Construction and resolutions of certain projectively normal curves

M.V. Catalisano\textsuperscript{a,*,} A. Gimigliano\textsuperscript{b}

\textsuperscript{a} Istituto Matematico di Ingegneria, Univ. di Genova, Piazzale Kennedy 1, Pad. D. 16129 Genova, Italy
\textsuperscript{b} Dip. di Matematica, Univ. di Bologna, Piazza di Porta S. Donato 5, 40127 Bologna, Italy

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Abstract

In this paper the (smooth irreducible) curves which can be constructed on cones over projectively normal curves \( C \subset \mathbb{P}^r \) are studied. It is shown that all smooth curves on such cones are projectively normal and a resolution for their homogeneous ideal is given, depending on the resolution of the ideal of \( C \). © 1999 Elsevier Science B.V. All rights reserved.

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0. Introduction

The aim of this paper is to give a construction for certain projectively normal (p.n.) curves in \( \mathbb{P}^n \), \( n \geq 3 \), which will also furnish a resolution of the homogeneous ideal of the curve.

The idea is to start from a projectively normal curve \( C \subset \mathbb{P}^r \), \( r \geq 2 \) and to construct, on a cone \( A \subset \mathbb{P}^{r+1} \) over \( C \), a (smooth, irreducible) curve \( C_m \) which passes through the vertex of \( A \) and meets its lines \( m \) times, not counting the vertex (hence \( C_m \) is an \( m:1 \) covering of \( C \)).

This kind of construction is due to Jaffe (see [5]) and it is a generalization of the classical "Cayley monoidal construction" (e.g., see [1, 3, 4]).

The aim of this work is, once a bound for \( m \) in order for \( C_m \) to exist is given, to prove that \( C_m \) is p.n. (see Proposition 1) and eventually to get a resolution (which will...
not always be minimal) of the ideal sheaf of \( C_m \) from the resolution of the ideal sheaf \( \mathcal{I} \) of \( C \) (see Proposition 2).

In the last section examples of this kind of construction are given.

1. Preliminaries

Let \( C \) be a smooth, irreducible, non-degenerate projectively normal curve in \( \mathbb{P}^r = \mathbb{P}^r_k \), where \( k \) is an algebraically closed field with \( \text{char} \, k = 0 \). Let \( \deg C = d \) and \( g(C) = g \).

It is quite easy to compute the degree and genus of the curve \( C_m \) described above (when it exists). On the blow-up \( X \) of the cone \( A \) at its vertex, we have that (by abuse of notation we will denote with \( C \) and \( C_m \) also their strict transforms on \( X \)):

\[
C \equiv C_0 + dF, \quad C_m \equiv mC_0 + (md + 1)F,
\]

where \( C_0 \) is the exceptional divisor and \( F \) a fiber on \( X \), while \( \equiv \) denotes numerical equivalence.

Thus, we have

\[
\deg C_m = C_m \cdot C = mC_0^2 + (md + 1) + md = md + 1,
\]

while \( 2g(C_m) - 2 = C_0^2 + C_m \cdot K_X = dm^2 + 2m + C_m \cdot (-2C_0 + (2g - 2 - d)F) = dm^2 + 2m - 2 + m(2g - 2 - d) = 2mg + 2d \left( \begin{array}{c} m \\ 2 \end{array} \right) - 2, \)

and so

\[
g(C_m) = mg + d \left( \begin{array}{c} m \\ 2 \end{array} \right).
\]

Let us recall the necessary "ingredients" which allow the Cayley-Jaffe construction of the curve \( C_m \) to work in our case (see [5, Section 2]):

(\*) We need a triple \((P, H, E)\), where \( P \) is a point on \( C \), \( H \) a reduced hyperplane section of \( C \) with \( P \in H \); \( E \in |mH + P| \) on \( C \) is made of distinct points and does not meet \( H \).

The curve \( C_m \) will appear as the section cut on \( A \) by a hypersurface \( G \) of degree \( m + 1 \) such that (as schemes): \( A \cap G = C_m \cup L_1 \cup \cdots \cup L_{d-1} \), where the \( L_i \)'s are the lines on \( A \) over the points of \( H - P \).

