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A reproducing kernel method for solving nonlocal fractional boundary value problems

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ABSTRACT

In our previous works, we proposed a reproducing kernel method for solving singular and nonsingular boundary value problems of integer order based on the reproducing kernel theory. In this letter, we shall expand the application of reproducing kernel theory to fractional differential equations and present an algorithm for solving nonlocal fractional boundary value problems. The results from numerical examples show that the present method is simple and effective.

1. Introduction

In this letter, we consider the following nonlocal fractional boundary value problem:

$$\begin{cases} D^{\alpha}u + a(x)u'(x) + b(x)u(x) = f(x), \quad 0 \le x \le 1, \\ u(0) = 0, u(1) = \sum_{i=1}^{m} \alpha_i u(\xi_i). \end{cases} \tag{1.1}$$

where $1 < \alpha \le 2, D^{\alpha}$ denotes the Caputo fractional derivative of order α, m is a positive integer, $0 < \xi_i < 1$, $a(x), b(x) \in C[0, 1]$ and $f(x)$ is given such that (1.1) satisfies the existence and uniqueness of the solutions. Here we only consider homogeneous boundary conditions $u(0) = 0, u(1) = \sum_{i=1}^{m} \alpha_i u(\xi_i)$ since nonhomogeneous boundary conditions can be reduced to the homogeneous boundary conditions easily.

Fractional differential equations arise in various fields of physics and engineering such as biophysics, blood flow phenomena, aerodynamics, electrodynamics of complex media, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, etc. In recent years, fractional differential equations, as an important research branch, have attracted much attention [1–8]. However, most of the papers and books on fractional calculus are devoted to the solvability and numerical solutions of initial value problems for fractional order differential equations. In contrast to the case for initial value problems, not much attention has been paid to the nonlocal fractional boundary value problems. Some recent results on the existence and uniqueness of nonlocal fractional boundary value problems can be found in [9–13]. However, discussion on numerical solutions of nonlocal fractional boundary value problems is rare.

The goal of this letter is to give an effective method for solving nonlocal fractional boundary value problems based on the reproducing kernel theory.

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The rest of the letter is organized as follows. In the next section, some definitions and lemmas for fractional derivatives are gathered. An algorithm for solving nonlocal fractional boundary value problems is introduced in Section 3. Some numerical examples are presented in Section 4. Section 5 ends this paper with a brief conclusion.

2. Preliminaries

In this section, we provide some basic definitions and properties of the fractional calculus theory which are useful in the following discussion. These definitions and properties can be found in [1,3,5,11–13], and references therein.

Definition 2.1. The Riemann–Liouville fractional order integral of order α of a function $f(t)$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad \alpha > 0,\ t > 0$$
$$I^0 f(t) = f(t). \tag{2.1}$$

Properties of the operator $I^\alpha f$ can be found in [1]. We mention only the following:

Lemma 2.1. *For $\alpha, \beta \geq 0$ and $\gamma \geq -1$, we have*

$$I^\alpha I^\beta f = I^{\alpha+\beta} f,$$
$$I^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} t^{\alpha+\gamma}.$$

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator proposed by Caputo.

Definition 2.2. The α-order Caputo derivative of $f(t)$ is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)d\tau, \quad \alpha > 0,\ t > 0 \tag{2.2}$$

where $m-1 < \alpha < m$, $m \in N$.

Also, we introduce one of its basic properties.

Lemma 2.2. *If $m-1 < \alpha < m$, $m \in N$, $\alpha > 0$, $t > 0$, $\gamma > 0$, then*

$$D^\alpha I^\alpha f(t) = f(t),$$
$$I^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{t^k}{k!}, \quad t > 0,$$
$$D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}.$$

3. The algorithm

Applying the operator I^α to both sides of (1.1) yields

$$u(x) - u'(0)x + I^\alpha[a(x)u'(x) + b(x)u(x)] = F(x), \tag{3.1}$$

where $F(x) = I^\alpha f(x)$.

