A Characterization of Odd Order Extensions of the Finite Simple Chevalley Groups $F_4(q)$, $q$ Odd

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INTRODUCTION

Let $p$ denote an odd prime integer and let $q = p^f$ where $f$ is a positive integer. Let $F_4(K) (=F_4(q))$ denote the finite simple Chevalley group of type $(F_4)$ over a field $K$ of $q$ elements. Then $F_4(K)$ has two conjugacy classes of involutions (cf., [8, Section 9]).

Let $D$ denote an automorphism of $K$. Then $D$ induces an automorphism of $F_4(K)$ (cf., [11, Section 10]). In fact, the cyclic subgroup of $\text{Aut}(K)$ generated by $D$ acts faithfully on $F_4(K)$ and one may form the natural semidirect product $\langle D \rangle F_4(K)$. If $\sigma$ is an odd ordered automorphism of $K$, then $\sigma$ centralizes a Sylow $2$-subgroup of $F_4(K)$ and $\langle \sigma \rangle F_4(K)$ is an odd ordered extension of $F_4(K)$ with trivial $2$-core. In fact, any odd ordered extension of $F_4(K)$ with trivial $2$-core is of this form (cf., [11, Section 10]).

Let $t$ be an involution in the center of a Sylow $2$-subgroup of $F_4(K)$ that is centralized by the odd ordered automorphism $\sigma$ of $K$. Then the centralizer $C(t)$ of $t$ in $\langle \sigma \rangle F_4(K)$ is a semidirect product $\langle \sigma \rangle \mathcal{C}$ where $\mathcal{C}$ denotes the centralizer of $t$ in $F_4(K)$ and $C(t)$ has trivial $2$-core. Moreover, $\mathcal{C}$ contains a Sylow $2$-subgroup of $\langle \sigma \rangle F_4(K)$ and $\mathcal{C}$ is isomorphic to $\text{Spin}(9, K)$ (the universal Chevalley group associated with a root system of type $(B_4)$ over the field $K$).

We shall prove the following result:

**Theorem.** Let $G$ be a finite group with an involution $t$ whose centralizer $C_G(t)$ in $G$ is such that:

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(a) $O(C_G(t)) = \{1\}$ and

(b) $C_G(t)$ has a normal subgroup $H$ of odd index $\rho$ such that $H \cong \text{Spin}(2n + 1, K)$ for some positive integer $n \geq 2$ and some finite field $K$ of odd characteristic (here $\text{Spin}(2n + 1, K)$ denotes the universal Chevalley group associated with a root system of type $(B_n)$ over the field $K$).

Then either

(i) $G = O(G) C_G(t) O_Y$

(ii) $n = 4$ and $G \cong \langle \sigma \rangle F_4(K)$ where $\sigma$ is an automorphism of $F_4(K)$ of order $\rho$ induced by an automorphism of order $\rho$ of the field $K$.

In particular, when $\rho = 1$, we obtain a characterization of $F_4(K)$.

The proof of the theorem follows a fairly standard pattern. In Section 1, we describe the construction and various properties of $\text{Spin}(2n + 1, K)$ for $n \geq 2$. Then, in Section 2, we study the fusion of involutions and we show that if Condition (i) of the theorem does not hold, then $n = 4$ and $G$ has two conjugacy classes of involutions. This section closes with the construction of the Weyl group of the $(B, N)$-pair to be constructed in Section 4. In Section 3, we determine the structure of the centralizer of an involution not conjugate in $G$ to $t$. In Section 4, we construct a $(B, N)$-pair and quote the proof of [3, Theorem C] to show that $G$ contains a subgroup $G_1 \cong F_4(K)$. Then a standard argument implies that $G$ satisfies Condition (ii) of the theorem.

Our notation is fairly standard and tends to follow that of [7]. In particular, if $K$ is a field, then $K^+$ will denote the additive group of $K$ and $K^\times$ will denote the multiplicative group of nonzero elements of $K$.

Let $X$ denote a finite group. If $A$ and $B$ are subsets of $X$ and $x \in X$ are such that $A^x = x^{-1} A x = B$, then we shall write $x : A \rightarrow B$. Moreover, if $x : A \rightarrow B$ and $x : B \rightarrow A$, we shall write $x : A \leftrightarrow B$. If $a, b \in X$, then we shall write $a \sim b$ if there is an $x \in X$ such that $x : a \rightarrow b$. Finally, if $A$ is a subset of $X$, then $\mathcal{I}(A)$ will denote the set of involutions of $A$.

1. SOME PROPERTIES OF SPIN GROUPS

Let $n$ denote an integer such that $n \geq 2$ and let $V$ denote a real Euclidean vector space of dimension $n$ with positive definite scalar product $(\cdot, \cdot)$ and orthonormal basis $\{e_1, e_2, \ldots, e_n\}$. As in [1], a root system $\Delta_n$ of type $(B_n)$ consists of the $2n^2$ roots:

$$\pm e_i (1 \leq i \leq n), \quad \pm e_i \pm e_j (1 \leq i < j \leq n).$$
Also \( \mathcal{B} = \{x_1 = e_1 - e_2, x_2 = e_2 - e_3, \ldots, x_{n-1} = e_{n-1} - e_n, x_n = e_n \} \) is a system of fundamental roots or base for this root system \( \Delta_n \). Relative to this base, the positive roots \( \Delta_n^+ \) of \( \Delta_n \) are:

\[
\Delta_n^+ = \begin{cases} 
\epsilon_i & (1 \leq i \leq n), \\
\epsilon_i - \epsilon_j & (1 \leq i < j \leq n), \\
\epsilon_i + \epsilon_j & (1 \leq i < j \leq n).
\end{cases}
\]

The set of negative roots \( \Delta_n^- = \{ -v | v \in \Delta_n^+ \} \) and \( \Delta_n = \Delta_n^+ \cup \Delta_n^- \).

As in [1], for \( v_1 \in V \) and \( 0 \neq v_2 \in V \), we write

\[
\langle v_1, v_2 \rangle = \frac{2(v_1, v_2)}{(v_2, v_2)}.
\]

Then if \( s_1, s_2 \in \Delta_n \), we have \( \langle s_1, s_2 \rangle \in \mathbb{Z} \) and we call \( \langle s_1, s_2 \rangle \) a Cartan integer. The \( n \times n \) integral matrix with \( (i, j) \) entry equal to \( \langle x_i, x_j \rangle \) for all \( 1 \leq i, j \leq n \) is called the Cartan matrix. This matrix is given on [1, p. 253].

For \( 0 \neq z \in V \), define \( \tilde{w}_z : V \to V \) by

\[
\tilde{w}_z(v) = v - \langle v, z \rangle z \quad \text{for all} \quad v \in V.
\]

Then \( \tilde{w}_z \) is an orthogonal transformation called the reflection associated to \( z \).

Set

\[
\tilde{W} = \langle \tilde{w}_s | s \in \Delta_n \rangle.
\]

Then \( \tilde{W} \) is called the Weyl group of \( \Delta_n \) and we have

\[
\tilde{W} = \langle \tilde{w}_\alpha | \alpha \in \mathcal{B} \rangle, \quad | \tilde{W} | = 2^{n!}
\]

and

\[
\tilde{w}_s(\Delta_n) = \Delta_n \quad \text{for all} \quad s \in \Delta_n.
\]

Also \( \tilde{W} \) has subgroups \( \tilde{W}_1, \tilde{W}_2 \) such that \( \tilde{W}_2 \triangleleft \tilde{W}, \tilde{W} = \tilde{W}_1 \tilde{W}_2, \tilde{W}_1 \cap \tilde{W}_2 = \{1\} \). \( \tilde{W}_2 \) is elementary abelian of order \( 2^n \), \( \tilde{W}_2 \) operates on \( V \) by sending \( \epsilon_i \mapsto (\pm 1) \epsilon_i \) for all \( 1 \leq i \leq n \) and where \( \tilde{W}_1 \) is the symmetric group on \( \{e_1, e_2, \ldots, e_n\} \).

For an arbitrary finite field \( K \), there exists a \( K \)-vector space \( M \) and elements \( x_s(k) \in GL(M, K) \) for each \( s \in \Delta_n \) and each \( k \in K \) such that we may take

\[
\text{Spin}(2n + 1, K) = \langle x_s(k) | s \in \Delta_n, k \in K \rangle.
\]
(cf., [11, Section 3]) and such that if we set

\[ w_s(k) = x_s(k) x_s(-k^{-1}) x_s(k) \quad \text{and} \quad h_s(k) = w_s(k) w_s(1)^{-1} \]

for each \( s \in \Delta_n \) and each \( k \in K^\times \), (1.1)

\[ \omega_s = w_s(1) \quad \text{for each} \quad s \in \Delta_n , \]

(1.2)

\[ \mathcal{X}(s) = \langle x_s(k) \mid k \in K \rangle \quad \text{for each} \quad s \in \Delta_n , \]

(1.3)

and

\[ L(s) = \langle \mathcal{X}(s), \mathcal{X}(-s) \rangle \quad \text{for any} \quad s \in \Delta_n , \]

(1.4)

then, for any \( r, s \in \Delta_n \), we have

\[ \omega_s^2 = h_s(-1), \]

(1.5)

\[ L(s) \cong SL(2, K) \quad \text{and} \quad Z(L(s)) = \langle h_s(-1) \rangle , \]

(1.6)

\[ x_s(k_1) x_s(k_2) = x_s(k_1 + k_2) , \]

(1.7)

for any \( k_1, k_2 \in K \), and the map \( k \mapsto x_s(k) \) is an isomorphism of \( K^\times \) onto \( \mathcal{X}(s) \), for any \( k_1, k_2 \in K \), if \( r + s \neq 0 \), then

\[ x_s(k_1) x_s(k_2) x_s(k_1)^{-1} x_s(k_2)^{-1} = \prod x_{ir+js} (c_{ij} k_1^{i} k_2^{j}) , \]

(1.8)

where the product on the right is taken over \( \{ir + js \in \Delta_n \mid i, j \text{are positive integers} \} \) arranged in some fixed order, and where the \( c_{ij} \) are integers that depend only on \( r, s \) and on the chosen ordering but not on \( k_1 \) or \( k_2 \),

\[ \omega_s x_s(k) \omega_s^{-1} = x_{\omega_s}(c k) , \]

(1.9)

where \( c = \pm 1 \) is as in [11, Lemma 19(a)] for all \( k \in K \),

\[ \omega_s h_s(k) \omega_s^{-1} = h_{\omega_s}(c k) \quad \text{for all} \quad k \in K^\times , \]

(1.10)

\[ h_s(k_1) x_s(k_2) h_s(k_1)^{-1} = x_s(k_1^{r+s} k_2) \]

(1.11)

for all \( k_1 \in K^\times, k_2 \in K \), and

\[ h_s(k_1) h_s(k_2) = h_s(k_1 k_2) \quad \text{for all} \quad k_1, k_2 \in K^\times \]

(1.12)

and the map \( k \mapsto h_s(k) \) is an isomorphism of \( K^\times \) onto \( \langle h_s(k) \mid k \in K^\times \rangle \).

Let

\[ q = |K| , \]

(1.13)

so that

\[ q = p^f , \]

(1.14)

where \( p \) is a prime integer and \( f \) is a positive integer.
Set
\[ H_s = \langle h_s(k) \mid k \in K^X \rangle \]  
for each \( s \in \Delta_n \) and \( H = \langle H_s \mid s \in \Delta_n \rangle \).

