f, h$. Let $a(b, h) = (1, 2)$. We put $a(v) \notin a\{c, g\}$, $a(x, y, z, u) \cap \{1, 2\} = \emptyset$, besides, if $a(a) = a(v) = a(e)$, then we moreover demand that $a(x) \neq a(z)$.

Subcase 2.2: $|W(v) \cap I^5_4| = 1$ (see Fig. 12).

Delete $x, y, z, u, v$, insert an edge $bi$, and identify $b, e$ and $g, i$. Let $a(b, i) = (1, 2), a(h) \in \{1, 3\}$. Either among the colors 4 and 5 there is $t$ represented in $a\{a, c, d, f\}$ at most once, then we put $a(v) = t$, and $a(y) \in \{3, 4, 5\}$ except the
case when \( a\{c, d, v\} = \{3, 4, 5\} \), or each of them is represented twice, for example, \( a(a, c, d, f) = (4, 4, 5, 5) \), then we color \( a(x, y, z, v) = (5, 3, 3, 4) \).

Subcase 2.3: \( |W(v) \cap I_4^5| = 2 \) (see Fig. 13a and 13b).

Delete \( x, y, z, u, v \), insert an edge \( ce \), and identify: in the first case \( c, j \), and \( e, g \), and in the second \( c, g, j \). Let \( a(c, e) = (1, 2) \). If some color \( t \in \{3, 4, 5\} \) is represented at most once in \( a\{a, b, d, f, h, i\} \), then we color \( a(v) = t \), and \( x, y, z, u \) with \( 3, 4, 5 \), except the case, when \( W(x) \) or \( W(z) \) contains all the colors \( 3, 4, 5 \). Let now each of the colors \( 3, 4, 5 \) is represented twice on six vertices \( a, b, d, f, h, i \). If \( a\{a, b\} = a\{h, i\} = \{3, 4\} \), then we put \( a(v) = 5 \), \( a(y) \neq a(h) \). Otherwise a color exists, for example \( 3 \), which is represented exactly once in \( \{a, b, h, i\} \), and \( a(a) = 3 \), then we put \( a(v) = 3 \). Let the second vertex colored with \( 3 \) is \( d \). We make \( a(y) \neq a(x) \).

Subcase 2.4: \( |W(v) \cap I_4^5| = 3 \) (see Fig. 14).

Delete \( x, y, z, u, v \) and identify \( a, d, f, i \). Let \( a(d, e) = (1, 2) \). If some \( t \in \{3, 4, 5\} \) is represented on the vertices \( b, c, g, h, k, j \) exactly once, then we put \( a(v) = t \); now if we color \( x, z, u \) with colors from \( \{3, 4, 5\} \) \( \setminus \{t\} \) besides the case, when all this colors are already represented on the neighbour vertices, then bichromatic cycles will not appear: those containing \( v \): because they can not leave the bound of the configuration for the second time: 1, 2-cycles: because some of \( x, z, u \) has a color \( t \) represented twice on it’s neighbour vertices, and for this reason is colored with one of \( 3, 4, 5 \). If all the colors \( 3, 4, 5 \) are represented exactly twice on the vertices \( b, c, g, h, j, k \) for example, \( a(b, c, g, h, j, k) = (3, 4, 4, 5, 5, 3) \), then we color \( a(x, z, u, v) = (2, 3, 4, 5) \). Let now the color \( 3 \) is absent on \( b, c, g, h, j, k \). If we put \( a(v) = 3 \), then bichromatic cycles through \( v \) would be impossible. But there is the only case, when we are forced to color all the \( x, z, u \) with 2, that is \( a(b, c) = a(g, h) = a\{j, k\} = \{4, 5\} \), and a short (i.e. enclosed into the configuration) 1, 2-cycle should inevitably arise. But in this exceptional case, if \( a(c) \neq a(g) \), we can not have simultaneously the two bichromatic
chains: 2, a(c)-chain from c to h, and 3, a(g)-chain from b to g. Let first is absent, then a(v) = a(c), a(x) = a(z) = 2, a(u) = 3. Let, conversely, a(c) = a(g) = 4, and k can be supposed to have color 4, then (x, v, u) are colored similarly either as (2, 4, 2), or as (3, 5, 3), and a(z) ∈ \{2, 3\} \setminus a(x).

Case 3: s(v) > 8.

By Lemma 3, v can not be adjacent to more, than \(\left\lfloor \frac{1}{2}s(v) \right\rfloor\) vertices of \(I^m_4\). But \(\left\lfloor \frac{1}{2}s(v) \right\rfloor < s(v) - 4\), when \(s(v) > 8\).

**Definition 6.** Call an amount \(e(v) = s(v) - 6\) a *contribution* of a vertex \(v \in I\); for \(c \in C\) we define a *contribution* to be \(e(c) = s(c) - 3\).

The purpose of the two succeeding sections is certain redistribution of amounts just defined for the vertices of \(\bar{I}\), i.e. the construction of a function \(e' : V(\bar{I}) \to R\), which satisfies
\[
\sum_{v \in \bar{I}} e'(v) = \sum_{v \in \bar{I}} e(v),
\]
and results in nonnegative *modified contribution* \(e'(v)\) for all vertices \(v \in \bar{I}\). We shall proceed the elucidation of the structure of \(G\) in a degree, necessary for a solution of the stated task.

The vertices of degree greater, than 6, are called *major*.

**Definition 7.** A vertex \(v \in I\), \(s(v) \geq 6\), is called 1-*weak*, if it is adjacent to exactly \(s(v) - 5\) vertices of \(I^m\).

**Definition 8.** A vertex \(v \in I\), \(7 \leq s(v) \leq 8\), is called 2-*weak*, if it is adjacent to exactly \(s(v) - 4\) vertices of \(I^5\).

**Definition 9.** We call a vertex *weak*, if it is either 1-weak, or 2-weak.

**Definition 10.** A major vertex of \(I\) is called *strong*, if it fails to be weak.

By Lemma 5, each major vertex of \(I\) is either weak, or strong.

