Series representations for some mathematical constants

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Received 13 June 2005
Available online 8 August 2005
Submitted by William F. Ames

Abstract

The authors apply a classical series identity involving the psi (or digamma) function with a view to deriving series representations for a number of known mathematical constants. Several closely-related consequences and results are also considered.

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Keywords: Series representations; psi (or digamma) function; Fractional calculus; Hurwitz (or generalized) zeta function; Special functions; Summation of series; Gauss hypergeometric function; Polygamma function; Normalized binomial mid-coefficients; Harmonic numbers; $\pi$; $\pi^2$; $\pi^3$; $\pi^4$; log 2; log 3; Catalan’s constant; Apéry’s constant

1. Introduction, definitions and preliminaries

In the significantly vast (and widely scattered) literature on fractional calculus (that is, calculus of derivatives and integrals of an arbitrary real or complex order), we find many
systematic (and historical) accounts of its theory and applications in a number of areas including (for example) ordinary and partial differential equations, special functions, integral equations, and summation of series. Many of the recent works (or the so-called serendipities) in fractional calculus, especially those in the area of summation of infinite series, did indeed revive (as illustrations emphasizing the usefulness of the underlying fractional calculus techniques) various special cases and consequences of either the following familiar summation theorem for the Gauss hypergeometric function (cf., e.g., [29, p. 282]; see also [1, p. 556, Entry 15.1.20]):

$$2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}$$

$$\left( \Re(c - a - b) > 0; \ c \notin \mathbb{Z}^- := \{0, -1, -2, \ldots\} \right)$$

(1.1)

or the following well-known (rather classical) result in the theory of the psi (or digamma) function $\psi(z)$ (cf., e.g., [15, p. 19, Eq. 1.7.4 (30)):

$$\sum_{n=1}^{\infty} \frac{(\nu)_n}{n \cdot (\lambda)_n} = \psi(\lambda) - \psi(\lambda - \nu) \quad (\Re(\lambda - \nu) > 0; \ \lambda \notin \mathbb{Z}^-_0),$$

(1.2)

where

$$\psi(z) := \frac{d}{dz}\left\{\log \Gamma(z)\right\} = \frac{\Gamma'(z)}{\Gamma(z)}$$

(1.3)

or, equivalently,

$$\log \Gamma(z) = \int_1^z \psi(\zeta) \, d\zeta,$$

(1.4)

and $(\lambda)_v$ denotes the Pochhammer symbol defined (for $\lambda, \nu \in \mathbb{C}$ and in terms of the Gamma function) by

$$(\lambda)_v := \frac{\Gamma(\lambda + v)}{\Gamma(\lambda)}$$

$$= \begin{cases} 1 & (v = 0; \ \lambda \in \mathbb{C}\setminus\{0\}), \\ \lambda(\lambda + 1)\cdots(\lambda + n - 1) & (v = n \in \mathbb{N} := \{1, 2, 3, \ldots\}; \ \lambda \in \mathbb{C}). \end{cases}$$

(1.5)

While the summation theorem (1.1) is attributed to Carl Friedrich Gauss (1777–1855), a finite-difference derivation of the series identity (1.2) can be found on pp. 251 and 261 in the 1924 classical monograph, entitled Vorlesungen über Differenzenrechnung, by Niels Erik Nörlund (1885–1981) (see, for example, Erdélyi et al. [15, p. 19]; see also Hansen [17, p. 126, Entry (6.6.34)], Magnus et al. [21, p. 17], Prudnikov et al. [23, p. 537, Entry 40], and Whittaker and Watson [29, p. 263, Example 43]). For a reasonably detailed historical account of the series identity (1.2), and of its numerous consequences and generalizations, one may refer to the work on the subject by Nishimoto and Srivastava [22], who also provided a number of relevant earlier references on summation of infinite series by means of fractional calculus. Many further developments on this subject are reported by (among others) Srivastava [26], Al-Saqabi et al. [2], and Aular de Durán et al. [5].
In our present investigation dealing with the series identity (1.2), we shall also make use of such other higher transcendental functions as the Riemann zeta function \( \zeta(s) \) as well as the Hurwitz (or generalized) zeta function \( \zeta(s,a) \) defined, when \( \Re(s) > 1 \), by (cf., e.g., [27, p. 88 et seq.])

\[
\zeta(s) := \zeta(s,1) \quad \text{and} \quad \zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad (\Re(s) > 1; \ a \notin \mathbb{Z}_0^-),
\]

(1.6)

which satisfies the following relationship:

\[
\psi^{(p)}(z) = \frac{d^p}{dz^p} \left\{ \psi(z) \right\} = (-1)^{p+1} p! \zeta(p+1, z) \quad (p \in \mathbb{N}; \ z \notin \mathbb{Z}_0^-)
\]

(1.7)

with the polygamma function \( \psi^{(p)}(z) \) defined by

\[
\psi^{(p)}(z) := \frac{d^{p+1}}{dz^{p+1}} \left\{ \log \Gamma(z) \right\} = \frac{d^p}{dz^p} \left\{ \psi(z) \right\} \quad (p \in \mathbb{N}_0; \ z \notin \mathbb{Z}_0^-).
\]