For convenience, if \( \alpha_i \in \mathcal{O} \) for \( 1 \leq i \leq n \), we set
\[ \omega_i = \omega_{\alpha_i}, \quad h_i(k) = h_{\alpha_i}(k) \]  
for \( k \in K^X \) and \( H_i = H_{\alpha_i} \). Then, we also have
\[ H \text{ is abelian of order } (q - 1)^n \]  
and
\[ H = H_1 \times H_2 \times \cdots \times H_n, \]  
\[ \langle H, w_s(k) \mid k \in K, s \in \Delta_n, i \leq i \leq n \rangle = \langle H, \omega_i \mid 1 \leq i \leq n \rangle \]  
and is the normalizer of \( H \) in \( \text{Spin}(2n + 1, K) \),
the centralizer of \( H \) in \( \text{Spin}(2n + 1, K) \) is \( H \) itself  
and
the map \( w_s \mapsto \omega_s H \) for each \( s \in \Delta_n \) induces an isomorphism of
the Weyl group \( \tilde{W} \) onto \( \langle H, \omega_s \mid s \in \Delta_n \rangle / H \).

Let \( K \) be an arbitrary subfield of \( K \) and for each \( s \in \Delta_n, \) set
\[ E(s) = \langle x_s(k) \mid k \in K \rangle \]  
and \( E = \langle x_s(s) \mid s \in \Delta_n \rangle \).

Then
\( \tilde{G} \) is a subgroup of \( \text{Spin}(2n + 1, K) \),
\( \tilde{G} \cong \text{Spin}(2n + 1, K) \)
and
\[ \langle h_s(k), \omega_s \mid s \in \Delta_n \text{ and } k \in K \rangle \subseteq \tilde{G}. \]

We shall require the following facts about the group \( \text{Aut}(\text{Spin}(4n, K)) \) with \( n \geq 2 \) and \( |K| \) odd (cf., [11, Section 10]).
Let \( A = \text{Aut}(\text{Spin}(4n, K)) \), let \( B = \text{Inn}(\text{Spin}(4n, K)) \) and let \( C \) denote the cyclic subgroup of \( A \) of order \( \log_v(|K|) \) induced by \( \text{Aut}(K) \) and set \( \overline{A} = A/B. \) Then
\[ C \cong C, \]  
if \( n > 2, \) then \( \overline{A} = \overline{X} \times \overline{C}, \)  
where \( X \) is a dihedral group of order 8,
\[ C \cong C, \]  
if \( n = 2, \) then \( \overline{A} = \overline{X} \times \overline{C}, \)  
where \( \overline{X} \cong \Sigma_4 \), the symmetric group on four symbols.
In both cases $X$ is generated by the images in $A$ of the group of "diagonal automorphisms" and the group of "graph automorphisms" of Spin$(4n, K)$. In both cases, the image in $A$ of the group of "diagonal automorphisms" forms a normal 4-subgroup of $X$ that is complemented in $X$ by the image in $A$ of the group of "graph automorphisms."

2. The Fusion of Involutions in $G$

From now on, we assume that $G$ is a finite group satisfying the hypotheses of the theorem. In particular, from now on, we assume that $K$ is a finite field of odd characteristic $p$.

We shall also assume:

$$G \neq O(G) C_G(t).$$

Thus it remains to prove that Condition (ii) of the theorem holds.

First, suppose that $n = 2$ and observe that Spin$(5, K) \cong Sp(4, K)$, so that $S \cong Sp(4, K)$ where $Z(S) = \langle t \rangle$. Let $T$ denote a Sylow 2-subgroup of $S$. Then $T$ is a Sylow 2-subgroup of $C_G(t)$. Since $Z(T) = \langle t \rangle$ (cf., [13, proof of Lemma 2.3]) it follows that $T$ is a Sylow 2-subgroup of $G$. Then [13, Lemma 2.3] yields a contradiction to (2.1).

Next suppose that $n = 3$. Then [10, Theorem 3.4] implies that $t \in Z^*(G)$, whence $G = O(G) C_G(t)$ and again we contradict (2.1).

Thus for the remainder of this paper, we assume that $n \geq 4$.

Clearly we have the following.

**Lemma 2.1.** $O^\ast(C_G(t)) = S$.

Noting that $Z(S) = \langle t \rangle$, the proof of [7, Lemma 1.5] yields the following lemma.

**Lemma 2.2.** $C_G(S) = \langle t \rangle$.

We shall now fix an isomorphism

$$\varphi: \text{Spin}(2n + 1, K) \to S.$$

Utilizing the structure of Aut(Spin$(2n + 1, K)$) (cf., [11, Section 10]), the action of Aut$(K)$ on Spin$(2n + 1, K)$ and the proof of [7, Lemma 1.6], we obtain the lemma.

**Lemma 2.3.** There exists a cyclic subgroup $A$ of $C_G(t)$ of order $p$ and a monomorphism $\beta: A \to \text{Aut}(K)$ such that
(i) if \( a \in \mathcal{U} \) and \( y \in \text{Spin}(2n + 1, K) \), then

\[ \varphi(y^a) = \varphi(y^{\vartheta(a)}) \]

and

(ii) \( C_\mathcal{G}(t) = \mathcal{S}\mathcal{U} \) and \( \mathcal{S} \cap \mathcal{U} = \{1\} \).

**Corollary 2.3.1.** If \( q = 3 \), then \( \mathcal{U} = \{1\} \) and \( C_\mathcal{G}(t) = \mathcal{S} \).

For convenience, we shall suppress the isomorphism \( \varphi \) and we will identify \( \mathcal{S} \) with \( \text{Spin}(2n + 1, K) \). Thus we shall assume that \( \mathcal{S} = \text{Spin}(2n + 1, K) \) is as defined in Section 1. Then, the fact that \( Z(\mathcal{S}) = \langle t \rangle \) yields:

\[ t = h_n(-1). \]  (2.2)

**Lemma 2.4.** If \( \tau \in \mathcal{J}(C_\mathcal{G}(t)) \) and \( \tau \neq t \), then there exists a 2-element \( \theta \in \mathcal{S} \) such that \( \theta : \tau \mapsto \tau t \).

**Proof.** Clearly \( \tau \in \mathcal{S} \) and, since \( \text{char}(K) \) is odd, \( \tau \) is conjugate in \( \mathcal{S} \) to an involution of \( H \) (cf., [8, Section 7]). Consequently we may assume that \( \tau \in H \). Thus \( \tau \) has form \( \tau = h_{i_1}(-1) h_{i_2}(-1) \cdots h_{i_r}(-1) t^e \) where \( r \geq 1, e \in \{0, 1\} \) and \( 1 \leq i_1 < i_2 < \cdots < i_r \leq n - 1 \). Let \( \nu = \prod_{j=i_r+1}^{r-1} \omega_j \). Then \( \theta = \omega_n^\nu \) is the required 2-element.

Applying [8, Sections 7 and 8], we may determine a set of representatives for the distinct conjugacy classes of involutions in \( \mathcal{S} \) and their centralizers in \( \mathcal{S} \). We conclude

\( \mathcal{S} \) contains \( \lfloor (n/2) \rfloor + 1 \) conjugacy classes of involutions \( \mathcal{R}_0 = \{t\}, \mathcal{R}_1, ..., \mathcal{R}_{\lfloor n/2 \rfloor} \) which can be indexed such that if \( \tau \in \mathcal{R}_j \) with \( 1 \leq j \leq \lfloor (n/2) \rfloor \), then \( O^\circ(C_\mathcal{S}(\tau)) \) is isomorphic to the central product \( \text{Spin}(4j, K) \rtimes \text{Spin}(2(n - 2j) + 1, K) \), where \( \text{Spin}(4, K) \cong SL(2, K) \times SL(2, K) \), \( \text{Spin}(3, K) \cong SL(2, K) \) and \( \text{Spin}(1, K) \cong SL(2, K) \).  (2.3)

Moreover, we have the following.

**Lemma 2.5.** Let \( \tau \in \mathcal{J}(C_\mathcal{G}(t)) = \mathcal{J}(\mathcal{S}) \) with \( \tau \neq t \) and set \( D = \langle t, \tau \rangle \). Then the following three conditions hold:

(i) if \( q > 3 \), then \( O^\circ(C_\mathcal{G}(D)) = C_\mathcal{S}(\tau) \) and if \( q = 3 \), then \( C_\mathcal{G}(D) = C_\mathcal{S}(\tau) \);  

(ii) exactly one of the following three conditions hold:

1. the 2-layer if \( C_\mathcal{S}(\tau) \) contains a unique component (cf., [5]) \( J \) such that \( Z(J) = \langle t \rangle \) and either \( J \cong \text{Spin}(2j + 1, K) \) for some integer \( 2 \leq j < n \) or \( J \cong SL(2, K) \) and \( q > 3 \).
(2) \( q = 3, \) \( O_2(O_2(C_2(\tau))) \) is a quaternion group of order 8 with \( Z(O_2(O_2(C_2(\tau)))) = \langle t \rangle, \)
(3) \( n \) is even, \( C_2(\tau) \cong \text{Spin}(2n, K) \) and \( Z(C_2(\tau)) = D; \)
(iii) if \( \tau_1 \in \mathcal{J}(C_2(t)) = \mathcal{J}(5) \) with \( t \neq \tau_1 \not\sim \tau \) in \( 5, \) then \( C_2(\tau) \not\cong C_2(\tau_1). \)

We can now prove the lemma.

**Lemma 2.6.** \( n = 4 \) and there exists an involution \( \tau \) in \( 5 \) with \( \tau \neq t \) such that \( \tau \sim t \) in \( G \) and \( C_2(\tau) \cong \text{Spin}(8, K). \) Moreover \( \tau \sim h_4(-1) h_3(-1) \) in \( 5. \)

**Proof.** By (2.1) and [14, (2B)], there exists an involution \( \tau \) in \( 5 \) such that \( \tau \neq t \) and \( \tau \sim t \) in \( G. \) Applying Lemma 2.4, we conclude that there is a 2-element \( \theta_1 \) in \( C_2(\tau) \) such that \( \theta_1 : t \mapsto tr. \) Setting \( D = \langle t, \tau \rangle, \) we conclude that \( N_2(D) \) induces the full symmetric group \( \Sigma_3 \) on \( D^* = \{t, \tau, \tau^2\} \) under conjugation. Thus there exists a 3-element \( y \in N_2(D) \) such that \( y \) is transitive on \( D^*. \) Since \( y \) normalizes \( C_2(D), \) Lemma 2.5(ii) implies that \( n \) is even, \( C_2(\tau) \cong \text{Spin}(2n, K) \) and \( Z(C_2(\tau)) = D. \)

From [8, Sections 7 and 8], conjugacy in \( 5 \) implies that we may assume that \( \tau = h_4(-1) h_3(-1) \cdots h_{n-4}(-1). \) Then \( [\tau, \mathfrak{A}] = \{1\} \) and \( C_2(D) = C_2(\tau) \mathfrak{A} \) where

\[
C_2(\tau) = \left\{ \mathfrak{A}(s) \mid s \in \mathcal{A}_n \cap \left( \sum_{i=1}^{n-1} Zx_i + Z2x_n \right) \right\} \cong \text{Spin}(2n, K).
\]

Clearly \( O(C_2(D)) = \{1\}. \) Since \( Z(C_2(\tau)) = D, \) the proof of [7, Lemma 1.5] yields \( C_2(C_2(\tau)) = D. \) Also \( C_2(\tau) = O^*(C_2(D)) \subseteq N_2(D); \) hence \( N_2(D)/D \) is isomorphic to a subgroup of \( \text{Aut}(C_2(\tau)) \) that contains \( \text{Inn}(C_2(\tau)) \). Note that \( (N_2(D)/D)/(C_2(\tau)/D) \cong N_2(D)/C_2(\tau) \) and that \( N_2(D)/C_2(\tau) \) has \( N_2(D)/C_2(D) \cong \Sigma_3 \) as a homomorphic image.

Suppose that \( n > 4. \) Then \( \text{Aut}(\text{Spin}(2n, K))/\text{Inn}(\text{Spin}(2n, K)) \) has the property that every element of odd order is central (cf., (1.22) and (1.23)). Since \( N_2(D)/C_2(D) \cong \Sigma_3 \) does not have this property, we obtain a contradiction. Thus \( n = 4 \) and the lemma follows.

