Decentralized observer-based tracker for analog systems with saturating actuator and state constraints

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Abstract

A weighted switching strategy and an inner-loop compensator are presented in this paper to design an observer-based tracker for a decentralized closed-loop cascaded system with a saturating actuator and state constraints. The LQR design methodology for the observer-based tracker is proposed to simplify the complexity of the decentralized control. The realizable sample-data controller with a low-gain property and a high design performance is realized through the digital redesign method. For obtaining a better design performance, evolutionary programming is then presented to tune the parameters of the tracker. Some examples are also presented to demonstrate the effectiveness of the proposed methodology.

1. Introduction

With the passage of time, many practical systems become more and more complicated. They have a common characteristic: they are multi-dimensional and sizable systems. In order to achieve the purpose of a multiplexer, we need to divide a system into several small parts to control each part and deal with it separately. Each small system operates separately, so that if a portion of the system goes bad, it would not influence the operation of the whole system. While designing this kind of controller, we must notice whether the decentralized system is stable or not. Here, the objective is to track the pre-specified references with as small a tracking error as possible. Considering a cascaded analog system with saturating actuators and state constraints, we operate this by the decentralized controller to effect the treatment of the multiplexer, such as Fig. 1 [1,2].

In general, there are some restrictions on the objective conditions in real operations for linear continuous-time systems. These are natural factors or artificial interference. If we neglect this restriction [3,4], it usually leads to the...
deterioration of the controlling performance in a closed-loop system. For this reason, we must consider the constraints on the system while designing the controller.

An anti-windup compensation (AWC) is combined to deal with input and state saturation with the controller [5]. Assume that it may be achieved through using the virtual sensors to measure the states of the system. The observer will be built to estimate the contacted cascaded system in the real design. The development of anti-windup controllers requires certain constraints on the controllers, such as the minimum phase condition and a bi-proper structure. In addition, the implementation of the developed analog anti-windup controllers using analog components is less reliable and less flexible. So, a new less restrictive strategy is required.

To have a better tracking performance, the linear-quadratic optimal tracker design methodology [6] for decentralized control [7,8] is applied in this paper. The design method for the linear-quadratic optimal regulator is one of the main design tools in modern control theory. The quadratic weightings in optimization theory are considered to be the important tuning parameters. We need to be adjusting them repeatedly until we reach the desired responses, so trial and error iterations are essential.

The amplitude of the analog control input would exceed the limits of the control saturation, so it cannot be realized in fact. However, the applied predictive digital redesign controller is in general a low gain controller that preserves the high performance quality of the original analog controller. In addition, the proposed EP-based digital redesign observer-based tracker provides a more mathematical analysis in taking advantage of evolutionary programming (EP) for global optimization. The EP-based minimal principle [9] has been utilized to tune system parameters for the digitally redesigned observer-based tracker to get a better tracking performance.

This article is made up of several parts. A weighted switching strategy and a digitally redesigned observer with the high-gain property to accurately estimate the internal states of the cascaded system have been developed for the continuous-time cascaded system with saturating actuators and state constraints. The selected switching levels and the internal state set-point are the critical elements for the performance of the system. We use an evolutionary programming (EP) optimal search algorithm [9–11] to solve for the variables. This methodology is aimed at driving the system out of the saturation region as soon as possible, and reducing the effect of distortion while the system is in the saturation region. In addition, we use the decentralized control to simplify a bigger and more complicated designed system. Besides, we introduce the optimization theory to get a controller with better performance. The estimator adds an adjustable parameter to enable it to have a better behavior. A cascaded analog system with saturating actuators and state constraints is described in Section 2. The continuous-time decentralized linear-quadratic tracker and observer are briefly discussed in Section 3. The design principles and procedures of the newly developed tracker, along with some necessary analysis, are presented in Section 4. Section 5 presents an illustrative example to demonstrate the effectiveness of the proposed procedure, and the paper concludes with some summary comments in Section 6.
2. **A cascaded analog system with saturating actuators and state constraints**

A cascaded analog system with state constraints in internal state \( z(t) \) is shown in Fig. 2. The dynamic equation of plant \( G_1(s) \) can be described as

\[
\begin{align*}
\dot{x}_1(t) &= A_1 x_1(t) + B_1 u_1(t), \\
y_1(t) &= C_1 x_1(t),
\end{align*}
\]

(1a)

and the dynamic equation of plant \( G_2(s) \) as

\[
\begin{align*}
\dot{x}_2(t) &= A_2 x_2(t) + B_2 u_2(t), \\
y_2(t) &= C_2 x_2(t),
\end{align*}
\]

(2a)

where \( x_1(t) \in \mathbb{R}^{m_1}, u_1(t) \in \mathbb{R}^{m_1}, y_1(t) \in \mathbb{R}^{l_1}, x_2(t) \in \mathbb{R}^{m_2}, u_2(t) \in \mathbb{R}^{m_2}, y_2(t) \in \mathbb{R}^{l_2} \) and system matrices \((A_1, B_1, C_1, A_2, B_2, \text{ and } C_2)\) have appropriate dimensions. Since \( G_1(s) \) and \( G_2(s) \) are cascaded, the output of \( G_1(s) \) and inputs of \( G_2(s) \) have the same dimensions, i.e., \( l_1 = m_2 \). Then, we assume that the composite system input \( u(t) = u_1(t) \) and the composite system output \( y(t) = y_2(t) \).

Next, consider a cascaded analog system with a nominal model described as

\[
\begin{align*}
Y(s) &= G_o(s) U(s), \\
Z(s) &= G_{o2}(s) U(s),
\end{align*}
\]

(3)

(4)

where \( G_o(s) \) is the open-loop transfer function used to generate output \( Y(s) \), and \( G_{o2}(s) \) is the transfer function to produce the internal state from control input \( U(s) \). It is assumed we are able to measure or estimate \( z(t) \) from the available data \( u(t) \) and \( y(t) \) by using virtual sensors, i.e., using an observer to construct an estimate \( \hat{z}(t) \) for \( z(t) \). Hence, the Laplace transform of the estimated \( \hat{z}(t) \) can be expressed as

\[
\hat{Z}(s) = T_{1z}(s) U(s) + T_{2z}(s) Y(s),
\]

(5)

where \( T_{1z}(s) \) and \( T_{2z}(s) \) are stable transfer functions that have a common denominator. Thus, the dynamic equations (1) and (2) can be rewritten as

\[
\begin{align*}
\dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t), \\
z(t) &= C_1 x_1(t),
\end{align*}
\]

(6a)

(6b)

and

\[
\begin{align*}
\dot{x}_2(t) &= A_2 x_2(t) + B_2 z(t), \\
y(t) &= C_2 x_2(t).
\end{align*}
\]

(7a)

(7b)

The dynamic equation of \( G_o(s) \) is described as

\[
\begin{align*}
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t), \\
y(t) &= \begin{bmatrix} 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},
\end{align*}
\]

(8a)

(8b)
which is also denoted by
\[
\dot{x}_o (t) = A_o x_o (t) + B_o u (t), \quad (9a)
\]
\[
y (t) = C_o x_o (t). \quad (9b)
\]
The dynamic equation of \(G_{oz} (s)\) is described by Eqs. (6), as follows,
\[
\dot{x}_1 (t) = A_1 x_1 (t) + B_1 u (t), \quad (6a)
\]
\[
z (t) = C_1 x_1 (t), \quad (6b)
\]
or
\[
\dot{x}_{oz} (t) = A_{oz} x_{oz} (t) + B_{oz} u (t), \quad (10a)
\]
\[
z (t) = C_{oz} x_{oz} (t), \quad (10b)
\]
where \(x_o (t) = [x_1^T \quad x_2^T (t)]^T \in \mathbb{R}^{n_1+n_2}, u (t) \in \mathbb{R}^{m_1}, y (t) \in \mathbb{R}^{l_2}, x_{oz} (t) \in \mathbb{R}^{n_2}, z (t) \in \mathbb{R}^{l_1}\) and system matrices \((A_o, B_o, C_o, A_{oz}, B_{oz}, \text{and } C_{oz})\) have appropriate dimensions.

