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Radicals of rings and pullbacks

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Abstract

Generalizing various concrete radicals in associative rings like the nilradical, the Jacobson radical, and so on, A.G. Kurosh and S.A. Amitsur introduced an abstract notion of radical in the early 1950s. The basic notions of their general radical theory can be characterized by properties which are "almost" categorical – in the sense that they can be conveniently defined in the category of rings or even in suitable categories of Ω-groups but not in general categories. Here we are going to characterize radicals of associative rings by means of pullbacks, a notion which is of a *purely* categorical nature. Throughout the paper we shall work in the category **C** of associative rings (not necessarily with identity), just calling them "rings". We hope that our two categorical characterizations of semisimple classes in **C** can provide natural general frameworks for radical theory, just as localizations do for torsion theories.

1. Introduction

For a general background on radicals of rings we refer to [1] or [6].

Recall, in particular, that radicals can be described in various classical ways: as functions which assign to every ring one of its ideals called its radical (or equivalently, as developed first by Hoehnke [2], the quotient by its radical), or in terms of radical classes, or again in terms of semisimple classes or, finally, as in the case of torsion theories, as pairs consisting of a radical class and a semisimple class. The one which we will mainly work with describes a *semisimple class* as a class **X** of rings which satisfies the following conditions:

(S) **X** is subdirectly closed, i.e., if A is a subdirect product of rings from **X**, then A is in **X**;

(H) **X** is hereditary, i.e., if X is in **X** and A is an ideal of X, then A is in **X**;

(E) **X** is closed under extensions, i.e., if A is a ring with an ideal X such that X and A/X are in **X**, then A is in **X**.

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The categorical meaning of condition (S) is clear: let $\mathbf{C}_s$ and $\mathbf{X}_s$ be the categories with objects all rings and the elements of $\mathbf{X}$, respectively, and with morphisms the surjective ring homomorphisms in both cases – then (S) is equivalent to

(S_0) the inclusion functor $\mathbf{X}_s \to \mathbf{C}_s$ has a left adjoint.

We will denote that left adjoint by I and the unit of the adjunction by η; so, for every ring A, the homomorphism $\eta_A : A \to IA$ has the universal property which says that for each surjective homomorphism $f: A \to X$ with X in $\mathbf{X}$ there exists a unique $\bar{f}: IA \to X$ making the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta A} & IA \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \bar{f}} & \\ & X & \end{array} \tag{1}$$

commute.

Of course if $\mathbf{X}$ satisfies also the conditions (H) and (E), i.e. if it is a semisimple class, then $\eta_A : A \to IA$ is just the canonical homomorphism $A \to A/RA$, where R is the corresponding radical.

The question we are interested in can now be formulated as follows: What are the categorical properties of I which correspond to (H) and (E)?

First we need the following observation. Certainly the surjective homomorphisms are not enough, and we will use $\mathbf{C}$ and $\mathbf{X}$ to denote the category of rings (with all homomorphisms) and its full subcategory corresponding to the class $\mathbf{X}$, respectively. If $f: A \to X$ is a morphism in $\mathbf{C}$ – not necessarily surjective and not necessarily with X in $\mathbf{X}$, but such that there exists a morphism $\bar{f}: IA \to X$ making the diagram (1) commute – then $\bar{f}$ is still uniquely determined, and we will say briefly that $\bar{f}$ is well defined. Furthermore, if $\alpha : A \to B$ is such that $\overline{\eta_B \alpha} : IA \to IB$ is well defined, then we will write $\overline{\eta_B \alpha} = I\alpha$ and say that $I\alpha$ is well defined. This agrees with the definition of I and extends it at least to all morphisms in $\mathbf{X}$.

Experience with torsion theories suggests now that (H) and (E) hold if and only if our extended functor I preserves some pullbacks. However, as the following example shows, we have to be careful; for I need not preserve *all* pullbacks, even where it is well defined.