Obviously, (1.1) and (3.1) are equivalent. Therefore, it suffices for us to solve (3.1). The reproducing kernel theory has important applications in numerical analysis, differential equations, probability and statistics, and so on [14–21]. We shall solve (3.1) in the reproducing kernel space.

To solve (3.1) using the reproducing kernel method presented in [14,16], it is necessary to construct a reproducing kernel space $W_2^3[0,1]$ in which every function satisfies the nonlocal boundary conditions of (1.1).

First, we construct the following reproducing kernel space.

The reproducing kernel Hilbert space $W^3[0,1]$ is defined as $W^3[0,1] = \{u(x) \mid u''(x)$ is an absolutely continuous real valued function, $u'''(x) \in L^2[0,1]$, $u(0) = 0\}$. The inner product and norm in $W^3[0,1]$ are given, respectively, by

$$(u(y), v(y))_{W_2^3} = u(0)v(0) + u'(0)v'(0) + u(1)v(1) + \int_0^1 u'''v'''dy$$

and

$$\|u\|_{W^3} = \sqrt{(u, u)_{W^3}}, \quad u, v \in W^3[0, 1].$$

By [14,16], it is easy to obtain the reproducing kernel

$$k_0(x, y) = \begin{cases} h_1(x, y), & y \le x, \\ h_1(y, x), & y > x, \end{cases} \tag{3.2}$$

where $h_1(x, y) = \frac{1}{120} y(-(x^2 - 1)y^4 + 5(x - 1)xy^3 - x(x^4 - 5x^3 + 10x^2 - 246x + 120)y - 120(x - 1)x)$.

Next, we construct a reproducing kernel space $W_2^3[0, 1]$ in which every function satisfies $u(0) = 0$ and $u(1) = \sum_{i=1}^{m} \alpha_i u(\xi_i)$.

$W_2^3[0, 1]$ is defined as $W_2^3[0, 1] = \{u(x) \mid u(x) \in W^3[0, 1], u(1) = \sum_{i=1}^{m} \alpha_i u(\xi_i)\}$.

Clearly, $W_2^3[0, 1]$ is a closed subspace of $W^3[0, 1]$ and therefore it is also a reproducing kernel space.

Put $L_1 u(x) = u(1) - \sum_{i=1}^{m} \alpha_i u(\xi_i)$. In the following theorem, the reproducing kernel of $W_2^3[0, 1]$ is introduced.

Theorem 3.1. *If $L_{1x} L_{1y} k_0(x, y) \neq 0$, then the reproducing kernel $k(x, y)$ of $W_2^3[0, 1]$ is given by*

$$k(x, y) = k_0(x, y) - \frac{L_{1x} k_0(x, y) L_{1y} k_0(x, y)}{L_{1x} L_{1y} k_0(x, y)}, \tag{3.3}$$

where the subscript x on the operator L_1 indicates that the operator L_1 applies to the function of x.

Proof. It is easy to see that $k(x, 0) = 0$ and $k(x, 1) = \sum_{i=1}^{m} \alpha_i k(x, \xi_i)$, and therefore $k(x, y) \in W_2^3[0, 1]$.

For every $u(y) \in W_2^3[0, 1]$, obviously, $L_{1y} u(y) = 0$ and

$$\begin{aligned} (u(y), k(x, y))_{W^3} &= (u(y), k_0(x, y))_{W^3} - \frac{L_{1y} k_0(x, y)}{L_{1x} L_{1y} k_0(x, y)} (u(y), L_{1x} k_0(x, y))_{W^3} \\ &= u(x) - \frac{L_{1y} k_0(x, y)}{L_{1x} L_{1y} k_0(x, y)} L_{1x} (u(y), k_0(x, y))_{W^3} \\ &= u(x) - \frac{L_{1y} k_0(x, y)}{L_{1x} L_{1y} k_0(x, y)} L_{1x} u(x) = u(x). \end{aligned}$$

That is, $k(x, y)$ has the "reproducing property". Thus, $k(x, y)$ is the reproducing kernel of $W_2^3[0, 1]$ and the proof is complete. □