The primary controller \(C_o (s)\) is a standard controller aimed at achieving the main control goal so that the plant output \(y (t)\) tracks a given reference \(r_y (t)\). The secondary controller \(C_{oz} (s)\) is to keep the internal state \(z (t)\) within prescribed bounds. This is achieved by the use of a secondary closed loop aimed at the regulation of the estimated state \(\dot{z} (t)\). This internal state regulation controller \(C_{oz} (s)\) is designed for tracking an internal state set-point \(z_{sp} (t)\) between the upper and lower bounds.

The control strategy is to switch between the controllers \(C_o (s)\) and \(C_{oz} (s)\). We assume that the upper bound is equal to the absolute value of the lower bound, i.e., the state constraint is symmetrical. The general structure of the proposed control scheme is shown in Fig. 3. The control input \(u (t)\) in Fig. 3 is composed of \(u_y (t)\) and \(u_z (t)\), combined by the weighting switch \(W\). Moreover, the design rule of the switch is based on the amplitude of \(\dot{z} (t)\) as observed from the virtual sensor.

Since the control objective is to keep \(|z (t)|\) bounded by a known constant \(z_{sat} > 0\), the approach can be implemented by using a switch with a dead-zone and saturation nonlinearity, and its switching levels \(z_l\) and \(z_h\) are chosen as \(0 \leq z_l < z_h \leq z_{sat}\). An illustration of switching levels for the relation between estimate \(\dot{z} (t)\) and the output is shown in Fig. 3. The internal state regulation control input \(u_z (t)\) is designed to track the internal state set-point \(z_{sp} (t)\), which is given by
\[
z_{sp} (t) = r_z \times \text{sign} (\dot{z} (t)), \quad (11)
\]
where \(r_z\) is a positive constant that satisfies \(r_z \leq z_{sat}\). The reason to design \(z_{sp} (t)\) in the form of Eq. (11) is based on the symmetry of the state constraint such that \(C_{oz} (s)\) will track negative \(z_{sp} (t)\), when \(\dot{z} (t)\) is negative.

One form of the switching strategy between \(C_o (s)\) and \(C_{oz} (s)\) is given in [5]. It relies on the use of the switching levels \(z_l\) and \(z_h\), with the plant control input \(u (t)\), a linear combination of \(u_y (t)\) and \(u_z (t)\) as follows
\[
u (t) = \lambda u_z (t) + (1 - \lambda) u_y (t), \quad (12)
\]
where \(\lambda\) is a blended proportion of \(u_y (t)\) and \(u_z (t)\). One way to determine \(\lambda\) that relies on the switching levels \(z_l\) and \(z_h\) is as follows
\[
\lambda = \begin{cases} 
0 & \text{for } |\dot{z} (t)| \leq z_l \\
|\dot{z} (t)| - z_l & \text{for } z_l < |\dot{z} (t)| < z_h \\
z_h - z_l & \text{for } |\dot{z} (t)| \geq z_h.
\end{cases} \quad (13)
\]

The reason for determining \(\lambda\) as shown in Eq. (13) is to ensure that the secondary controller \(C_{oz} (s)\) takes over from the primary controller \(C_o (s)\), while the amplitude of internal state \(|\dot{z} (t)|\) grows over \(z_h\) and reverts back to the original controller \(C_o (s)\) when \(|\dot{z} (t)|\) falls below \(z_l\).

An original anti-windup scheme united with bi-proper controllers is used to deal with the problem of the saturating actuator and state constraints [5]. These bi-proper controllers are transformed into the feedback form to constrain all
dynamics of the controller, but this presupposes that all controllers are minimum phase and bi-proper. The developed anti-windup controllers are analogical. The above restrictions are improved in our proposed design methodology.

We reconsider the problem of state saturation from the control scheme with input and state saturations as shown in Fig. 3. Since the controllers $C_o(s)$ and $C_{oz}(s)$ are aimed at tracking the reference input $r_y(t)$ and the internal state set-point $z_{sp}(t)$, respectively, the control input would be given in the general form of an optimal tracking control as in Eq. (10). Assume that the forms of the feedback tracking controller and the internal state regulation controller in Fig. 4 are respectively given as

$$u_y(t) = -K_{co} x_o(t) + E_{co} r_y(t),$$  \hspace{1cm} (14)$$

and

$$u_z(t) = -K_{coz} x_{oz}(t) + E_{coz} z_{sp}(t),$$  \hspace{1cm} (15)$$

where $K_{co} \in R^{m_1 \times (n_1+n_2)}$ and $E_{co} \in R^{m_1 \times p_1}$ are the feedback and forward gains for $u_y(t)$ to track the reference input $r_y(t) \in R^{p_1}$. Also, $K_{coz} \in R^{m_1 \times n_1}$ and $E_{coz} \in R^{m_1 \times p_2}$ are feedback and forward gains for $u_z(t)$ to track the internal state set point $z_{sp} \in R^{p_2}$. Substituting (14) and (15) into (12), one has

$$u(t) = \lambda u_z(t) + (1 - \lambda) u_y(t)$$

$$= \lambda \left(-K_{coz} x_{oz}(t) + E_{coz} z_{sp}(t) \right) + (1 - \lambda) \left(-K_{co} x_o(t) + E_{co} r_y(t) \right)$$

$$= -\left[ (1 - \lambda) \begin{bmatrix} x_o(t) \\ x_{oz}(t) \end{bmatrix} \right] + \left[ (1 - \lambda) E_{co} \lambda E_{coz} \right] \begin{bmatrix} r_y(t) \\ z_{sp}(t) \end{bmatrix}$$

$$= -K_{ce} x_e(t) + E_{ce} r_e(t),$$  \hspace{1cm} (16)$$
where
\[ K_{ce} = [(1 - \lambda) K_{co} \quad \lambda K_{coz}] \in R^{m_1 \times (n_1 + n_2 + n_1)}, \quad E_{ce} = [(1 - \lambda) E_{co} \quad \lambda E_{coz}] \in R^{m_1 \times (p_1 + p_2)}, \]

the equivalent reference input \( r_e (t) \) is \([ry(t) \quad zsp(t)] \in R^{p_1 + p_2}\), and the equivalent system state variable \( x_e (t) = [\hat{x}_o (t) \quad \hat{z}_{sp} (t)] \in R^{n_1 + n_2 + n_1}\).

Eq. (16) is a composite control input with a weighted switching behavior, which can be interpreted as the numerical equivalent of the traditional control input, rather than a real controller having the values of gains. The weighting ratio, \( \lambda \), is a nonlinear function of \( \hat{z} (t) \) and depends on how close the state is to the saturation boundary. Hence, the transformation of controller (16) from the continuous-time domain to discrete-time domain is fairly complicated. In a sampled-data control system such as this one, the controller is digital and the plant is analog. As such, the continuous-time control gains, \( K_{ce} \) and \( E_{ce} \), need to be transformed to their discrete-time equivalents.