(1.8)

The following result will also be required in our investigation. Indeed, for the psi (or digamma) function \( \psi(z) \), it is known that [27, p. 14, Eq. 1.2 (7)]

\[
\psi(z+n) = \psi(z) + \sum_{k=1}^{n} \frac{1}{z+k-1} \quad (z \in \mathbb{C}\\mathbb{\setminus}\mathbb{Z}_0^-),
\]

(1.9)

which, for \( z = 1 \), yields the following special case for the \( n \)th harmonic number \( H_n \):

\[
H_n := \sum_{k=1}^{n} \frac{1}{k} = \gamma + \psi(n+1),
\]

(1.10)

where \( \gamma \) denotes the Euler–Mascheroni constant defined by

\[
\gamma := \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) = -\psi(1)
\]

\[
\cong 0.57721\ 56649\ 01532\ 8606\ 06512\ldots.
\]

(1.11)

In view of this last relationship (1.11), an obvious special case of the series identity (1.2) when

\[
\lambda = 1 \quad \text{and} \quad \nu = 1 - t \quad (0 < t < 1)
\]

would yield the following formula which plays a key role in our investigation:

\[
\psi(t) = -\gamma - \frac{1}{\Gamma(1-t)} \sum_{n=1}^{\infty} \frac{\Gamma(n+1-t)}{n \cdot n!} \quad (0 < t < 1).
\]

(1.12)

Applying the identity (1.9), we find from (1.12) by differentiation that

\[
\psi'(t) = \sum_{n=1}^{\infty} A_n(t) \quad \text{with} \quad A_n(t) := \frac{\Gamma(n+1-t)}{\Gamma(1-t)} \sum_{k=1}^{n} \frac{1}{k-t}
\]

(1.13)
\[
\psi''(t) = \sum_{n=1}^{\infty} \frac{B_n(t)}{n \cdot n!}
\]

with
\[
B_n(t) := \frac{\Gamma(n+1-t)}{\Gamma(1-t)} \left[ \sum_{k=1}^{n} \frac{1}{(k-t)^2} - \left( \sum_{k=1}^{n} \frac{1}{k-t} \right)^2 \right].
\]

We note that the infinite series in (1.13) and (1.14) converge uniformly in every closed interval contained in (0,1), so that the term-by-term differentiation is justified.

In [18] it is shown that (1.12) and (1.13) can be used to get good numerical approximations for certain values of \(\psi(t)\) and \(\psi'(t)\).

Such mathematical constants as (for instance) \(\pi\) and \(e\) have attracted the attention of researchers for centuries. Their fascination has led to the discovery of numerous remarkable series, product, and integral representations for many of these mathematical constants. In Sections 2 and 3 of the present paper, we apply the formulas (1.12), (1.13) and (1.14), as well as other results, in order to obtain (presumably new) series expansions for \(\pi\), \(\pi^2\), \(\pi^3\), \(\pi^4\), \(\log 2\), \(\log 3\), Catalan’s constant
\[
G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \cong 0.91596 55941 77219015 \ldots, \tag{1.15}
\]
and Apéry’s constant
\[
\zeta(3) \cong 1.20205 69031 59594 28540 \ldots.
\]

2. Series representations

1. The following elegant series representations for \(\pi\), \(\pi^3\), and \(\pi^4\) were given by Lehmer [20], Zucker [30], and Borwein and Borwein [8], respectively:
\[
\begin{align*}
\frac{\pi}{2} &= \sum_{n=2}^{\infty} \frac{1}{2^n \mu_n}, \quad \frac{7\pi^3}{216} = \sum_{n=0}^{\infty} \frac{\mu_n}{2^{2n}(2n+1)^3}, \quad \text{and} \quad \frac{\pi^4}{32} = \sum_{n=1}^{\infty} \left( \frac{h_n}{n} \right)^2, \tag{2.1}
\end{align*}
\]
where \(\mu_n\) denotes the normalized binomial mid-coefficient defined by
\[
\mu_n := 2^{-2n} \binom{2n}{n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} = \frac{\left( \frac{1}{2} \right)_n}{n!} = \frac{\Gamma(n+\frac{1}{2})}{n! \sqrt{\pi}}
\]
and
\[
h_n := \sum_{k=1}^{n} \frac{1}{2k-1} = H_{2n} - \frac{1}{2} H_n = \frac{1}{2} \left[ \psi\left( n + \frac{1}{2} \right) - \psi\left( \frac{1}{2} \right) \right]
\]
where
\[
\psi\left( n + \frac{1}{2} \right) = \frac{\Gamma(n+1)}{\Gamma(n)} = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)}
\]
in terms of the \( n \)th harmonic number \( H_n \) given already by (1.10). The third formula in (2.1) is a corrected version of a result found earlier by De Doelder [14, p. 138, Eq. (22)] (see also [8, p. 1198]).