**Corollary 2.6.1.** \( O(G) = \{1\}. \)

**Proof.** Clearly

\[
O(G) = \langle O(G) \cap C_2(\delta) \mid \delta \in D^* \rangle \quad \text{where} \quad D = \langle t, h_4(-1) h_3(-1) \rangle.
\]

But \( O(G) \cap C_2(\delta) \subseteq O(C_2(\delta)) = \{1\}, \) since \( \delta \sim t \) in \( G \) and \( O(C_2(t)) = \{1\}, \) for each \( \delta \in D^*. \)
Consequently, we have

\[ n - 4. \]  \hspace{1cm} (2.4)

Set

\[ t_1 = h_1(-1) h_2(-1), \quad D = \langle t, t_1 \rangle, \]

\[ v = h_1(-1)t, \quad \Delta = \Delta_4, \]

and

\[ \mathfrak{g} = C_8(t_1) = C_8(D). \] \hspace{1cm} (2.5)

Then

\[ H \subseteq \mathfrak{g} \subseteq \mathfrak{g}, \] \hspace{1cm} (2.6)

and

\[ [t_1, \mathfrak{U}] - [v, \mathfrak{U}] - 1, \quad C_6(D) - C_8(t_1)\mathfrak{U} - \mathfrak{g}\mathfrak{U}, \]

\[ O^2(C_6(D)) = \mathfrak{g} \quad \text{and} \quad C_6(t, v) = C_8(v)\mathfrak{U}. \] \hspace{1cm} (2.7)

For convenience of notation, set

\[ a = e_1, \quad b = e_2, \quad c = e_3 \quad \text{and} \quad d = e_4. \] \hspace{1cm} (2.8)

Set

\[ \Delta(t_1) = \{ s \in \Delta \mid (s, s) = 2 \} \]

and

\[ \Delta(v) = \{ \pm d, \pm c, \pm(c - d), \pm(c + d) \}. \] \hspace{1cm} (2.9)

From [8, Section 9], we conclude

\[ \Delta(t_1) \] is a root system of type \((D_4)\) with base

\[ \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4^* = c + d \} \]

and Dynkin diagram:

\[ \begin{tikzpicture}
  \node (alpha1) at (0,0) [circle, fill=black] {$\alpha_1$};
  \node (alpha2) at (1,0) [circle, fill=black] {$\alpha_2$};
  \node (alpha3) at (2,0) [circle, fill=black] {$\alpha_3$};
  \node (alpha4) at (0,1) [circle, fill=black] {$\alpha_4^*$};
  \draw (alpha1) -- (alpha2);
  \draw (alpha2) -- (alpha3);
\end{tikzpicture} \] \hspace{1cm} (2.10)

and \( \mathfrak{g} = C_8(t_1) = \langle \mathfrak{X}(s) \mid s \in \Delta(t_1) \rangle \cong \text{Spin}(8, K) \), and

\[ \Delta(v) \] is a root system of type \((B_2)\) with base \( \{ \alpha_3, \alpha_4 \} \) and Dynkin diagram

\[ \alpha_4 \longleftrightarrow \alpha_3. \] \hspace{1cm} (2.11)
Set
\[ R = \langle x \mid s \in A(v) \rangle. \] (2.12)

Then
\[ R \cong Sp(4, K), \quad Z(R) = \langle t \rangle \] (2.13)
and all involutions of \( R - \{t\} \) are conjugate in \( R \) to \( h_3(-1) \).

We also have
\[ [L(a + b), L(a - b)] = [L(a + b)L(a - b), R] = \{1\} \] (2.14)
and
\[ Z(L(a - b)) = \langle vt \rangle \quad \text{and} \quad Z(L(a + b)) = \langle v \rangle. \] (2.15)

Let \( \kappa \) denote a generator of the cyclic group \( K \) and \( q = |K| = p^f \)
where \( p \) is an odd prime integer and \( f \) is a positive integer. (2.16)

Then we have
\[ C_{i_0}(v) = L(a + b)L(a - b)L(c + d)L(c - d)H_2 \subseteq C_{i_0}(v) \]
\[ = L(a + b)L(a - b)RH_2, \] (2.17)

where \( H_2 = \langle h_3(\kappa) \rangle, \ h_3(\kappa) \not\in L(a + b)L(a - b)R \) and
\[ h_3(\kappa)^2 \in L(a + b)L(a - b)L(c + d)L(c - d) \subseteq L(a + b)L(a - b)R. \]
\[ O^\nu(C_G(D, v)) = C_{i_0}(v), \quad O^\nu(C_G(t, v)) = C_{i_0}(v) \] (2.18)
and
\[ O^\mu(C_G(v)) = L(a + b)L(a - b)L(c + d)L(c - d) \]
and
\[ O^\mu(C_{i_0}(v)) = L(a + b)L(a - b)R. \] (2.19)

We can now prove the following lemma.

**Lemma 2.7.** (i) \( C_G(t) \) has three conjugacy classes of involutions with representatives \( t, t_1, \) and \( v; \) and

(ii) \( t \sim t_1 \sim v \) in \( G. \)

**Proof.** From [8, Sections 7 and 8] and the fact that \( C_{i_0}(t_1) \cong C_{i_0}(v), \)
we obtain (i). By Lemma 2.6, we have \( t \sim t_1 \) in \( G. \) Assume that \( v \sim t \) in \( G. \)
Noting that
\[ C_G(t, v) \not\cong C_G(t, t_1) \quad \text{and} \quad \langle t \rangle = Z(R) \text{ char } R \text{ char } C_{i_0}(v) \text{ char } C_G(t, v), \]
the argument of [14, (2A)] yields a contradiction, and the lemma follows.
Lemma 2.8. Let $S$ be a Sylow 2-subgroup of $\mathcal{S}$. Then $S$ is a Sylow 2-subgroup of $G$ and $Z(S)$ is cyclic with $\Omega_1(Z(S)) = \langle t \rangle$.

Proof. Assume that $\langle t, \tau \rangle \leq Z(S)$ with $\tau$ an involution not $t$. Clearly we may assume that $\tau = t_1$ or $\tau = v$. Then $S \leq C_\mathcal{S}(t_1)$ or $S \leq C_\mathcal{S}(v)$. However, the power of 2 appearing in the prime factorizations of

$$|C_\mathcal{S}(t_1)| = |\text{Spin}(8, K)|$$

and

$$|C_\mathcal{S}(v)| = |\text{SL}(2, K)^2| |\text{Sp}(4, K)|$$

are both less than $|S|$ which is impossible. Thus $Z(S)$ is cyclic and $\Omega_1(Z(S)) = \langle t \rangle$. Since $S$ is clearly a Sylow 2-subgroup of $C_\mathcal{S}(t)$, this forces $S$ to be a Sylow 2-subgroup of $G$.

Corollary 2.8.1. $G$ has two conjugacy classes of involutions represented by $t$ and $v$.

Since all involutions in $H$ with the exception of $t, t_1$ and $tt_1$ are conjugate in $\mathcal{S}$ to $v$, we have the following.

Corollary 2.8.2. $\text{ccl}_G(t) \cap H = D^*$ and $D$ is strongly closed in $H$ with respect to $G$.

Lemma 2.9. Let $T$ be a Sylow 2-subgroup of $C_\mathcal{S}(v)$. Then $T$ is a Sylow 2-subgroup of $C_\mathcal{S}(v)$ and $Z(T) = \langle t, v \rangle$.

Proof. Noting that $C_\mathcal{S}(v) \cap C_\mathcal{S}(L(a + b)) = L(a - b)R < C_\mathcal{S}(v)$ and $C_\mathcal{S}(v)/L(a - b)R \cong \text{PGL}(2, K)$, it follows that $Z(T) \subseteq L(a + b) L(a - b)R$. Then $Z(T) \subseteq Z(T) \cap (L(a + b) L(a - b)R) = \langle t, v \rangle$. Since $\langle t, v \rangle \subseteq Z(T)$, it follows that $Z(T) = \langle t, v \rangle$. Finally $\text{ccl}_G(t) \cap Z(T) = \{t\}$ and $T$ is clearly a Sylow 2-subgroup of $C_\mathcal{S}(v, t)$. Thus $T$ is a Sylow 2-subgroup of $C_\mathcal{S}(v)$.

Lemma 2.10. There exists an element $y \in N_G(D)$ of order 3 such that

(i) for each $s \in \Delta(t_1)$, $\mathcal{X}(s)^y = \mathcal{X}(\gamma(s))$ where $\gamma$ denotes the unique linear automorphism of the root system $\Delta(t_1)$ such that $\gamma: \alpha_2 \to \alpha_2$ and $\gamma: \alpha_1 \to \alpha_1, \alpha_1 \to \alpha_1^* \to \alpha_1$ (cf., (2.10));

(ii) $y: t_1 \to t \to tt_1 \to t_1$;

(iii) $y \in N_G(H) \cap N_G(\mathcal{H})$;

(iv) $[y, \mathcal{U}] \subseteq H$; and

(v) $y \in C_G(v)$.

Proof. Set $N_G(D) = N_G(D)/D$. In the proof of Lemma 2.6, we showed $C_G(C_\mathcal{S}(t_1)) = D$. $\mathcal{Y} = C_\mathcal{S}(t_1) < N_G(D)$ and $N_G(D)$ acts faithfully by conjugation on $\mathcal{Y} = C_\mathcal{S}(t_1)$ as a group of automorphisms of $\mathcal{Y}$ with $\mathcal{Y}$ corre-
sponding to $\text{Inn}(\mathfrak{g})$. Here, $\mathfrak{g}$ is cyclic of odd order $\rho$ and acts as a group of "field automorphisms" on $\mathfrak{g} = C_8(t_1) (= \text{Spin}(8, K))$. Thus (1.24) implies that $C_G(D)/C_G(t_1) = \mathfrak{g}/\mathfrak{h} \subseteq Z(N_G(D)/\mathfrak{g})$.

Note that $\mathfrak{h} \cong \mathfrak{g} \cong C_8(D)/C_8(t_1)$ is cyclic of order $\rho$ and $N_G(D)/C_G(D) \cong N_G(D)/C_8(D) \cong \Sigma_4$. Thus $N_G(D)/\mathfrak{g}$ has a normal 2-complement of index 2; let $Y_1$ be its inverse image in $N_G(D)$ so that $C_G(D) \cap Y_1 \leq N_G(D)$, $|N_G(D)/Y_1| = 2$ and $|Y_1/C_G(D)| = 3$. From the above, it follows that $Y_1/\mathfrak{g}$ is abelian of odd order $3\rho$. It now follows that $N_G(D)$ contains a normal subgroup $Y_2$ such that $\mathfrak{g} \subseteq Y_2 \leq N_G(D)$ and $Y_2/\mathfrak{g} \cong \Sigma_3$, $Y_2 \cap C_G(D) = \mathfrak{g}$ and $N_G(D) = C_G(D) Y_2$. Let $Y$ denote the unique normal subgroup of $Y_2$ of index 2 such that $\mathfrak{g} = C_8(t_1) \subseteq Y_2$. Then $Y/\mathfrak{g} = (N_G(D)/\mathfrak{g})'$ and $|Y/\mathfrak{g}| = 3$. Comparing this with

$$\text{Out}(\text{Spin}(8, K))' = (\text{Aut}(\text{Spin}(8, K))/\text{Inn}(\text{Spin}(8, K)))'$$

as described in (1.24) and using the fact that $\Sigma_4$ has exactly one conjugacy class of elements of order 3 and that the automorphism $\gamma$ of the root system $\Delta(t_1)$ induces an element of order 3 in $\text{Out}(\text{Spin}(8, K))$, it follows that there exists an element $y \in Y - C_8(t_1)$ such that $y^3 \in D$ and such that $\gamma$ induces an automorphism on $\mathfrak{g} = C_8(t_1) = \text{Spin}(8, K)$, that corresponds to a (possibly trivial) "diagonal automorphism" of $\text{Spin}(8, K)$ followed by the "graph automorphism" of $\text{Spin}(8, K)$ induced by the automorphism $\gamma$ of the root system $\Delta(t_1)$. Clearly $y \in N_G(D) - C_G(D)$ and $y^3 \in D$ so that $y$ is transitive on $D^\ast$. Thus $y^3 = 1$ and (i) holds. Moreover, it is clear that (iii) holds. Since $y: h_3(-1) h_4(-1) \to h_3(-1) h_4(-1) = t$, (ii) holds. Since $Z(L(a + b)) = \langle v \rangle$ and $a + b = a_1 + 2a_2 + a_3 + a_4^*$, (v) holds. Now $[y, \mathfrak{h}] \subseteq Y_2 \cap C_G(D) = C_8(t_1)$ so that $[\mathfrak{g}, \mathfrak{h}] \subseteq C_8(t_1) = \mathfrak{g}$. Also if $a \in \mathfrak{h}$, then $[\mathfrak{g}, a]$ corresponds to a diagonal automorphism which is also an inner automorphism since $C_8(t_1) = \mathfrak{g}$ corresponds to $\text{Inn}(C_8(t_1))$. Hence $[y, a] \in H$ for all $a \in \mathfrak{h}$ and (iv) holds, thereby completing the proof of this lemma.