As the behavior of the weighting switch is not involved with the digital redesign of the controller (16), the feedback and forward gains can be designed first without considering \( \lambda \). So, one has
\[ K_{ce} = [K_{co} \quad K_{coz}] \in R^{m_1 \times (n_1 + n_2 + n_1)} \] (17)

and
\[ E_{ce} = [E_{co} \quad E_{coz}] \in R^{m_1 \times (p_1 + p_2)}. \] (18)

By using the prediction-based digital redesign method [12,15], one has the discrete-time feedback and forward gains \( K_{de} \) and \( E_{de} \) as follows
\[ K_{de} = (I_m + K_e H_e)^{-1} K_{ce} G_e = \begin{bmatrix} K_{do} & K_{doz} \end{bmatrix} \in R^{m_1 \times (n_1 + n_2 + n_1)}, \] (19)
\[ E_{de} = (I_m + K_{ce} H_e)^{-1} E_{e} = \begin{bmatrix} E_{do} & E_{doz} \end{bmatrix} \in R^{m_1 \times (p_1 + p_2)}, \] (20)
\[ r^*_e (kT) = r_e (kT + T) = \begin{bmatrix} r^*_y (kT) \\ z^*_sp (kT) \end{bmatrix} \in R^{p_1 + p_2}, \] (21)

where \( A_e = \begin{bmatrix} A_o & 0 \\ 0 & A_{oz} \end{bmatrix}, B_e = \begin{bmatrix} B_o & B_{oz} \end{bmatrix}, G_e = e^{A_e T}, \) and
\[ H_e = \begin{cases} (G_e - I_n) A_e^{-1} B_e, & \text{if } A_e^{-1} \text{ exist.} \\ \left( I_n T + \frac{A_e T^2}{2!} + \frac{A_e^2 T^3}{3!} + \frac{A_e^3 T^4}{4!} + \cdots \right) B, & \text{if } A_e^{-1} \text{ doesn’t exist.} \end{cases} \]

The weighting factor \( \lambda \) can be calculated from the observed states \( \hat{z} (t) \) by using (13). The main difference in \( \lambda \) between the continuous-time controller and the discrete-time controller is that in the discrete-time controller, \( \lambda \) only changes at the sampling instance. Considering \( \lambda \), the latter is given as
\[ u (kT) = \lambda (kT) u_z (t) + (1 - \lambda (kT)) u_y (t) \]
\[ = \lambda (kT) \left( -K_{doz} \hat{x}_{oz} (kT) + E_{doz} z^*_sp (kT) \right) + (1 - \lambda (kT)) \left( -K_{do} \hat{x}_o (kT) + E_{co} r^*_y (kT) \right) \]
\[ = -\left( (1 - \lambda) K_{doz} \quad \lambda K_{doz} \right) \begin{bmatrix} \hat{x}_o (kT) \\ \hat{z}_{sp} (kT) \end{bmatrix} + \left( (1 - \lambda) E_{doz} \quad \lambda E_{doz} \right) \begin{bmatrix} r^*_y (kT) \\ z^*_sp (kT) \end{bmatrix} \]
\[ = -K_{de} x_e (kT) + E_{de} r^*_e (kT). \] (22)

Since the process of digital redesign is not directly carried out from (16), Eq. (22) is only an approximation of (16). From the above deduction, the augmented digital redesigned control system [2] is shown in Fig. 5, where \( G_{do} = G_o - L_{do} C_o G_o, H_{do} = H_o - L_{do} C_o H_o, L_{do} = (G_o - I_n) A_o^{-1} L_{co} \left[ I_n + C_o \left( G_o - I_n \right) A_o^{-1} L_{co} \right]^{-1}, \) \( G_o = e^{A_o T}, H_o = (G_o - I_n) A_o^{-1} B_o, L_{co} = P_o C_o^T R_o^{-1}, \) and \( P_o \) is the positive-definite and symmetric solution of the following Riccati equation
\[ A_o P_o + P_o A_o^T - P_o C_o^T R_o^{-1} C_o P_o + Q_o = 0. \] (23)
3. Decentralized tracker for linear systems

Here, we consider four conditions for designing the trackers, which present the design of the decentralized tracker for continuous-time systems in Section 3.1, for discrete-time systems in Section 3.2, the design of a decentralized observer-based tracker for continuous-time systems in Section 3.3, and for discrete-time systems in Section 3.4, sequentially.

3.1. Decentralized tracker for continuous-time systems

First, consider the class of systems with two decentralized control agents of the form

\[
\dot{x}_c(t) = Ax_c(t) + B_1 u_{c1}(t) + B_2 u_{c2}(t),
\]

\[
y_{ci}(t) = C_i x_c(t), \quad \text{for } i = 1, 2,
\]

where \( x_c(t) \in \mathbb{R}^n \) is the state, \( u_{ci}(t) \in \mathbb{R}^{m_i} \) is the input, \( y_{ci}(t) \in \mathbb{R}^{p_i} \) is the output of the \( i \)-th control agent (\( i = 1, 2 \)), \( B = [B_1 \ B_2] \), \( U_c(t) = \begin{bmatrix} u_{c1}^T(t) & u_{c2}^T(t) \end{bmatrix}^T \) and \( (A, B_1, C_1) \) are system matrices of appropriate dimensions (Fig. 6). Indeed, the proposed design methodology works for multi-decentralized control agents.
The optimally decentralized control law is needed to track a reference trajectory \( r(t) = [r_1^T(t) \ r_2^T(t)]^T \) and minimize the following performance indices

\[
J_i = \frac{1}{2} \int_0^\infty \left\{ [C_i x_c(t) - r_i(t)]^T Q_i [C_i x_c(t) - r_i(t)] + u_{ci}^T(t) R_{ii} u_{ci}(t) + u_{cj}^T(t) R_{ij} u_{cj}(t) \right\} dt,
\]

for \( i = 1, 2, i \neq j, R_{ij} > 0, R_{ij} > 0 \) and \( Q_i \geq 0 \). We assume \( C_1 \) and \( C_2 \) have full row ranks and \( R_{ij} \) is assumed to be closed to zero, and \( u_{cj}(t) \) is constrained in the system with the saturating actuator. So \( u_{cj}^T(t) R_{ij} u_{cj}(t) \to 0 \). The control agents \( u_{ci}(t) \) are dependent on the state vector \( x_c(t) \) through the relation

\[
u_c(t) = -K_c x_c(t) + E_c r(t).
\]

Considering a weighting parameter \( \alpha \) on the quadratic performance index \( J = J_1 + \alpha J_2 \), one has

\[
J = \frac{1}{2} \int_0^\infty \left\{ [C_1 x_c(t) - r_1(t)]^T Q_1 [C_1 x_c(t) - r_1(t)] + [C_2 x_c(t) - r_2(t)]^T \alpha Q_2 [C_2 x_c(t) - r_2(t)] + u_{c1}^T R_{11} u_{c1} + u_{c2}^T \alpha R_{22} u_{c2} \right\} dt
\]

\[
= \frac{1}{2} \int_0^\infty \left\{ [C_1 x_c(t) - r_1(t)]^T Q_1 [C_1 x_c(t) - r_1(t)] + [C_2 x_c(t) - r_2(t)]^T \alpha Q_2 [C_2 x_c(t) - r_2(t)] + U_c^T R U_c \right\} dt,
\]

where \( U_c(t) = \begin{bmatrix} u_{c1}(t) \\ u_{c2}(t) \end{bmatrix} \), and \( R = \begin{bmatrix} R_{11} & 0 \\ 0 & a R_{22} \end{bmatrix} \). If \( J_1 \) plays a more important role, we can set \( \alpha \ll 1 \). Otherwise, we set \( \alpha \gg 1 \). Here, we assume \( \alpha = 1 \). Next, let us use the optimal state-feedback control law to derive the optimally decentralized controller as follows:

\[
\dot{\lambda} = \frac{\partial H}{\partial x_c} = C_1^T Q_1 C_1 x_c + \alpha C_2^T Q_2 C_2 x_c - C_1^T Q_1 r_1 - \alpha C_2^T Q_2 r_2 + A^T \lambda
\]

\[
= C^T Q C x_c + A^T \lambda - C^T Q r,
\]

where \( C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \), \( Q = \begin{bmatrix} \phi_1 & 0 \\ 0 & \alpha \phi_2 \end{bmatrix} \), and \( r(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix} \).