**Definition 11.** A vertex of \(I^4\) is called *particular*, if it is adjacent to two nonadjacent vertices of \(I^6\).

**Definition 12.** A major 1-weak vertex is called *special*, if it is adjacent to a single 4-vertex, which is a particular one.

By Lemma 4, a special vertex may have degree 7, or 8; we shall call them 7-special and 8-special, respectively.

**Definition 13.** A strong vertex \(v \in I\) is called *singular* for \(v' \in I^5 \cap W(v)\), if it is adjacent to exactly \(s(v) - 6\) vertices of \(I^m\), including a particular vertex, \(v_1\) and \(d_{W(v)}(v_1, v') = 3\).

2.2. *The formation of nonnegative modified contributions for the vertices of \(I^5\)*

Each 5-vertex \(v \in I\) receives 1 from every adjacent: (a) vertex of \(C\); (b) strong vertex, which is not singular for it; (c) 1-weak 8-vertex, which is adjacent to three vertices of \(I^5\), if \(v\), besides this, is adjacent to four vertices of \(I^6\). If it is not adjacent to \(C\), it receives \(\frac{1}{2}\) from every adjacent singular for it vertex, major 1-weak nonspecial vertex, and 2-weak 8-vertex. It receives \(\frac{1}{2}\) from every adjacent 2-weak 7-vertex, and \(\frac{1}{4}\) from 8-special vertex.

**Proposition 1.** Every 5-vertex of \(I\) has a nonnegative modified contribution.

**Lemma 6.** Let \(z \in I^5\), \(W(z) \cap C = \emptyset\), \(y \in I^6\), and \(v\) is 2-weak, then a face \([yzv]\) in \(G\) is impossible.

**Proof.** Case 1. \(s(v) = 8\).

The statement follows from Lemma 4.
mean, that there is 4-cycle be shown, that in Lemma 8. reducible (by Lemma 4, situation is symmetric on and y is adjacent a, b, c, d, e, f, g, h, i, j marks 1 and 2, this transformation does not result in loops and multiple edges. All possible colorings of the vertices appear while identifying vertices, which are connected by chains of the length 3 along the configurations. It is checked in a similar fashion, that in all succeeding configurations, loops and multiple edges can not Remark 2. It is checked in a similar fashion, that in all succeeding configurations, loops and multiple edges can not be a contradiction to the definition of C. Let \( I^\circ \in \bar{I} \), then obviously, \( a, g \in C \). One of the chains, which arise while dividing C by a and g, has the length at most 3, because \( |C| \leq 6 \). But then if the other one is replaced by avyg, an element of \( S_5 \cup S_6 \), imbedded into \( \bar{I} \) is obtained, which is impossible.

**Remark 2.** It is checked in a similar fashion, that in all succeeding configurations, loops and multiple edges can not appear while identifying vertices, which are connected by chains of the length 3 along the configurations.

So, let we obtained \( a(a, e, f) = (1, 2, 3) \).

(1) \( a(h) = 2 \): take \( a(z) = 3 \). If some color of \{3, 4, 5\}, for example, 4, is represented in \( a\{b, c, j, k\} \) at most once, then it is enough to put \( a(v, y) = (4, 5) \). Let, conversely, \( a\{b, c\} = a\{j, k\} = \{4, 5\} \), then we color \( a(x, y, u, v) = (2, 5, 3, 4) \).

(2) \( a(h) = 4 \): take \( a(y) = 5 \). If some color from \{2, 3\} is absent in \( a\{b, c\} \) or \( a\{j, k\} \), for example, \( 2 \notin \{b, c\} \) (the situation is symmetric on \{b, c\} and \{j, k\}), then we put \( a(v, z) = (2, 3) \), and color u with 5 only if forcedly, i.e. when \( a\{j, k\} = \{3, 4\} \). Let, conversely, \( a\{b, c\} = a\{j, k\} = \{2, 3\} \), then make \( a(x, z, u, v) = (5, 3, 5, 4) \).

**Lemma 7.** A vertex of \( I^5 \) can not be surrounded by five vertices of \( I^6 \).

**Proof.** (see Fig. 16). Delete x, y, z, u, v, w, insert an edge ce, and identify a, c, i, and e, g. By Lemma 4 and Remarks 1 and 2, this transformation does not result in loops and multiple edges. All possible colorings of the vertices a, b, c, d, e, f, g, h, i, j are considered, which are not equivalent under bichromatic interchange and mirror reflection, and sequences of these colorings are given (Table 1) to the acyclic colorings of G.

**Lemma 8.** If \( v \in I^2 \), and there is a chain yzd in W(v), where z, d \( \in I^6 \), and y is a major 1-weak vertex, then s(y) = 8, and y is adjacent, besides v, with two vertices of \( I^5 \).

**Proof.** We must show, that each of the seven configurations, obtained from the configuration on the Fig. 17 by “planting” of one or two vertices of \( I^m \) (the case when they both have degree 5 is the only exception) on the edges ab, and bc, is reducible (by Lemma 4, \( s(v) \leq 8 \)).

Delete options \( x_1, x_2 \), and the vertices y, z, v. If there is a 5-vertex between \( x_1, x_2 \), then we identify those vertices adjacent to it on the bound of the configuration (i.e. a, b, or b, c); if 5-vertex is absent, then we do the same with some
Table 1

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of 4-vertices of \{x_1, x_2\} \neq \emptyset. Afterwards we identify also a, d, f. By Remarks 1 and 2, loops and multiple edges do not appear. Those vertices adjacent to \(x_1\) and not shown at Fig. 17, are denoted by \(x'_i, x''_i\), where \(i = 1, 2\).

Case 1: No 5-vertices between \(x_1, x_2\).

We shall assume, that \(x_1\) is an identified 4-vertex, and let \(a(a, h, g) = (1, 2, 3)\).

(1) \(a(b) = 2\).

(1.1) \(a(e) \in \{2, 3\}: \) demand that \(a(v) \in \{2, 3\}\), then \(a(y) \in \{4, 5\}\backslash a(x'_1)\).