Now we want to determine a bound on \( m \) which guarantees the existence of a smooth curve \( C_m \). Since \( mH \) is very ample on \( C \), the only problem in (\*) could be that \( P \) is fixed for \( |mH + P| \). Observe that we can always find \( E, H, P \) as required when \( mH \) is non-special; in fact in this case the addition of a generic point \( P \) to \( mH \) will give again a non-special divisor, so (by R.R. Theorem), \( h^0(C, \mathcal{O}_C(mH + P)) = h^0(C, \mathcal{O}_C(mH)) + 1 \), and \( P \) is not fixed for \( |mH + P| \).

From the exact sequence

\[
0 \to \mathcal{I}(m) \to \mathcal{O}_{\mathbb{P}^r}(m) \to \mathcal{O}_C(m) \to 0
\]

we get that \( h^1(C, \mathcal{O}_C(mH)) = h^2(\mathbb{P}^r, \mathcal{I}(m)) \).
If a resolution of $\mathcal{I}$ is

$$0 \rightarrow \mathcal{F}_{r-1} \rightarrow \mathcal{F}_{r-2} \rightarrow \cdots \rightarrow \mathcal{F}_1 \rightarrow \mathcal{I} \rightarrow 0,$$

where $\mathcal{F}_i = \bigoplus_{j=1}^{k_i} \mathcal{O}_{\mathbb{P}^r}(-x_{i,j})$, $i = 1, \ldots, r-1$, then (twisting by $\mathcal{O}_{\mathbb{P}^r}(m)$ and considering the short exact sequences into which the resolution above decomposes) it is easy to see that we have $h^2(\mathbb{P}^r, \mathcal{I}(m)) = 0$ whenever $h^i([\mathbb{P}^r, \mathcal{F}_{r-1}(m))] = 0$, which we have for $m \geq \max\{x_{r-1,j}\} - r$.

One could also consider the more direct fact that $h^i(C, \mathcal{O}_C(mH)) = 0$ for $md \geq 2g - 1$; we prefer to give a bound on $m$ with respect to the $x_{i,j}$'s since we are concerned with studying the graded Betti numbers (the two bounds differ only in few cases).

Better bounds for $m$, when a specific curve $C$ is given, can be found with ad hoc considerations (e.g., see last section).

It can also be noticed that we can view $\mathbb{P}^r \subset \mathbb{P}^{r+1}$ as the hyperplane $\{w_{r+1} = 0\}$ and $H = \{l = 0\}$, so that $C_m$ is given on $A$ by a hyperurface of equation $lw_{r+1}^m - g = 0$ (which cuts $C_m \cup L_1 \cup \cdots \cup L_{d-1}$, the $L_i$'s being the lines over the points of $H - P$ (e.g. see [5]). In this case the ideal sheaf $\mathcal{I}_m$ is (e.g. see [4])

$$(\mathcal{I}_m, lw_{r+1}^m - g); (g).$$

This implies that when we apply this construction in the case $r = 2$, the curve $C_m$ turns out to be directly linked to a complete intersection ($A$ in a surface in $\mathbb{P}^3$), hence it is p.n. and (by the Hilbert–Burch Theorem) determinantal: its ideal sheaf $\mathcal{I}_m$ is generated by global sections which come from the maximal minors of a $(2 \times 3)$-matrix of forms, whose degrees must be

$$\begin{pmatrix} d-1 & d-1 & m \\ 1 & 1 & m-d+2 \end{pmatrix}.$$ 

A resolution of $\mathcal{I}_m$ is given by the Eagon–Northcott complex:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-m-d) \oplus \mathcal{O}_{\mathbb{P}^3}(-m-2) \rightarrow \mathcal{O}_{\mathbb{P}^3}(-d) \oplus \left( \bigoplus_{j=1}^{2} \mathcal{O}_{\mathbb{P}^3}(-m-1) \right) \rightarrow \mathcal{I}_m \rightarrow 0.$$ 

Notice that the resolution above (hence the ideal generation) could be non-minimal (when $C_m$ is a complete intersection, e.g. for $d = 3$, $m = 1$).

