DENSE EMBEDDINGS OF NOWHERE LOCALLY COMPACT SEPARABLE METRIC SPACES

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Nowhere locally compact separable metric spaces are characterized (up to homeomorphism) as precisely the dense subsets of the separable Hilbert space and those of dimension at most \( n \) are characterized (up to homeomorphism) as precisely the dense subsets of the \( n \)-dimensional Menger-Nöbeling space.

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Introduction

The obvious necessary conditions for a metric space to be densely embeddable in the separable Hilbert space \( l_2 \) are separability and nowhere local compactness. In [10], Curtis asks whether or not these conditions are sufficient. In this paper, we answer Curtis' question affirmatively.

In [10], Curtis constructs dense embeddings of \( \sigma \)-compact, nowhere locally compact metric spaces into Hilbert space. In [4], Curtis' result is extended and generalized in two distinct ways: first, it is shown that any dense map of a \( \sigma \)-compact, nowhere locally compact metric space into Hilbert space is strongly approximable by dense embeddings, and second, this result is extended to finite-dimensional analogs of Hilbert space. In this paper we prove the following Theorem.

**Theorem.** Let \( n \in \mathbb{N} \cup \{0, \infty\} \). Let \( X \) be a nowhere locally compact separable metric space of dimension at most \( n \) and let \( Z \) be a complete separable LC\(^n-1\)-space if \( 0 < n < \infty \) and a complete separable ANR if \( n = \infty \). If \( Z \) satisfies the discrete \( n \)-cells property, then for every map \( f : X \to Z \) such that \( f(X) \) is dense in \( Z \) and for every open cover \( \mathcal{U} \) of \( Z \), there exists an embedding \( g : X \to Z \) such that \( g(X) \) is dense in \( Z \) and \( g \) is \( \mathcal{U} \)-close to \( f \).
In [4], with the additional assumption that $X$ is $\sigma$-compact and $Z$ is an ANR (rather than $\text{LC}^{n-1}$ whenever $n < \infty$), a slightly stronger version of the Theorem is proved. There it is proved that the collection of dense embeddings of a $\sigma$-compact $X$ into $Z$ forms a dense $G_\delta$-subset of the collection of dense maps of $X$ into $Z$ topologized by the limitation topology. Such a result is not possible in our setting, for when one removes $\sigma$-compactness from the hypothesis, it is not necessarily true that the collection of dense embeddings of $X$ into $Z$ forms a $G_\delta$-subset of the collection of dense maps of $X$ into $Z$.

The following corollaries are consequences of the Theorem.

**Corollary 1.** Let $n$ and $Z \neq 0$ be as in the Theorem with the additional assumption that $Z$ is connected if $n \neq 0$. Then $Z$ admits a dense embedding of every nowhere locally compact separable metric space $X$ of dimension at most $n$.

The next corollary answers Curtis' question and characterizes the nowhere locally compact separable metric spaces as precisely the dense subsets of Hilbert space.

**Corollary 2.** A space $X$ is a nowhere locally compact separable metric space if and only if $X$ densely embeds into the separable Hilbert space $l_2$.

We also obtain characterizations of finite-dimensional, nowhere locally compact separable metric spaces.

**Corollary 3.** Let $n \in \mathbb{N} \cup \{0\}$. A space $X$ is a nowhere locally compact separable metric space of dimension at most $n$ if and only if $X$ densely embeds into the $n$-dimensional Menger-$\text{Nöbeling}$ space.

Because $l_2$ densely embeds into the Hilbert cube and the $n$-dimensional Menger-$\text{Nöbeling}$ space densely embeds into the $n$-dimensional Menger universal space, we have the following corollary.

**Corollary 4.** Let $n \in \mathbb{N} \cup \{0\}$. A nowhere locally compact separable metric space $X$ (of dimension at most $n$) embeds densely in the Hilbert cube ($n$-dimensional Menger universal space).

Section 2 of this paper introduces preliminary material needed for the proof of the Theorem. Section 3 contains the statement and proof of the main lemma that allows us to prove the Theorem. The proof of the Theorem appears in Section 4 and applications appear in Section 5. In particular, the corollaries are proved in Section 5.

1. **Terminology and notation**

All spaces except for function spaces are separable and metrizable and continuous functions are called maps or mappings. If $K$ is an abstract simplicial complex, then
$K^{(n)}$ denotes its $n$-skeleton and $|K|$ a standard geometric realization equipped with the metric topology [15]. Since all complexes in this paper are locally finite, the metric topology coincides with the Whitehead topology. If $f$ and $g$ are maps of a space $X$ into a space $Y$ and $\mathcal{U}$ is a collection of subsets of $Y$, then $g$ is called a $\mathcal{U}$-approximation to $f$ or $g$ is said to be $\mathcal{U}$-close to $f$ provided for every $x \in X$, $(f(x), g(x))$ is contained in some member of $\mathcal{U}$. Given families $\mathcal{A}$ and $\mathcal{B}$ of subsets of a space $Y$, we say that $\mathcal{A}$ refines $\mathcal{B}$ and we write $\mathcal{A} < \mathcal{B}$ provided each member of $\mathcal{A}$ is contained in some member of $\mathcal{B}$ and, if $f: X \to Y$ is a map, we let $f^{-1}(\mathcal{A}) = \{f^{-1}(A)\}_{A \in \mathcal{A}}$. Observe that we do not assume that $\bigcup \mathcal{A} = \bigcup \mathcal{B}$ if $\mathcal{A} < \mathcal{B}$. We say that $\mathcal{A}$ star-refines $\mathcal{B}$ if $\text{st} \mathcal{A} = \{\text{st}(A, \mathcal{A}) | A \in \mathcal{A}\}$ refines $\mathcal{B}$ where $\text{st}(A, \mathcal{A})$ is the union of all elements of $\mathcal{A}$ that hit $A$, and $\mathcal{A}$ double star-refines $\mathcal{B}$ if $\text{st}^2 \mathcal{A} = \text{st}(\text{st} \mathcal{A})$ refines $\mathcal{B}$. $\mathcal{A}$ is star-finite provided each $A \in \mathcal{A}$ meets at most finitely many members of $\mathcal{A}$.