(1.2) \(a(e) = 4: \) if \(a(x'_1) \neq 3\), then \(a(y, z, v) = (3, 5, 2)\); moreover, if \(a(x'_2) = 3\), then \(a(x_2) = 4\), else \(a(y, z, v) = (4, 5, 3)\), and in case if \(a(x'_2) = 4\), then \(a(x_2) = 5\).

(2) \(a(b) = 4\).

(2.1) \(a(e) = 2: \) if \(a(x'_1) \neq 3\), then we take \(a(y, z, v) = (3, 5, 4)\), moreover, if \(a(x'_2) = 3\), then \(a(x_2) = 2\); but if \(a(x'_1) = 3\), then \(a(y, z, v) = (5, 4, 3)\).

(2.2) \(a(e) = 4: \) if \(a(z) = 5; a(y) \in \{2, 3\}\backslash a(x'_1)\); if arise \(a(y) = a(x'_2)\), then we demand in addition \(a(x_2) \neq 5\).

(2.3) \(a(e) = 5: \) if \(a(x'_1) \neq 5\), then \(a(y, z, v) = (5, 4, 3)\); else if \(a(x'_1) = 5\), then \(a(y, z, v) = (2, 5, 4)\), and if \(a(x'_2) = 2\), then \(a(x_2) = 3\).

Case 2: \(s(x_1) = 5\).

Let \(a(a, h, g) = (1, 2, 3)\).

(1) \(a(b) = 2\).

(1.1) \(a(e) \in \{2, 3\}: \) put \(a(y, z) = (5, 4)\); demand: if \(5 \in a[x'_1, x''_1]\), then \(a(x) \neq 1\), and if \(a(x'_2) = 5\), then \(a(x_2) \neq a(x_1)\).

(1.2) \(a(e) = 4: \) if \(3 \notin a[x'_1, x''_1]\), then \(a(y, z, v) = (3, 5, 2)\), and if \(a(x'_2) = 3\), then \(a(x_2) = 4\); if \(4 \notin a[x'_1, x''_1]\), then \(a(y, z, v) = (4, 5, 3)\), and if \(a(x'_2) = 4\), then \(a(x_2) = 5\). Let now \(a[x'_1, x''_1] = \{3, 4\}\); if \(a(x'_2) \neq 4\), then \(a(x_1, y, z, v) = (5, 4, 5, 3)\), otherwise \(a(x_1, y, z, v) = (5, 3, 4, 5)\).

(2) \(a(b) = 4\).

(2.1) \(a(e) = 2; a(y, z, v) = (5, 4, 3)\); \(x_1\) is colored with 1 only if forcedly, i.e. when \(5 \notin a[x'_1, x''_1]\); if \(a(x'_2) = 5\), then \(a(x_2) \neq a(x_1)\).

(2.2) \(a(e) = 4: \) if \(a[x'_1, x''_1] \neq \{2, 3\}\), then \(a(y) \in \{2, 3\}\backslash a(x'_2), a(z) = 5\), and if \(a(y) \in a[x'_1, x''_1]\), then \(a(x_2) \neq \{2, 3\}\). Let \(a[x'_1, x''_1] = \{2, 3\}\); if there is no 2, 4-chain \(h, e\), then \(a(y, z, v) = (5, 4, 2)\), else 3, 5-chain from \(g\) to \(x'_1, x''_1\) is absent, and we put \(a(x_1, y, z, v) = (5, 3, 5, 2)\), and in case \(a(x'_2) = 3; a(x_2) = 2\).

(2.3) \(a(e) = 5: \) if \(a[x'_1, x''_1] \neq \{2, 3\}\), then \(a(y) \in \{2, 3\}\backslash a[x'_1, x''_1], a(z, v) = (5, 4)\); if we have \(a(x'_2) = a(y)\), then \(a(x_2) \in \{2, 3\}\). Let \(a[x'_1, x''_1] = \{2, 3\}\); put \(a(x_1, y, z, v) = (1, 5, 4, 3)\) and if \(a(x'_2) = 5\), then \(a(x_2) = 2\).

**Proof of Proposition 1.** Let \(v\) be any 5-vertex of \(I\), and \(e'(v)\) it’s modified contribution.

(1) If \(v\) is adjacent to \(C\), or strong vertex, which is not singular for it, then \(e'(v) \geq -1 + 1 = 0\).

Note, that if \(v\) is adjacent to a special or singular for it vertex \(w\), and \(u \in W(w) \cap I^b\) is adjacent to a 4-vertex of \(W(w)\), then that vertex \(v', \) “joint”, adjacent to \(u, w, u\) (see Fig. 18), by Lemma 4, is not a 6-vertex, not special, and by Lemma 6 not 2-weak. So \(v'\) is a major 1-weak and contributes to \(e'(v)\) at least \(\frac{1}{2}\); for this reason, if \(w\) is singular for \(v\), then \(e'(v) \geq -1 + 2\left(\frac{1}{2}\right) = 0\).
(2) Let \( v \) is surrounded by weak vertices.

(2.1) If \( v \) is adjacent to an 8-special vertex \( w \), and \( u \) is one of it’s joint, and \( u' \) is a vertex, dual to \( u \) with respect to the edge \( vw \), then \( u' \) is not, by Lemma 4, 6-vertex, and obviously is not 7-special, so \( e'(v) \geq -1 + 2 \frac{1}{2} + \frac{1}{2} = 0 \).

(2.2) Let there are no 8-special vertices in \( W(v) \), but there are two 7-special vertices \( w_1 \) and \( w_2 \), then if their joint vertices do not coincide, we have \( e'(v) \geq -1 + 2 \frac{1}{2} = 0 \). Suppose, we have a configuration, shown at the Fig. 19.

By Lemma 6, in \( W(v') \) there is a chain of 7 vertices, among which at most one vertex of \( I^m \) is possible, namely, \( v \). It is easily seen, that \( v' \) can not be 1-weak, which contradicts to the assumption made in (2).