From now on, let $y$ denote an element of order 3

$$y = \text{denote an element of order } 3 \quad (2.20)$$

as in the above lemma.

From Corollary 2.8.2, we have

\textbf{Corollary 2.10.1.} \quad $N_G(H) = \bigcup_{i=0}^{2} (N_G(H) \cap C_G(t)) y^i$, where the union is disjoint.

For simplicity of notation, set

$$u = \omega_k. \quad (2.21)$$
Then

\[ u^2 - t, \quad u : L(c + d) \leftrightarrow L(c - d), \]
\[ [u, L(a + b)] = [u, L(a - b)] = \{1\}, \]
\[ u \in N_G(H), \quad u : t_1 = h_1(-1) h_3(-1) \leftrightarrow h_1(-1) h_3(-1) h_4(-1) = t_1 t \]
\[ u \text{ centralizes } v = h_1(-1) t \text{ and} \]
\[ [u, \mathfrak{H}] = \{1\} = [u, L(b - c)] \]
\[ (\text{since } [L(\alpha_1), L(\alpha_2)] = \{1\} \text{ and } \alpha_2 = b - c). \quad (2.22) \]

Set

\[ z = u^v. \quad (2.23) \]

Then (2.22) and Lemma 2.10 yield

\[ z^2 = t t_1, \quad z : L(a - b) \leftrightarrow L(c + d), \]
\[ [z, L(a + b)] = [z, L(c - d)] = \{1\}, \]
\[ z \in N_G(H), \quad z : t \leftrightarrow t_1, \]
\[ z \text{ centralizes } v, \quad [z, \mathfrak{H}] \subseteq H \text{ and} \]
\[ [z, L(b - c)] = \{1\}. \quad (2.24) \]

Note that

\[ \text{if } s \in \Delta, \text{ then } H \cap L(s) = H_s. \quad (2.25) \]

Also \( \langle H_s, \omega_s \rangle \subseteq N_G(H) \cap L(s) \subseteq N_{L(s)}(H \cap L(s)) = N_{L(s)}(H_s) = \langle H_s, \omega_s \rangle, \) so that

\[ \text{if } s \in \Delta, \text{ then } N_G(H) \cap L(s) = \langle H_s, \omega_s \rangle. \quad (2.26) \]

Since \([u, H_s] = \{1\} \) and \( y \in N_G(H) \cap N_G(L(b - c)) \) (where \( \alpha_2 = b - c \)), we conclude that \( y \) normalizes \( H_s \) so

\[ [u, H_s] = [z, H_s] = \{1\}. \quad (2.27) \]

We have

\[ N_G(H) \cap C_G(t) = N_{\mathfrak{g}}(H)^{\mathfrak{H}}, \quad (2.28) \]

and

\[ u - \omega_1 \in N_{\mathfrak{g}}(H) - C_G(D). \quad (2.29) \]

Set

\[ N_1 = N_{\mathfrak{g}}(H) \quad \text{and} \quad N_2 = N_{\mathfrak{g}}(H). \quad (2.30) \]

Then, since \( \mathfrak{g} \cong \text{Spin}(8, K) \) and \( \mathfrak{h} \cong \text{Spin}(9, K), \) we have

\[ N_1/H \text{ is isomorphic to the Weyl group of a root system of type} \]
\( (D_4) \) and \( |N_1/H| = 2^6 \cdot 3 \) (cf., [1, p. 257 (X)]), and \( N_2/H \) is isomorphic to the Weyl group of a root system of type \( (B_4) \) and \( |N_2/H| = 2^7 \cdot 3 \) (cf., [1, p. 253 (X)]).

(2.31)

Also

\[ N_1 \text{ is a normal subgroup of } N_2 \text{ of index 2 and } N_2 = \langle N_1, u \rangle. \]

(2.32)

Clearly Corollary 2.8.2 yields

\[ D \triangleleft N_G(H), \]

(2.33)

whence

\[ C_G(D) \cap N_G(H) = N_1 \mathfrak{U} \triangleleft N_G(H). \]

(2.34)

But \( \langle u, z \rangle \subseteq N_G(H) \cap N_G(N_1) \) and \( \langle u, z \rangle \) is transitive on \( D^e \), so that

\[ N_G(H) = \langle N_2, \mathfrak{U}, u, z \rangle. \]

(2.35)

Since \( \mathfrak{U} \) also normalizes \( N_1 \), we have

\[ N_1 \triangleleft N_G(H). \]

(2.36)

**Lemma 2.11.** \( (zu)^8 \in \langle v \rangle \) and \( zu \) acts transitively on \( D^e \).

**Proof.** Clearly \( \langle z, u \rangle \subseteq N_G(H) \cap C_G(v) \cap N_G(D) \) and \( zu \) acts transitively on \( D^e \), so that \( (zu)^8 \in C_G(D) \). Thus \( \langle z, u \rangle \) normalizes

\[ C_\beta(v) = L(a + b)L(a - b)L(c + d)L(c - d), \]

where \( h_\beta = \langle h_\beta(z) \rangle \) and \( h_\beta(z)^8 \in L(a + b)L(a - b)L(c + d)L(c - d) = O_8(C_\beta(v)) \). Hence \( \langle z, u \rangle \) normalizes

\[ O_8(C_\beta(v)) = L(a + b)L(a - b)L(c + d)L(c - d). \]

Then the argument on the first half of [13, p. 501] together with (2.22) and (2.24) yield \( (zu)^8 \in C_G(O^8(C_\beta(v))) \). Then (2.17) and (2.27) imply that \( (zu)^8 \in C_\beta(C_\beta(v)) \). But \( C_G(D, v) = C_\beta(v) \mathfrak{U}, C_\beta(v) \) is normal and of odd index in \( C_G(D, v), Z(C_\beta(v)) = \langle D, v \rangle \) and \( O(C_G(D, v)) = \{1\} \). A standard argument implies that \( C_G(C_\beta(v)) = \langle D, v \rangle \). Hence \( (zu)^8 \in \langle D, v \rangle \). Since \( zu \) centralizes \( v \) and \( zu \) is transitive on \( D^e \), we have \( (zu)^8 \in \langle v \rangle \) and we are done.

Set

\[ N_3 = \langle H, z, u \rangle. \]

(2.37)

Since \( \langle u^3, z^3, v \rangle \subseteq H, (zu)^8 \in H, [u, \mathfrak{U}] = \{1\} \) and \( [z, \mathfrak{U}] \subseteq H \), we have
Lemma 2.12. (i) $N_3/H \cong \Sigma_3$;
(ii) $C_G(t) \cap N_3 = H\langle u \rangle$;
(iii) $C_G(D) \cap N_3 = H$; and
(iv) $\mathcal{U}$ normalizes $N_3$ and acts trivially on $N_3/H$.

Set
$$N = N_1N_3.$$ (2.38)

Then

Lemma 2.13. (i) $N \leqslant N_G(H) = N\mathcal{U}$ and $N \cap \mathcal{U} = \{1\}$;
(ii) $N_1 \cap N_3 = H$;
(iii) $N_3/H$ is a split extension of $N_1/H$ by $N_3/H$; and
(iv) $C_G(D) = N_1$ and $C_G(t) = N_2$.

Proof. Since $H \subseteq N_1 \cap N_3 = N_1 \cap C_G(D) \cap N_3 = N_1 \cap H = H$, we have (ii) and (iii). Also $\mathcal{U}$ normalizes $N$, $N_G(H) = \langle N, \mathcal{U} \rangle$ and $N \cap \mathcal{U} = N \cap C_G(t) \cap \mathcal{U} = (N_1 \langle u \rangle) \cap \mathcal{U} \subseteq N \cap \mathcal{U} = \{1\}$ so that (i) holds. Clearly $C_G(D) = N_1C_G(D) = N_1H = N_1$ and $C_G(t) = N_1C_G_3(t) = N_1H\langle u \rangle = N_2$ so that (iv) also holds.

Set
$$\overline{N} = N/H.$$ (2.39)

Lemma 2.14. (i) $\overline{N}_1 = \langle \omega_1, \omega_2, \omega_3, \omega_{c+d} \rangle$, $\overline{N}_2 = \langle \omega_1, \omega_2, \omega_3, \bar{u} \rangle$ and $\overline{N} = \langle \omega_2, \omega_3, \bar{u}, \bar{x} \rangle$;

(ii) $\bar{u} : \omega_3 \leftrightarrow \omega_{c+d}$ and $\bar{x} : \omega_1 \leftrightarrow \omega_{c+d}$;

(iii) the elements of $\{\omega_1, \omega_2, \omega_3, \omega_{c+d}, \bar{u}, \bar{x}, \omega_1\omega_3, \omega_1\omega_{c+d}, \omega_2\omega_{c+d}, \omega_3\omega_{c+d}, \omega_1\omega_2\omega_{c+d}, \omega_1\omega_{c+d}, \omega_2\omega_{c+d} \}$ have order 2 in $\overline{N}$, the elements of $\{\omega_1\omega_2, \omega_2\omega_3, \omega_2\omega_{c+d}, \omega_3\omega_{c+d} \}$ have order 3 in $\overline{N}$ and $\omega_2\omega_3$ has order 4 in $\overline{N}$.

(iv) $\overline{N}$ is isomorphic to the Weyl group of a root system of type $(F_4)$.

(v) $[\omega_3, \langle u, \bar{x} \rangle] = \{1\}$, $[\omega_3, \bar{x}] = 1$, $\bar{x}\omega_3 = \omega_3\bar{x}$ and $\bar{x}\omega_3 = \omega_3\bar{x}$.

Proof. Since $u : L(c - d) \leftrightarrow L(c + d)$ and $z : L(a - b) \leftrightarrow L(c + d)$, (2.26) yields (ii). Then (i) follows immediately. Applying (1.20) and (2.21), the orders of all elements in $\overline{N}_2$ are determined. Moreover, $\bar{x}$ has order 2 and $\bar{x} \neq \bar{u}$ by (2.24). Thus $\bar{u}\bar{x}$ has order 3 by Lemma 2.11. Note that

$y$ normalizes $H$ and $N \leqslant N_G(H)$.