For (3.1), letting $Lu(t) = u(t) - u'(0)t + I^{\alpha}[a(t)u(t)]$, it is clear that $L : W_2^3[0, 1] \to W_2^1[0, 1]$ is a bounded linear operator (for the definition of $W_2^1[0, 1]$ and its reproducing kernel, refer to [14]). Put $\varphi_i(x) = \overline{k}(x_i, x)$ and $\psi_i(x) = L^* \varphi_i(x)$ where $\overline{k}(x_i, x)$ is the reproducing kernel of $W_2^1[0, 1]$ and L^* is the adjoint operator of L. The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W_2^3[0, 1]$ can be derived from the Gram–Schmidt orthogonalization process applied to $\{\psi_i(x)\}_{i=1}^{\infty}$:

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik} \psi_k(x), \quad (\beta_{ii} > 0, \ i = 1, 2, \ldots). \tag{3.4}$$

Theorem 3.2. *For (3.1), if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_2^3[0, 1]$ and $\psi_i(x) = L_t k(x, t)|_{t=x_i}$.*

Proof. Note that

$$\begin{aligned} \psi_i(x) &= (L^* \varphi_i)(x) = ((L^* \varphi_i)(t), k(x, t)) \\ &= (\varphi_i(t), L_t k(x, t)) = L_t k(x, t)|_{t=x_i}. \end{aligned}$$

Clearly, $\psi_i(x) \in W_2^3[0, 1]$.

For each fixed $u(x) \in W_2^3[0, 1]$, let $(u(x), \psi_i(x)) = 0$, $(i = 1, 2, \ldots)$, which means that

$$(u(x), (L^* \varphi_i)(x)) = (Lu(\cdot), \varphi_i(\cdot)) = (Lu)(x_i) = 0. \tag{3.5}$$

Note that $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$. Hence, $(Lu)(x) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1}. So the proof of the Theorem 3.2 is complete. □

Theorem 3.3. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$ and the solution of (3.1) is unique on $W_2^3[0, 1]$, then the solution of (3.1) is*

$$u(x) = \sum_{j=1}^{\infty} A_j \overline{\psi}_j(x), \tag{3.6}$$

where $A_j = \sum_{l=1}^{j} \beta_{jl} F(x_l)$.

Proof. Applying Theorem 3.2, it is easy to see that $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is the complete orthonormal basis of $W_2^3[0, 1]$.

Note that $(w(x), \varphi_i(x)) = w(x_i)$ for each $w(x) \in W_2^1[0, 1]$; hence we have

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty} (u(x), \overline{\psi}_i(x)) \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} (u(x), L^* \varphi_k(x)) \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} (Lu(x), \varphi_k(x)) \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} (F(x), \varphi_k(x)) \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F(x_k) \overline{\psi}_i(x). \end{aligned} \tag{3.7}$$

and the proof of the theorem is complete. □

The approximate solution $u_n(x)$ can be obtained by taking finitely many terms in the series representation of $u(x)$ and

$$u_n(x) = \sum_{j=1}^{n} A_j \overline{\psi}_j(x). \tag{3.8}$$

Remark. Since $W_2^3[0, 1]$ is a Hilbert space, it is clear that $\sum_{i=1}^{\infty} (\sum_{k=1}^{i} \beta_{ik} f(x_k))^2 < \infty$. Therefore, the sequence u_n is convergent in norm.

Lemma 3.1. *If $u(x) \in W_2^3[0, 1]$, then there exists a constant c such that $|u(x)| \le c\|u(x)\|_{W_2^3}$, $|u'(x)| \le c\|u(x)\|_{W_2^3}$.*

From Lemma 3.1, by the convergence of $u_n(x)$ in the sense of norm, it is easy to obtain the following theorem.

Theorem 3.4. *The approximate solution $u_n(x)$ and its derivatives $u_n'(x)$ are both uniformly convergent.*

4. Numerical examples

In this section, two numerical examples are provided to show the accuracy of the present method. All computations are performed by Mathematica 5.1. Results obtained by the method are compared with the exact solution for each example and they are found to be in good agreement with each other.