A non-empty space $X$ is $\text{LC}^n$ for some $n \in \mathbb{N} \cup \{0\}$ provided for each $x \in X$ and neighborhood $U$ of $x$, there exists a neighborhood $V$ of $x$ contained in $U$ such that any mapping of the $i$-sphere $S^i$ into $V$ for $0 \leq i \leq n$ extends to a mapping of the $(i+1)$-ball $B^{i+1}$ into $U$. We find it convenient to say that a space is $\text{LC}^{-1}$ provided it is non-empty. ANR (respectively, AR) denotes absolute neighborhood retract (respectively, absolute retract) for the class of metrizable spaces. By a complete space, we mean a topologically complete space.

2. Preliminaries

A collection $\mathcal{D}$ of subsets of a space $Z$ is discrete in $Z$ provided every point in $Z$ has a neighborhood that meets at most one member of $\mathcal{D}$. A space $Z$ satisfies the discrete $n$-cells property for some $n \in \mathbb{N} \cup \{0, \infty\}$ if for each map $f: \bigoplus_{i=1}^{\infty} I^i \to Z$ of the countable free union of $n$-cells into $Z$ (in-cell = Hilbert cube) and each open cover $\mathcal{U}$ of $Z$, there exists a $\mathcal{U}$-approximation $g: \bigoplus_{i=1}^{\infty} I^i \to Z$ to $f$ for which $\{g(I^i)\}_{i=1}^{\infty}$ is discrete in $Z$. The separable Hilbert space $l_2$ satisfies the discrete $\infty$-cells property, usually referred to as the discrete approximation property [16], and we shall show in Section 5 that the $n$-dimensional Menger–Nöbeling space satisfies the discrete $n$-cells property. For examples of complete separable $(n+1)$-dimensional ANR’s that satisfy the discrete $n$-cells property, see [5] and for further results about discrete cells properties, see [16, 1, 11, 5, 6, 7].

For spaces $X$ and $Z$, $\text{cov}(Z)$ denotes the family of open coverings of $Z$ and $C(X, Z)$ denotes the set of all maps from $X$ to $Z$ equipped with the limitation topology in which a subset $U$ of $C(X, Z)$ is open if and only if, for every $f \in U$, there exists $\mathcal{U} \in \text{cov}(Z)$ such that $B(f, \mathcal{U}) \subseteq U$. Here $B(f, \mathcal{U})$ denotes the set $\{g \in C(X, Z) | g$ is $\mathcal{U}$-close to $f\}$ and each $f \in C(X, Z)$ has $\{B(f, \mathcal{U}) | \mathcal{U} \in \text{cov}(Z)\}$ as a basis of (in general, not open) neighborhoods. The reader should be warned that in general, a set of the form $B(f, \mathcal{U})$ is not open in the limitation topology, though it does have non-empty interior. In this topology a subset $E$ of $C(X, Z)$ is dense
if and only if, for every \( f \in C(X, Z) \) and cover \( \mathcal{U} \in \text{cov}(Z) \), there exists a map \( g \in E \) \( \mathcal{U} \)-close to \( f \). Whenever \( Z \) is metrizable, the limitation topology coincides with the topology of uniform convergence with respect to all metrics for \( Z \). Let \( Z \) be metrizable and let \( \text{Metr}(Z) \) denote the collection of all bounded compatible metrics for \( Z \). An alternative description of the limitation topology on \( C(X, Z) \) is that a subset \( U \) of \( C(X, Z) \) is open if and only if, for every \( f \in U \), there exists a metric \( d \in \text{Metr}(Z) \) such that \( B(f, d) \subset U \), where \( B(f, d) = \{ g \in C(X, Z) \mid d(f, g) < 1 \} \). \( D(X, Z) \) denotes the subspace of \( C(X, Z) \) that consists of all maps \( f \) in \( C(X, Z) \) with dense image in \( Z \) and \( \text{Emb}(X, Z) \) denotes the subspace of \( C(X, Z) \) that consists of all embeddings. In the subspace topology on \( E \subset C(X, Z) \), a subset \( U \) of \( E \) is open if and only if, for each \( f \in U \), there exists a cover \( \mathcal{U} \in \text{cov}(Z) \) such that \( B(f, \mathcal{U}) \cap E = U \). Furthermore, a subset \( D \subset E \) is dense in \( E \) provided, for every \( f \in E \) and \( \mathcal{U} \in \text{cov}(Z) \), there exists \( g \in D \) \( \mathcal{U} \)-close to \( f \). The reader is advised to consult [8] for details concerning the limitation topology.