Some of the main properties of \( \mu_n \) are collected in [3,6]. Many interesting facts about \( H_n \) can be found in [25]. Furthermore, series representations for \( \pi^4 \) involving \( H_n \) are considered in (for example) \[8,14,24\].

An application of (1.13) leads us to a series representation for \( \pi^2 \) involving \( \mu_n \) and \( h_n \) defined by (2.2) and (2.3), respectively. Indeed, using the last expression in (2.2), together with the fact that
\[
\psi\left(\frac{1}{2}\right) = \log \frac{\pi}{2} - \frac{3}{2} \log 2 \quad \text{and} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},
\]
we get
\[
\pi^2 = \sum_{n=1}^{\infty} \frac{\mu_n h_n}{n}.
\]

2. We know that (see, for example, [27, p. 20])
\[
\psi\left(\frac{1}{4}\right) = -\gamma - \frac{\pi}{2} - 3 \log 2 \quad \text{and} \quad \frac{\Gamma\left(n + \frac{3}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \left(\frac{3}{4}\right)_n = \frac{1}{4^n} \prod_{k=1}^{n} (4k - 1),
\]
so that (1.12) (with \( t = \frac{1}{4} \)) yields the following companion of the first formula in (2.1):
\[
\frac{\pi}{2} = -3 \log 2 + \sum_{n=1}^{\infty} \frac{p_n}{n},
\]
where
\[
p_n := \prod_{k=1}^{n} \left(1 - \frac{1}{4k}\right).
\]
Furthermore, in terms of \( q_n \) defined by
\[
q_n := \prod_{k=1}^{n} \left(1 - \frac{3}{4k}\right),
\]
by applying [27, p. 20]
\[
\psi\left(\frac{3}{4}\right) = -\gamma + \frac{\pi}{2} - 3 \log 2
\]
and (1.12) (with \( t = \frac{3}{4} \)), we get
\[
\frac{\pi}{2} = 3 \log 2 - \sum_{n=1}^{\infty} \frac{q_n}{n}.
\]

Numerous remarkable representations for \( \pi \) are given in [9,11].
3. The following interesting connections of the Catalan constant $G$ defined by (1.15) with $\pi^2$, $\psi'(\frac{1}{4})$, and $\psi'(\frac{3}{4})$ were given by Kölbig [19]:

$$\psi'(\frac{1}{4}) = \pi^2 + 8G \quad \text{and} \quad \psi'(\frac{3}{4}) = \pi^2 - 8G. \quad (2.11)$$

Making use of (2.11), together with the second formula in (2.6) and

$$\frac{\Gamma(n + \frac{1}{4})}{\Gamma(\frac{1}{4})} = \left(\frac{1}{4}\right)_n = \frac{1}{4^n} \prod_{k=1}^{n} (4k - 3), \quad (2.12)$$

we find from (1.13) (with $t = \frac{1}{4}$ and $t = \frac{3}{4}$, respectively) that

$$\frac{\pi^2}{2} = \sum_{n=1}^{\infty} \frac{\sigma_n}{n}, \quad (2.13)$$

where

$$\sigma_n := p_n \sum_{k=1}^{n} \frac{1}{4k - 1} + q_n \sum_{k=1}^{n} \frac{1}{4k - 3}, \quad (2.14)$$

and

$$4G = \sum_{n=1}^{\infty} \frac{\delta_n}{n}, \quad (2.15)$$

where

$$\delta_n := p_n \sum_{k=1}^{n} \frac{1}{4k - 1} - q_n \sum_{k=1}^{n} \frac{1}{4k - 3}, \quad (2.16)$$

$p_n$ and $q_n$ being given by (2.8) and (2.9), respectively.

Further series expansions as well as product and integral representations for the Catalan constant $G$ can be found in (for example) [12,27], and [28, pp. 199–200].

4. In his celebrated paper [4], Roger Apéry (1916–1994) used the following known formula (see, for details, [27, p. 280]):

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n n^3 \mu_n} \quad (2.17)$$

to prove the irrationality of $\zeta(3)$.

In view of (1.7) (with $p = 2$ and $z = \frac{1}{2}$), we have [27, p. 96]

$$\psi''\left(\frac{1}{2}\right) = -2\zeta\left(3, \frac{1}{2}\right) = -14\zeta(3), \quad (2.18)$$

so that (1.14) yields the following series representation for the Apéry constant $\zeta(3)$:
\[ \zeta(3) = \frac{2}{7} \sum_{n=1}^{\infty} \frac{\mu_n \omega_n}{n}, \quad (2.19) \]

where \( \mu_n \) is given by (2.2) and

\[ \omega_n := \left( \sum_{k=1}^{n} \frac{1}{2k-1} \right)^2 - \sum_{k=1}^{n} \frac{1}{(2k-1)^2}. \quad (2.20) \]

A detailed list of references on Apéry’s constant can be found in [16].