(2.3) Let there is exactly one 7-special vertex \( y \) in \( W(v) \); it’s joint vertex \( d \) contributes \( \frac{1}{2} \) to the \( e'(v) \), hence we should exclude the cases, when \( v \) besides \( y \) and \( d \) is adjacent either to two 6-vertices of \( I \) and 2-weak 7-vertex \( w \), or to three vertices of \( I^6 \). But in the first one \( w \) should be adjacent to some 6-vertex of \( I \), which is impossible by Lemma 6. In the second case (see Fig. 20) we should have a configuration, which has been already shown to be reducible (Lemma 8, Case 1).
(2.4) Let now $v$ is not adjacent to special vertices. In view of Lemma 7, we are to consider the following cases.

(2.4.1) $v$ is adjacent to four 6-vertices and major vertex $w$. By Lemma 6, $w$ is not 2-weak. Then, By Lemma 8, $s(w) = 8$, and $w$ is adjacent, besides $v$, to else two 5-vertices. As we remember, in such a situation 5-vertex $v$ receives 1 of modifying contribution, and again $e'(v) = -1 + 1 = 0$.

(2.4.2) $v$ is adjacent to one, two, or three 6-vertices. Let $v_1v_2\cdots v_l$ is a maximal chain of 6-vertices in $W(v)$. Then $v_1$ and $v_{l+1}$ are two distinct major vertices, which are not 2-weak by Lemma 6, because they have 6-neighbours of $I$ $v_1$ and $v_{l+1}$, respectively; hence, $e'(v) \geq -1 + 2\left(\frac{1}{4}\right) = 0$.

(2.4.3) If $W(v)$ consists of major vertices, each of them contributes at least $\frac{1}{4}$ to $e'(v)$, and $e'(v) \geq -1 + 5\left(\frac{1}{4}\right) > 0$.

This completes the proof of the Proposition 1.

2.3. Formation of nonnegative modified contribution for the vertices of $I^4$

Definition 14. A major vertex of $I$ is called fat, if it is adjacent to at least two vertices of $I^5$.

Definition 15. A major 1-weak vertex of $I$ is called quasistrong, if it is adjacent to a vertex of $I^5$, which in turn, is adjacent to $C$.

Definition 16. 1-weak 7-vertex is called bad for a particular vertex $v$, if it is adjacent to two vertices of degree 4, one of which is $v$.

If $v \in I^4$, then every adjacent vertex, except those of $I^6$, contributes at least $\frac{1}{4}$ to $e'(v)$. Moreover, $v$ receives at least 1 from every adjacent strong vertex, or vertex of $C$, as well as from quasistrong, fat, or 7-special vertex. $v$ receives 1 from weak 8-vertex provided that $v$ is particular, and this 8-vertex is not special. At last, if $w$ is strong and adjacent to less, then $s(w)$-6 vertices of $I^m$, or it is singular, or 8-special, or one of $C$, then it transmits $\frac{3}{2}$ to every particular neighbour.

Proposition 2. If $v \in I^4$, then $e'(v) \geq 0$.

Lemma 9. 4-vertex $v \in I$ can not be adjacent to 6-vertex $y \in I$ and two nonadjacent 1-weak nonfat vertices $x, z$ except when both $x$ and $z$ are quasistrong.

Proof. Each configuration of the type “4-vertex $v \in I$, adjacent to 6-vertex $y \in I$ and two nonadjacent 1-weak vertices $x, z$” can be generated from “4-vertex $v \in I$, adjacent to three 6-vertices of $I$” by “planting” of minor vertices $x_1, x_2, x_3, z_1, z_2, z_3$ of $I$ to the edges $ab, bc, ac, ah, hg, gf$, respectively (see Fig. 21). This results from lemmas, proved in the Section 2.1.

Let $w$ denotes any of the vertices $x_1, x_2, x_3, z_1, z_2, z_3$. Then the set of vertices, adjacent to $w$ and not shown at Fig. 21, consists of $w'$, if $s(w) = 4$, and of $w', w''$, if $s(w) = 5$.

Remark 3. By Remarks 1 and 2, among edges and chains of the length 2, connecting the pairs of vertices of $a, b, c, d, f, g, h$, only 2-chains, connecting $b, c$ and $g, h$ are possible. Moreover, if such chains exist, then an intermediate vertex $l^o$ of each of them does not belong to $I$. At last, $C$ should then be divided by two representatives of $\{b, c\}$ and $\{g, h\}$ into two chains of the length 3.

Delete $x, y, z, v, x_1, z_1$ from $G$ and identify $a, d, f$.

If $\{x_i\}$ contains 5-vertex (unique, because $x$ is not fat), then we identify the two vertices of $W(x)$, adjacent to it, and other vertices of $\{x_i\}$, if present, replace by edges, whose endpoints belong to $W(x)$. Let $\{x_i\}$ consists of 2-vertexes. If $\{x_i\}$ contains only one 4-vertex, then it is replaced by an edge. If at least two, and $x_2$ is one of them, then similarly $x_2$ is identified, and others are replaced by edges. If there are exactly two vertices, $x_1$ and $x_2$, then in case when there are no 2-chains from $b$ to $\{g, h\}$, we identify $x_1$ and replace $x_3$ by an edge. In the remaining case, in view of Remark 3, there are no 2-chains from $c$ to $\{g, h\}$, then we identify $x_3$ and replace $x_1$ by an edge.
The similar work is done with \( \{z_i\} \).

Remark, that the described transformation does not create loops and multiple edges. Indeed, if we identify a vertex from \( \{b, c\} \) with a vertex of \( \{g, h\} \), and both are adjacent to 5-vertices of \( I \), then because \( x, z \) cannot be quasistrong simultaneously by assumption, at least one of contracted vertices is one of \( I \), so a 6-cycle of \( S_6 \) arise imbedded into \( I \), which is impossible by the definition of \( C \).

Case 1: Neither \( x \), nor \( z \) are contracted.

Let \( a(a, h, g) = (1, 2, 3) \). In the circumstances, when there are no contractions at \( x \) (or at \( z \)), the general rule acts: \( x, y, z, v \) being colored, we should not color \( x_i \) by colors of \( a(a, b, c) \) but forcedly.

1. \( a(b, c) = \{2, 3\} \).
2. \( a(e) = 2 \): we put \( a(x, y, z, v) = (4, 3, 5, 2) \).