Given \( \mathcal{U} \in \text{cov}(X) \), a map \( f : X \to Z \) is a \( \mathcal{U} \)-map if there exists \( \mathcal{V} \in \text{cov}(Z) \) with \( f^{-1}(\mathcal{V}) \) refining \( \mathcal{U} \). \( f \) is called a near-\( \mathcal{U} \)-map if each \( z \in f(X) \) has an open neighborhood \( V \) with \( f^{-1}(V) \) contained in some member of \( \mathcal{U} \). Equivalently, \( f \) is a near-\( \mathcal{U} \)-map if there is a collection \( \mathcal{V} \) of open subsets of \( Z \) covering \( f(X) \) for which \( f^{-1}(\mathcal{V}) \) refines \( \mathcal{U} \).

**Lemma 2.1.** (a) Let \( f : X \to Z \) be a map and \( \mathcal{U} \in \text{cov}(X) \). \( f \) is a near-\( \mathcal{U} \)-map if and only if \( f : X \to f(X) \) is a \( \mathcal{U} \)-map.

(b) For any \( \mathcal{U} \in \text{cov}(X) \), the collection of \( \mathcal{U} \)-maps in \( C(X, Z) \) is open in \( C(X, Z) \).

(c) Let \( d \) be any metric for \( X \) and for each \( n \in \mathbb{N} \), let \( \mathcal{U}_n \in \text{cov}(X) \) such that \( \text{diam}_d U < 2^{-n} \) for all \( U \in \mathcal{U}_n \). Let

\[
\mathcal{F}_n = \{ f \in C(X, Z) \mid f \text{ is a } \mathcal{U}_n\text{-map} \},
\]

\[
\mathcal{G}_n = \{ f \in C(X, Z) \mid f \text{ is a near-} \mathcal{U}_n\text{-map} \}.
\]

Then

\[
\bigcap_{n=1}^{\infty} \mathcal{F}_n \subset \bigcap_{n=1}^{\infty} \mathcal{G}_n = \text{Emb}(X, Z).
\]

The proof of Lemma 2.1 is omitted. Part (a) is trivial and parts (b) and (c) are straightforward. In order to prove the results of [4] it is necessary to use the fact that if \( f \) and \( g \) are maps of a closed subset \( A \) of a space \( X \) into an ANR \( Z \) such that \( f \) extends to a map defined on \( X \), then if \( g \) is close enough to \( f \), \( g \) also extends with control to a map defined on \( X \). This is an easy consequence of Borsuk's Homotopy Extension Theorem and of the fact that sufficiently close maps into an ANR are homotopic via small homotopies [15]. We need the \( LC^{n-1} \) version of this result for the proofs in this paper.

**Lemma 2.2.** Let \( n \in \mathbb{N} \cup \{0, \infty\} \) and let \( Z \) be an \( LC^{n-1} \)-space if \( n < \infty \) and an ANR if \( n = \infty \). For every \( \mathcal{U} \in \text{cov}(Z) \), there exists \( \mathcal{V} \in \text{cov}(Z) \) refining \( \mathcal{U} \) such that, if \( f, g : A \to Z \)
are \( V \)-close maps of a closed subset \( A \) of a space \( X \) of dimension at most \( n \) such that \( f \) extends to a map \( \bar{f} : X \to Z \), then \( g \) extends to a map \( \bar{g} : X \to Z \) with \( \bar{g} \) \( U \)-close to \( \bar{f} \).

Although there is a rich source of results about \( LC^{n-1} \)-spaces in the literature (e.g., [3, 15]), we failed to find an explicit statement of Lemma 2.2 (except in the case that \( X \) and \( Z \) are compact, in which case Lemma 2.2 is a direct consequence of [2, Proposition 2.1.4]). The proof of Lemma 2.2 uses standard techniques from the theory of \( LC^{n-1} \)-spaces as presented, for example, in [15]. The lemma can be proved by first reducing it to the case where \( X - A \) is a polyhedron (use the 'nerve replacement trick' [15, p. 53] and the fact that \( X \) is \( LC^{n-1} \)), then extending \( g \) over a polyhedral neighborhood \( U \) of \( A \) [15, p. 150] and thus arriving at the case where \( Bd U \) is a subpolyhedron of \( X - A \), and finally proving the statement for polyhedral pairs by induction on \( n \) and applying this to \( (X - U, Bd U) \). This outline for the proof of Lemma 2.2 was suggested by the referee and is essentially the outline noted by M. Bestvina in [2].

In order to prove the results of this paper, we need the \( LC^{n-1} \) version of the main result of [4] that is stated below.

**Theorem 2.3.** Let \( n \in \mathbb{N} \cup \{0, \infty\} \). Let \( X \) be a \( \sigma \)-compact, nowhere locally compact metric space of dimension at most \( n \) and let \( Z \) be a complete separable \( LC^{n-1} \)-space if \( n < \infty \) and a complete separable \( ANR \) if \( n = \infty \). If \( Z \) satisfies the discrete \( n \)-cells property, then \( \text{Emb}(X, Z) \cap D(X, Z) \) is a dense \( G_\delta \)-subset of \( D(X, Z) \).

The proof of Theorem 2.3 is essentially the same as the proof of the \( ANR \) version of this theorem that appears in [4] with the main modification being that Lemma 2.2 of this paper is used at several points in place of Borsuk's Homotopy Extension Theorem and the fact that sufficiently close maps into an \( ANR \) are homotopic via small homotopies. We include an appendix at the end of this paper that contains a discussion of the necessary changes needed in [4] in order to prove Theorem 2.3.

Finally, we need the following lemma.

**Lemma 2.4.** Let \( Z \) be a complete separable \( LC^{n-1} \)-space (ANR) that satisfies the discrete \( n \)-cells property for some \( n \in \mathbb{N} \cup \{0\} \) (discrete approximation property). If \( Y \) is an open subset of \( Z \), then \( Y \) is a complete separable \( LC^{n-1} \)-space (ANR) that satisfies the discrete \( n \)-cells property (discrete approximation property).