5. Using the formulas [27, p. 20]:

\[ \psi\left(\frac{1}{3}\right) = -\gamma - \frac{\pi}{2\sqrt{3}} - \frac{3}{2} \log 3, \quad \psi\left(\frac{2}{3}\right) = -\gamma + \frac{\pi}{2\sqrt{3}} - \frac{3}{2} \log 3, \]

\[ \frac{\Gamma(n + \frac{1}{3})}{\Gamma\left(\frac{1}{3}\right)} = \left(\frac{1}{3}\right)_n = \frac{1}{3^n} \prod_{k=1}^{n} (3k - 2), \quad \frac{\Gamma(n + \frac{2}{3})}{\Gamma\left(\frac{2}{3}\right)} = \left(\frac{2}{3}\right)_n = \frac{1}{3^n} \prod_{k=1}^{n} (3k - 1), \]

and (1.12) (with \( t = \frac{1}{3} \) and \( t = \frac{2}{3} \)), we get

\[ 3 \log 3 = \sum_{n=1}^{\infty} \frac{r_n + s_n}{n}, \quad (2.21) \]

and

\[ \frac{\pi}{\sqrt{3}} = \sum_{n=1}^{\infty} \frac{r_n - s_n}{n}, \quad (2.22) \]

respectively, \( r_n \) and \( s_n \) being given by

\[ r_n := \prod_{k=1}^{n} \left(1 - \frac{1}{3k}\right) \quad \text{and} \quad s_n := \prod_{k=1}^{n} \left(1 - \frac{2}{3k}\right). \quad (2.23) \]

3. Further remarks and observations

In this section, we first derive a mild generalization of the series representation (2.5) for \( \pi^2 \), which we developed in Section 2. Indeed, in view of the obvious derivative formula:

\[ \frac{\partial}{\partial a} \left\{ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \right\} = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \left[ \psi(a + n) - \psi(a) \right] \frac{z^n}{n!} \]

\[ \left( |z| < 1; |z| = 1 \text{ and } \Re(c - a - b) > 0; \ c \notin \mathbb{Z}_0^- \right), \quad (3.1) \]

the classical Gauss summation theorem (1.1) yields the following summation identity [24, p. 595, Eq. (1.8)]:
\[
\sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} \left[ \psi(a+n) - \psi(a) \right]
= \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \left[ \psi(c-a) - \psi(c-a-b) \right]
= \frac{\Gamma(1-a) \Gamma(1+b)}{\Gamma(1-a+b)} \left( \frac{\psi(1-a+b) - \psi(1-a)}{b} \right)
\quad (\Re(c-a-b) > 0; \; c \notin \mathbb{Z}_0^-).
\] (3.2)

A summation identity, equivalent to (3.2), would follow upon differentiating both sides of (1.1) partially with respect to \(b\). Moreover, by similarly involving the denominator parameter \(c\) instead, we obtain [24, p. 595, Eq. (1.10)]

\[
\sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} \left[ \psi(c+n) - \psi(c) \right]
= \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \left[ \psi(c-a) + \psi(c-b) - \psi(c) - \psi(c-a-b) \right]
\quad (\Re(c-a-b) > 0; \; c \notin \mathbb{Z}_0^-).
\] (3.3)

In its special case when \(c = b + 1\), the summation identity (3.2) becomes

\[
\sum_{n=1}^{\infty} \frac{(a)_n}{(b+n) \cdot n!} \left[ \psi(a+n) - \psi(a) \right]
= \frac{\Gamma(1-a) \Gamma(1+b)}{\Gamma(1-a+b)} \left( \frac{\psi(1-a+b) - \psi(1-a)}{b} \right)
\quad (\Re(a) < 1; \; b \notin \mathbb{Z}_0^-).
\] (3.4)

which, when \(b \to 0\), yields the following alternative form of (1.13):

\[
\sum_{n=1}^{\infty} \frac{(a)_n}{n \cdot n!} \left[ \psi(a+n) - \psi(a) \right] = \psi'(1-a) \quad (\Re(a) < 1).
\] (3.5)

By virtue of (2.2), (2.3), and the first formula in (2.4), our series representation (2.5) for \(\pi^2\) is an immediate consequence of the summation identity (3.5) when \(a = \frac{1}{2}\).

Next we turn to the summation identity (3.3). By setting \(c = b + 1\) and proceeding as in the derivation of (3.4), we find from (3.3) that

\[
\sum_{n=1}^{\infty} \frac{(a)_n}{(b+n) \cdot n!} \left[ \psi(b+n+1) - \psi(b+1) \right]
= \frac{\Gamma(1-a) \Gamma(1+b)}{\Gamma(1-a+b)} \left( \frac{\psi(1-a+b) - \psi(1-a)}{b} - \gamma + \psi(1+b) \right)
\quad (\Re(a) < 1; \; b \notin \mathbb{Z}_0^-),
\] (3.6)

which, in the limit case when \(b \to 0\), leads us to the following summation formula involving the \(n\)th harmonic number \(H_n\) given by (1.10):

\[
\sum_{n=1}^{\infty} \frac{(a)_n}{n \cdot n!} H_n = \psi'(1-a) - \frac{\pi^2}{6} \quad (\Re(a) < 1).
\] (3.7)
For $a = \frac{1}{2}$, (3.7) yields yet another series representation for $\pi^2$ in the form:
\[
\frac{\pi^2}{3} = \sum_{n=1}^{\infty} \frac{\mu_n H_n}{n}.
\] (3.8)
where $\mu_n$ is the normalized binomial mid-coefficient given by (2.2).