\[
0 = \frac{\partial H}{\partial U_c(t)} = R U_c(t) + B^T \lambda(t),
\]

which implies

\[
U_c(t) = -R^{-1} B^T \lambda(t).
\]

Substituting (30) into (24) yields

\[
\dot{x}_c(t) = A x_c(t) + B U_c(t) = A x_c(t) - B R^{-1} B^T \lambda(t),
\]

and the optimally decentralized controller is given by

\[
U_c(t) = \begin{bmatrix} u_{c1}(t) \\ u_{c2}(t) \end{bmatrix} = -K_c x_c(t) + E_c r(t),
\]

where

\[
K_c = R^{-1} B^T S,
\]

\[
E_c = -R^{-1} B^T \left[ (A - B K_c)^{-1} \right]^T C^T Q,
\]

in which the analog state-feedback gain \( K_c = \begin{bmatrix} k_{c1} \\ k_{c2} \end{bmatrix} \in R^{m \times n} \), \( m = m_1 + m_2 \), forward gain \( E_c \in R^{m \times m} \), and \( S \) is the positive-definite and symmetric solution of the Riccati equation

\[
A^T S + SA^T - SBR^{-1} B^T S + C^T Q C = 0.
\]
3.2. Decentralized tracker for discrete-time systems

Consider the discrete-time decentralized control system as shown in Fig. 7. By sampling theory, it is important to calculate the computational time of the proposed control methodology and the physical mechanism constraint on the range of sampling time. If the sampling frequency is equal or less than the maximum frequency of the system (\( f_{\text{max}} \), freq. of system \( \times 2 \)), the sampled-data will lose the information of the original system. As a result, if the analog controller (32) is directly implemented using a microprocessor with a relatively long sampling period, the digitally designed system often exhibits poor inter-sampling behavior, such as high ripple effects and over/under shoots etc. To overcome this drawback, the prediction-based digital redesign method [6] is often used to improve the inter-sampling behavior. The derivation of the prediction-based digital controller for the digitally decentralized sampled-data control system is formulated as follows:

The equivalent sampled-data model of system (24) is shown in Fig. 8, consisting of the form:

\[
\begin{align*}
\dot{x}_d (t) &= Ax_d (t) + B_1 u_{d1} (t) + B_2 u_{d2} (t) \\
&= Ax_d (t) + BU_d (t), \quad x_d (0) = x_0, \quad (36a) \\
y_{di} (t) &= C_i x_d (t), \quad \text{for } i = 1, 2, \quad (36b)
\end{align*}
\]

where the piecewise-constant continuous input \( U_d (t) \) satisfies

\[
U_d (t) = U_d (kT), \quad \text{for } kT \leq t \leq (k + 1) T, \quad (37)
\]

and \( T > 0 \) is the sampling period. The piecewise-constant control law is of the form

\[
U_d (kT) = \begin{bmatrix} u_{d1} (kT) \\ u_{d2} (kT) \end{bmatrix} = -K_d x_d (kT) + E_d r^* (kT), \quad (38)
\]

where \( K_d = \begin{bmatrix} K_{d1} \\ K_{d2} \end{bmatrix} \in R^{n \times n} \) and \( E_{di} \in R^{m \times m} \) are respective digital state-feedback gain and digital forward gain, which are to be determined from the analog gains, \( K_c \) and \( E_c \), in (32). A zero-order-hold device is used in (38). We set \( r^* (kT) \) as a sequence of reference input vectors, which equals \( r (kT + T) \) in order to track the original reference input \( r (kT) \).

From (36) and (38), the sampled-data controlled closed-loop system is

\[
\begin{align*}
\dot{x}_d (t) &= Ax_d (t) - BK_d x_d (kT) + BE_d r^* (kT), \quad x_d (0) = x_0, \quad (39a) \\
y_{di} (t) &= C_i x_d (t), \quad \text{for } kT \leq t \leq (k + 1) T. \quad (39b)
\end{align*}
\]

The digital redesign problem is thus reduced to finding the digital controller gains \( K_d \) and \( E_d \) in (38) from the analog controller gains \( K_c \) and \( E_c \) in (39), so that the sampled-data closed-loop state \( x_d (t) \) in (39) closely matches the continuous-time closed-loop state \( x_c (t) \) at all the sampling instants for a given \( r (t) \). If \( A \) is invertible,

\[
\begin{align*}
G &= e^{AT}, \quad (40a) \\
H &= \left( I_n T + A \frac{T^2}{2!} + A^2 \frac{T^3}{3!} + \cdots \right) B = (G - I_n) A^{-1} B. \quad (40b)
\end{align*}
\]
The state \( x_d (t) \) at \( t = kT + T \) in (40) can be evaluated as

\[
x_d (kT + T) = Gx_d (kT) + HU_d (kT).
\]

It is necessary to make \( U_d (kT) = U_c (kT + T) \) so that we can obtain the predicted state \( x_d (kT + T) = x_c (kT + T) \) under the assumption of \( x_d (kT) = x_c (kT) \). This results in the prediction-based digital controller

\[
U_d (kT) = U_c (kT + T) = -K_c x_c (kT + T) + E_c r (kT + T) \\
= -K_c x_d (kT + T) + E_c r (kT + T),
\]

where the state \( x_c (kT + T) \) needs to be predicted based on the available causal signals \( x_d (kT) \) and \( U_d (kT) \). Substituting (41) into (42) yields

\[
U_d (kT) = (I_m + K_c H)^{-1} [-K_c Gx_d (kT) + E_c r (kT + T)].
\]

Therefore, the desired sampled-data state-feedback control law (38) is obtained from Eq. (43) as

\[
U_d (kT) = -K_d x_d (kT) + E_d r^* (kT),
\]

where \( G = e^{AT} \), \( H = (G - I_m) A^{-1} B \), \( r^* (kT) = r (kT + T) \),

\[
K_d = \begin{bmatrix} K_{d1} \\ K_{d2} \end{bmatrix} = (I_m + K_c H)^{-1} K_c G,
\]

and

\[
E_d = (I_m + K_c H)^{-1} E_c.
\]

3.3. Decentralized observer-based tracker for continuous-time systems

Consider the linear observable continuous-time decentralized system

\[
\dot{x}_c (t) = Ax_c (t) + B_1 u_{c1} (t) + B_2 u_{c2} (t) = Ax_c (t) + B \begin{bmatrix} u_{c1} (t) \\ u_{c2} (t) \end{bmatrix},
\]

\[
y_{c1} (t) = C_1 x_c (t),
\]

where \( x_c (t) \in \mathbb{R}^n \) is the state vector, \( u_{ci} (t) \in \mathbb{R}^{m_i} \) is the control input, \( y_{ci} (t) \in \mathbb{R}^{p_i} \) is the measurable output, and \( (A, B_i, C_i) \) are system matrices of appropriate dimensions.