The considerations in the next section can be viewed as a generalization of what we have just seen for $r = 2$ to the case $r \geq 3$, where we no longer have liaison to get that $C_m$ is p.n. nor the Hilbert–Burch theorem to find a resolution.

Let us notice that the ideal of such $C_m$ do not arise, in general, by taking the Hilbert–Burch matrix of the base curve $C$ in $\mathbb{P}^3$, viewing it over the coordinate ring of $\mathbb{P}^4$, and adding a suitable column (e.g., see Section 3).
2. Projective normality and a resolution of $C_m$

With notation as in Section 1, let us consider curves $C_m$, $m > 0$, on $X$, with $C_m \in |mC_0 + (md + 1)F| = |mC + F|$. Let $V = H^0(X, \mathcal{O}_X(C)),$ and $S = S(V) = k[x_0, x_1, x_2, \ldots, x_{r+1}]$, so we view $\mathbb{P}^{r+1}$ as $\text{Proj}(S)$.

Let $B = \bigoplus_q H^0(X, \mathcal{O}_X(qC))$; then notice that a resolution of $B$ as an $S$-module is known (it corresponds to a resolution of $\bigoplus_q H^0(X, \mathcal{O}_X(qC))$), and it has the same graded Betti numbers as those of $C \subseteq \mathbb{P}^r$. Hence, we have

$$0 \to F_{r-1} \to F_{r-2} \to \cdots \to F_1 \to S \to B \to 0,$$

where $F_i = \bigoplus_{j=1}^{i} S(-x_i, i), i = 1, \ldots, r-1$.

We want to show, when $C_m$ is irreducible and smooth, that its image in $\mathbb{P}^{r+1}$ (which is isomorphic to $C_m$, since $C_m \cdot C_0 = 1$, and which we will denote by $C_m$ again) is a projectively normal curve. Moreover, we look for a resolution of $A_m = \bigoplus_q H^0(X, \mathcal{O}_{C_m}(qC))$ as an $S$-module (if $C_m$ is p.n. $A_m$ is its ring of coordinates).

Let us make several remarks.

Remark 1. Let $A_F = \bigoplus_q H^0(X, \mathcal{O}_F(qC)) = \bigoplus_q H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(q))$; a resolution of $A_F$ as an $S$-module is well known (by viewing $F$ as a line in $\mathbb{P}^{r+1}$):

$$0 \to L_r \to L_{r-1} \to \cdots \to L_1 \to \mathcal{O} \to A_F \to 0,$$

where $L_i = \bigoplus_{j=0}^{i} S(-i, j), i = 1, \ldots, r$.

Remark 2. From the exact sequence

$$0 \to \mathcal{O}_X(qC - F) \to \mathcal{O}_X(qC) \to \mathcal{O}_F(qC) \to 0$$

since, $\forall q \geq -1$, we have $H^1(F, \mathcal{O}_F(qC)) = H^1(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(q)) = 0$, and since $\mathcal{O}_X(qC) \to \mathcal{O}_F(qC)$ is surjective at the $H^0$-level, it follows that $H^1(X, \mathcal{O}_X(qC - F)) = H^1(X, \mathcal{O}_X(qC)), \forall q \geq -1$.

Remark 3. Let $q \geq 0$; from the exact sequence

$$0 \to \mathcal{O}_X((q - 1)C) \to \mathcal{O}_X(qC) \to \mathcal{O}_C(qC) \to 0,$$

since $C \subseteq \mathbb{P}^r$ is p.n., and $h^2(X, \mathcal{O}_X((q - 1)C)) = h^0(X, \mathcal{O}_X((1 - q)C + K_X)) = 0$, we get exactness at $H^1$-level, i.e.

$$h^1(X, \mathcal{O}_X(qC)) = h^1(X, \mathcal{O}_X((q - 1)C)) + h^1(C, \mathcal{O}_C(qC)).$$

Remark 4. $\forall q \leq -1$, $H^1(X, \mathcal{O}_X(qC - F)) = 0$.