Also $\omega_2\bar{x} = \omega_2\bar{u} = (\omega_2\bar{u})^y = \omega_2H$ by (2.26) since $y$ normalizes $L(x_2)$. Thus $\omega_2\bar{x}$ has order 2 and similarly $\omega_2\bar{x}$ has order 2. This
proves (iii). We have $|\bar{N}| = |\bar{N}_1| |\bar{N}_3| = 273^2$ (since $|\bar{N}_3| = 6$ and $|\bar{N}_1| = 2^3$ by (2.31)), $\bar{N} = \langle \bar{\omega}_3, \bar{\omega}_5, \bar{u}, \bar{z}\rangle$ and these involutions $\bar{\omega}_3, \bar{\omega}_5, \bar{u}, \bar{z}$ satisfy (iii). On the other hand, the Weyl group of a root system of type $(F_4)$ also has order $273^2$, is generated by 4 involutions satisfying relations [6, (2.2)] and (with these involutions) forms a Coxeter system in the sense of [1, Chapter 4, Definition 31]. Now (iv) is immediate.

We conclude this section with the lemma.

**Lemma 2.15.** There exists an element $w_0 \in N_1 - H$ such that

(i) $w_0^2 = 1$;

(ii) $Z(N) = Z(N_1) = Z(N_3) = \langle \bar{w}_0 \rangle$; and

(iii) $X(s)^{w_0} = X(-s)$ for all $s \in A$.

**Proof.** Applying (2.31) and [1, p. 257 (X) and p. 253 (X)], there exists an element $w_0 \in N_1 - H$ satisfying (i) and (iii) and such that $\langle \bar{w}_0 \rangle = Z(\bar{N}_1) = Z(\bar{N}_3)$. But the structure of $\bar{N}$ forces $Z(\bar{N})$ to be in $\bar{N}_1$ and, since $\bar{N}_1 \vartriangleleft \bar{N}$, (ii) holds.

### 3. The Structure of $C_G(\nu)$

In this section, we determine the structure of $C_G(\nu)$.

Set

$$J = C_G(L(a + b)).$$

Clearly

$$\langle z, L(a - b) \times R \rangle \subseteq J.$$  

**Lemma 3.1.** (i) $J \cong Sp(6, K)$ and $Z(J) = \langle \nu \rangle$;

(ii) $C_J(t) = L(a - b) \times R$; and

(iii) $J = \langle L(a - b), R, z \rangle$.

**Proof.** Since $\langle \nu \rangle = Z(L(a + b)) \subseteq Z(J)$, we have

$$C_J(t) = C_G(t, \nu) \cap C_G(L(a + b)) = L(a - b) \times R$$

by (2.14); thus (ii) holds. Let $J_1$ be any subgroup of $J$ containing $\langle L(a - b), R, z \rangle$ and set $J_1 = J_1/\langle \nu \rangle$. Since $C_{J_1}(t) = C_J(t)$, we have $O(C_{J_1}(t)) = \{1\}$. Also $\langle z, u, D \rangle \subseteq J_1$ and $\langle z, u \rangle$ acts transitively on $D^*$. Now the argument used to prove Corollary 2.6.1 yields $O(J_1) = \{1\}$; whence $O(J_1) = \{1\}$. Since $t \not\sim vt$ in $G$, we have

$$C_{J_1}(t) = C_{J_1}(t, \nu) = L(a - b) \times \bar{R}$$
and \( \tilde{x} : i \to i_1 \neq i \). Then [13, Theorem] implies that \( \tilde{\mathcal{J}}_1 \cong \text{PSp}(6, K) \). Since \( \langle v \rangle \subseteq (L(a - b)R) \subseteq \mathcal{J}_1 \), we have \( \mathcal{J}_1 = \tilde{\mathcal{J}}_1 \). But \( \text{Sp}(6, K) \) is the unique covering group of \( \text{PSp}(6, K) \) so that \( \mathcal{J}_1 \cong \text{Sp}(6, K) \). Now (i) and (iii) are immediate.

From [14, (1A)], we have

\[ \mathcal{J} \text{ has 3 conjugacy classes of involutions represented by } v, t, vt. \quad (3.3) \]

Clearly

\[ \mathcal{H} \supseteq N_G(L(a + b)) \cap N_G(\mathcal{J}), \quad (3.4) \]

and

\[ [L(a + b), \mathcal{J}] = \{1\}, \quad (3.5) \]

\[ \mathcal{J} \cap L(a + b) = \langle v \rangle = Z(\mathcal{J}) = Z(L(a + b)) \]

and

\[ L(a + b) \mathcal{J} = L(a + b) \ast \mathcal{J}. \]

Note also that

\[ H_2 = \langle h_2(\kappa) \rangle \text{ is normalized by } \mathfrak{H} \text{ and } \]

\[ h_2(\kappa)^2 = h_2(\kappa^2) \in L(a + b) L(a - b) R \subseteq L(a + b) \mathcal{J}. \quad (3.6) \]

Since \( \mathfrak{H} \) is of odd order and \( H_2 \) is cyclic of order \( q - 1 \), we have

\[ [\mathfrak{H}, O_q(H_2)] = \{1\}. \quad (3.7) \]

Also

\[ L(a + b) \mathcal{J} \mathfrak{H} = L(a + b) \mathcal{J} H_2. \quad (3.8) \]

**Lemma 3.2.** (i)

\[ N_G(L(a + b)) = L(a + b) \mathcal{J} H_2 \mathfrak{H} \quad \text{and} \quad (L(a + b) \mathcal{J} H_2) \cap \mathfrak{H} = \{1\}; \]

(ii) \( L(a + b) \mathcal{J} \lhd N_G(L(a + b)) \);

(iii) \( O^2(N_G(L(a + b))) = L(a + b) \mathcal{J} H_2 \);

(iv) \( O^2(N_G(L(a + b))) = L(a + b) \mathcal{J} \mathfrak{H} \text{ is of index 2 in } N_G(L(a + b)); \)

(v) \( N_G(L(a + b)) \cap C_G(t) = C_G(v, t) = C_G(v) \mathfrak{H}; \) and

(vi) \( N_G(L(a + b)) \subseteq C_G(v). \)

**Proof.** Clearly \( \mathcal{J} \lhd N_G(L(a + b)) \) and \( v \not\sim t \not\sim vt \) in \( G \). Then, the Frattini argument and (3.3) yield \( N_G(L(a + b)) = \mathcal{J}(N_G(L(a + b)) \cap C_G(t)). \)

On the other hand, \( N_G(L(a + b)) \subseteq C_G(v) \) since \( \langle v \rangle \) char \( L(a + b) \), so that

\[ N_G(L(a + b)) \cap C_G(t) \subseteq C_G(v) \cap C_G(t) \]

\[ -L(a + b)L(a - b)R \mathfrak{H} H_2 \subseteq N_G(L(a + b)). \]
Thus (v) holds and \( N_c(L(a + b)) = L(a + b) \mathcal{J} H_2 \mathfrak{A} \). Moreover,

\[
(L(a + b) \mathcal{J} H_2) \cap \mathfrak{A} = (L(a + b) \mathcal{J} H_2) \cap C_c(t) \cap \mathfrak{A} = (L(a + b) C_c(t) H_2) \cap \mathfrak{A} \subseteq \mathfrak{H} \cap \mathfrak{A} = \{1\}
\]

so that (i), (ii) and (vi) hold. Also

\[
L(a + b) \mathcal{J} H_2 = L(a + b) \mathcal{J} O_2(H_2) \subseteq N_c(L(a + b)),
\]

\[
L(a + b) \mathcal{J} \mathfrak{A} \lhd N_c(L(a + b)) \text{ and } \vert N_c(L(a + b)) : L(a + b) \mathcal{J} \mathfrak{A} \vert = 2. \text{ On the other hand } \langle L(a + b), \mathcal{J}, \mathfrak{A} \rangle \subseteq O^2(N_c(L(a + b))) \text{ and } \langle L(a + b), O_2(H_2), \mathcal{J} \rangle \subseteq O^2(N_c(L(a + b)))
\]

since \( O_2(H_2) \) acts as a "diagonal outer automorphism" on \( L(a + b) \) even when \( q = 3 \). Thus (iii) and (iv) hold, and we are done.

Since \( C_c(v, t) = C_c(v, vt) \), (3.3) implies

\[
\text{if } g \text{ is an involution of } \mathcal{J} \text{ and } g \neq v, \text{ then } \]

\[
C_c(g) \cap C_c(v) = C_c(g) \cap N_c(L(a + b)). (3.9)
\]

Since \( L(a + b) \) has exactly one conjugacy class of elements \( x \) such that \( x^2 = v \) and similarly for \( \mathcal{J} \) by [14, (1A)], we have

All involutions of \( L(a + b) \times \mathcal{J} - \mathcal{J} \) are conjugate in \( L(a + b) \mathcal{J} \) and are of form \( \lambda = \lambda_1 \lambda_2 \) where \( \lambda_1 \in L(a + b) \), \( \lambda_2 \in \mathcal{J} \) and \( \lambda_1^2 = \lambda_2^2 = v \). (3.10)

Let

\[
\lambda = \lambda_1 \lambda_2 \quad \text{where} \quad \lambda_1 \in L(a + b), \quad \lambda_2 \in L(a + b) \times R \subseteq \mathcal{J} \cap \mathfrak{H},
\]

and

\[
\lambda_1^2 = \lambda_2^2 = v. (3.11)
\]

**Lemma 3.3.**

(i) \( v \not\sim t \not\sim \lambda \) in \( G \); and

(ii) \( \lambda \not\sim vt \) in \( C_c(v) \).

**Proof.** Let \( K \) denote a quadratic extension field of \( K \) and view \( \mathfrak{H} = \text{Spin}(9, K) \) as a subgroup of \( \overline{\mathfrak{H}} = \text{Spin}(9, \overline{K}) \) as in (1.21). Since \( q^2 \equiv 1 \pmod{4} \), there exists an element \( \xi \) in \( K^\times \) such that \( \xi^q = -1 \). Note that \( [L(r), L(s)] = \{1\} \) and \( L(r) \subseteq C_{\overline{\mathfrak{H}}}(v) \) for all

\[
r, s \in \{a + b, a - b, c + d, c - d\} \quad \text{with} \quad r \neq s.
\]
Also \( \lambda_1 \sim \omega_{a+b} \) in \( L(a+b) \) so that we may assume \( \lambda_1 = \omega_{a+b} \). Similarly by [14, (1A)], we may assume that 
\[ \lambda_2 = \omega_{a+b} \cdot h_{a+b}(\xi) h_{a+b}(\xi) h_{c+d}(\xi) h_{c+d}(\xi) = m \in L(a+b)L(a-b)L(c+d)L(c-d) \subseteq C_\mathfrak{g}(v). \]
But \( m = h_1(-1) h_2(-1)t \) in \( L(a+b)L(a-b)L(c+d)L(c-d) \) so that \( \lambda \sim h_1(-1) h_2(-1)t \) and \( \lambda v \sim h_2(-1) h_2(-1) v = h_2(-1) \) in \( C_\mathfrak{g}(v) \). Since \( h_1(-1) h_2(-1)t \sim v \sim h_2(-1) \) in \( \mathfrak{g} \), it follows that \( \lambda \sim v \sim \lambda v \) in \( \mathfrak{g} \). Hence (i) holds. If \( \lambda \sim vt \) in \( C_c(v) \), then \( \lambda v \sim i \) in \( C_c(v) \). Since \( \lambda v \sim v \) in \( \mathfrak{g} \), (ii) also holds.

**Lemma 3.4.** \( C_c(v) = N_c(L(a+b)) \).

**Proof.** We already know that \( N_c(L(a+b)) \subseteq C_c(v) \) by Lemma 3.3. We shall now apply the clever argument used to prove [9, (3.3)].