Whenever the system state \( x_c (t) \) cannot be measured, the optimal continuous-time observer (48) is proposed in this section to have the estimated state \( \hat{x}_c (t) \),

\[
\dot{\hat{x}}_c (t) = A \hat{x}_c (t) + B_1 u_{c1} (t) + B_2 u_{c2} (t) + L_{c1} [y_{c1} (t) - C_1 \hat{x}_c (t)] + L_{c2} [y_{c2} (t) - C_2 \hat{x}_c (t)],
\]

Fig. 8. Digitally decentralized sampled-data control system.
where \( \hat{x}_c(t) \in \mathbb{R}^n \) is the estimate of the actual state \( x_c(t) \), and the observer gain matrices are to be determined in order to satisfy some specified optimal criterion in (53). A block diagram of the decentralized observer-based tracker for linear continuous-time systems is shown in Fig. 9, and its detailed derivation is given as follows.

From Eq. (48), one has

\[
\dot{\hat{x}}_c(t) = Ax_c(t) + Bu_c(t) + B_2 u_{c\alpha}(t) \quad \text{(49)}
\]

where \( L_c \) is an analogous observer gain with an appropriate dimension. Define the observer error equation as

\[
\tilde{x}_c(t) = x_c(t) - \hat{x}_c(t) \quad \text{(50)}
\]

which implies

\[
\dot{\tilde{x}}_c(t) = \tilde{x}_c(t) - \hat{\dot{x}}_c(t) \quad \text{(51)}
\]

Substituting (47a), (49) and (50) into (51) yields

\[
\dot{\tilde{x}}_c(t) = (A - L_c C) \tilde{x}_c(t) \quad \text{(52)}
\]

The optimal performance index to be minimized for the individually decentralized control agents is

\[
J_i = \int_0^\infty \left[ \tilde{x}_c^T(t) Q_i \tilde{x}_c(t) + u_{ci}^T(t) R_{ii} u_{ci}(t) + u_{cj}^T(t) R_{ij} u_{cj}(t) \right] dt, \quad \text{(53)}
\]

where \( Q_i \geq 0, R_{ii} > 0, R_{ij} > 0 \), and \( R_{ij} \) is assumed to be close to zero, and \( u_{cj}(t) \) is constrained in the system with the saturating actuator. So \( u_{cj}^T(t) R_{ij} u_{cj}(t) \rightarrow 0 \).

Comparing Eqs. (53) and (25), one can see that

\[
(A - L_c C)^T = A^T - C^T L_c^T, \quad \text{(54)}
\]

which has the structure of a state-feedback controller. With the dual property of linear systems, the optimal observer gain \( L_c \) can be found by designing the optimal control gain \( K_c \) for the dual system, via \( A = A^T, B = C^T \) and \( \alpha = 1 \).

As a result, the optimal observer gain \( L_c \) that minimizes the overall performance index \( J \)

\[
J = J_1 + J_2 = \int_0^\infty \left[ \tilde{x}_c^T(t) Q_o \tilde{x}_c(t) + U_c^T(t) R U_c(t) \right] dt, \quad \text{(55)}
\]
with \( Q_o = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \quad U = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \) and \( R = \begin{bmatrix} R_{11} & 0 \\ 0 & R_{22} \end{bmatrix} \) is obtained as

\[
L_c = K_c^T = PC^TR^{-1},
\]

(56)

where \( P \) is the positive definite and symmetric solution of the Riccati equation

\[
AP + PA^T - PC^TR^{-1}CP + Q_o = 0.
\]

(57)

3.4. Decentralized observer-based tracker for sampled-data systems

Define the discrete-time state estimation error as

\[
\tilde{x}_d(kT + T) = x_d(kT) - \hat{x}_d(kT)
\]

(58)

such that the discrete-time error dynamics match the continuous-time error dynamics at each sampling instant \( \tilde{x}_d(kT) \approx \tilde{x}_d(t)|_{t=kT} \), or equivalently, assuming that the continuous-time observer is asymptotically stable, the original state and the digital state match \( \tilde{x}_d(kT) \approx \tilde{x}_d(t)|_{t=kT} \approx x_c(t)|_{t=kT} \).

Using the dual property once again, we can find the discrete-time error dynamics of (52) from (41) and (44), as follows

\[
\tilde{x}_d(kT + T) = (G - MN)\tilde{x}_d(kT)
\]

(59)

where

\[
G = e^{A^T}, \quad M = (G - I)A^{-1}L_c, \quad N = (I + CM)^{-1}CG.
\]

(60)

Further defining \( L_d = M(I + CM)^{-1} \), one can write \( MN = L_dCG \), and by substituting (58) into (59), we get

\[
x_d(kT + T) - \hat{x}_d(kT + T) = (G - L_dCG)[x_d(kT) - \hat{x}_d(kT)].
\]

(61)

Besides, by substituting the following identities into (61)

\[
x_d(kT + T) = Gx_d(kT) + Hu_d(kT),
\]

(62a)

\[
y_d(kT) = Cx_d(kT),
\]

(62b)

\[
CGx_d(kT) = Cx_d(kT + T) - CHu_d(kT) = y_d(kT + T) - CHu_d(kT)
\]

(62c)

and solving the result for \( \hat{x}_d(kT) \), one obtains the new digitally redesigned observer for system (49)

\[
\hat{x}_d(kT + T) = G_d\hat{x}_d(kT) + H_du_d(kT) + L_dy_d(kT + T)
\]

(63a)

or

\[
\hat{x}_d(kT) = G_d\hat{x}_d(kT - T) + H_du_d(kT - T) + L_dy_d(kT),
\]

(63b)

where

\[
L_d = (G - I_n)A^{-1}L_c\left[I_n + C(G - I_n)A^{-1}L_c\right]^{-1}, \quad G_d = G - L_dCG,
\]

(64)

\[
H_d = H - L_dCH,
\]

with \( G = e^{A^T}, \quad H = (G - I_n)A^{-1}B, \quad L_c = PC^TR^{-1}, \) and \( P \) is the solution in (57).

A rule of thumb to select an appropriate initial estimation of the state is

\[
\hat{x}_d(0) = C^T\left( C^TC \right)^{-1}y(0).
\]

(65)

The practically implementable full-order observer-based tracker for the sampled-data system is then shown in Fig. 10.
4. Improved decentralized observer-based tracker: An evolutionary programming approach

Due to the over-excited or under-excited output measurements, the decentralized observer-based tracker is not functioning properly. In the proposed system, the output matrix $C$ affects the decentralized observer-based tracker. One may wonder if we could appropriately weigh the measurable output signals; i.e., weigh the output matrix $C$, so that the decentralized observer-based tracker can work properly.

In order to improve the performance of the tracker, tuning the observer-based tracker with evolutionary programming (EP) may be the optimum method. In this section, a tuning of the output matrix $C$ via an EP-based tuning algorithm is applied to establish an effective tracker. For the tracker to work properly, $[C_1^T C_2^T]^T$ is weighted as $[(\xi_1 C_1) \ (\xi_2 C_2)]^T$, and the block diagram of the proposed methodology is shown in Fig. 11.

Now, let’s study how the evolutionary programming can be applied to the decentralized observer-based tracker. First, we should review what the quasi-random sequences are.