This comes from the Kodaira Vanishing Theorem (note that $-qC + F$ is ample).
Remark 5. A generic curve \( r \in |mC| \) on \( X \) is smooth and irreducible, and its image in \( \mathbb{P}^{r+1} \) (which we still will denote by \( r \)) is cut out on \( X \) by a hypersurface of degree \( m \), so it is projectively normal.

Proposition 1. Let \( C, A \) be as in Section 1. Then all the smooth, irreducible curves on \( A \) are projectively normal (in particular, \( \forall m \geq \max \{a_{i,j}\} - r \) there is a smooth irreducible curve \( C_m \) on \( A \) which is p.n).

Proof. It is easy to check that the only smooth irreducible curves on \( A \) are those of type \( C_m, m \geq 0 \) or \( \Gamma_m, m \geq 1 \), which come from divisors of type \( mC_0 + (md + 1)F \), or \( mC_0 + mdF \) on \( X \), respectively (those have intersection 0 or 1 with \( C_0 \); all the others will be singular at the vertex).

We already noticed that the curves \( \Gamma_m \) are p.n. (Remark 5), so let us consider the curves \( C_m \); it is enough to show that one curve in \( |C_m| \) on \( A \) is p.n. to have that all the smooth curves in the class are p.n. In fact, \( C_m \) is p.n. if and only if
\[
\text{h}^1(\mathbb{P}^{r+1}, \mathcal{I}_{C_m}(q)) = 0, \forall q \in \mathbb{Z};
\]
and
\[
\text{h}^1(\mathbb{P}^{r+1}, \mathcal{I}_{C_m}(q)) = \text{h}^1(\mathbb{P}^{r+1}, \mathcal{I}_{C_m}(q)),
\]
where \( \mathcal{I}_{C_m} = \mathcal{O}_A(-C_m) \) is the ideal sheaf of \( C_m \) in \( \mathcal{O}_A \).

Since \( A \) is p.C.M., we have that
\[
\text{h}^1(\mathbb{P}^{r+1}, \mathcal{I}_A(q)) = \text{h}^2(\mathbb{P}^{r+1}, \mathcal{O}_A(q)) = 0, \forall q \in \mathbb{Z},
\]
and we get that
\[
\text{h}^1(\mathbb{P}^{r+1}, \mathcal{I}_{C_m}(q)) = \text{h}^1(\mathbb{P}^{r+1}, \mathcal{I}_{C_m}(q)),
\]
where the second depends only on the class of \( C_m \) as a divisor on \( A \).

Now let us show that \( C_m \) is p.n.; in the linear system \( |C_m| = |\Gamma_m + F| \) we can consider (\( \forall m \geq 1 \)) a reduced and reducible curve \( C_m = \Gamma_m \cup F \) given by a smooth, irreducible \( \Gamma_m \) and a line \( F \), with \( \Gamma_m \cap F \) given by \( m \) distinct points.

Since \( \forall q \geq m \) the sequence
\[
0 \to \mathcal{O}_{\Gamma_m \cup F}(qC) \to \mathcal{O}_{\Gamma_m}(qC) \oplus \mathcal{O}_F(qC) \to \mathcal{O}_{\Gamma_m \cap F}(qC) \to 0,
\]
is exact both at the \( H^0 \) and at the \( H^1 \) level, then, for \( q \geq m \), we get
\[
\text{h}^0(X, \mathcal{O}_{C_m}(qC)) = \text{h}^0(\Gamma_m, \mathcal{O}_{\Gamma_m}(qC)) + q - m + 1,
\]
\[
\text{h}^1(X, \mathcal{O}_{C_m}(qC)) = \text{h}^1(\Gamma_m, \mathcal{O}_{\Gamma_m}(qC)).
\]

Now consider the exact sequence
\[
0 \to \mathcal{O}_X((q - m)C - F) \to \mathcal{O}_X(qC) \to \mathcal{O}_{C_m}(qC) \to 0.
\]
We want to show that \( (3) \) is \( H^0 \)-exact, \( \forall q \in \mathbb{Z} \). If \( q - m < 0 \), this is true by Remark 4; when \( q \geq m \), consider the exact sequence

\[
0 \to \mathcal{O}_X((q - m)C) \to \mathcal{O}_X(qC) \to \mathcal{O}_{T_m}(qC) \to 0.
\]  

We have that \( (4) \) is exact at the \( H^0 \) and \( H^1 \) level, so, from Remark 2, and by (1) and (2) we have that also \( (3) \) is exact at the \( H^0 \) and \( H^1 \) level.