**Proof.** We prove the case for \( n \in \mathbb{N} \cup \{0\} \), the remaining case being similar. Obviously \( Y \) is a complete separable \( LC^{n-1} \)-space and Lemma 1.1 of [16] guarantees that \( C(\bigoplus_{i=1}^{\infty} I^*_i, Y) \) is a Baire space. Write \( Y = \bigcup_{k=1}^{\infty} Y_k \) where for each \( k \in \mathbb{N} \), \( Y_k \) is open in \( Y \) and \( Cl_Y Y_k \subset Y_{k+1} \). For each \( k \in \mathbb{N} \), let \( U_k \) be the subset of \( C(\bigoplus_{i=1}^{\infty} I^*_i, Y) \) that consists of all maps \( f \) for which each point in \( Cl_Y Y_k \) has a neighborhood open in \( Y \) that meets at most one member of \( \{ f(I^*_i) \}_{i=1}^{\infty} \). It suffices to prove that each \( U_k \) is both open and dense in \( C(\bigoplus_{i=1}^{\infty} I^*_i, Y) \), for then \( \bigcap_{k=1}^{\infty} U_k \) is dense in
C(\bigoplus_{i=1}^{\infty} I^*_n, Y) and it is clear that \{f(I^*_n)\}_{i=1}^{\infty} is discrete in Y for every map 
f \in \bigcap_{k=1}^{\infty} U_k.

Let \(f \in U_k\) and for each point \(y \in \text{Cl}_Z Y_k\), choose an open neighborhood \(V_y\) of \(y\) 
that meets at most one member of \(\{f(I^*_n)\}_{i=1}^{\infty}\) and choose an open set \(W\) in \(Y\) such that
\[
\text{Cl}_Z Y_k \subset W \subset \text{Cl}_Z W \subset V = \bigcup \{V_y \mid y \in \text{Cl}_Z Y_k\}.
\]

Let \(V\) be an open cover of \(Y\) by a locally finite star-refinement of the open cover 
\(\{V_y \mid y \in \text{Cl}_Z Y_k\} \cup \{Y - \text{Cl}_Z W\}\) of \(Y\). Then \(B(f, V)\) is contained in \(U_k\), hence \(U_k\) is 
open in \(C(\bigoplus_{i=1}^{\infty} I^*_n, Y)\).

Now assume that \(f \in C(\bigoplus_{i=1}^{\infty} I^*_n, Y)\). For an open cover \(U\) of \(Y\) such that
\(\text{st}(Y, U) \cap \text{st}(Y - Y_{k+1}, U) = \emptyset\), let \(V\) be the open cover of \(Y\) guaranteed by Lemma 2.2. Since \(Z\) satisfies the discrete \(n\)-cells property, there is a map \(\tilde{f} \in C(\bigoplus_{i=1}^{\infty} I^*_n, Z)\) such that \(\{\tilde{f}(I^*_n)\}_{i=1}^{\infty}\) is discrete in \(Z\) and \(\tilde{f}\) is \(V\)-close to \(f\) where \(V = V \cup \{Z - \text{Cl}_Z Y_{k+1}\}\), an open cover of \(Z\). Since \(g|f^{-1}(\text{Cl}_Z Y_{k+1})\) then must be \(V\)-close to 
\(f|f^{-1}(\text{Cl}_Z Y_{k+1})\), which extends to the map \(f\), \(g|\) extends to a map \(g \in C(\bigoplus_{i=1}^{\infty} I^*_n, Y)\) such that \(g\) is \(U\)-close to \(f\). Then \(g \in U_k \cap B(f, U)\), hence \(U_k\) is dense 
in \(C(\bigoplus_{i=1}^{\infty} I^*_n, Y)\). \(\square\)

3. Factoring a map through a \(\sigma\)-compact, nowhere locally compact space

In this section we state and prove the main lemma (Lemma 3.2) that is used in 
the proof of the main result in Section 4. We begin with a preliminary lemma.

Lemma 3.1. Let \(n \in \mathbb{N} \cup \{0, \infty\}\) and let \(Z\) be an LC\(^{n-1}\)-space if \(n < \infty\) and an ANR if \(n = \infty\). Then for every \(\mathcal{U} \in \text{cov}(Z)\), there exists \(\mathcal{V} \in \text{cov}(Z)\) refining \(\mathcal{U}\) such that, for 
every map \(f : A \to Z\) where \(A\) is a nowhere locally compact separable metric space of 
dimension at most \(n\), there exists a \(\mathcal{U}\)-approximation \(g : A \to Z\) to \(f\) for which the closure 
in \(Z\) of \(g(A)\) contains \(\text{st}(f(A), \mathcal{V})\).

Proof. Let \(\mathcal{V}\) be the cover promised in Lemma 2.2 and without loss of generality 
assume that \(\mathcal{V}\) is locally finite. Let \(\mathcal{V}_0 = \{V \in \mathcal{V} \mid V \cap \text{f}(A) \neq \emptyset\}\) and choose a discrete 
family \(\{P(V) \mid V \in \mathcal{V}_0\}\) of non-empty open subsets of \(A\) such that \(P(V) \subset f^{-1}(V)\) for 
each \(V \in \mathcal{V}_0\). Since \(A\) is a nowhere locally compact we can find for each \(V \in \mathcal{V}_0\) a 
countable discrete (in \(A\)) set \(D(V) \subset P(V)\). Observe that the set \(D = \bigcup \{D(V) \mid V \in \mathcal{V}_0\}\) is a discrete collection of points in \(A\) that forms a closed subset 
of \(A\).