4.1. Quasi-random sequences (QRS)

Suppose that the natural numbers are expressed in the scale of notation with $R$, so that

$$n = a_0 + a_1 R + a_2 R^2 + \cdots + a_m R^m, \quad 0 \leq a_i \leq R.$$  \hspace{1cm} (66)

Write the digits of these numbers in the reverse order, preceded by a decimal point. This gives the number

$$\phi_R(n) = a_0 R^{-1} + a_1 R^{-2} + \cdots + a_m R^{-m-1}.$$  \hspace{1cm} (67)

Halton [13] extended the two-dimensional result of Van Der Corput [14] to $\rho$-dimensions, where $R_1, R_2, \ldots, R_\rho$ are mutually co-prime. We show a binary scale and an illustrative case in Tables 1 and 2.

4.2. Tuning observer-based tracker

The minimal principle of evolutionary programming (EP) to search the “best” nominal controller is applied to the proposed method. The developed EP algorithm is described as follows:

(1) Individual population:

According to Section 4.1, $\phi_R(n) < 1$, to satisfy this range, scaling any varying parameter $\varepsilon$ from its range $[\underline{\varepsilon} \ \bar{\varepsilon}]$ to $[0 \ 1]$ is required. Let the interval real (IR) matrix $X \in IR^{n \times m}$ be a set of degenerate real matrices defined by

$$X = [L, U] = \{[x_{ij}] \mid l_{ij} \leq x_{ij} \leq u_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m\},$$ \hspace{1cm} (68)
Table 1
Natural numbers in binary scale

<table>
<thead>
<tr>
<th>n (decimal)</th>
<th>(Binary)</th>
<th>φ₂(n) (binary)</th>
<th>(Decimal)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.1</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>0.01</td>
<td>0.25</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>0.11</td>
<td>0.75</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>0.001</td>
<td>0.125</td>
</tr>
<tr>
<td>5</td>
<td>101</td>
<td>0.101</td>
<td>0.625</td>
</tr>
<tr>
<td>6</td>
<td>110</td>
<td>0.011</td>
<td>0.375</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Table 2
Quasi-random sequences

<table>
<thead>
<tr>
<th>φₐ(n)</th>
<th>R = 2</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>11</th>
<th>13</th>
<th>17</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>N = 1</td>
<td>0.5000</td>
<td>0.3333</td>
<td>0.2000</td>
<td>0.1429</td>
<td>0.0909</td>
<td>0.0769</td>
<td>0.0588</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td>0.2500</td>
<td>0.6667</td>
<td>0.4000</td>
<td>0.2857</td>
<td>0.1818</td>
<td>0.1538</td>
<td>0.1176</td>
<td>...</td>
</tr>
<tr>
<td>3</td>
<td>0.7500</td>
<td>0.1111</td>
<td>0.6000</td>
<td>0.4286</td>
<td>0.2727</td>
<td>0.2308</td>
<td>0.1765</td>
<td>...</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>49</td>
<td>0.5469</td>
<td>0.5309</td>
<td>0.9680</td>
<td>0.0029</td>
<td>0.4876</td>
<td>0.7870</td>
<td>0.8893</td>
<td>...</td>
</tr>
<tr>
<td>50</td>
<td>0.2969</td>
<td>0.8642</td>
<td>0.0160</td>
<td>0.1458</td>
<td>0.5785</td>
<td>0.8639</td>
<td>0.9481</td>
<td>...</td>
</tr>
</tbody>
</table>

Fig. 11. Continuous-time system with tuning full-order observer-based tracker.

where $L$ and $U$ are constant real matrices. We introduce the variable $\epsilon_{ij}, \ 0 \leq \epsilon_{ij} \leq 1$ such that

$$x_{ij} = l_{ij} + \epsilon_{ij} (u_{ij} - l_{ij}), \quad (69)$$

and use the notation $\varepsilon = [\epsilon_{11}, \ldots, \epsilon_{1m}, \epsilon_{21}, \ldots, \epsilon_{2m}, \ldots, \epsilon_{nm}].$

Then the interval matrix $X$ can be denoted as $X(\varepsilon)$. Let $\epsilon_{11} = \phi_2(n), \ \epsilon_{12} = \phi_3(n), \ \epsilon_{13} = \phi_5(n),$ and so on, to construct the desired initial population of size $N$ (e.g., $N = 50$).

(2) Objective function:

Assign to each $P_i, i = 1, \ldots, N$, an objective function score. Arrange $P_i, i = 1, \ldots, N$, in descending order, starting from the best one generated from the objective function score. An evolutionary programming technique is
Fig. 12. The output responses of the continuous-time system: (a) without the observers, and (b) with the observers, for $r_1(t) = \cos(t)$ and $r_2(t) = \sin(t)$; The unanticipated failure occurs during $t = 3–6$ s and starts the proposed methodology at $t = 6$ s.

Fig. 13. The inputs of the continuous-time system: (a) without the observers, and (b) with the observers, for $r_1(t) = \cos(t)$ and $r_2(t) = \sin(t)$; The unanticipated failure occurs during $t = 3–6$ s and starts the proposed methodology at $t = 6$ s.

Fig. 14. The internal states of the continuous-time system: (a) without the observers, and (b) with the observers, for $r_1(t) = \cos(t)$ and $r_2(t) = \sin(t)$; The unanticipated failure occurs during $t = 3–6$ s and starts the proposed methodology at $t = 6$ s.
Fig. 15. The output responses of the discrete-time system: (a) without the observers, and (b) with the observers, for \( r_1(t) = \cos(t) \) and \( r_2(t) = \sin(t) \); The unanticipated failure occurs during \( t = 3-6 \) s and starts the proposed methodology at \( t = 6 \) s.

Fig. 16. The inputs of the discrete-time system: (a) without the observers, and (b) with the observers, for \( r_1(t) = \cos(t) \) and \( r_2(t) = \sin(t) \); The unanticipated failure occurs during \( t = 3-6 \) s and starts the proposed methodology at \( t = 6 \) s.

Fig. 17. The internal states of the discrete-time system: (a) without the observers, and (b) with the observers, for \( r_1(t) = \cos(t) \) and \( r_2(t) = \sin(t) \); The unanticipated failure occurs during \( t = 3-6 \) s and starts the proposed methodology at \( t = 6 \) s.
Fig. 18. The output responses of the sampled-data system: (a) without the EP-based observers, and (b) with the EP-based observers, for $r_1(t) = \cos(t)$ and $r_2(t) = \sin(t)$.

Fig. 19. The inputs of the sampled-data system: (a) without the EP-based observers, and (b) with the EP-based observers, for $r_1(t) = \cos(t)$ and $r_2(t) = \sin(t)$.

first proposed to minimize the objective function (OF) score

$$\text{OF} := E\left[ e(k)^T e(k) \right] \approx \frac{1}{k_f} \sum_{i=1}^{N} \sum_{k=1}^{k_f} e^2(k),$$

where $k_f$ is the final time step of interest.

(3) Fitness function:

Assign each sorted $P_i, i = 1, \ldots, N$, a fitness function score to weight those high-quality individuals in the pool of individuals, based on the obtained objective function scores:

For the maximal principle, use

$$\text{FF}(\text{OF}(P_i)) = \left( \frac{\bar{\beta} - \beta}{\text{OF}(P_i) - \text{OF}(P_j)} \right) (\text{OF}(P_i) - \text{OF}(P_j)) + \beta.$$
Fig. 20. The internal states of the sampled-data system: (a) without the EP-based observers, and (b) with the EP-based observers, for \( r_1(t) = \cos(t) \) and \( r_2(t) = \sin(t) \).