Example 4.1. Consider the following nonlocal fractional boundary value problem:

$$\begin{cases} D^{1.3} u(x) + \cos(x) u'(x) + 2u(x) = f(x), & 0 \le x \le 1, \\ u(0) = 0, u(1) = u(1/8) + 2u(1/2) + \dfrac{31}{49} u(7/8) \end{cases}$$

where $f(x) = 2x^2 + 2x\cos(x) + \frac{\Gamma(3)}{\Gamma(1.7)} x^{0.7}$. The exact solution is given by $u(x) = x^2$.

Using the present method, taking $x_i = \frac{i}{n}$, $i = 1, 2, \ldots, n$, $n = 10$, the numerical results are as given in Fig. 1.

Example 4.2. Consider the following nonlocal fractional boundary value problem:

$$\begin{cases} D^{1.6} u(x) + \sinh(x) u(x) = f(x), & 0 \le x \le 1, \\ u(0) = 0, u(1) = u(1/10) + u(1/2) + \dfrac{9397}{8704} u(9/10), \end{cases}$$

where $f(x) = \sinh(x)(x^2 + x^3) + \frac{\Gamma(3)}{\Gamma(1.4)} x^{0.4} + \frac{\Gamma(4)}{\Gamma(2.4)} x^{1.4}$. It is easy to verify that the exact solution is $u(x) = x^2 + x^3$.

Using the present method, taking $x_i = \frac{i}{n}$, $i = 1, 2, \ldots, n$, $n = 20$, the numerical results are as shown in Fig. 2.

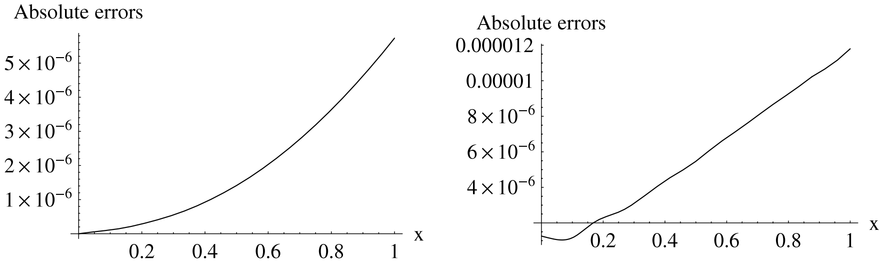

Fig. 1. Absolute errors $|u_{10}(x) - u(x)|$, $|u'_{10}(x) - u'(x)|$ for Example 4.1.

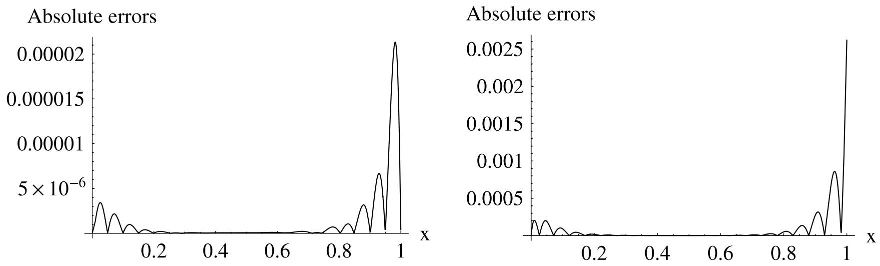

Fig. 2. Absolute errors $|u_{20}(x) - u(x)|$, $|u'_{20}(x) - u'(x)|$ for Example 4.2.

5. Conclusion

In this letter, we introduce an algorithm for solving nonlocal fractional boundary value problems. The major advantage of the proposed method resides in its simplicity in dealing with essential boundary conditions. Also, the approximate solution obtained by the present method and its derivative are both uniformly convergent. The results from numerical examples show that the present method is an accurate and reliable analytical technique for treating nonlocal fractional boundary value problems.

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