Define \(g_0 : D \to Z\) by requiring that \(g_0\) maps \(D(V)\) onto a dense subset of \(V\) for 
each \(V \in \mathcal{V}_0\). Then \(g_0\) is continuous and \(\mathcal{V}\)-close to \(f|D\) (since \(D(V) \subset f^{-1}(V)\) for
each $V \in \mathcal{V}_0$). Since $f|D$ extends to the map $f$ defined on $A$, our choice of $\mathcal{V}$ ensures that $g_0$ extends to a map $g: A \to Z$ that is $\mathcal{U}$-close to $f$. Clearly, $\operatorname{im}(g)$ contains the dense subset $\operatorname{im}(g_0)$ of $\operatorname{st}(f(A), \mathcal{V})$. □

**Lemma 3.2.** Let $n \in \mathbb{N} \cup \{0, \infty\}$. Let $X$ be a nowhere locally compact separable metric space of dimension at most $n$ and let $Z$ be a complete separable $\mathcal{LC}^{n-1}$-space if $n < \infty$ and a complete separable ANR if $n = \infty$. If $Z$ satisfies the discrete $n$-cells property, then for each $\mathcal{U} \in \operatorname{cov}(X)$, $\mathcal{W} \in \operatorname{cov}(Z)$, and $f \in D(X,Z)$, there exists a $\sigma$-compact, nowhere locally compact metric space $Y$ of dimension at most $n$ and maps $\mu \in D(X,Y)$ and $\nu \in D(Y,Z)$ such that $\mu$ is a $\mathcal{U}$-map, $\nu$ is a (dense) embedding, and $\nu \circ \mu$ is $\mathcal{W}$-close to $f$.

**Proof.** The proof for $n = 0$ is left as an exercise for the reader. In this case, $Y$ can be chosen to be of the form $C \times Q$ where $C$ is a discrete countable space and $Q$ denotes the rationals. The proof for $n > 0$ follows.

**Step 1.** Let $\mathcal{V}$ be as asserted in Lemma 3.1 for the cover $\mathcal{W}$ of $Z$. Construct maps $X \to u|K| \to ^{v} Z$ where $u$ is a $\mathcal{U}$-map, $K$ is a locally finite countable simplicial complex of dimension at most $n$, and $v \circ u$ is $\mathcal{V}$-close to $f$. The construction of $K$, $u$, and $v$ involves standard constructions in the theory of ANR's and $\mathcal{LC}^{n-1}$-spaces. See for example the Appendix of [4]. We claim that without loss of generality we may assume that $u$ has dense image in $|K|$. Indeed, by passing to a subdivision if necessary, we may assume three properties for $K$:

1. $K$ refines $v^{-1}(\mathcal{V})$ in the sense that for each $\sigma$ in $K$, $|\sigma|$ is contained in some member of $v^{-1}(\mathcal{V})$,
2. if $u': X \to |K|$ is a map for which each $\{u(x), u'(x)\}$ for $x \in X$ is contained in $|\sigma|$ for some $\sigma$ in $K$, then $u'$ is a $\mathcal{U}$-map (Lemma 2.1(b)), and
3. the interior of each principal simplex $\sigma$ in $K$ meets the image of $u$.

Since $X$ is nowhere locally compact and $K$ is locally finite, we may choose a sequence $\{N_\sigma\}_\sigma$ indexed by the principal simplices $\sigma$ of $K$ where each $N_\sigma$ is the closure in $X$ of an open subset of $u^{-1}(\operatorname{Int} |\sigma|)$ such that $\{N_\sigma\}_\sigma$ forms a discrete family. We now redefine $u$ by requiring $u(N_\sigma)$ to be dense in $|\sigma|$. This is accomplished as follows. Choose a countable subset of $\operatorname{Int} X N_\sigma$ that is discrete in $X$ and let $u_\sigma$ map this countable set onto a dense subset of $|\sigma|$. Define $u_\sigma$ on $\operatorname{Fr} X N_\sigma$ by requiring $u_\sigma = u$ on $\operatorname{Fr} X N_\sigma$ and use the fact that $|\sigma|$ is an AR to extend $u_\sigma$ on $N_\sigma$. Now redefine $u$ on each $N_\sigma$ by requiring $u|N_\sigma = u_\sigma$. Since $\{N_\sigma\}_\sigma$ is discrete, the redefined $u$ is continuous and obviously has dense image in $|K|$. Also, for the redefined $u$, $v \circ u$ is $\mathcal{V}$-close to $f$ (from (1)) and $u$ is a $\mathcal{U}$-map (from (2)). By using a star-refinement of $\mathcal{V}$ in place of $\mathcal{V}$, we may assume that $v \circ u$ is $\mathcal{W}$-close of $f$.