Let $\mathbf{C}$ be the category of commutative rings (an easier case!) and $\mathbf{X}$ the category of commutative rings without nonzero nilpotent elements – the semisimple class of the usual nilradical. Consider the pullback

$$\begin{array}{ccc} x^2K[x] & \longrightarrow & 0 \\ {\scriptstyle \text{inclusion}}\downarrow & & \downarrow \\ xK[x] & \xrightarrow[\text{canonical map}]{} & xK[x]/x^2K[x] \end{array} \tag{2}$$

where K is, say, a field and $K[x]$ the polynomial ring. Its image under I is

$$\begin{array}{ccc} x^2K[x] & \longrightarrow & 0 \\ {\scriptstyle \text{inclusion}}\big\downarrow & & \big\| \\ xK[x] & \longrightarrow & 0 \end{array} \tag{3}$$

and so I does not preserve that pullback.

Nevertheless the conditions (H) and (E) (together) have an equivalent form in terms of pullbacks. Actually we propose two such conditions (conditions (b) and (c) in Theorem 3.1) one of which – namely (b) – says in fact that I must preserve those pullbacks

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ B & \xrightarrow[g]{} & Y \end{array} \tag{4}$$

in which g is surjective and X and Y are in $\mathbf{X}$.

2. Conditions on pullbacks

We shall use only very elementary facts from category theory. They can be found e.g. in [4].

Let $\mathbf{C}$ be the category of (associative) rings, $\mathbf{X}$ a full subcategory of $\mathbf{C}$ satisfying condition (S_0), and I and η as above (we will identify $\mathbf{X}$ with the class of its objects). $\mathbf{C}$ has pullbacks, i.e., any pair

$$B \xrightarrow{g} Y \xleftarrow{\varphi} X$$

of morphisms with the same codomain can be completed to a pullback

$$\begin{array}{ccc} B \times_Y X & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ B & \xrightarrow[g]{} & Y \end{array} \tag{5}$$

Consider the following conditions, in which (4) is used as an arbitrary pullback in $\mathbf{C}$ with g surjective and X in $\mathbf{X}$, and notice that the surjectivity of g implies here that f is also surjective because pullbacks of rings lie over pullbacks of sets:

(S_1) if $g: B \to Y$ is the same as $\eta_B: B \to IB$ then $\bar{f}: IA \to X$ is an isomorphism;

(C_1) if B and Y are in $\mathbf{X}$, then A is in $\mathbf{X}$;

(S_2) if Y is in $\mathbf{X}$, then $I\alpha: IA \to IB$ is well defined and

$$\begin{array}{ccc} IA & \xrightarrow{\bar{f}} & X \\ {\scriptstyle I\alpha}\downarrow & & \downarrow{\scriptstyle \varphi} \\ IB & \xrightarrow[\bar{g}]{} & Y \end{array} \tag{6}$$

is a pullback;

(C_2) if B is in $\mathbf{X}$, then A is in $\mathbf{X}$;

(S_3) if $\bar{g}: IB \to Y$ is well defined, then $I\alpha: IA \to IB$ is well defined and (6) is a pullback.

Proposition 2.1. (S_2) $\Leftrightarrow$ (S_1) & (C_1).

Proof. The implications (S_2) $\Rightarrow$ (S_1) and (S_2) $\Rightarrow$ (C_1) are obvious. In order to prove (S_1) & (C_1) $\Rightarrow$ (S_2), decompose the pullback (4) as

$$\begin{array}{ccccc} A & \xrightarrow{\langle \eta_B \alpha, f \rangle} & IB \times_Y X & \xrightarrow{\text{proj}_2} & X \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \text{proj}_1} & & \downarrow{\scriptstyle \varphi} \\ B & \xrightarrow[\eta_B]{} & IB & \xrightarrow[\bar{g}]{} & Y \end{array} \tag{7}$$

where both squares are pullbacks. Applying (C_1) to the right-hand square, we obtain $IB \times_Y X \in \mathbf{X}$. After that applying ($S_1$) to the left-hand square, we conclude that $\langle \eta_B \alpha, f \rangle : IA \to IB \times_Y X$ is an isomorphism. On the other hand, since $\langle \eta_B \alpha, f \rangle$ is well defined, $I\alpha = \overline{\eta_B \alpha} = \text{proj}_1 \overline{\langle \eta_B \alpha, f \rangle}$ is also well defined, and $\overline{\langle \eta_B \alpha, f \rangle} = \langle I\alpha, \bar{f} \rangle$. Therefore, $\langle I\alpha, \bar{f} \rangle : IA \to IB \times_Y X$ is an isomorphism and so (6) is a pullback. □

Exactly the same arguments prove also the following.