Let conversely, from symmetry considerations, all four \( a, b, c \), \( t \) are present, then we put \( a(x, y, z, v) = (5, 3, 5, 4) \), and \( \{x_i\} \), \( \{z_i\} \) color in spite of the general rule by colors from \( a(a, b, c) \), \( a(a, h, g) \), if it is possible.

2. \( a(b, c) = \{2, 4\} \).
3. \( a(e) = 2 \): make \( a(x, y, z, v) = (3, 5, 4, 2) \).
4. \( a(e) \in \{4, 5\} \): put \( a(x, y) = (3, 2) \), then demand that \( a(z) \in \{4, 5\}\{a(e)\} \) \( v \) is colored forcedly.

Case 2: There is a contraction at \( z \), but not at \( x \).

Let \( a(a, b, c) = (1, 2, 3) \). Define \( M_z \) as a set of colors \( t \neq 1 \), satisfying: (a) \( t \notin a(g, h) \); (b) \( t \) is represented at most once in \( a\{z_i', z_i''\} \); (c) \( t \) does not coincide with a color \( a(z_i') \), if \( z_i' \) is a contracted 4-vertex. Clearly, \( M_z \) is always nonempty, moreover, if \( |M_z| = 1 \), then \( t \in M_z \) is absent in \( a\{z_i', z_i''\} \) at all. Remark also, that while coloring \( z \) by a color from \( M_z \) and such admissible coloring of \( \{z_i\} \), that every \( z_{i^{\phi}} \) is not colored with an element of \( a(a, g, h) \) but forcedly, i.e. when \( z_{i^{\phi}} \) is 5-vertex, which has all other colors in it's neighbourhood, bichromatic cycles, avoiding \( a, f, y, v \), can not appear.

1. \( a\{g, h\} \ni 2 \).
2. \( a(e) \in \{2, 3\} \): if \( M_z \cap \{4, 5\} \neq \emptyset \), then we take successively \( a(z) \in M_z \cap \{4, 5\} \), \( a(x) \in \{4, 5\}\{a(z)\} \) \( y, v \) are colored forcedly, and \( x_i \) by general rule from Case 1), else \( 3 \notin a\{z_i', z_i''\} \) and put \( a(x, y, z) = (5, 4, 3) \).

1. \( a(e) = 4 \).
2. \( a(x, y, z, v) = (5, 2, 3, 4) \).
3. \( 5 \notin a\{z_i', z_i''\} \); \( a(x, y, z, v) = (5, 3, 5, 4) \).
4. \( 4 \notin a\{z_i', z_i''\} \); \( a(x, y, z, v) = (5, 3, 4, 2) \).
(1.2.4) Let $3 = a(z'_{i_1})$; if $z_{i_1}$ is a contracted 4-vertex, and $a(z'_{i_2}) = 4$, then put $a(x, y, z, v) = (5, 3, 4, 2)$, $a(z_{i_2}) = 5$; else if $s(z_{i_1}) = 5$, and $a(z'_{i_2}) = 3$, we color $z_{i_1}$ by a color of $\{4, 5\} \setminus a(z'_{i_1})$, and $z_{i_2}$ by a color of $\{4, 5\} \setminus a(z'_{i_2})$.

(2) $a\{g, h\} \ni 4$.

(2.1) $a(e) = 2$: if $M_x \cap \{3, 5\} \neq \emptyset$, we put $a(x, v) = (4, 2)$, $a(z) \in M_x \cap \{3, 5\}$, else $a(x, y, z, v) = (5, 3, 2, 4)$.

(2.2) $a(e) = 4$: if $M_x \cap \{2, 3\} \neq \emptyset$, then $a(x, v) = (5, 4)$, $a(z) \in M_x \cap \{2, 3\}$, else $a(x, y, z, v) = (5, 3, 5, 4)$.

(2.3) $a(e) = 5$: if $M_x \cap \{2, 3\} \neq \emptyset$, then $a(x, v) = (4, 5)$, $a(z) \in M_x \cap \{2, 3\}$, else $a(x, y, z, v) = (4, 2, 5, 3)$.

Case 3: Both $x, z$ are contracted.

Let $a(a) = 1$, $a\{b, c\} \ni 2$; $M_x$ is defined similarly as $M_z$.

(1) $a(h, g) \ni 2$.

(1.1) $a(e) = 2$.

(1.1.1) A color of $\{3, 4, 5\}$, for example $3$, is absent at $\{x'_i, x''_i\}$: if $M_x \ni 3$, we color $a(x, y, z, v) = (3, 4, 3, 5)$.

But if a short 1, 2-cycle arise, that is $\{x_i\}$ consists of a single 5-vertex, which is forcedly colored with 1 or 2, we put $a(x, y, z, v) = (5, 4, 2, 3)$, $a(x_i) = 3$. Let $M_x \ni 4$, then $a(x, y, z, v) = (3, 4, 5, 2)$, and if a short 1, 2-cycle inevitably arise, we color $a(x, y) = (5, 3)$, and $x_i$ with $3$. A case $M_x \ni 5$ is equivalent to the just considered one.

(1.1.2) $a(x'_i, x''_i)$ and $a(z'_i, z''_i)$ both contain $\{3, 4, 5\}$.

(1.1.2.1) Let $x_{i_1}$ be a contracted 4-vertex, and $a(x'_{i_1}) = 3$, so there are also $x_{i_2}$ and $x_{i_3}$, and $a(x'_{i_2}) = 4$, $a(x''_{i_2}) = 5$. Demand that $a(z) \in M_x$, $a(x) \in \{4, 5\} \setminus a(z)$.

(1.1.2.2) Now $\{x_i\}$ and $\{z_i\}$ both contain a 5-vertex. Colors $t \in M_x$, $r \in M_z$, $t \neq r$, can be find, because else $M_x = M_z$ and this sets have cardinality one, but in this case some color should be absent at $\{x'_i, x''_i\}$ which contradicts to the assumption made. We let $a(x, z) = (t, r)$.

(1.2) $a(e) = 3$.