**Step 2.** We construct a $\sigma$-compact, nowhere locally compact ANR $Y$ of dimension at most $n$ that contains $|K|$ as a strong deformation retract of $Y$. Let $\bar{D}$ be a dendrite (= compact 1-dimensional AR) whose endpoints are dense and let $D$ denote the complement in $\bar{D}$ of all but one endpoint $d_0$ of $\bar{D}$. Choose a sequence $\{d_i\}_{i=1}^\infty$ of
points in $D - \{d_0\}$ that converges to $d_0$ and let $D_i$ denote the component of $D - \{d_i\}$ that contains $d_0$. Each $D_i$ is a $\sigma$-compact, nowhere locally compact AR. Let $\{s_i\}_{i=1}^\infty$ be a countable dense subset of $|K|$ and define the subspace $Y$ of $|K| \times D$ by

$$Y = |K| \times \{d_0\} \cup \bigcup_{i=1}^\infty \{s_i\} \times D_i.$$ 

$Y$ is a $\sigma$-compact, nowhere locally compact ANR (since locally contractible) of dimension $\dim K$ and, by identifying $|K|$ with $|K| \times \{d_0\}$, we consider $|K|$ as a subspace of $Y$. Since $d_0$ is a strong deformation retract of $D_i$ for each $i$ (in fact, by the restriction to $D_i \times [0, 1]$ of a contraction of $D$ to the point $d_0$ keeping $d_0$ fixed), $|K|$ is a strong deformation retract of $Y$ with respect to the retraction $r: Y \rightarrow |K|$ defined by $r = \text{id}$ on $|K|$ and $r(\{s_i\} \times D_i) = \{s_i\}$ for each $i$. Observe that there is a strong deformation retraction from $\text{id}_Y$ to $r$ so that the image of $r^{-1}(A) \times [0, 1]$ is contained in $r^{-1}(A)$ for any subset $A$ of $|K|$.

**Step 3.** Define an extension $\tilde{v}: Y \rightarrow Z$ of $v$ by $\tilde{v} = v \circ r$. Let $\mathcal{A}$ be an open cover of $|K|$ such that for each $\mathcal{A}$-approximation $u': X \rightarrow |K|$ to $u$, $u'$ is a $\mathcal{U}$-map (Lemma 2.1(b)) and $v \circ u'$ is $\mathcal{W}$-close to $v \circ u$. Observe that if $\mu: X \rightarrow Y$ is $r^{-1}(\mathcal{A})$-close to $u$ (considered as a map into $Y$), then $\mu$ is a $\mathcal{U}$-map and $\tilde{v} \circ \mu$ is $\mathcal{W}$-close to $\tilde{v} \circ u$. Let $\mathcal{B}$ be a locally finite open cover of $|K|$ that refines $\mathcal{A}$ such that $\mathcal{B}$-close maps are $\mathcal{A}$-homotopic. Observe that $r^{-1}(\mathcal{B})$ is a locally finite open cover of $Y$ such that $r^{-1}(\mathcal{B})$-close maps into $Y$ are $r^{-1}(\mathcal{A})$-homotopic. This follows from the last observation of the preceding paragraph. An examination of the proof of Lemma 3.1 now shows that there exists a map $\mu: X \rightarrow Y$ such that $\mu$ is $r^{-1}(\mathcal{A})$-close to $u$ and for which $\text{Cl}_{Y}(\mu(X))$ contains $\text{st}(u(X), r^{-1}(\mathcal{B}))$. Since $u(X)$ is dense in $|K|$ (Step 1), $\mu(X)$ is dense in $Y = \text{st}(u(X), r^{-1}(\mathcal{B}))$, that is, $\mu \in D(X, Y)$. Furthermore, since $\mu$ is $r^{-1}(\mathcal{A})$-close to $u$, $\mu$ is a $\mathcal{U}$-map and $\tilde{v} \circ \mu$ is $\mathcal{W}$-close to $\tilde{v} \circ u$.

**Step 4.** Since $Y$ is nowhere locally compact, there is a $\mathcal{W}$-approximation $\tilde{\nu}: Y \rightarrow Z$ to $\tilde{v}$ for which $\text{Cl}_{Z} \tilde{\nu}(Y)$ contains $\text{st}(\tilde{v}(Y), \mathcal{Y})$. Since $\tilde{v}(Y) = v(|K|) \cup vu(X)$, since $v \circ u$ is $\mathcal{Y}$-close to $f$ and $f$ has dense image in $Z$, we infer that $\text{st}(\tilde{v}(Y), \mathcal{Y})$ is dense in $Z$, hence $\tilde{v} \in D(Y, Z)$. By Theorem 2.3, there is a map $v \in \text{Emb}(Y, Z) \cap D(Y, Z)$ that is $\mathcal{W}$-close to $\tilde{v}$. Since each map $\nu \circ \mu$, $\tilde{v} \circ \mu$, $\tilde{v} \circ u = v \circ u$, $f$ is $\mathcal{W}$-close to the next, it follows that $v \circ \mu$ is $\text{st}^3 \mathcal{W}$-close to $f$. By replacing $\mathcal{W}$ by a triple star-refinement of $\mathcal{W}$, we may assume that $v \circ \mu$ is $\mathcal{W}$-close to $f$.

**4. Proof of the main result**

Throughout this section, $n$ denotes a fixed element of $\mathbb{N} \cup \{0, \infty\}$ and $X$ denotes a nowhere locally compact separable metric space of dimension at most $n$. $Z$ denotes a complete separable LC$^{n-1}$-space if $n < \infty$ and a complete separable ANR if $n = \infty$ and we assume that $Z$ satisfies the discrete $n$-cells property. The main result that is stated in the introduction may be restated as follows.
Theorem. \( \text{Emb}(X, Z) \cap D(X, Z) \) is dense in \( D(X, Z) \).

Lemma 3.2 provides a quick proof of the following result.

Lemma 4.1. For every \( \mathcal{U} \in \text{cov}(X) \), the collection of near-\( \mathcal{U} \)-maps in \( D(X, Z) \) is dense in \( D(X, Z) \).