Proposition 2.2. (S_3) $\Leftrightarrow$ (S_1) & (C_2).

Now we shall see how these conditions are connected with the conditions (H) and (E).

Proposition 2.3. (C_1) $\Rightarrow$ (H).

Proof. Suppose X is in **X** an A and ideal in X. Consider the pullback

$$\begin{array}{ccc} X\times_{X/A}X & \xrightarrow{\text{proj}_2} & X \\ {\scriptstyle\text{proj}_1}\downarrow & & \downarrow{\scriptstyle\text{canonical map}} \\ X & \xrightarrow[\text{canonical map}]{} & X/A \end{array} \tag{8}$$

Since **X** satisfies condition (S_0), it satisfies condition (S), too, and so $X\times_{X/A}X$ is in **X**. Now applying condition (C_1) to the pullback

$$\begin{array}{cccc} a & A & \longrightarrow & 0 \\ \downarrow & \downarrow & & \downarrow \\ (0,a) & X\times_{X/A}X & \xrightarrow[\text{proj}_1]{} & X \end{array} \tag{9}$$

we conclude that A is in **X**. □

Proposition 2.4. (S_2) ⇒ (E).

Proof. Let X be an ideal in a ring A such that X and A/X are in **X**. Applying (S_2) to the pullback

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ {\scriptstyle\text{inclusion}}\downarrow & & \downarrow \\ A & \xrightarrow[\text{canonical map}]{} & A/X \end{array} \tag{10}$$

we obtain the pullback

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ IA & \longrightarrow & A/X \end{array} \tag{11}$$

which together with the previous one gives the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & A & \longrightarrow & A/X & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\eta_A} & & \| & & \\ 0 & \longrightarrow & X & \longrightarrow & IA & \longrightarrow & A/X & \longrightarrow & 0 \end{array} \tag{12}$$

with exact rows. By the standard 5-lemma in homological algebra η_A must be an isomorphism and so A is in **X**. □

Proposition 2.5. (H) & (E) $\Rightarrow$ (C_2).

Proof. Consider the pullback (4) again – with g surjective and X and B in **X** as in (C_2). The kernel $\mathrm{Ker} f$ is isomorphic to $\mathrm{Ker} g$, which is in **X** by (H). Now, since f is surjective and $\mathrm{Ker} f$ as well as X are in **X**, A is in **X** by (E). □

What we really need is the following lemma, which is a formal consequence of these propositions:

Lemma 2.6. *Suppose that* **X** *satisfies condition* (S_1). *Then the conditions*

$$(H)\&(E), (S_2), (S_3), (C_1), (C_2)$$

are equivalent.

Proof. We have (C_1) $\Rightarrow$ (H) by 2.3 and

$$(C_1) \overset{\text{by 2.1}}{\Longleftrightarrow} (S_2) \overset{\text{by 2.4}}{\Longrightarrow} (E).$$

Therefore (C_1) $\Rightarrow$ (H)&(E), and we have

$$\begin{array}{ccc} (H)\&(E) & \overset{\text{by 2.5}}{\Longrightarrow} (C_2) \overset{\text{by 2.2}}{\Longrightarrow} & (S_3) \\ \Uparrow & & \Downarrow \text{trivially} \\ (C1) & \underset{\substack{\text{trivially} \\ \text{(by 2.1)}}}{\Longleftarrow} & (S_2) \end{array}$$ □

3. The Theorem

Let **X** be a semisimple class of rings and A an arbitrary ring. We set

$$RA = \mathrm{Ker}\,\eta_A$$

and

$$\mathbf{R} = \{A \in \mathbf{C} \mid RA = A\} = \{A \in \mathbf{C} \mid IA = 0\}.$$

The class **R** is called the *radical class* corresponding to the semisimple class **X**, and its elements the *radical rings* (with respect to **X**, whose elements are called the *semisimple rings*). In particular, RA is a radical ring for each ring A, i.e.,

$$RRA = RA. \qquad (13)$$

Moreover, RA is the largest ideal in A with this property, i.e.,

$$K \lhd A \ \&\ RK = K \quad \Rightarrow \quad K \subset RA, \tag{14}$$

and RA is called the *radical of A* (with respect to **X**). In other words, RA can be defined by a universal property using only the class **R**. And this makes possible to recover **X** from **R** by

$$\mathbf{X} = \{A \in C \mid RA = 0\},$$

which is one of the basic connections in Kurosh–Amitsur radical theory.