(1.2.1) $\{(4, 5), (5, 4)\} \cap M_x \times M_z \neq \emptyset$, for example, $4 \in M_x$, $5 \in M_z$: let $a(x, z) = (4, 5)$. If a short 1, 2-cycle inevitably arise, i.e. $\{x_i\}$ and $\{z_i\}$ both consist of a single 5-vertices, which can be colored only with 1 or 2, then we recolor $x$ with 5, $y$ with 4, and $x_i$ color with 4.

(1.2.2) $\{(3, 4), (3, 5), (4, 3), (5, 3)\} \cap M_x \times M_z \neq \emptyset$, for example, $3 \in M_x$, $5 \in M_z$. If this case is not reducible to the previous one, then $4 \notin M_z$. If the color 3 is present at a 4-vertex $x_{i_1}$—noncontracted—we take $a(x_{i_1}) = 5$, $a(x, y, z, v) = (3, 4, 5, 2)$. If $a(x'_{i_1}) = 3$, and $s(x'_i) = 5$, we can not do the same only if $a(x'_i, x''_i) = (3, 5)$. Then put $a(x, y, z, v) = (4, 2, 5, 3)$, and if $a(x''_{i_1}) = a(x''_{i_3}) = 4$, then demand in addition $a(x'_{i_2}) \neq a(x_{i_3})$.

(1.2.3) $\{(3, 3), (4, 4), (5, 5)\} \cap M_x \times M_z \neq \emptyset$: if $(t, t) \in M_x \times M_z$, where $t \in \{4, 5\}$, we put $a(x, v, z) = (t, 3, t)$. At last, let $M_x \times M_z = \{(3, 3)\}$, then if there is a contracted vertex $x_{i_1}$ of degree 4, we put $a(x) \in \{4, 5\} \setminus a(x_{i_1})$, $a(x, v) = (2, 3)$, else $s(x_{i_1}) = 5$, and we put $a(x, y, z, v) = (5, 4, 3, 2)$, $a(x_{i_1}) = 3$, and that $x_{i_2}$, which has $a(x'_{i_2}) = 5$, color with 4.

(2) $a\{g, h\} \ni 3$.

(2.1) $a(e) = 2$.

(2.1.1) $\{(4, 5), (5, 4), (3, 5), (5, 3)\} \cap M_z \times M_z \neq \emptyset$: take one of these pairs, $x$ color with the first element of it, and $z$ with the second one.

(2.1.2) $\{(4, 5), (5, 4)\} \ni M_x \times M_z \neq \emptyset$: for example, $(4, 4) \in M_x \times M_z$. In view of previously proved, let $M_x \cap \{3, 5\} = \emptyset$, i.e. $4 \notin a(x'_i, x''_i)$; we take $a(x, y, z, v) = (4, 3, 4, 5)$.

(2.1.3) $(t, 2) \in M_x \times M_z$: where $t \in \{3, 4, 5\}$: in view of previously done, $2 \in a(z'_{i}, z''_{i})$; put $a(x, z) = (t, 2)$, $a(y) \in \{4, 5\}$. If a short 1, 2-cycle arise, we interchange colors of $x$ and $v$, and $x_i$ color with an element of $\{3, 4, 5\}$.

(2.2) $a(e) = 4$.

(2.2.1) $\{(3, 2), (3, 5), (5, 2)\} \cap M_x \times M_z \neq \emptyset$: see (2.1.1).

(2.2.2) $\{(4, 5), (5, 4)\} \cap M_x \times M_z \neq \emptyset$: for example, $(4, 5) \in M_x \times M_z$: put $a(x, y, z, v) = (4, 3, 5, 2)$. Let $a(x''_{i_1}) = 4$. We can not color $x_{i_1}$ with 5 only if $s(x_{i_1}) = 5$, and $a(x''_{i_1}) = 5$. Then recolor $a(x, y) = (3, 2)$, and if $a(x''_{i_2}) = a(x'_{i_3})$, demand that $a(x_{i_2}) \neq a(x_{i_3})$.
Lemma 10. If \( z \) is bad for a 4-vertex \( v \), and \( x \) is a strong vertex of \( W(z) \), adjacent to exactly \( s(x) - 6 \) vertices of \( I^m \), then \( x \) is singular.

Proof. Suppose on the contrary, that \( s(x_1) = s(x_3) = 4, 4 \leq s(x_2) \leq 5 \) (see Fig. 22). The vertices adjacent to \( x_2 \) and not shown at the Fig. 22, we denote by \( x_2' \) and, if \( s(x_2) = 5, x_2'' \).

Delete \( x, y, z, u, v, z', \{x_i\} \) from \( G \) and insert an edge \( eg \). Further, if \( s(x_2) = 5 \), identify \( a, b \) and replace \( x_1, x_3 \) by edges \( aj, bc \). Let \( \{x_i\} \) has no 5-vertex, then if \( |\{x_i\}| = 1 \), or \( |\{x_i\}| = 2 \), but \( x_2 \) is absent, we only insert edges.

In the opposite case identify \( a, b \) and replace remaining \( x_i \) by edges. At last, identify \( c, e, h, g \). In view of Remarks 1 and 2, loops and multiple edges under this transformation do not appear.

We may assume that \( a(c) = 1 \) in the coloring obtained, further, if \( a, b \) were not contracted, then \( a(a, b) = (2, 3) \), else \( a(a) = 2 \), and \( a(x_2') \in \{1, 3\} \).

At once color \( v \) with 1. Clearly, if \( a(x) \neq a(z) \), short cycles do not arise, except the case when \( x_1, x_3 \) are absent, \( s(x_2) = 5, x_2 \) is colored with 2 and either \( a(d) = a(z) = a(i) = 2 \), or \( a(y) = a(u) = 2 \).

Case 1: Either \( x_2 \) is absent, or \( s(x_2) = 4 \).

1. There is \( t \in \{4, 5\} \) which is absent at \( \{d, i\} \cup \{x'_i\} \); put \( a(x) = t \); \( z \) color with a color different from \( a(x) \) and represented at most once at \( \{d, i\} \); moreover, if \( |\{a, g, d, i\}| = 4 \), we can choose also \( a(z) \neq a(f) \). If it so happened \( a(z) = a(d) \), we put \( a(y) \neq a(g) \) (from the coloring point of view, the configuration is symmetric in respect to the horizontal axis).