This implies that for the embedded curve \( C_m \subseteq \mathbb{P}^{r+1} \) the map

\[
H^0(\mathbb{P}^{r+1}, \mathcal{O}_{\mathbb{P}^{r+1}}) \to H^0(\mathbb{P}^{r+1}, \mathcal{O}_{C_m})(q)
\]

is surjective, hence \( H^1(\mathbb{P}^{r+1}, \mathcal{I}_{C_m}(q)) = 0, \forall q \in \mathbb{Z} \).

Then, when we choose an \( m \) such that one (hence the generic) curve \( C_m \subseteq mC + F \) is irreducible and smooth, we will have again that \( H^1(\mathbb{P}^{r+1}, \mathcal{I}_{C_m}(q)) = 0, \forall q \), since this property is an open one in the Hilbert scheme of curves of given genus and degree of \( \mathbb{P}^{r+1} \), and the reduced curve we have considered before is a deformation of the generic curve in \( H^0(\mathbb{P}^{r+1}, \mathcal{O}_A(C_m)) \). Thus, \( C_m \subseteq \mathbb{P}^{r+1} \) is projectively C.M., hence projectively normal. \( \square \)

Now, let \( C_m \subseteq A \subseteq \mathbb{P}^{r+1} \) be a (smooth, irreducible, p.n.) curve as before, and consider the exact sequence

\[
0 \to \mathcal{O}_X((q - m)C - F) \to \mathcal{O}_X((q - m)C) \to \mathcal{O}_F((q - m)C) \to 0.
\]

Since resolutions of \( A_F \) (see Remark 1) and of \( B = \bigoplus_q H^0(X, \mathcal{O}_X(qC)) \) are known, and the above sequence is exact at the \( H^0 \) level (see Remark 2), in order to find a resolution of \( C_m = \bigoplus_q H^0(X, \mathcal{O}_X((q - m)C - F)) \) as \( S \)-module, we can consider the short exact sequences which fit into the following diagram:

\[
\begin{array}{c}
0 \\
\uparrow \\
0 \to C_m \to B(-m) \to A_F(-m) \to 0 \\
\uparrow \\
0 \to S(-m) \to S(-m) \to 0 \\
\uparrow \\
N \to K \\
\uparrow \\
0 \quad 0 \\
\end{array}
\]

The resolution for \( N \) (resp. \( K \)) comes from the rest of the resolution for \( B(-m) \) (resp. \( A_F(-m) \)). We thus have (from the Snake Lemma) a short exact sequence

\[
0 \to N \to K \to C_m \to 0
\]

and we know the resolution of \( N \) and of \( K \). The mapping cone construction then gives
a resolution for $C_m$:

\[ 0 \rightarrow C_m \rightarrow B(-m) \rightarrow A_m(-m) \rightarrow 0 \]

(We thank the referee for suggesting this proof, which is more direct than the one we had originally given.)

Then consider the exact sequence

\[ 0 \rightarrow \mathcal{O}_X((q - m)C - F) \rightarrow \mathcal{O}_X(qC) \rightarrow \mathcal{O}_{C_m}(qC) \rightarrow 0 \]

(which is exact at the $H^0$-level since $C_m$ is p.n.) and again use a mapping cone in order to compute a resolution (may be not minimal) of $A_m$:

\[ 0 \rightarrow C_m \rightarrow B \rightarrow A_m \rightarrow 0 \]
So one gets (via sheafifying the resolution of $A_m$):