Our theorem uses these few well-known facts from radical theory as well as Lemma 2.6.

Theorem 3.1. *Let* **X** *be a class of associative rings which satisfies condition* (S_0) *(equivalent to* (S)*). Then the following conditions are equivalent*:

(a) **X** *is a semisimple class, i.e., it satisfies* (H) *and* (E);

(b) **X** *satisfies* (S_2) *(equivalently* (S_1) *and* (C_1)*)*;

(c) **X** *satisfies* (S_3) *(equivalently* (S_1) *and* (C_2)*)*.

Proof. Since the conditions (S_2) and (S_3) are stronger than (S_1), by Lemma 2.6 it suffices to prove that every semisimple class **X** satisfies (S_1).

So, we have to prove that if **X** is a semisimple class and

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\varphi} \\ B & \xrightarrow[\eta_B]{} & IB \end{array} \tag{15}$$

is a pullback with X in **X**, then $\bar{f}: IA \to X$ is an isomorphism.

Compare the kernels $RA = \operatorname{Ker} \eta_A$ and $\operatorname{Ker} f$. On the one hand, we have $RA \subset \operatorname{Ker} f$ since $\bar{f}$ is well defined. On the other hand, since (15) is a pullback, $\operatorname{Ker} f \cong \operatorname{Ker} \eta_B = RB$, which is a radical ring by (13), and so $\operatorname{Ker} f \subset RA$ by (14). Therefore, $\operatorname{Ker} f = RA$, and since f and η_A are surjections, this means that $\bar{f}$ is an isomorphism. □

Remark 3.2. The proof of (a) $\Rightarrow$ (S_1) above uses the same arguments as the proof of [3, Theorem 3.1].

Remark 3.3. Since our proofs are more categorical than ring-theoretical, let us briefly describe possible levels of generalization for them. Propositions 2.1 and 2.2 obviously can be proved in general categories: we only need the existence of some pullbacks and we have to choose a "good" class of morphisms in place of surjections. In order to have the equivalence (S) $\Leftrightarrow$ (S_0), this good class should give a factorization system in the ground category **C** – like that given for rings by the surjections together with the injections. Lemma 2.6 depends on the strong property that in the situation

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & 0
\end{array}
$$

the broken arrow is an isomorphism (see (12)). This property need not hold even in an exact category, although it does hold in any variety of Ω-groups and in abelian categories, where radical theory has been developed (for abelian categories, radicals are just torsion theories). Finally, Theorem 3.1 uses (13) and (14), i.e., it makes real use of the radical theory of associative rings. Moreover, the fact that any semisimple class satisfies (H) is not true even for non-associative rings. Theorem 3.1 holds, however, in those classes of Ω-groups where semisimple classes are characterized as the classes satisfying conditions (S), (H) and (E); for instance, in groups, lattice-ordered groups, alternative rings, Jordan algebras over a field of characteristic $\neq 2$ (see [1]).

Remark 3.4. In the special case when $X = 0$, Condition (S_1) reads as follows: if $A = \operatorname{Ker} \eta_B$ then $IA = 0$. This is a very well-known condition in radical theory – in our presentation above, it is just (13) – which together with (S), (E) and a weaker version of (H) characterizes semisimple classes in any variety of Ω-groups (see [1, Theorem 3.3.4] or [6, Theorem 13]). In this general case (S_1) does not follow from (S), (E) and (H). Thus adding the weak version of (S_1) as above to (S), (E) and (H), we get a characterization of all hereditary semisimple classes in any variety of Ω-groups.

It would be interesting to see how conditions (S_2) and (S_3) relate to semisimple classes in structures other than Ω-groups, e.g., whether they fit into the approach presented in [5].

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