2. There is \( t \in \{4, 5\} \) represented at \( \{d, i\} \cup \{x'_i\} \) exactly once: if just at \( \{x'_i\} \), then the solution of (1) is valid (respective \( x_i \) is colored with 4 or 5). Let \( t = 4 = a(d) \); put \( a(x, y) = (4, 5); a(z) \in \{2, 3\} \). If we obtained \( a(z) = a(i) \), then make \( a(u) \neq a(g) \); if else \( a(z) = a(f) \), put \( a(z') \neq a(u) \).

3. 4 and 5 are represented at \( \{d, i\} \cup \{x'_i\} \) twice.

\( a(x'_i) = a(x'_i) = 4 \); \( x \) can not be colored with 4 without such a corollary, that after coloring of only \( x_1, x_3 \), bichromatic cycles inevitably appear, only in the case when \( a \) and \( b \) were not contracted and there were chains: 4,5-between \( x'_i \) and \( x'_3 \), 2,4-between \( b \) and \( x'_i \), and 3,4-between \( a \) and \( x'_3 \). But then we color \( x \) with 5 and in coloring \( y \) and \( u \) with 2 or 3, bichromatic cycles do not appear. Put \( a(z) \in \{2, 3\}, a(u) \in \{2, 3\}, a(y) = 4 \). Let now \( x \) can be colored with 4 and in certain coloring of \( \{x_i\} \) bichromatic cycles do not appear. We take \( a(z) \in \{2, 3\} \).
(3.2) \( a(x_1') = 4 \), \( a(x_2') = 5 \): if \( a, b \) are not contracted, we put \( a(x_1, x) = (3, 4) \), \( a(z) \in \{2, 3\} \), and that of \( y, u \) adjacent to a vertex of \([d, i]\) colored with 4, we color with 5. Let \( a(a, b) = (2, 3) \). If there is no 4, 3-chain \( b, x_1' \), then repeat the solution just given. Else there is no 2, 5-chain \( a, x_3 \) and the argument is repeated symmetrically.

**Case 2:** \( s(x_2) = 5 \).

(1) Some \( t \in \{4, 5\} \) is absent at \([d, i] \cup \{x_1', x_2''\} \): the proof given in (1) of the Case 1 is invalid only if a short 1, 2-cycle appear which passes \( y, u \). This means that \( x_1 \) and \( x_3 \) are absent, \( x_2 \) is forcedly colored with 1, and \( y, u \) with 2. If \( a(x) \neq 2 \), it is enough to interchange the color of \( z \) with the color 2 of \( y \) and \( u \). Else we interchange the colors of \( x \) and \( z \), and color \( x_2 \) differently from 2.

(2) \( t \in \{3, 4, 5\} \) is represented at \([d, i] \cup \{x_1', x_2''\} \) exactly once: repeat the proof of (2), Case 1.

(3) 3, 4, and 5 are represented twice at \([d, i] \cup \{x_1', x_2''\} \): let \( a(x_2', x_2'') = (3, 4) \), put \( a(x) = 3 \), \( a(x_2) = 5 \). If the second vertex colored with 3 is \( x_i' \), then put \( a(x_i) = 4 \), and act further as in (1) of the Case 1. If \( a(d) = 3 \), then put \( a(y) = 4 \) and act further as in (2) of the Case 1.

This completes the proof of Lemma 10.

**Proof of the Proposition 2.** Let \( v \in I^4 \), and \( W(v) = \{v_1 v_2 v_3 v_4\} \).

(1) \( W(v) \cap I^6 = \emptyset \). Each of four vertices of \( W(v) \) contributes to \( e'(v) \) at least \( \frac{1}{2} \) and \( e'(v) \geq 2 + 4(\frac{1}{2}) = 0 \).

(2) \( |W(v) \cap I^6| = 1 \). By Lemma 9, either one of \( v_2, v_4 \) belongs to \( C \), either is strong or fat and contributes 1 to \( e'(v) \) and then \( e'(v) \geq 2 + 1 + 2(\frac{1}{2}) = 0 \), or they are both quasistrong and contribute by 1 each.

(3) \( |W(v) \cap I^6| = 2 \).

(3.1) \( v_1, v_2 \in W(v) \cap I^6 \). By Lemma 9, each of \( v_3, v_4 \) is either strong, or fat, else belongs to \( C \), so contributes 1 to \( e'(v) \) and \( e'(v) \geq 2 + 2(1) = 0 \).

(3.2) \( v_1, v_3 \in W(v) \cap I^6 \). Suppose, there is a vertex \( v_2 \) among \( v_2, v_4 \), which fails to belong to \( C \), either be strong or special, or to have degree 8, i.e. \( v_2 \) is a bad vertex (if they both are not bad, then they contribute 1 each and \( e'(v) \geq 2 + 2(1) = 0 \)). Remark, that in this case \( v_4 \) can not be bad by Lemma 9.

(3.2.1) \( v_4 \in C \): \( v \) receives \( \frac{3}{4} \) from \( v_4 \) and \( e'(v) \geq 2 + \frac{3}{2} + \frac{1}{2} = 0 \).

(3.2.2) \( v_4 \) is strong: if it is adjacent to less than \( s(v_4) - 6 \) vertices of \( I^m \), then it transmits \( \frac{3}{2} \) to \( v \) and \( e'(v) \geq 2 + \frac{3}{2} + \frac{1}{2} = 0 \). In the opposite case, in view of Lemma 10, \( v_4 \) is singular and again transmits \( \frac{3}{2} \) to \( v \).

(3.2.3) \( v_4 \) is weak: by Lemma 9, \( v_4 \) is 8-special and gives \( \frac{3}{4} \) to \( v \).

(4) \( |W(v) \cup I^6| < 3 \) by Lemma 9.

### 2.4. Completion the proof of the Theorem 2

The Euler formula for the plane graph \( \bar{I} \) looks as follows:

\[
\sum_{v \in I} (s(v) - 6) = -6 - 2|C|,
\]

or

\[
\sum_{v \in C} (s(v) - 3) + \sum_{v \in I} (s(v) - 6) = |C| - 6,
\]

(1)

where by \( s(v) \) of a vertex \( v \in C \) we mean it’s degree in \( \bar{I} \), but not in \( G \).