**Proposition 2.** Let $C_m$ be as above, then a resolution (maybe non-minimal) of the ideal sheaf $I_{C_m} \subseteq \mathcal{O}_{\mathbb{P}^r+1}$ is

$$
0 \to \mathcal{O}_{\mathbb{P}^r+1}(-m-r) \oplus \bigoplus_{j=1}^{\beta_{r-1}} \mathcal{O}_{\mathbb{P}^r+1}(-m - \alpha_{r-1,j}) \to \mathcal{O}_{\mathbb{P}^r+1}(-m-r+1) \oplus \bigoplus_{j=1}^{\beta_{r-1}} \mathcal{O}_{\mathbb{P}^r+1}(-m - \alpha_{r-1,j}) \oplus \bigoplus_{j=1}^{\beta_{r-2}} \mathcal{O}_{\mathbb{P}^r+1}(-m - \alpha_{r-2,j}) \to \ldots
$$

$$
\to \left( \bigoplus_{j=1}^{\beta_2} \mathcal{O}_{\mathbb{P}^r+1}(-m-2) \right) \oplus \left( \bigoplus_{j=1}^{\beta_2} \mathcal{O}_{\mathbb{P}^r+1}(-\alpha_{2,j}) \right) \oplus \left( \bigoplus_{i=1}^{\beta_1} \mathcal{O}_{\mathbb{P}^r+1}(-m - \alpha_{1,i}) \right)
$$

$$
\to \left( \bigoplus_{j=1}^{\beta_1} \mathcal{O}_{\mathbb{P}^r+1}(-m-1) \right) \oplus \left( \bigoplus_{i=1}^{\beta_1} \mathcal{O}_{\mathbb{P}^r+1}(-\alpha_{1,i}) \right) \to \mathcal{I}_{C_m} \to 0.
$$

3. **Examples**

3.1. **Curves in $\mathbb{P}^4$**

May be the most interesting case of our construction is for $r = 3$, since the resolutions of p.n. curves in $\mathbb{P}^3$ is well known, while they are not so easy to compute for p.n. curves in $\mathbb{P}^4$.

By the Hilbert–Burch Theorem, every projectively normal curve $C \subseteq \mathbb{P}^3$ is determinantal, and so its ideal sheaf $\mathcal{I}$ is generated by global sections which come from the maximal minors of a $\rho \times (\rho+1)$-matrix of forms, and a resolution of $\mathcal{I}$ is given by the Eagon–Northcott complex

$$
0 \to \bigoplus_{j=1}^{\rho} \mathcal{O}_{\mathbb{P}^3}(-n_j) \to \bigoplus_{i=1}^{\rho+1} \mathcal{O}_{\mathbb{P}^3}(-d_i) \to \mathcal{I} \to 0.
$$

When $C_m \subseteq \mathbb{P}^4$ exists, we get a resolution

$$
0 \to \mathcal{O}_{\mathbb{P}^4}(-m-3) \oplus \left( \bigoplus_{j=1}^{\rho} \mathcal{O}_{\mathbb{P}^4}(-m-n_j) \right).
$$
\[ \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^4}(-m - 2) \bigoplus_{j=1}^{p+1} \mathcal{O}_{\mathbb{P}^4}(-m - d_i) \bigoplus_{j=1}^{\rho} \mathcal{O}_{\mathbb{P}^4}(-n_j) \]

\[ \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^4}(-m - 1) \bigoplus_{j=1}^{p+1} \mathcal{O}_{\mathbb{P}^4}(-d_i) \rightarrow \mathcal{O}_{\mathbb{P}^4} \rightarrow \mathcal{O}_{C_m} \rightarrow 0. \]

In general, \( C_m \) can be constructed when \( m \geq \max_j \{n_j\} - 3 \), but there are cases when the construction can be carried on also for smaller values of \( m \) (for which \( mH \) is special); for example, let \( C \) be a \( C^2_3 \); in this case \( |H| \) has degree \( 7 = 2g - 3 \), and index of speciality = 1, hence its hyperplane divisors can be viewed as obtained from a canonical one minus one point, so if one adds that point, say \( P \), to a divisor \( H \), the system \( |H + P| = |K_C| \) has again index of speciality = 1, i.e. \( P \) is not fixed for it and condition (*) holds, hence we can construct a curve \( C_1 = C^2_3 \subset \mathbb{P}^4 \) on a cone over \( C \).

Notice that \( C_1 \) is a canonical curve and is non-trigonal (otherwise it would have \( \infty^1 \) trisecants and \( C \) should have a node, being its projection from one point), so it is generated by quadrics (it is actually the complete intersection of three quadrics) and it is immediate to check that the resolution shown above is not minimal for it.