For the minimal principle, use

\[
FF(OF(P_i)) = \left[ \left( \frac{\bar{\beta} - \beta}{OF(P_i) - OF(P_i)} \right) (OF(P_i) - OF(P_i)) + \beta \right]^{-1}.
\] (72)

This function linearly maps the real-valued space \([OF(P_i), OF(P_i)]\) to any appropriate specified space \([\bar{\beta}, \bar{\beta}]\), where \( \beta > 0 \), for weighting the objective function scores. Hence, the better an individual is, the higher the objective function score that it will have.

(4) Probability function:

Calculate the probability function score of each \( P_i, i = 1, \ldots, N \), using the fitness function score

\[
PF(FF(P_i)) := PF(P_i) = \frac{FF(P_i)}{\sum_{i=1}^{N} FF(P_i)}.
\] (73)

(5) Mutation:

Mutate each \( P_i, i = 1, \ldots, N \), based on statistics to double the population size from \( N \) to \( 2N \); assign \( P_{i+N} \) the following value

\[
P_{i+N,j} := P_{i,j} (1 + \text{sgn}(N(0,1))) \gamma (1 - PF(P_i)) ,
\] (74)

where \( P_{i,j} \) is the \( j \)th element in the \( i \)th individual, \( N(\mu, \sigma^2) \) is the Gaussian random variable with mean \( \mu \) and variance \( \sigma^2 \), \( \gamma \) is a weighting factor for the percentage change of \( P_{i,j} \), and \( \text{sgn}(\cdot) \) is the standard sign function.

Whenever \( P_{i+N,j} \notin [P_j, \bar{P}_j] \), some modification is required:

\[
P_{i+N,j} := \begin{cases} P_j & \text{if } P_{i+N,j} < P_j \\ \bar{P}_j & \text{if } P_{i+N,j} > \bar{P}_j. \end{cases}
\] (75)

Properly adjusting the weighting factor \( \gamma \) can possibly avoid the undesired situation \( P_{i+N,j} \notin [P_j, \bar{P}_j] \). It is notable that \( \gamma \) heavily dominates the convergence rate of the EP.

(6) Selection:

Calculate the objective function score of each \( P_i, i = 1, \ldots, N \). Rank the objective function scores of \( P_i, i = 1, \ldots, 2N \). Recording \( P_i, i = 1, \ldots, 2N \), in descending order, starting from the best individual in the pool of the population. The first \( N \) individuals are selected for the next generation, in which the top one of each generation,
denoted $P^*_g,t$, always survives and is selected for the next generation. Whenever $P^*_g,t$ is no longer the best during the evolutionary process, update it by the newly generated best one.

5. An illustrative example

Consider a cascaded MIMO system with input and state constraints. The dynamic equations of plant $G_1(s)$ are given as

$$
\begin{align*}
\dot{x}_1(t) &= A_1x_1(t) + B_1u_1(t), \\
y_1(t) &= C_1x_1(t), \\
x_1(0) &= x_{10} = [0, 0, 0]^T.
\end{align*}
$$

and the dynamic equation of plant $G_2(s)$ as

$$
\begin{align*}
\dot{x}_2(t) &= A_2x_2(t) + B_2u_2(t), \\
y_2(t) &= C_2x_2(t), \\
x_2(0) &= x_{20} = [0, 0]^T.
\end{align*}
$$

Condition 1. A tracker has good performance in the beginning, but the input is broken by external factors in the middle stage. Then, we observe the changes in the performance of the tracker.

$$
A_1 = \begin{bmatrix}
-1 & 2 & 0 \\
0 & -1 & 3 \\
0 & 1 & -2
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
1 & 0 \\
1 & -1 \\
0 & 1
\end{bmatrix}, \quad C_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix},
$$

$$
A_2 = \begin{bmatrix}
-2 & 0 & 0 \\
0 & -3 & 0
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
1 & 2 \\
2 & 1
\end{bmatrix}, \quad C_2 = \begin{bmatrix}
15 & 10 \\
-15 & 10
\end{bmatrix}.
$$

Let four components of the controller output $u(t)$ and $z(t)$ be constrained to lie in the ranges of $[-50, 50]$ and $[-0.4, 0.4]$, respectively. $z_{h1} = z_{h2} = 0.4$, $z_{l1} = z_{l2} = 0.2$. The system matrices of $G_o(s)$ and $G_oz(s)$ are

$$
A_o = \begin{bmatrix}
-1 & 2 & 0 & 0 & 0 \\
0 & -1 & 3 & 0 & 0 \\
0 & 1 & -2 & 0 & 0 \\
1 & 2 & 0 & -2 & 0 \\
2 & 1 & 0 & 0 & -3
\end{bmatrix}, \quad B_o = \begin{bmatrix}
1 & 0 \\
1 & -1 \\
0 & 1 \\
0 & 0 \\
0 & 0
\end{bmatrix}, \quad C_o = \begin{bmatrix}
0 & 0 & 0 & 15 & 10 \\
0 & 0 & 0 & -15 & 10
\end{bmatrix},
$$

$$
A_{oz} = \begin{bmatrix}
-1 & 2 & 0 \\
0 & -1 & 3 \\
0 & 1 & -2
\end{bmatrix}, \quad B_{oz} = \begin{bmatrix}
1 & 0 \\
1 & -1 \\
0 & 1
\end{bmatrix}, \quad C_{oz} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}.
$$

The reference inputs $r(t) = [\cos(t) \ \sin(t)]^T$ and $r_z(t) = [0.32 \ 0.2]^T$ are applied. We build a controller as follows, by choosing $Q_1 = Q_2 = 7 \times 10^5$ and $R_1 = R_2 = 1$ to track the reference input $r(t)$. Assume $u_1(t)$ is destroyed during $t = 3–6$ s and start the proposed methodology at $t = 6$, so one can get the associated analog gains

$$
K_{co1} = \begin{bmatrix}
238.43 & 1.9989 & 1.3368 \times 10^{-3} & 12344 & 8161.7
\end{bmatrix}, \quad E_{co1} = 836.66,
$$

$$
K_{co2} = \begin{bmatrix}
6.3435 \times 10^{-16} & -180.41 & -2.9956 & -12279 & 8096.7
\end{bmatrix}, \quad E_{co2} = 836.66,$$

$$
K_{coz1} = \begin{bmatrix}
835.66 & 1.994 & 6.6558 \times 10^{-3}
\end{bmatrix}, \quad E_{coz1} = 836.66,$$

$$
K_{coz2} = \begin{bmatrix}
-1.9048 \times 10^{-16} & -835.67 & -2.9861
\end{bmatrix}, \quad E_{coz2} = -836.66.
$$