Let a vertex \( v \in I \) is adjacent to \( k \) particular vertices and \( l \) other vertices of \( I^m \).

If \( v \in C \), then by Corollary 4.1, \( s(v) \geq 4 \), and by Lemmas 3 and 4, \( 2k + l \leq \frac{1}{2} s(v) - 2 \); \( v \) has given at most \( \frac{3}{4} k + l \) to it’s \( I^5 \)-neighbours, and \( e'(v) \geq s(v) - 3 - \frac{3}{2} k - l \geq s(v) - 3 - 2k - l \geq s(v) - 3 - \frac{3}{2} (s(v) - 2) \geq 0 \).

Let \( v \) is a major vertex of \( I \). If \( v \) is 2-weak, then it’s positive contribution is distributed among \( I^5 \)-neighbours, and \( e'(v) = 0 \). If it is 1-weak, and \( k = 0 \), then \( e'(v) \geq s(v) - 6 - \frac{1}{2} (s(v) - 5) \geq 0 \) (if \( s(v) = 8 \), and \( v \) is adjacent to three 5-vertices of \( I \), moreover, in \( W(v) \) there is a chain \( v_1 v_2 v_3 \), where \( v_1, v_3 \in I^6, v_2 \in I^5 \), then 1 is transmitted by Lemmas 3 and 4, to the single vertex \( v_2 \), and two other 5-neighbours are given by \( \frac{1}{4} \), so \( e'(v) = 2 - 1 - 2 (\frac{1}{2}) = 0 \). If \( k = 1 \) and \( v \) is bad, then \( e'(v) = +1 - 2 (\frac{1}{2}) = 0 \); if 7-special, then \( e'(v) = +1 - 1 - 0 = 0 \); if 8-special, then \( e'(v) = +2 - \frac{3}{2} - 2 (\frac{1}{4}) = 0 \). But \( s(v) \leq 8 \) by Lemmas 3, 4, and 5; \( k \) does not exceed 1 by the same lemmas.
Let $v$ is strong. If it is not adjacent to particular vertices, then gives 1 to every $I^m$-neighbour and $e'(v) \geq s(v) - 6 - (s(v) - 6) = 0$. Let $v$ is adjacent to a particular vertex $v'$. If $|W(v) \cap I^m| = s(v) - 6$, and $v$ is not singular, then it transmits 1 to each neighbour vertex of $I^m$, and $e'(v) \geq 0$. If $v$ is singular for $v'$, then it gives $\frac{2}{3}$ to $v'$ and $\frac{1}{2}$ to the vertex $v'' \in W(v) \cap I^5$, which makes $v$ particular for $v'$, and 1 to each other $I^m$-neighbour, hence $e'(v) \geq s(v) - 6 - \frac{2}{3} - \frac{1}{2} - (s(v) - 8) = 0$. But by Lemmas 3 and 4, $v$ can not be singular, be adjacent to more then one particular vertex. Let now $k + l \leq s(v) - 7$. For $k \leq 2$, it follows $e'(v) \geq s(v) - 6 - \frac{2}{3}k - l \geq 0$. Let $k \geq 3$. By Lemmas 3 and 4, $2k + l \leq \frac{1}{3}s(v)|$, i.e. $4k + 2l \leq s(v)$. Add the last inequality with $2k + 2l \leq 2s(v) - 14$ and divide the result by $4$: $\frac{3}{2}k + l \leq \frac{1}{3}s(v) - \frac{7}{2}$. Hence $e'(v) \geq s(v) - 6 - \frac{3}{2} - \frac{2}{3}k - l \geq 6 + \frac{5}{2}k - \frac{5}{2} > 0$.

At last, by Propositions 1 and 2, for every $v \in I^m$, $e'(v) \geq 0$.

Let us show, that at least one vertex of $\bar{I}$ has strictly positive modified contribution. Clearly, each vertex $v \in C$ possess this property, provided that $s(v) > 5$. If $v \in C^5$, then $e'(v) = 0$ only if $|W(v) \cap I^m| = 2$, but then some $v_1 \in W(v) \cap I^m$ is adjacent to two vertices of $C$ and a vertex $v' \in W(v) \cap I$, which has $s(v') > 6$ by Lemma 4, hence $e'(v_1) > 0$. Let all the vertices of $C$ have degree 4, and each of them is adjacent by Lemma 3 to a single vertex $v_1 \in I^m$. Again by Lemma 4, each of the so defined vertices $v_1$ has $e'(v_1) > 0$.

Now from (1) we have a contradiction

$$0 \geq |C| - 6 = \sum_{v \in \bar{I}} e(v) = \sum_{v \in \bar{I}} e'(v) > 0,$$

which proves the Theorem 2.

3. Conclusion

It seems to me, that the two ideas of the just given proof can find application in the solution of some other difficult planar graph coloring problems.

Till now, as soon as I know, the reducibility of single configurations being 1-neighbourhoods of vertices have been usually proved. But in some cases such means may turn out to be insufficient for the construction of the desirable redistribution of Euler contributions.

Then, firstly, one should try to introduce the concept of a “weak” vertex being one, adjacent to sufficiently many minor vertices. The weak vertex should be thought of as a generalization of a minor vertex in a sense, that one should try to generate from the already known reducible configurations the whole families of reducible configurations by the substitution of weak vertices instead of minor ones. There is a hope to prove the reducibility of the whole family at once. In Lemma 9 it is easily seen some inner classification of the family, arising from the needs of the reducibility proof itself.

Secondly, if the contraction of configurations is prevented from by loops, or other forbidden subgraphs, then instead of the whole graph, an admissible subgraph should be considered, as in the proof we passed from $G$ to $\bar{I}$.

I think the following strengthened variant of the Grünbaum’s conjecture to be the truth (compare [2]).

**Conjecture.** The vertices of every planar graph can be 5-colored in such a way, that each $k$-chromatic subgraph is $k$-degenerated, for all $1 \leq k \leq 4$ simultaneously.

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References