First, we claim that no involution of \( N_c(L(a+b)) \) is conjugate in \( C_c(v) \) to an involution of \( \mathfrak{g} \). For, assume that \( x \in N_c(L(a-b)) \) is such that \( x^g = t \) for some \( g \in C_c(v) \). Since \( t \) is a central involution in \( N_c(L(a+b)) \) by Lemma 2.9 (using \( C_\mathfrak{g}(v) \subseteq N_c(L(a+b)) \)), the conjugacy class of \( t \) in \( N_c(L(a+b)) \) contains an odd number of involutions. Thus \( x \) centralizes some involution, say \( i \), with \( i \sim t \) in \( N_c(L(a+b)) \). Similarly \( x \) centralizes some involution \( j \) with \( j \sim vt \) in \( N_c(L(a+b)) \). Then \( i^g, j^g \in C_c(t) \cap C_c(v) = N_c(L(a+b)) \cap C_c(v) \). Also \( i \not\sim j \) in \( G \) so that \( \langle ij \rangle \) contains an involution \( k \). By Lemma 3.2(iv) and the fact that \( O^\mathfrak{g}(L(a+b), \mathfrak{g}) \subseteq L(a+b) \mathfrak{g} \), at least one of \( i^g, j^g, k^g \), say \( k^g \), lies in \( L(a+b) \mathfrak{g} \). Moreover, \( i, j \in \mathfrak{g} \) since \( t, vt \in \mathfrak{g} \cap N_c(L(a+b)) \) and hence \( h \in \mathfrak{g} \). Then Lemma 3.3 and (3.3) imply that \( h^g \in \mathfrak{g} \). But \( h^g \sim h \) in \( C_c(v) \) so that (3.3) implies that there exists an element \( g' \in \mathfrak{g} \) such that \( h^g \sim h \). But then \( gg' \in C_c(h) \cap C_c(v) \subseteq N_c(L(a+b)) \) so that \( g \in N_c(L(a+b)) \). Since \( t \in \mathfrak{g} \cap N_c(L(a+b)) \) while \( x \in N_c(L(a+b)) \) is not conjugate in \( \mathfrak{g} \), we have a contradiction. Similarly \( x \not\sim vt \) in \( C_c(v) \), and our claim is proved.

Now let \( c \in C_c(v) \). Since \( t \) and \( (vt)^c \) are not conjugate in \( G \), \( \langle t(vt)^c \rangle \) contains an involution \( \tau \) such that \( t, (vt)^c \in C_c(\tau) \) and either \( \tau t \sim t \) or \( \tau t \sim vt \) in \( C_c(v) \) by [14, (21)]. Since \( \tau \in C_c(v, t) = N_c(L(a+b)) \cap C_c(t) \), the claim above forces \( \tau t \sim t \) in \( \mathfrak{g} \). Hence \( \tau \in \mathfrak{g} \) and \( (vt)^c \in C_c(\tau, v) \subseteq N_c(L(a+b)) \). Since \( vt \in \mathfrak{g} \), the claim above forces \( (vt)^c \) to lie in \( \mathfrak{g} \). Hence, as above, there is an element \( c' \in \mathfrak{g} \) such that \( (vt)^c \sim vt \) and, as above, \( c \in N_c(L(a+b)) \). Then \( C_c(v) \subseteq N_c(L(a+b)) \) and the proof is complete.

4. The \( (BN) \)-pair

In this section we construct a \((B, N)\)-pair of type \( F_4(K) \) and apply the proof of [3, Theorem C] to show that \( G \) has a subgroup \( G_1 \) isomorphic to \( F_4(K) \). A standard argument then forces \( G \) to satisfy condition (ii) of the theorem.
Let
\[ \mathcal{P}_1 = \mathcal{X}(a + b) \mathcal{X}(a + c) \mathcal{X}(a + d) \mathcal{X}(b + c) \mathcal{X}(b + d) \mathcal{X}(c + d) \mathcal{X}(a - b) \]
\[ \cdot \mathcal{X}(a - c) \mathcal{X}(a - d) \mathcal{X}(b - c) \mathcal{X}(b - d) \mathcal{X}(c - d), \]  
(4.1)
\[ \mathcal{P} = \mathcal{P}_1 \mathcal{X}(a) \mathcal{X}(b) \mathcal{X}(c) \mathcal{X}(d), \]  
(4.2)
\[ \mathcal{L} = \mathcal{X}(a + b) \mathcal{X}(a + c) \mathcal{X}(a + d) \mathcal{X}(a - c) \mathcal{X}(a - d) \mathcal{X}(b + c) \mathcal{X}(b - c) \]
\[ \cdot \mathcal{X}(b + d) \mathcal{X}(b - d) \mathcal{X}(a) \mathcal{X}(b), \]  
(4.3)
and
\[ \mathcal{L}_1 = \mathcal{L} \cap \mathcal{P}. \]  
(4.4)

Then we have
\[ \mathcal{P}_1 \text{ is a Sylow } p\text{-subgroup of } \mathfrak{F} = C_{\mathfrak{G}}(t_1), \]  
(4.5)
\[ |\mathcal{P}_1| = q^{12} \quad \text{and} \quad N_{\mathfrak{G}}(\mathcal{P}_1) = \mathcal{P}_1 \mathcal{H}, \]  
(4.6)
\[ \mathcal{P} \text{ is a Sylow } p\text{-subgroup of } \mathfrak{G}, \quad |\mathcal{P}| = q^{16} \]  
(4.6)
\[ \text{and } N_{\mathfrak{G}}(\mathcal{P}) = \mathcal{P} \mathcal{H} \text{ and } \mathfrak{G} = \mathcal{P} \mathcal{N}_2 \mathcal{P}, \]  
(4.6)
\[ Z(\mathcal{P}_1) = Z(\mathcal{P}) = \mathcal{X}(a + b), \]  
(4.7)
\[ \mathcal{L} \text{ is a normal subgroup of } \mathcal{P} \text{ and } |\mathcal{L}| = q^{11}, \]  
(4.8)
\[ \mathcal{L}_1 = \mathcal{X}(a + b) \mathcal{X}(a + c) \mathcal{X}(a + d) \mathcal{X}(a - c) \mathcal{X}(a - d) \mathcal{X}(b + c) \]
\[ \cdot \mathcal{X}(b - c) \mathcal{X}(b + d) \mathcal{X}(b - d), \]  
(4.9)
and \( \mathcal{L}_1 \) is an elementary abelian normal subgroup of \( \mathcal{P}_1 \) with \( |\mathcal{L}_1| = q^8 \), and
\[ \mathcal{H} \mathfrak{H} \text{ normalizes } \mathcal{P}, \mathcal{P}_1, \mathcal{L} \text{ and } \mathcal{L}_1. \]  
(4.10)

Let
\[ M = O(C_\mathfrak{G}(\mathcal{X}(a + b))). \]  
(4.11)
Clearly
\[ M \trianglelefteq N_\mathfrak{G}(\mathcal{X}(a + b)) \quad \text{and} \quad \mathcal{X}(a + b) \subseteq M, \]  
(4.12)
and
\[ \mathcal{J} \subseteq C_\mathfrak{G}(\mathcal{X}(a + b)) \text{ and } \mathcal{H} \mathfrak{H} \text{ normalizes } M. \]  
(4.13)

Let
\[ Q_1 \text{ and } Q_2 \text{ be Sylow 2-subgroups of } L(a - b) \text{ and } L(c - d) \]  
respectively.
(4.14)

Then \( Q_1 \) and \( Q_2 \) are isomorphic generalized quaternion 2-groups, \( Z(Q_1) = \langle vt \rangle \) and \( Z(Q_2) = \langle h_3(-1) \rangle \). Set
\[ Q_3 = Q_2^u. \]  
(4.15)
Thus $Q_3$ is a Sylow 2-subgroup of $L(c + d)$, $Z(Q_3) = \langle h_3(-1)t \rangle$ and $u: Q_3 \leftrightarrow Q_3$. Also

\[
\langle Q_2, Q_3, u \rangle \text{ is a Sylow 2-subgroup of } R \text{ and } Z(\langle Q_2, Q_3, u \rangle) = \langle t \rangle.
\]

(4.16)

Set

\[
T := \langle Q_1, Q_2, Q_3, u \rangle.
\]

(4.17)

Then

\[
T = Q_1 \times \langle Q_2, Q_3, u \rangle \text{ is a Sylow 2-subgroup of both } L(a - b) \times R \text{ and } \mathcal{J},
\]

(4.18)
of both $L(a - b) \times R$ and $\mathcal{J}$, and

\[
Z(T) = Z(Q_1) \times Z(\langle Q_2, Q_3, u \rangle) = \langle v \rangle \times \langle t \rangle.
\]

**Lemma 4.1.** (i) $N_G(\mathfrak{X}(a + b)) \cap C_G(t) = N_G(\mathfrak{X}(a + b))\mathfrak{X}$ where $N_G(\mathfrak{X}(a + b)) = 2(L(a - b) \times R)H$ and $2 \cap ((L(a - b) \times R)H) = \{1\}$;

(ii) $\mathfrak{X} < N_G(\mathfrak{X}(a + b)) \cap C_G(t)$;

(iii) $C_G(\mathfrak{X}(a + b)) \cap C_G(t) = 2(L(a - b) \times R)$; and

(iv) $T$ is a Sylow 2-subgroup of $C_G(\mathfrak{X}(a + b))$.

**Proof.** Since $Z(\mathcal{P}) = \mathfrak{X}(a + b)$, we have $\mathcal{P}H = N_G(\mathcal{P}) \subseteq N_G(\mathfrak{X}(a + b))$.

Thus $N_G(\mathfrak{X}(a + b))$ contains the Borel subgroup $\mathcal{P}H$ of $\mathcal{S}$ and hence $N_G(\mathfrak{X}(a + b)) = \mathcal{P}H(\omega_1, \omega_3, u)\mathcal{P}$ by [12, Théorème 2]. Then, it readily follows that $N_G(\mathfrak{X}(a + b)) = 2(L(a - b) \times R)H$ and (i) holds. Also $\mathfrak{X} = O_{\mathfrak{X}}(N_G(\mathfrak{X}(a + b)))$ and $N_G(\mathfrak{X}(a + b)) \leq N_G(\mathfrak{X}(a + b)) \cap C_G(t)$ so that (ii) holds. Since $C_G(\mathfrak{X}(a + b)) = H_1 \times H_3 \times H_4 \subseteq L(a - b) \times R \subseteq \mathcal{J} \subseteq C_G(\mathfrak{X}(a + b))$, an easy calculation yields (iii). Finally, $T$ is a Sylow 2-subgroup of $C_G(\mathfrak{X}(a + b)) \cap C_G(t)$, $Z(T) = \langle v, t \rangle$ and $c_{\mathcal{C}}(t) \cap Z(T) = \{t\}$, whence (iv) also holds.

Since $D = \langle t, t_d \rangle$ normalizes $\mathfrak{X}$,

\[
D \subseteq H \cap R \quad \text{and} \quad \mathfrak{X} \cap ((L(a - b) \times R)H) = \{1\},
\]

it follows that

\[
N_G(\mathfrak{X}(a + b)) \cap C_G(D) = N_G(\mathfrak{X}(a + b)) = C_G(D)(L(a - b) \times C_G(D))H.
\]

Since $C_G(D) = \mathfrak{X}^1$ and $C_G(D) = L(c - d) \times L(c + d)$, we have:

**Lemma 4.2.** (i) $N_G(\mathfrak{X}(a + b)) \cap C_G(D) = N_G(\mathfrak{X}(a + b))\mathfrak{X}$ where

\[
N_G(\mathfrak{X}(a + b)) = \mathfrak{X}^1(L(a - b) \times L(c - d) \times L(c + d))H;
\]
(ii) \( \mathcal{A}_1 \triangleleft N_G(\mathfrak{X}(a + b)) \cap C_G(D) \); and
(iii) \( C_G(\mathfrak{X}(a + b)) \cap C_G(D) = \mathcal{A}_1(L(a - b) \times L(c - d) \times L(c + d)) \).