Hence there are cases when the resolution we gave is actually non-minimal, anyway we notice that it will surely be minimal (for trivial reasons) when \( m \geq \max_j \{n_j\} \), and \( d_i \neq 3, n_j, \forall i,j \).

3.2. Curves over complete intersections in \( \mathbb{P}^3 \)

Let \( C \subset \mathbb{P}^3 \) be the complete intersection of two surfaces of degree \( a, b \); then the resolution of \( C \) is

\[ 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-a - b) \rightarrow \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(-b) \rightarrow \mathcal{I} \rightarrow 0, \]

hence \( mH \) is non-special for \( m \geq a + b - 3 \) and for these values of \( m \) we can obtain a curve \( C_m \) on \( A \).

A resolution of \( C_m \) is

\[ 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-a - b - m) \oplus \mathcal{O}_{\mathbb{P}^3}(-m - 3) \]

\[ \rightarrow \mathcal{O}_{\mathbb{P}^3}(-b - m) \oplus \mathcal{O}_{\mathbb{P}^3}(-a - m) \oplus \mathcal{O}_{\mathbb{P}^3}(-a - b) \oplus \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^3}(-m - 2) \]

\[ \rightarrow \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(-b) \bigoplus \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^3}(-m - 1) \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{O}_{C_m} \rightarrow 0. \]

It is easy to check that this resolution has to be minimal when \( a, b \geq 4 \) and \( m \geq a + b \), but those bounds are far from being sharp, as the following examples will show.
Let us consider the case \( a = b = 2 \): when \( m = 1 \) (hence \( C \) and \( C_1 \) are elliptic normal curves), the resolution above is not minimal, since the minimal one is

\[
0 \to \mathcal{O}_{\mathbb{P}^4}(-5) \to \bigoplus_{i=1}^{5} \mathcal{O}_{\mathbb{P}^4}(-3) \to \bigoplus_{i=1}^{5} \mathcal{O}_{\mathbb{P}^4}(-2) \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{C_1} \to 0.
\]

For each \( m \geq 2 \), instead, the resolution is minimal; this is obvious for \( m = 2 \) and \( m \geq 4 \), while for \( m = 3 \) we have that \( C_3 = C_{13}^{15} \subseteq \mathbb{P}^4 \) is a curve of degree 13 and genus 15 for which we get a resolution:

\[
0 \to \mathcal{O}_{\mathbb{P}^4}(-7) \oplus \mathcal{O}_{\mathbb{P}^4}(-6) \to \left( \bigoplus_{i=1}^{5} \mathcal{O}_{\mathbb{P}^4}(-5) \right) \oplus \mathcal{O}_{\mathbb{P}^4}(-4) \to \left( \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^4}(-4) \right) \oplus \left( \bigoplus_{i=1}^{2} \mathcal{O}_{\mathbb{P}^4}(-2) \right) \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{C_3} \to 0.
\]

Is this resolution minimal? One could think that maybe it is not; that there is a redundant addendum \( \mathcal{O}_{\mathbb{P}^4}(-4) \) (i.e. a redundant syzygy and a redundant generator in degree 4). So the question is: can two forms of degree 4 (and two of degree 2) be enough in order to generate the homogeneous ideal of \( C_3 \)? The answer, which implies that the above resolution is minimal, is "No". One can check this in the following way: we can work on the ideal of a generic hyperplane section \( Z \) of \( C_3 \) (which will have the same graded Betti numbers since \( C_3 \) is p.n); \( Z \) is reduced and consists of 13 points in \( \mathbb{P}^3 \) which are contained in an elliptic curve \( E_4 \), hence on two quadric surfaces.

Consider a smooth quadric \( Q \) containing \( Z \) and a plane model of it: then the hyperplane sections of \( Q \) are given on the plane by the conics through two points \( P_1, P_2 \), while \( Z \) corresponds to 13 other points \( P_3, \ldots, P_{15} \).