Proof. Given \( \mathcal{U} \in \text{cov}(X) \), \( \mathcal{W} \in \text{cov}(Z) \), and \( f \in D(X, Z) \), Lemma 3.2 guarantees maps \( \mu \) and \( \nu \) such that \( \nu \circ \mu \) is \( \mathcal{W} \)-close to \( f \) and has dense image in \( Z \). Since \( \mu \) is a \( \mathcal{U} \)-map and \( \nu \) is an embedding, \( \nu \circ \mu \) is a near-\( \mathcal{U} \)-map.

Proof of Theorem. Let \( f \in D(X, Z) \) and let \( d \) be an arbitrary bounded compatible metric for \( Z \). It suffices to find \( g \in \text{Emb}(X, Z) \cap D(X, Z) \) with \( d(f, g) \leq 1 \). Without loss of generality, we may assume in addition that \( d \) is a complete metric for \( Z \).

Fix a compatible metric for \( X \) and for each \( k \in \mathbb{N} \), let \( \mathcal{U}_k \) denote the collection of all open balls in \( X \) of radius at most \( 2^{-k} \). Apply Lemma 4.1 to get a near-\( \mathcal{U}_1 \)-map \( g_1 \in D(X, Z) \) with \( d(f, g_1) < 2^{-1} \). By the definition of a near-\( \mathcal{U}_1 \)-map there is an open subset \( Z_1 \) of \( Z \) that contains \( g_1(X) \) for which \( g_1 : X \to Z_1 \) is a \( \mathcal{U}_1 \)-map. Let \( d_i \in \text{Metr}(Z_i) \) be such that any map \( g : X \to Z_i \) with \( d_i(g_1, g) \leq 1 \) is again a \( \mathcal{U}_1 \)-map (Lemma 2.1(b)) and assume in addition that \( d_i \) is complete and \( d_i(a, b) \geq d(a, b) \) for all \( a, b \in Z_1 \) (otherwise replace \( d_i \) by \( d_i + \text{restriction of } d \)). According to Lemma 2.4, we may repeat the above process with \( \mathcal{U}_2 \) in place of \( \mathcal{U}_1 \), \( g_1 \) in place of \( f \), and \( (Z_1, d_1) \) in place of \( (Z, d) \) to obtain a corresponding triple \( (Z_2, g_2, d_2) \) with \( d_2(g_1, g_2) < 2^{-2} \).

Continuing in this way employing Lemma 2.4 at each step, we obtain for each \( i \in \mathbb{N} \), a triple \( (Z_i, g_i, d_i) \) such that, with \( (Z_0, g_0, d_0) = (Z, f, d) \), the following conditions hold.

1. \( g_i(X) \) is a dense subset of \( Z_{i-1} \) and \( Z_i \) is an open neighborhood of \( g_i(X) \) in \( Z_{i-1} \);
2. \( g_i \in D(X, Z_i) \) is a \( \mathcal{U}_i \)-map with \( d_{i-1}(g_i, g_{i-1}) \leq 2^{-i} \);
3. \( d_i \in \text{Metr}(Z_i) \) is complete and \( d_i(a, b) \geq d_{i-1}(a, b) \) for all \( a, b \in Z_i \) and, for any map \( g : X \to Z_i \), if \( d_i(g_i, g) \leq 1 \), then \( g \) is a \( \mathcal{U}_i \)-map.

It is a straightforward exercise to show using the completeness of the metrics \( d_i \) that there is a map \( g : X \to \bigcap_{i=1}^\infty Z_i \) satisfying \( d_i(g, g) \leq 1 \) for each \( i \in \mathbb{N} \). By (3), \( g \) is a \( \mathcal{U}_i \)-map as a map into \( \bigcap_{i=1}^\infty Z_i \) for each \( i \) and hence by Lemma 2.1(c), \( g \) is a homeomorphism onto its image. Moreover by (1) and (2), each \( g_i \) has dense image in \( Z_{i-1} \) and hence in \( Z \) and therefore \( g \) has dense image in \( Z \) [4, Lemma 1.3]. Finally, \( d(f, g) = d_0(g_0, g) \leq 1 \).

5. Applications

In this section, we outline proofs of Corollaries 1 through 4 that are stated in the introduction.
Proof of Corollary 1. For $n > 0$, it suffices to show that there is at least one map $X \rightarrow Z$ whose image is dense in $Z$. The proof is exactly the proof of Corollary in Introduction of [4] with the exception that Lemma 2.2 of this paper is used in place of Borsuk's Homotopy Extension Theorem and the fact that close maps into an ANR are homotopic. For $n = 0$, the proof is found in Section 3 of [4].

Proof of Corollary 2. By [16], $l_2$ is characterized as the topologically unique complete separable AR that satisfies the discrete approximation property. Apply Corollary 1.