We find from (3.7) by differentiation that
\[
\sum_{n=1}^{\infty} \frac{(a)_n}{n \cdot n!} H_n \cdot \left[ \psi(a + n) - \psi(a) \right] = -\psi''(1 - a) \quad (\Re(a) < 1).
\] (3.9)

Using (2.2), (2.3), and (2.18), we get from (3.9) (with $a = \frac{1}{2}$) another series representation for Apéry’s constant as follows:
\[
\zeta(3) = \frac{1}{7} \sum_{n=1}^{\infty} \frac{\mu_n h_n H_n}{n}.
\] (3.10)

This last result (3.10) may be viewed as a companion of the following series representations:
\[
\zeta(3) = \sum_{n=1}^{\infty} \frac{H_n}{(n + 1)^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n}{n^2} = \frac{4}{7} \sum_{n=1}^{\infty} \frac{h_n}{n^2}.
\] (3.11)

The first result in (3.11) and its obvious variant given by the second result in (3.11) were proven by Leonhard Euler (1707–1783) in his 1775 paper (see, for details, [7, p. 252 et seq.]; see also [24, p. 602]). The third result in (3.11) was derived by De Doelder [14, p. 134, Eq. (15)].

We now recall the following relationships proven by Kölbig [19]:
\[
\psi''\left(\frac{1}{4}\right) = -2\pi^3 - 56\zeta(3) \quad \text{and} \quad \psi''\left(\frac{3}{4}\right) = 2\pi^3 - 56\zeta(3).
\] (3.12)

By using (3.12), (1.9), the second formula in (2.6), and (2.12), we find from (3.9) (with $a = \frac{1}{4}$ and $a = \frac{3}{4}$, successively) that
\[
\pi^3 = \sum_{n=1}^{\infty} \frac{\delta_n H_n}{n},
\] (3.13)
where $\delta_n$ is defined in (2.16). Furthermore, since
\[
\lim_{a \to 0} \{(a)_n \cdot \left[ \psi(a + n) - \psi(a) \right]\} = (n - 1)!,
\] (3.14)
in its limit case when $a \to 0$, (3.9) yields Euler’s result given by the second formula in (3.11).

From (3.9) we find by further differentiation that
\[
\sum_{n=2}^{\infty} \frac{H_n}{n \cdot n!} B_n(1 - a) = -\psi'''(1 - a) \quad (\Re(a) < 1),
\] (3.15)
where $B_n(1 - a)$ is defined as in (1.14). Since
\[
\lim_{{a \to 0}} \{B_n(1 - a)\} = -2 \cdot (n - 1)! \cdot H_{n-1} \quad (n \in \mathbb{N}\backslash\{1\})
\]
and
\[
\psi'''(1) = 6 \zeta(4) = \frac{\pi^4}{15},
\]
in its limit case when $a \to 0$, (3.15) yields
\[
\frac{\pi^4}{30} = \sum_{n=2}^{\infty} \frac{H_{n-1} H_n}{n^2}.
\]
(3.16)

Using (3.16) in conjunction with Euler’s formula (see, for example, [24, p. 602, Eq. (4.13)]):
\[
\frac{\pi^4}{72} = \sum_{n=1}^{\infty} \frac{H_n}{n^3},
\]
(3.17)
we get the following known companion of the third formula in (2.1) [17, p. 366, Entry (55.8.2)] (see also [24, p. 603]):
\[
\frac{17\pi^4}{360} = \sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2.
\]
(3.18)
By means of a different method, the series representation (3.18) was rediscovered by (for example) Borwein and Borwein [8, p. 1192, Eq. (3)]. Moreover, in its equivalent form:
\[
\frac{11\pi^4}{360} = \sum_{n=1}^{\infty} \left(\frac{H_n}{n + 1}\right)^2,
\]
(3.19)
the known result (3.18) was also derived markedly differently by De Doelder [14, p. 129, Eq. (9)] (see, for details, [24, p. 603]).