The other quadric containing \( Z \) gives on the plane a quartic curve \( D \) passing with multiplicity 2 through \( P_1, P_2 \) and simply through \( P_3, \ldots, P_{15} \); the quartic surfaces through \( Z \) correspond to the linear system of curves of degree eight containing \( P_3, \ldots, P_{15} \) and having \( P_1, P_2 \) as (at least) quadruple points. A simple computation shows that this linear system contains at least \( 45 - 33 = 12 \) independent curves; among them at most nine are of type \( D \cup D' \), where \( D' \) is another quartic containing \( P_1, P_2 \) as double points, hence there must be at least three independent curves not composed with \( D \); those correspond to the three quartic surfaces through \( Z \) in \( \mathbb{P}^3 \) which are needed as generators.

Hence the resolution is minimal.

### 3.3. Bielliptic curves

A bielliptic curve is a (smooth, irreducible, non-hyperelliptic) curve \( \mathcal{C} \) endowed with a \( 2:1 \) morphism \( \phi: \mathcal{C} \to E \), where \( E \) is a (smooth, irreducible) elliptic curve.

Let us consider an elliptic normal curve \( E_{r+1} \subseteq \mathbb{P}^r \); for \( m = 2 \), we get that we can obtain two bielliptic curves \( \Gamma_2 \) and \( C_2 \) in \( \mathbb{P}^{r+1} \) on a cone over \( E_{r+1} \). \( \Gamma_2 \) is
actually a canonical bielliptic curve $C_{r+2}^{2r+2}$, and its resolution is known [6] (see also [2]).

Also $C_2$ is bielliptic and p.n. It has degree $2r+3$ and genus $r+3$ and it can be viewed as the projection of a canonical bielliptic curve $C_{r+3}^{2r+4} \subseteq \mathbb{P}^{r+2}$ from one of its points.

Since the resolution of the ideal of $E_{r+1}$ is known (it has the same Betti numbers as $r+1$ generic points in $\mathbb{P}^r$), one can compute a resolution for these curves using Proposition 2.

For $r = 3$, hence $C_2 = C_6^{10} \subseteq \mathbb{P}^4$, we get the minimal resolution

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^4}(-6) \oplus \mathcal{O}_{\mathbb{P}^4}(-5) \rightarrow 5 \bigoplus \mathcal{O}_{\mathbb{P}^4}(-4)
\rightarrow \left( \bigoplus \mathcal{O}_{\mathbb{P}^4}(-3) \right) \oplus \left( \bigoplus \mathcal{O}_{\mathbb{P}^4}(-2) \right) \rightarrow \mathcal{I}_{C_2} \rightarrow 0.$$

For $r = 4$, so $C_2 = C_7^{11} \subseteq \mathbb{P}^4$, we get

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^4}(-7) \oplus \mathcal{O}_{\mathbb{P}^4}(-6) \rightarrow 10 \bigoplus \mathcal{O}_{\mathbb{P}^4}(-5)
\rightarrow \left( \bigoplus \mathcal{O}_{\mathbb{P}^4}(-4) \right) \oplus \left( \bigoplus \mathcal{O}_{\mathbb{P}^4}(-3) \right)
\rightarrow \left( \bigoplus \mathcal{O}_{\mathbb{P}^4}(-3) \right) \oplus \left( \bigoplus \mathcal{O}_{\mathbb{P}^4}(-2) \right) \rightarrow \mathcal{I}_{C_2} \rightarrow 0.$$

In this case the resolution could be non-minimal; the question is whether we need the four cubics among the generators.

The five quadrics intersect in the elliptic normal cone $A$ on $E_5$, hence a cubic surface will cut on it a curve $D = C_2 \cup L_1 \cup \ldots \cup L_4$, where the $L_i$'s and the tangent line to $C_2$ at $V$, the vertex of the cone, cuts on $E_5$ a hyperplane divisor (this is the Cayley–Jaffe construction; see [5]).

Hence, the curve $D$ has, at $V$, a tangent space of dimension 4 and the intersection with any other cubic surface through $C_2$ can lower this dimension only by one, so we need at least another three cubic surfaces to get the tangent line of $C_2$.

Thus, we need four cubic forms to generate the ideal of $C_2$, and the resolution is minimal.

References