Next we prove:

**Lemma 4.3.** (i) \( C_G(\mathfrak{X}(a + b)) = M \mathcal{J} \) with \( M \cap \mathcal{J} = \{1\} \); (ii) \( M \mathcal{J} H \leq N_G(\mathfrak{X}(a + b)) = M \mathcal{J} H \mathfrak{U} \) and \( (M \mathcal{J} H) \cap \mathfrak{U} = \{1\} \); (iii) \( C_M(v) = \mathfrak{X}(a + b) \) and \( N_G(\mathfrak{X}(a + b)) \cap C_G(v) = \mathfrak{X}(a + b) \mathcal{J} H \mathfrak{U} \); (iv) \( C_M(t) = 2 \); and (v) \( C_M(D) = 2_1 \).

**Proof.** Lemmas 3.2 and 3.4 imply that \( N_G(\mathfrak{X}(a + b)) \cap C_G(v) = \mathfrak{X}(a + b) \mathcal{J} H \mathfrak{U} \) with
\[
\mathfrak{X}(a + b) \mathcal{J} H \leq N_G(\mathfrak{X}(a + b)) \cap C_G(v) \quad \text{and} \quad (\mathfrak{X}(a + b) \mathcal{J} H) \cap \mathfrak{U} = \{1\}.
\]
Then \( C_G(\mathfrak{X}(a + b)) \cap C_G(v) = \mathfrak{X}(a + b) \mathcal{J} \) since \( C_H(\mathfrak{X}(a + b)) \subseteq \mathcal{J} \). Setting \( L = C_G(\mathfrak{X}(a + b)) \) and \( \mathcal{L} = L/M \), we have \( C_L(v) = C_L(\mathfrak{X}(a + b)) = \mathcal{J} \approx \mathcal{J} \approx \text{Sp}(6, K) \) (since \( \mathfrak{X}(a + b) \subseteq M \)). Also
\[
Z(\mathcal{J}) = \langle \bar{v} \rangle \quad \text{and} \quad T \cong T = \overline{Q}_1 \times \langle \overline{Q}_2, \overline{Q}_3, \bar{u} \rangle
\]
is a Sylow 2-subgroup of \( L \). But \( Z(Q_1) = \langle \bar{v}t \rangle \) and \( Z(\langle \overline{Q}_2, \overline{Q}_3, \bar{u} \rangle) = \langle t \rangle \), thus [2, Corollary 4.3] implies that \( \bar{v} \in Z(L) \). Thus
\[
C_G(\mathfrak{X}(a + b)) = M(C_G(\mathfrak{X}(a + b)) \cap C_G(v)) = M \mathcal{J}
\]
and
\[
N_G(\mathfrak{X}(a + b)) = M(N_G(\mathfrak{X}(a + b)) \cap C_G(v)).
\]
Since \( O(\mathcal{J}) = \{1\} \), we have \( M \cap \mathcal{J} = \{1\} \). Also \( C_G(\mathfrak{X}(a + b)) \cap C_G(v) = C_M(v) \mathcal{J} - \mathfrak{X}(a + b) \mathcal{J} \) and \( \mathfrak{X}(a + b) \subseteq C_M(v) \), so that \( C_M(v) = \mathfrak{X}(a + b) \).

Then
\[
(M \mathcal{J} H) \cap \mathfrak{U} = (M \mathcal{J} H) \cap C_G(v) \cap \mathfrak{U} = (C_M(v) \mathcal{J} H) \cap \mathfrak{U} = (\mathfrak{X}(a + b) \mathcal{J} H) \cap \mathfrak{U} = \{1\}
\]
by Lemma 3.2(i). We have proved (i), (ii) and (iii). Since \( v \) inverts \( 2/\mathfrak{X}(a + b) \) and \( Z(L) = Z(\mathcal{J}) = \langle \bar{v} \rangle \), we have \( 2_1 \subseteq 2 \subseteq M \). Also
\[
L(a - b) \times L(c - d) \times L(c + d) \subseteq L(a - b) \times R \subseteq \mathcal{J} \quad \text{and} \quad \mathcal{J} \cap M = \{1\};
\]
so that Lemma 4.1(iii) and Lemma 4.2(iii) imply (iv) and (v), and we are done.
We have
\[ u: \mathfrak{X}(a + b) \leftrightarrow \mathfrak{X}(a - d), \]
\[ u: \mathfrak{X}(b + d) \leftrightarrow \mathfrak{X}(b - d), \]
\[ u: \mathfrak{X}(c + d) \leftrightarrow \mathfrak{X}(c - d), \]
and \( u \) normalizes \( \mathfrak{X}(a + b), \mathfrak{X}(a - b), \mathfrak{X}(a + c), \mathfrak{X}(a - c), \mathfrak{X}(b + c), \mathfrak{X}(b - c), \mathfrak{X}(a), \mathfrak{X}(b) \) and \( \mathfrak{X}(c) \).

Since \( z = u^y \) with \( y \) as in Lemma 2.10, we also have
\[ x: \mathfrak{X}(a + c) \leftrightarrow \mathfrak{X}(a - c), \]
\[ x: \mathfrak{X}(a - c) \leftrightarrow \mathfrak{X}(b + d), \]
\[ x: \mathfrak{X}(a - b) \leftrightarrow \mathfrak{X}(c + d), \]
and \( z \) normalizes \( \mathfrak{X}(a + b), \mathfrak{X}(c - d), \mathfrak{X}(a + c), \mathfrak{X}(b - d), \mathfrak{X}(a + d) \) and \( \mathfrak{X}(b - c) \).

Thus
\[ \langle u, z \rangle \text{ normalizes } \mathcal{P}_1 \text{ and } \mathcal{P}'_1. \]

We can now prove the following.

**Lemma 4.4.** (i) \( M = \mathcal{P}_1 \mathfrak{X}(a) \mathfrak{X}(b) \mathfrak{X}(a)^x \mathfrak{X}(b)^x \mathfrak{X}(a)^{zu} \mathfrak{X}(b)^{zu} \) and \( |M| = q^{15}; \)

(ii) \( C_M(t_1) = \mathcal{P}_1 \mathfrak{X}(a)^x \mathfrak{X}(b)^x \) and \( C_M(tt_1) = \mathcal{P}_1 \mathfrak{X}(a)^{zu} \mathfrak{X}(b)^{zu}; \) and

(iii) \( \mathcal{P}_1 \leq M. \)

**Proof.** Clearly \( D = \langle t, t_1 \rangle \) normalizes \( M \) and

\[ \langle \ast \rangle \quad \mathcal{P}_1 \triangleleft C_M(t) = \mathcal{P}_1 \mathfrak{X}(a) \mathfrak{X}(b). \]

Also \( \langle z, u \rangle \subseteq J \subseteq C_G(\mathfrak{X}(a + b)) \) so that \( \langle z, u \rangle \) also normalizes \( M. \) Conjugating \( \langle \ast \rangle \) by \( z \) and \( zu \) yields (ii), \( \mathcal{P}_1 \triangleleft C_M(t_1) \) and \( \mathcal{P}_1 \triangleleft C_M(tt_1). \) Then a lemma of R. Brauer (cf., [13, top of p. 510]) yields (i) and (iii).

Since \( \langle z, u \rangle \) normalizes \( H \mathfrak{U} \) and \( H \mathfrak{U} \) normalizes \( \mathfrak{X}(s) \) for any \( s \in \Delta, \) we have

\[ H \mathfrak{U} \text{ normalizes } \mathfrak{X}(r)^g \text{ for } r \in \{a, b, c, d\} \text{ and } g \in [s, su]. \]

Let
\[ \mathcal{P}_2 = \mathfrak{X}(a - b) \mathfrak{X}(c + d) \mathfrak{X}(c - d) \mathfrak{X}(e) \mathfrak{X}(d)^x \mathfrak{X}(e)^x \mathfrak{X}(d)^{zu} \mathfrak{X}(e)^{zu} \mathfrak{X}(d)^{zu}. \]
Lemma 4.5. (i) $P_2$ is a Sylow $p$-subgroup of $I$ normalized by $H$ and $|P_2| = q^6$;

(ii) $C_{P_2}(t) = \langle a - b \rangle \langle c + d \rangle \langle c \rangle \langle d \rangle$,
    $C_{P_2}(t_1) = \langle a - b \rangle \langle c + d \rangle \langle c \rangle \langle c \rangle^u \langle d \rangle^u$,
    $C_{P_2}(t t_1) = \langle a - b \rangle \langle c + d \rangle \langle c \rangle \langle c \rangle^u \langle d \rangle^u$, and
    $C_{P_2}(D) = \langle a - b \rangle \langle c + d \rangle \langle c \rangle$.

Proof. First observe that $C_{G}(\langle a - b \rangle \langle c \rangle \langle d \rangle) \cap C_{P_2}(t) =$ \langle a - b \rangle \langle c \rangle \langle c \rangle \langle c \rangle \langle d \rangle$.
Then an argument similar to the proof of [13, Lemma 2.2] implies that $\theta$ is elementary abelian and is the normal 2-complement of

$$C_{G}(\langle a - b \rangle \langle c \rangle \langle d \rangle \langle c \rangle)$$

Since $C_{G}(\langle a - b \rangle) \cap C_{G}(t) = \langle a - b \rangle \langle c \rangle \langle d \rangle$ is a Sylow 2-subgroup of this group with $Z(T_1) = \langle v, t \rangle$. Thus $t$ is strongly closed in $Z(T_1)$ and hence $T_1$ is a Sylow 2-subgroup of $C_{G}(\langle a - b \rangle)$. Then [2, Theorem 4.2] implies that $C_{G}(\langle a - b \rangle) = \langle v \rangle \times R$ where $X = O(C_{G}(\langle a - b \rangle))$. Clearly $C_{X}(t) = \langle a - b \rangle$ so $t$ inverts $X/\langle a - b \rangle$.
As in the proof of [13, Lemma 2.4], we have $C_{X}(t_1) = \langle a - b \rangle \langle c \rangle \langle c \rangle \langle d \rangle$ and $C_{X}(t t_1) = \langle a - b \rangle \langle c \rangle \langle c \rangle \langle d \rangle$.
Again, by Brauer's lemma, we have

$$X = \langle a - b \rangle \langle c \rangle \langle c \rangle \langle d \rangle \langle c \rangle \langle d \rangle$$

Since $C_{G}(\langle a - b \rangle) = \langle v \rangle \times (XR)$, where $XR = RX$ and $X \cap R = \{1\}$, and since $\langle c \rangle \langle d \rangle \langle c \rangle \langle d \rangle \langle c \rangle \langle d \rangle$ is a Sylow $p$-subgroup of $R$, (i) follows. Since $D$ normalizes $R$ and $X$, (ii) follows also.

Set

$$U = M_{P_2}. \quad \quad \quad (4.24)$$

Lemma 4.6. (i) $U$ is a Sylow $p$-subgroup of $C_G(\langle a + b \rangle)$ which is normalized by $H \Psi$ and $|U| = q^{24}$;

(ii) $C_U(A) = C_M(A) C_{P_2}(A)$ for all subgroups $A \subseteq H \Psi$;

(iii) $C_H(U) = \{1\}$. 


Proof. We already know (i) and (ii). Since \( \mathcal{P} \subseteq U \) and \( C_H(\mathcal{P}) = \langle t \rangle \) (as can be seen in \( \mathcal{S} \)), we have \( C_H(U) \subseteq \langle t \rangle \). But \( t \) does not centralize \( U \), so that (iii) also holds.

Set
\[
B = UH.
\]  
(4.25)

Clearly

\[ U < B, \ U \text{ is a normal Sylow } p\text{-subgroup of } B \text{ and } H \text{ is a Hall } p'\text{-subgroup of } B. \]  
(4.26)

**Lemma 4.7.**  \( B \cap N = H < N. \)

**Proof.** Clearly \( H \subseteq B \cap N \) and \( [U \cap N, H] \subseteq [U, H] \cap [N, H] \subseteq U \cap H = \{1\} \). Thus \( U \cap N \subseteq U \cap C_N(H) \subseteq U \cap \mathcal{S} \cap C_G(H) \) by Lemma 2.13. But \( C_G(H) = H \) and hence \( U \cap N \subseteq U \cap H = \{1\} \). Then \( B \cap N = (U \cap N)H = H \), and we are done.