Proof of Corollary 3. By Corollary 1, it suffices to prove that the $n$-dimensional Menger-Nöbeling space $N_n^{2n+1}$ is a complete separable LC$^{n-1}$-space that satisfies the discrete $n$-cells property. Recall that $N_n^{2n+1}$ is the subspace of Euclidean $(2n + 1)$-space $\mathbb{R}^{2n+1}$ consisting of all points that have at most $n$ rational coordinates and hence $N_n^{2n+1}$ is the complement of a countable union of $n$-dimensional hyperplanes in $\mathbb{R}^{2n+1}$, each of which is parallel to a hyperplane spanned by $n$ standard coordinate vectors in $\mathbb{R}^{2n+1}$. It follows that $N_n^{2n+1}$ is a complete separable space. Let $F$ denote the complement of $N_n^{2n+1}$ in $\mathbb{R}^{2n+1}$. Then $F$ is a $\sigma$-$Z_n$-set in $\mathbb{R}^{2n+1}$ (i.e., $F$ is $\sigma$-compact and maps of the $n$-cell into $\mathbb{R}^{2n+1}$ can be approximated by maps that miss $F$), which implies that $N_n^{2n+1}$ is LC$^{n-1}$. To prove that $N_n^{2n+1}$ satisfies the discrete $n$-cells property, it suffices to prove that $F$ is LC$^{n-1}$ rel $\mathbb{R}^{2n+1}$ [5], meaning that for every $x \in \mathbb{R}^{2n+1}$ and neighborhood $U$ of $x$ in $\mathbb{R}^{2n+1}$ there exists a neighborhood $V$ of $x$ contained in $U$ such that every map $S^k \rightarrow V \cap F$ for $0 \leq k < n$ is null-homotopic in $U \cap F$ (it is proved that this suffices in [5] in case $F$ is a $\sigma$-$Z$-set; however, the proof used only the fact that a $\sigma$-$Z$-set is automatically a $\sigma$-$Z_n$-set). The proof of this fact uses standard techniques found for example in [14, 12, 9].

Proof of Corollary 4. The proof is similar to the proof of Corollary 3. The $n$-dimensional Menger universal space $\mu^n$ contains a 'natural pseudo-boundary' (the analog in $\mu^n$ of the 'endpoints' of the standard Cantor set). This pseudo-boundary $A$ is a $\sigma$-$Z_n$-set that is LC$^{n-1}$ rel $\mu^n$, implying that $\mu^n - A$ is a complete separable LC$^{n-1}$-space and satisfies the discrete $n$-cells property.

Appendix

Lemma 2.4 is proved by combining Lemmas 1.2, 1.3, and 2.1 of [4] with Theorem 2.2 of [4]. The only modification necessary is that Theorem 2.2 of [4] must be proved in case $Z$ is LC$^{n-1}$ rather than an ANR whenever $n < \infty$. In the proof of Theorem 2.2, the principal use of the ANR assumption was to obtain extensions of maps via the use of Borsuk's Homotopy Extension Theorem in conjunction with the fact that close maps into ANR's are homotopic. This must be replaced by the use of Lemma 2.2 of this paper whenever $Z$ is LC$^{n-1}$ rather than an ANR. There are three places in the proof of Theorem 2.2 where the hypothesis that $Z$ is an ANR is used:
(1) the fact that \( st^2 \mathcal{V} \)-close maps into \( Z \) are \( \mathcal{U} \)-homotopic is used with Borsuk's Homotopy Extension Theorem to obtain an extension of \( h \),

(2) the fact that \( \mathcal{A} \)-close maps into \( Z \) are \( \mathcal{B} \)-homotopic is used as in (1) to obtain an extension of \( g \), and

(3) Lemma 1.1 of [4] is applied to obtain the embedding \( g \).

The following modifications should be made:

(1) use Lemma 2.2 of this paper to choose \( \mathcal{V} \) so that \( st^2 \mathcal{V} \)-close maps from a closed subspace of a space of dimension at most \( n \) extend to \( \mathcal{U} \)-close maps provided one of the maps extends;

(2) similarly, use Lemma 2.2 to obtain \( \mathcal{A} \);

(3) Lemma 1.1 of [4] must be proved in case \( Z \) is LC\(^{-1}\) whenever \( n < \infty \). For this, only Lemma C and Theorem (1) of [4, Appendix] must be modified. In the first paragraph of the proof of Lemma C, the inductive assumption allows us to assume that \( f|K^{(n-1)} \) is a \( \mathcal{U} \)-map because \( Z \) is LC\(^{-1}\) and hence Lemma 2.2 of this paper applies. The existence of the map \( g \) in the second paragraph follows from two applications of Lemma 2.2, the first to obtain \( g \) so that \( \{g(C_\sigma)\} \) is discrete, and the second so that (*) holds. The remainder of the proof remains unchanged.

The proof of Theorem (1) is exactly as in [4] except that \( \mathcal{U} \) is chosen so that every partial \( \mathcal{U} \)-realization of any complex of dimension at most \( n \) extends to a full \( \mathcal{U} \)-realization [15, Theorem 4.1, p. 56]. A standard argument is then used to obtain \( \phi \), which is possible since \( K \) has dimension \( n \).

Addendum

H. Torunczyk was kind enough to read a preliminary version of this paper and he has made the following remark. In the set \( C(X, Z) \) one may consider a topology in which each \( f \in C(X, Z) \) has the family \( \{B(f, \mathcal{U})\}_\mathcal{U} \) as a basis of (not necessarily open) neighborhoods, where \( \mathcal{U} \) runs over all families of open (in \( Z \)) covers of \( f(X) \). For this topology Lemma 1.2 of [4] also holds and this provides an alternative proof of Theorem modulo Lemma 4.1 and of the fact that, in this modified topology, the set of dense embeddings \( \text{Emb}(X, Z) \cap D(X, Z) \) forms a dense \( G_\delta \)-subset of \( D(X, Z) \). The technique used to prove that Lemma 1.2 of [4] holds in case the modified topology on \( C(X, Z) \) is used is similar to the technique used in the proof of Theorem that is presented in Section 4. See [8] for details concerning this modified limitation topology.

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control convergence in the proof of the theorem in Section 4. The referee observed that a clearer and more concise proof could be presented if metrics rather than open covers were used to control convergence.

References