Similarly, we also construct the observers by choosing $Q_{o1} = Q_{o2} = 7 \times 10^5 I_5$ and $R_{o1} = R_{o2} = 1$, giving the associated analog gains

$$
L_{o1} = \begin{bmatrix}
1289.1 & 1177.4 & 523.7 & 553.05 & 678.79
\end{bmatrix}^T,$$

and $L_{o2} = \begin{bmatrix}
-925.66 & -1106 & -503.28 & -1014.4 & -13.393
\end{bmatrix}^T$. 
The gains of the digital system for the sampling period $T = 0.02$ s are
\[ K_{d1} = \begin{bmatrix} 68.263 & 1.9460 & 5.3628 \times 10^{-2} & 1.0287 \times 10^{3} & 666.71 \end{bmatrix}, \quad E_{d1} = 72.571, \]
\[ K_{d2} = \begin{bmatrix} 7.7331 \times 10^{-17} & -61.948 & -2.8569 & -1467.2 & 948.34 \end{bmatrix}, \quad E_{d2} = 104.05, \]
\[ K_{d1c} = \begin{bmatrix} 45.718 & 1.9383 & 6.1342 \times 10^{-2} \end{bmatrix}, \quad E_{d1c} = 46.698, \]
\[ K_{d2c} = \begin{bmatrix} -1.0315 \times 10^{-17} & -45.282 & -2.8473 \end{bmatrix}, \quad E_{d2c} = -46.221, \]
and the associated digital observer gains are
\[ L_{d1} = \begin{bmatrix} 8.5421 & 7.7637 & 3.4503 & 3.6469 & 4.4968 \end{bmatrix} \times 10^{-2}, \]
\[ L_{d2} = \begin{bmatrix} -6.2334 & -7.3731 & -3.3495 & -6.7502 & -0.15853 \end{bmatrix} \times 10^{-2}. \]

From the above, one has the following observation. The proposed decentralized controller can take apart the system after design to give several smaller parts. It has the capability of a multiplexer. When the inputs of some parts of the system are broken, the others are not influenced entirely. Hence, it can reduce the risk effectively to do it in this way. In addition, when the system has more constraints to reduce the performance of the tracker, we apply the digital redesign method to lower the gain, which the controllers need, so that it doesn’t consume a large amount of energy to get better performance out of the controller.

**Condition 2.** The input is never broken off, but the performance of the tracker is not good.

\[ A_1 = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 3 \\ 0 & 1 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \]
\[ A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 15 \\ 10 \\ 15 \end{bmatrix}. \]

Let four components of the controller output $u(t)$ and $z(t)$ be constrained to lie in the ranges of $[-50, 50]$ and $[-0.4, 0.4]$, respectively. $z_{h1} = z_{h2} = 0.4$, $z_{l1} = z_{l2} = 0.2$. The system matrices of $G_o(s)$ and $G_{oz}(s)$ are

\[ A_o = \begin{bmatrix} -1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 1 & 2 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 & -3 \end{bmatrix}, \quad B_o = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_o = \begin{bmatrix} 0 & 0 & 15 & 10 \\ 0 & 0 & -10 & 15 \end{bmatrix}, \]
\[ A_{oz} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 3 \\ 0 & 1 & -2 \end{bmatrix}, \quad B_{oz} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad C_{oz} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}. \]

The reference inputs $r(t) = [\cos(t) \sin(t)]^T$ and $r_z(t) = [0.32 \ 0.2]^T$ are applied. We build a controller as follows, by choosing $Q_1 = Q_2 = 7 \times 10^5$ and $R_1 = R_2 = 1$ to track the reference input $r(t)$.

\[ K_{co1} = \begin{bmatrix} 238.43 & 1.9989 & 1.3368 \times 10^{-3} & 12344 & 8161.7 \end{bmatrix}, \quad E_{co1} = 836.66, \]
\[ K_{co2} = \begin{bmatrix} 3.7712 \times 10^{-17} & -91.281 & -2.9872 & -8009.4 & 11755 \end{bmatrix}, \quad E_{co2} = 836.66, \]
\[ K_{co1c} = \begin{bmatrix} 835.66 & 1.994 & 6.6558 \times 10^{-3} \end{bmatrix}, \quad E_{co1c} = 836.66, \]
\[ K_{co2c} = \begin{bmatrix} -1.9048 \times 10^{-16} & -835.67 & -2.9861 \end{bmatrix}, \quad E_{co2c} = 836.66. \]

Similarly, we also construct the observers by choosing $Q_{o1} = Q_{o2} = 7 \times 10^5 I_5$ and $R_{o1} = R_{o2} = 1$, giving the associated analog gains
\[ L_{c1} = \begin{bmatrix} 1289.1 & 1177.4 & 523.7 & 553.05 & 678.79 \end{bmatrix}^T, \]
\[ L_{c2} = \begin{bmatrix} -1697.5 & -1418.2 & -625.71 & -1402.8 & 70.256 \end{bmatrix}^T. \]
The gains of the digital system for the sampling period $T = 0.02$ s are
\[
K_{do1} = \begin{bmatrix} 68.263 & 1.9460 & 5.3628 \times 10^{-2} & 1.0287 \times 10^{3} & 666.71 \end{bmatrix}, \quad E_{do1} = 72.571,
\]
\[
K_{do2} = \begin{bmatrix} 9.7566 \times 10^{-15} & -46.088 & -2.8262 & -2031.1 & 2922 \end{bmatrix}, \quad E_{do2} = 220.83,
\]
\[
K_{doc1} = \begin{bmatrix} 45.718 & 1.9383 & 6.1342 \times 10^{-2} \end{bmatrix}, \quad E_{doc1} = 46.698,
\]
\[
K_{doc2} = \begin{bmatrix} -1.0315 \times 10^{-17} & -45.282 & -2.8473 \end{bmatrix}, \quad E_{doc2} = -46.221,
\]
and the associated digital observer gains are
\[
L_{d1} = \begin{bmatrix} 8.5421 & 7.7637 & 3.4503 & 3.6469 & 4.4968 \end{bmatrix}^{T} \times 10^{-2}
\]
and
\[
L_{d2} = \begin{bmatrix} -11.472 & -9.5517 & -4.2120 & -9.42 & 0.36426 \end{bmatrix}^{T} \times 10^{-2}.
\]

It is desired to find the minimum value of the cost. The results obtained by EP are summarized in Table 1, in which the parameters are $N = 50$, $\beta = [1, 10]$, $\gamma = 0.1$ and $\varepsilon = 0.1$. After EP estimating, we can get the values for the switching levels $z_l$ and $z_h$, and the parameters for adjusting the performance of the observer-based tracker as:
\[
[z_{l1} \quad z_{h1}] = \begin{bmatrix} 0.3658 & 0.0330 \end{bmatrix}, \quad [z_{h2} \quad z_{l2}] = \begin{bmatrix} 0.3034 & 0.1155 \end{bmatrix}, \quad \text{and} \quad \zeta = \begin{bmatrix} 1.0123 & 1.0845 \\ 1.0044 & 1.9795 \end{bmatrix}.
\]

We can add observers to a phenomenon as the above. The application of the EP can effectively tune the parameter $\zeta$ to improve the performance of the trackers. Thus, if a system has bad trackers at the beginning, we can apply this method to refine the performance of the trackers. This is a convenient method to save the time and work in finding the parameter $\zeta$ (see Figs. 12–20).

6. Conclusion

A weighted switching strategy and an inner-loop compensator for a cascaded system with actuators and state saturation are presented in this paper to design the observer-based tracker for a decentralized closed-loop cascaded system. To avoid the complexity of centralized control, this paper proposes a decentralized tracker design method for MIMO systems via the LQR design methodology. Through the digital redesign method, a realizable sample-data controller with a low-gain property can be determined which can have the same high design performance as the originally superior analog one in theory as possible. Although only decentralizing the closed-loop system with two control agents has been considered, it can be extended for the case with more agents easily. This observer-based digital redesign method not only accurately and quickly estimates all the states of the system, but also tolerates the small disturbance and numerical errors presented. When the analog control signals are somewhat high, the proposed equivalent digital control signals are relatively small. The evolution programming (EP) algorithm is further presented to tune the measurable output signals to improve the performance of the designed system.

References