Let \( w_0 \) be as in Lemma 2.15
\( \) (4.27)

and set
\[ V = U^{w_0}. \]  
(4.28)

Since \( w_0^2 \in H \), we have
\[ w_0 : U \leftrightarrow V. \]  
(4.29)

**Lemma 4.8.**  \( U \cap V = \{1\}. \)

**Proof.** Set \( X = U \cap V \) and observe that \( H \) normalizes \( X \). Also \( w_0 \in N_1 = C_N(D) \) so that \( C_X(t) = (C_U(t)) \cap (C_V(t))^{w_0} = \mathcal{P} \cap \mathcal{P}^{w_0} = \{1\} \) in \( \mathcal{S} \) by Lemma 2.15(iii) and [11, Corollary 3 of Lemma 18]. Since \( C_U(t_1) = \mathcal{P}^z \) and \( [w_0, z] \in H \), we have \( C_X(t_1) = \mathcal{P}^z \cap \mathcal{P}^{w_0} = (\mathcal{P} \cap \mathcal{P}^{w_0})^z = \{1\} \) and similarly \( C_X(t_t) = \{1\} \). But \( X = \langle C_X(\tau) \mid \tau \in D^{\infty} \rangle \) and the lemma follows.

Set
\[ \mathcal{U} = \{ \mathfrak{X}(s), \mathfrak{X}(r)^x \mid s \in \Delta^+, r \in \{a, b, c, d\} \text{ and } x \in \{z, zu\} \} \]  
(4.30)

and
\[ \mathcal{V}^c = \{ \mathfrak{X}(s), \mathfrak{X}(r)^z \mid s \in \Delta^c, r \in \{-a, -b, -c, -d\} \text{ and } x \in \{z, zu\} \}. \]  
(4.31)

Clearly
\[ U = \langle \mathcal{U} \rangle \text{ and } V = \langle \mathcal{V}^c \rangle. \]  
(4.32)

Set
\[ \mathcal{Y} = \mathcal{U} \cup \mathcal{V}^c \text{ and } \mathcal{P} = \{\omega_3, \omega_2, u, z\}. \]  
(4.33)
Lemma 4.9. (i) \( N \), acting by conjugation, permutes the elements of \( \mathcal{Y} \) and \( H \) fixes each element of \( \mathcal{Y} \). Thus we may view \( \tilde{N} = N/H \) as permuting the elements of \( \mathcal{Y} \);

(ii) \( \omega_4 : \mathcal{X}(b - c) \rightarrow \mathcal{X}(c - b) \), \( \omega_3 : \mathcal{X}(c - d) \rightarrow \mathcal{X}(d - c) \),

\[ u : \mathcal{X}(d) \rightarrow \mathcal{X}(-d) \]

and \( z : \mathcal{X}(d)^{zu} \rightarrow \mathcal{X}(-d)^{zu} \);

(iii) \( \omega_2 , \omega_3 , u , z \) each send exactly one element of \( \mathcal{Y} \) into an element of \( \mathcal{Y} \) (these are given in (ii)).

Proof. Clearly \( H \) fixes each element of \( \mathcal{Y} \) so that, by Lemma 2.14(i), it suffices to examine the action of the elements of \( \mathcal{S} \) on \( \mathcal{Y} \). By (1.9), we know that \( \omega_2 , \omega_3 \) and \( u = \omega_4 \) permute the elements of \( \{ \mathcal{X}(s) \mid s \in \Delta \} \) with the action given by (1.9). Since \( [\omega_3 , u] \) and \( [\omega_2 , z] \) lie in \( H \) (Lemma 2.14(iii)), we have \( \mathcal{X}(r)^{zuo} = (\mathcal{X}(r)^{uo})^z \) for all \( r \in \{a, b, c, d\} \) and \( x \in \{z, uz\} \) and the statements made about \( \omega_3 \) hold. Similarly using Lemma 2.14(v), one checks that the statements about \( \omega_2 , u \) and \( z \) also hold and we are done.

Since \( U = O_2(B) \) and \( U^w \cap V \neq \{1\} = U \cap V \) for \( \omega \in \mathcal{S} \), we have

Corollary 4.9.1. \( \omega B \omega \neq B \) for all \( \omega \in \mathcal{S} \).

Lemma 4.10. (i) \( \mathcal{X}(b - c)^{uo} \subseteq B \cup B\omega_2 \mathcal{X}(b - c) \);

(ii) \( \mathcal{X}(c - d)^{uo} \subseteq B \cup B\omega_2 \mathcal{X}(c - d) \);

(iii) \( \mathcal{X}(d)^{u} \subseteq B \cup Bu \mathcal{X}(d) \); and

(iv) \( \mathcal{X}(d)^{zu} \subseteq B \cup Bu \mathcal{X}(d)^{zu} \).

Proof. Since \( L(b - c) = \mathcal{X}(b - c) H_a \cup \mathcal{X}(b - c) H_a \omega_2 \mathcal{X}(b - c) \), (i) holds. A similar argument using \( L(c - d) \) and \( L(d) \) yields (ii) and (iii). Finally \( \mathcal{X}(d)^{zu} = \mathcal{X}(-d)^{zu} \) by Lemma 4.9(ii) and

\[ \mathcal{X}(-d) \subseteq L(d) = \mathcal{X}(d) H_a \cup \mathcal{X}(d) H_a u \mathcal{X}(d). \]

Thus

\[ \mathcal{X}(d)^{zu} = \mathcal{X}(-d)^{zu} \subseteq \mathcal{X}(d)^{tu} H_d^u \cup \mathcal{X}(d)^{zu} H_d^u u \mathcal{X}(d)^{zu}. \]

Since \( u^{zu} \in uzu \mathcal{H} II \sim z \mathcal{H} II \sim H_x \), (iv) follows.

Lemma 4.11. If \( \omega \in N \) and \( \omega \in \mathcal{S} \), then \( \omega B \omega \subseteq B \omega B \cup B \omega \omega B \).

Proof. Since \( U = M\mathcal{P}_2 \) with \( M \cap \mathcal{P}_2 = \{1\} \), every element of \( U \) has a unique expression as a product of an element of \( M \) and an element of \( \mathcal{P}_2 \). But Lemma 4.4(i) and Lemma 4.5(i) imply that every element of \( U = M\mathcal{P}_2 \) has a unique expression as a product of elements from each subgroup of \( \mathcal{U} \) in the order prescribed by: \( U = M\mathcal{P}_2 \), (4.23) and Lemma 4.4(i). Calling this unique expression the “standard form” of an element of \( U \), the analog

Set

$$G_1 = BNB.$$ (4.34)

We conclude the proof of the theorem with:

**Lemma 4.12.** (i) $G_1$ is a normal simple subgroup of $G$ and $G_1 \cong F_4(K)$;

(ii) $G = G_1 \mathcal{H}$ where $G_1 \cap \mathcal{H} = \{1\}$; and

(iii) $G$ satisfies condition (ii) of the theorem.

**Proof.** Applying [1, Théorème 1 of Chapter IV Section 2], Lemmas 2.14, 4.7, 4.11 and Corollary 4.9.1, we conclude that $\langle B, N \rangle = BNB = G_1$ and that $(B, N)$ is a Tits system for $G$. We claim that $F(B) = U$ (the Fitting subgroup of $B$). For, $U = O_p(B)$ so that $U \subseteq F(B)$. If $U \neq F(B)$, then $F(B) = U \times X$ where $\{1\} \neq X$ is a normal $p'$-subgroup of $B$. Since $H$ is a Hall $p'$-subgroup of $B$, $X \subseteq C_H(U) = \{1\}$ by Lemma 4.6(iii) which is a contradiction; hence $F(B) = U$. Next note that $J' = J$, $J' = J$, $J = P^pN^p \subseteq G_1$ and $J' = \langle C_p(t), z \rangle \subseteq G_1$. Thus $\langle J, J' \rangle \subseteq G_1'$. But $G_1 \subseteq \langle J, z \rangle$ and hence $G_1 = G_1' = \langle J, J' \rangle$. Since $J = \langle \varphi_h | h \in J \rangle$ and $J = \langle \varphi_j | j \in J \rangle$, we have $G_1 = \langle U^g | g \in G_1 \rangle$. Also $\bigcap_{h \in J} H^h = \langle t \rangle$ and $t^x \neq t$, so that $\bigcap_{h \in G_1} H^h = \{1\}$. Setting $Z = \bigcap_{h \in G_1} B^h$, we have $O_p(Z) \triangleleft G_1$ and $O_p(Z) \subseteq U$ so that $O_p(Z) \subseteq \bigcap_{h \in G_1} U^h = \{1\}$ by Lemma 4.8. Thus $Z$ is a $p'$-group. Then $Z \subseteq H$; whence $Z \subseteq \bigcap_{h \in G_1} H^h = \{1\}$ and $Z = \{1\}$. Applying [1, Théorème 5 of Chapter IV, Section 7], we conclude that $G_1$ is simple. Moreover, setting $X = \bigcap_{x \in H} B^x$, we have $H \subseteq X$ and $O_p(X) = \{1\}$ by Lemma 4.8. This forces $X = H = R \cap N$, so that $(R, N)$ is saturated. Observing that we are in case (a) of [3, (11E)] by Lemma 4.9(iii), the proof of [3, Theorem C] implies that $G_1 \cong F_4(K)$.  

On the other hand, $\mathcal{H}$ normalizes $B$ and $N$ and hence $\mathcal{H}$ normalizes $G_1$. Also

$$G_1 \cap \mathcal{H} \subseteq N_{G_1}(U) \cap \mathcal{H} = B \cap \mathcal{H} = B \cap C_G(t) \cap \mathcal{H} = (C_G(t)H) \cap \mathcal{H} \cap \mathcal{H} = \{1\},$$

so that $G_1 \cap \mathcal{H} = \{1\}$. Since $\langle J, J', \mathcal{H} \rangle \subseteq G_1 \mathcal{H}$, we have $\langle C_G(t), C_G(v) \rangle \subseteq G_1 \mathcal{H}$ by Lemmas 2.3(ii), 3.2(i) and 3.4. Also $G_1 \cong F_4(K)$ so that $G_1$ and hence $G_1 \mathcal{H}$ has two conjugacy classes of involution which must be represented by $t$ and $v$. By Lemma 2.8, if $S$ is a Sylow 2-subgroup of $J$, then $S$ is a

1*Note added in proof:* Alternatively, (i) easily follows from the main result of J. Tits, Buildings and $(B, N)$-pairs of spherical type, unpublished.
Sylow 2-subgroup of $G$ and $N_G(S) \subseteq C_G(t) \subseteq G_t$. Since $G$ has two conjugacy classes of involutions, [4, Theorem 9.2.1] implies that $G_t\mathcal{U}$ contains the set, $\mathcal{J}(G)$, of all involutions of $G$. But then $\langle \mathcal{J}(G) \rangle \subseteq G_1$, whence $G_1 = \langle \mathcal{J}(G) \rangle$ since $G_1$ is simple. This implies that $G_1 \leq G$ and $G = G_1 C_G(t) = G_1 \mathcal{J} \mathcal{U} = G_t \mathcal{U}$ by the Frattini argument and hence both (i) and (ii) hold. Finally $|G/G_1| = |\mathcal{U}| = \rho$ and $O(G) = \{1\}$ by Corollary 2.6.1, so that the results of [11, Section 10] force $G$ to satisfy condition (ii) of the theorem.

This concludes the proof of the theorem.

REFERENCES

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