Since
\[
\psi'''\left(\frac{1}{2}\right) = 6 \zeta\left(4, \frac{1}{2}\right) = \pi^4 \quad \text{and} \quad \frac{B_n\left(\frac{1}{2}\right)}{n!} = -4 \mu_n \omega_n,
\]
where $\omega_n$ is defined by (2.20), we can deduce yet another series representation for $\pi^4$ from the formula (3.15) (with $a = \frac{1}{2}$) as follows:
\[
\frac{\pi^4}{4} = \sum_{n=1}^{\infty} \frac{\mu_n \omega_n H_n}{n}.
\]
(3.20)

Some other interesting consequences and applications of the results considered here are being recorded as follows.
1. By setting $b = 1$ in (3.6) and making use of (1.9), we get
\[
\sum_{n=1}^{\infty} \frac{(a)_n}{(n+1)!} (H_{n+1} - 1) = \frac{a}{(1 - a)^2}.
\] (3.21)

In view of the summation identity [20]:
\[
\sum_{n=1}^{\infty} \frac{\mu_n}{n+1} = 1,
\] (3.22)
a special case of (3.21) when $a = \frac{1}{2}$ yields the following relative of (3.8):
\[
\sum_{n=1}^{\infty} \frac{\mu_n H_{n+1}}{n+1} = 3.
\] (3.23)

2. For $a = b = c - 1 = \frac{1}{2}$, (3.3) yields
\[
\frac{1}{2} (\log 4 - 1)\pi = \sum_{n=1}^{\infty} \frac{\mu_n (h_{n+1} - 1)}{2n+1}.
\] (3.24)

Now, by observing that the power series in
\[
P(x) := \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^{2n}}{2n+1} = \frac{\arcsin(2x)}{2x} \quad (|x| < \frac{1}{2})
\]
converges also when $x = \frac{1}{2}$, we conclude from Abel’s theorem that
\[
P\left(\frac{1}{2}\right) = \frac{\pi}{2},
\]
which leads us to the following summation identity:
\[
\frac{\pi}{2} - 1 = \sum_{n=1}^{\infty} \frac{\mu_n}{2n+1}.
\] (3.25)

From (3.24) and (3.25), we obtain
\[
\frac{\pi}{2} \log 2 = \sum_{n=1}^{\infty} \frac{n \mu_n h_n}{(2n-1)^2}.
\] (3.26)

3. Using the limit relation (3.14), we find from (3.2) that
\[
\sum_{n=1}^{\infty} \frac{(b)_n}{nb \cdot (c)_n} = \frac{\psi(c) - \psi(c - b)}{b} \quad (\Re(c - b) > 0; \ b \neq 0; \ c \notin \mathbb{Z}_0),
\]
which, when $b \to 0$, yields
\[
\sum_{n=1}^{\infty} \frac{(n-1)!}{n \cdot c_n} = \psi'(c) \quad (\Re(c) > 0).
\] (3.27)

In view of (2.11), by setting \(c = \frac{1}{4}\) and \(c = \frac{3}{4}\) in (3.27), we obtain
\[
2\pi^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{1}{q_n} + \frac{1}{p_n} \right)
\text{ and } 16G = \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{1}{q_n} - \frac{1}{p_n} \right),
\] (3.28)
respectively, \(p_n\) and \(q_n\) being given by (2.8) and (2.9).

4. The reflection formula for the psi (or digamma) function states that
\[
\psi(1-z) - \psi(z) = \pi \cot(\pi z),
\] (3.29)
which, upon differentiation, yields
\[
\psi'(1-z) + \psi'(z) = \pi^2 \csc^2(\pi z)
\] (3.30)
and
\[
\psi''(1-z) - \psi''(z) = 2\pi^3 \csc^2(\pi z) \cot(\pi z).
\] (3.31)

Using (3.29) and (1.12), we find that
\[
\pi \cot(\pi t) = \sum_{n=1}^{\infty} \frac{(1-t)n - (t)n}{n \cdot n!} \quad (0 < t < 1),
\] (3.32)
which, in the special case when \(t = \frac{1}{4}\), leads to
\[
\pi = \sum_{n=1}^{\infty} \frac{p_n - q_n}{n},
\] (3.33)
where \(p_n\) and \(q_n\) are defined by (2.8) and (2.9), respectively. If, on the other hand, we combine (2.7) and (2.10), we obtain the following counterpart of (3.33):
\[
6\log 2 = \sum_{n=1}^{\infty} \frac{p_n + q_n}{n}.
\] (3.34)

In view of (1.13), a special case of (3.30) when \(z = \frac{1}{3}\) yields (2.13). Moreover, by applying (3.30) (with \(z = \frac{1}{3}\)) and (1.13), we get
\[
\frac{4\pi^2}{9} = \sum_{n=1}^{\infty} \frac{\epsilon_n}{n},
\] (3.35)
where
\[
\epsilon_n := r_n \sum_{k=1}^{n} \frac{1}{3k - 1} + s_n \sum_{k=1}^{n} \frac{1}{3k - 2}
\] (3.36)
with \(r_n\) and \(s_n\) given by (2.23).
By using (3.31) (with $z = 1/3$) and (1.14), we obtain
\[ \frac{8\sqrt{3}}{81}\pi^3 = \sum_{n=1}^{\infty} \frac{\theta_n}{n}, \tag{3.37} \]
where
\[ \theta_n := s_n \left[ \frac{1}{(3k-2)^2} - \left( \sum_{k=1}^{n} \frac{1}{3k-2} \right)^2 \right] \]
\[ - r_n \left[ \frac{1}{(3k-1)^2} - \left( \sum_{k=1}^{n} \frac{1}{3k-1} \right)^2 \right], \tag{3.38} \]
with $r_n$ and $s_n$ given by (2.23).

By differentiating both sides of (3.31) once more, we get
\[ \psi'''(1-z) + \psi'''(z) = 2\pi^4 \csc^2(\pi z) \left[ 1 + 3 \cot^2(\pi z) \right]. \tag{3.39} \]
If we set $z = 1/4$ in (3.39) and apply (3.15), we obtain
\[ \pi^4 = \sum_{n=2}^{\infty} \frac{H_n \vartheta_n}{n}, \tag{3.40} \]
where
\[ \vartheta_n := p_n \left[ \left( \sum_{k=1}^{n} \frac{1}{4k-1} \right)^2 - \sum_{k=1}^{n} \frac{1}{(4k-1)^2} \right] \]
\[ + q_n \left[ \left( \sum_{k=1}^{n} \frac{1}{4k-3} \right)^2 - \sum_{k=1}^{n} \frac{1}{(4k-3)^2} \right], \tag{3.41} \]
with $p_n$ and $q_n$ given in (2.8) and (2.9), respectively.

**Remark.** It is easily seen from (3.29) that
\[ \psi^{(N)}(1-z) + (-1)^{N+1} \psi^{(N)}(z) = (-1)^N \pi \frac{d^N}{dz^N} \left\{ \cot(\pi z) \right\} \quad (N \in \mathbb{N}_0), \]
in which the right-hand side is expressible as a polynomial of degree $N + 1$ in $\cot(\pi z)$. Thus, whenever this polynomial can be evaluated in a closed form, we are led to a series representation of the class exemplified already by (3.35), (3.37), and (3.40) above.

5. By setting $c = 2a$ in (3.2), we have
\[ \sum_{n=1}^{\infty} \frac{(a)_n}{(2a)_n} \frac{(b)_n}{n!} \frac{\psi(a+n) - \psi(a)}{b} \]
\[ = \frac{\Gamma(2a) \Gamma(a-b)}{\Gamma(a) \Gamma(2a-b)} \frac{\psi(a) - \psi(a-b)}{b} \]
\[ (\Re(a-b) > 0; \ a \notin \mathbb{Z}_0^+; \ b \neq 0). \tag{3.42} \]
We now let $b \to 0$ in (3.42). We thus find that
\[
\sum_{n=1}^{\infty} \frac{(a)_n}{n \cdot (2a)_n} \left[ \psi(a + n) - \psi(a) \right] = \psi'(a) \quad (\Re(a) > 0).
\] (3.43)

For $a = 1$, (3.43) yields the following counterpart of the first two series expansions given in (3.11):
\[
\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{H_n}{n(n + 1)}.
\] (3.44)

which is comparable with a known summation identity:
\[
\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{H_n}{n \cdot 2^n}.
\] (3.45)

The summation identity (3.44) follows also as a special case of the following known result [17, p. 361, Entry (55.2.7)]:
\[
\sum_{n=2}^{\infty} \frac{1}{n(x + n)} \left[ \psi(x + n) - \psi(x + 1) \right] = \frac{1}{x^2} \left[ \frac{\pi^2}{6} x - \gamma - \psi(x + 1) \right] \quad (x \notin \mathbb{Z})
\] (3.46)

when $x = 1$. On the other hand, since [17, p. 61, Entry (5.12.44)]
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{1}{2} \right)^n = \frac{\pi^2}{12} - \frac{1}{2} (\log 2)^2,
\]

the summation identity (3.45) is an obvious special case of the following known result [17, p. 63, Entry (5.13.15)]:
\[
\sum_{n=1}^{\infty} H_n \frac{x^n}{n} = \frac{1}{2} \left[ \log(1 - x) \right]^2 + \sum_{n=1}^{\infty} \frac{x^n}{n^2} \quad (|x| < 1)
\] (3.47)

when $x = \frac{1}{2}$. Formula (3.45) was, in fact, posed as a problem in [13].

6. It is known that [17, p. 360, Entry (55.1.2); p. 366, Entry (55.9.5)]
\[
\psi^{(N)}(mx) = \delta_{N,0} \log m + \frac{1}{m^{N+1}} \sum_{k=0}^{m-1} \psi^{(N)} \left( x + \frac{k}{m} \right),
\] (3.48)

where $\delta_{N,0}$ is the Kronecker symbol. Using (3.48) with $N = 1$, we get
\[
\sum_{k=1}^{m-1} \psi' \left( \frac{k}{m} \right) = \frac{1}{6} (m^2 - 1) \pi^2 \quad (m \in \mathbb{N} \setminus \{1\}),
\]

which, in conjunction with (1.13), yields
\[
\sum_{k=1}^{m-1} \sum_{n=1}^{\infty} \frac{A_n\left(\frac{k}{m}\right)}{n \cdot n!} = \frac{1}{6} (m^2 - 1) \pi^2 \quad (m \in \mathbb{N}\setminus\{1\}).
\]

Formula (2.5) is a special case of (3.49) when \( m = 2 \). By setting \( m = 3 \) in (3.49), we obtain (3.35).

7. By applying (3.48) with \( N = 2 \), we find that
\[
\sum_{k=1}^{m-1} \frac{\psi''\left(\frac{k}{m}\right)}{n} = -2(m^3 - 1) \xi(3) \quad (m \in \mathbb{N}\setminus\{1\}),
\]
which, together with (1.14), leads us to the following summation identity:
\[
\sum_{k=1}^{m-1} \sum_{n=1}^{\infty} \frac{B_n\left(\frac{k}{m}\right)}{n \cdot n!} = -2(m^3 - 1) \xi(3) \quad (m \in \mathbb{N}\setminus\{1\}).
\]

Formula (2.19) is a special case of (3.50) when \( m = 2 \). Moreover, for \( m = 3 \) in (3.50), we get
\[
\frac{52}{9} \xi(3) = \sum_{n=1}^{\infty} \frac{\alpha_n + \beta_n}{n},
\]
where
\[
\alpha_n := r_n \left[ \left( \sum_{k=1}^{n} \frac{1}{3k - 1} \right)^2 - \sum_{k=1}^{n} \frac{1}{(3k - 1)^2} \right]
\]
and
\[
\beta_n := s_n \left[ \left( \sum_{k=1}^{n} \frac{1}{3k - 2} \right)^2 - \sum_{k=1}^{n} \frac{1}{(3k - 2)^2} \right]
\]
with \( r_n \) and \( s_n \) given by (2.23). Formula (3.51) is related to (3.37).

8. By applying the following known result [14, p. 136, Eq. (19)]:
\[
\pi G - \frac{7}{4} \xi(3) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{h_n}{n^2}
\]
and the third formula in (3.11), we obtain
\[
\frac{1}{2} \pi G = \sum_{n=1}^{\infty} \frac{h_{2n-1}}{(2n - 1)^2}.
\]
9. By letting $a \to c$ in (3.2), we find that
\[
\sum_{n=1}^{\infty} \frac{(b)_n}{n!} \left[ \psi(c+n) - \psi(c) \right] = -\frac{\Gamma(c) \Gamma(-b)}{\Gamma(c-b)} \quad (\Re(b) < 0; \ c \notin \mathbb{Z}_0^-).
\] (3.56)

In its special cases when $c = -b = \frac{1}{2}$ and $c = -2b = 1$, if we appropriately apply the definitions (2.2) and (1.5), we obtain
\[
\frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{\mu_n h_n}{2n-1}
\] (3.57)
and
\[
2 = \sum_{n=1}^{\infty} \frac{\mu_n H_n}{2n-1},
\] (3.58)
respectively.

The formulas in (3.57) and (3.58) may be looked upon as counterparts of those in (2.5) and (3.8), respectively. A comparison of (2.5) with (3.57) would yield
\[
\frac{\pi^2}{4} = \left( \sum_{n=1}^{\infty} \frac{\mu_n h_n}{2n-1} \right)^2 = \sum_{n=1}^{\infty} \frac{\mu_n h_n}{n}. \quad (3.59)
\]

10. The general problem of evaluating double sums of the type:
\[
S_{l,m} := \sum_{n=1}^{\infty} \frac{1}{n^l} \sum_{k=1}^{n} \frac{1}{k^m} \quad (l \in \mathbb{N}\setminus\{1\}; \ m \in \mathbb{N})
\] (3.60)
was first proposed in a letter from Christian Goldbach (1690–1764) to Euler in 1742, and Euler was successful in obtaining closed-form sums in several cases (for details, see [7, p. 253] and [10, p. 2]; see also [27, p. 138, Problem 37; p. 157, Proposition 3.7] for some recent developments involving the general sum $S_{l,m}$ ($l \in \mathbb{N}\setminus\{1\}; \ m \in \mathbb{N}$). In particular, we recall that (cf., e.g., [24, p. 603, Eq. (4.20)])
\[
S_{2,2} = \left[ \zeta(2) \right]^2 - \frac{3}{4} \zeta(4) = \frac{7\pi^4}{360} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n-1} \frac{1}{k^2} + \zeta(4). \quad (3.61)
\]
It is easily proven by the principle of mathematical induction that
\[
H_n^2 = 2 \sum_{k=1}^{n} \frac{H_k}{k} - \sum_{k=1}^{n} \frac{1}{k^2} \quad (n \in \mathbb{N}). \quad (3.62)
\]
Making use of (3.61), (3.62), and (3.18), we get
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{H_k}{k} = \frac{17\pi^4}{720} + \frac{1}{2} S_{2,2} = \frac{\pi^4}{30}. \quad (3.63)
\]
Further series representations, analogous to the ones presented in this paper, can indeed be developed in a similar manner.
Acknowledgments

The work of the third-named author was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353.

References

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