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Factorization and Imbedding for General Linear Boundary Value Problems*

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A new direction in the theory of general linear boundary value problems is explored. The starting point is an explicit Volterra factorization of the Green's matrix (and related kernels) associated with the problem. This result leads to (1) imbedding of the boundary value problems, (2) initial value algorithms for their solution, and (3) comparison theorems relating two different boundary value problems with a common boundary condition. Extensions and connections with earlier work in this area are presented.

1. INTRODUCTION

In this paper we explore a new direction in the theory of general linear boundary value problems. Our method of approach is through the formalism of factorization and imbedding. The starting point is an explicit Volterra factorization of the Green's matrix (and related kernels) associated with the problem. This leads to a natural imbedding of the boundary value problem into a family of such problems. The imbedding in turn generates initial value algorithms for the solution to the original boundary value problem. In addition, the factorization and imbedding formalism yields comparison theorems relating the solutions of two different boundary value problems

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with a common boundary condition. The results presented in this paper extend our earlier work in this area found in [2,4,7].

We are concerned with general linear boundary value problems of the following type:

$$y' = (A + C)y + p, \; By = \xi, \tag{1.1}$$

where y and p are regulated mappings of an interval $[a, b]$ into a Banach space E; A and C are regulated mappings of $[a, b]$ into the space of bounded linear operators on E; B is a bounded linear operator from the space of regulated maps into E and ξ is an element of E.

To give the spirit of our theory, we describe briefly, but in some detail, the nature of one of our basic results. The factorization theory leads us naturally to an imbedding of the problem (1.1) into a one-parameter family of such problems. These can be described as follows: Let $\Phi_A(t, s)$ be the fundamental matrix defined by

$$\frac{\partial \Phi_A}{\partial t}(t, s) = A(t)\, \Phi_A(t, s), \qquad \Phi(s, s) = I,$$

where I is the identity operator on E. For $\tau \in (a, b]$, define the one-parameter family of boundary operators $B_A(\tau)$ by

$$B_A(\tau)y = B\{(1 - h(\cdot - \tau))y(\cdot)\} + B\{h(\cdot - \tau)\Phi_A(\cdot, \tau)\}y(\tau),$$

where h is the Heaviside step function

$$\begin{aligned} h(t) &= 1, \; t \geqslant 0 \\ &= 0, \; t < 0. \end{aligned}$$

The imbedded family of problems are now given by

$$y' = (A + C)y + p, \; B_A(\tau)y = \xi. \tag{1.2a}$$

It is shown that on the interval (a, τ) the solution of (1.2a) coincides with the solution of the problem,

$$y' = (A + C_\tau)y + p_\tau\,, \qquad By = \xi, \tag{1.2b}$$

where $C_\tau(t) = C(t)$, $p_\tau(t) = p(t)$ for $a \leqslant t \leqslant \tau$, and $C_\tau(t) = p_\tau(t) = 0$ for $\tau < t \leqslant b$. When $\tau = a$, (0.2b) is just the problem:

$$y' = Ay, \qquad By = \xi, \tag{1.3}$$

which is assumed solvable. On the other hand, for $\tau = b$, (1.2a) and (1.2b) are the same as (1.1). The imbedding (1.2b), thus, allows us to pass "smoothly" from a solvable problem (1.3) to the desired problem (1.1). A result which is characteristic of our theory can now be stated. Equations (1.2a,b) have a unique solution for every $\tau \in [a, b]$ if and only if the following Riccati initial value problem has a solution on $[a, b]$:

$$R' = (A + C)R - RDCR, \; R(a) = \Psi, \qquad \text{for some} \quad \Psi \in L(E), \tag{1.4}$$

where $\Psi\xi$ is the value at a of the solution to (1.3) and where

$$D(\tau) = B\{h(\cdot - \tau)\, \Phi_A(\cdot, \tau)\}.$$

In this case, the solution $y(t, \tau)$ to (1.2a) is given by

$$y(t, \tau) = R(t)\, U(t, \tau)\, \xi + R(t)\, w(t, \tau) + v(t),$$

where U, w, v satisfy the initial value equations

$$\frac{\partial U}{\partial \tau} = -UD(\tau)\, C(\tau)\, R(\tau), \qquad U(t, t) = I, \tag{1.5a}$$

$$\frac{\partial w}{\partial \tau} = -U(t, \tau)\, D(\tau)(C(\tau)\, v(\tau) + p(\tau)), \qquad w(t, t) = 0, \tag{1.5b}$$

$$\frac{\partial v}{\partial \tau} = (A + C)v + p - RD(Cv + p), \qquad v(a) = 0. \tag{1.5c}$$

The algorithm implied by this result is as follows: To determine the solution of (1.1), $y(t_0, b)$, for a fixed $t = t_0$ and given b, integrate (1.4) and (1.5c) up to the point t_0; then adjoin (1.5ab) and integrate the entire system (1.4), (1.5) to the end point b.

The one-pass nature of this algorithm makes it suitable for "real-time" problems as found in the theory of optimal filtering and control [9].

In the special case

$$A = 0, \qquad y = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad By = \begin{pmatrix} u(a) \\ v(b) \end{pmatrix} \tag{1.6}$$

Eqs. (1.4) and (1.5) give the formalism of "invariant imbedding" [7]. Our results generalize this setting in various directions. Apart from the increased generality of the boundary conditions in (1.6), the flexibility of having non-zero A gives promise of circumventing the "critical-length" problem (cf. [10]). This critical-length problem is analogous to a similar situation arising in the solution of linear algebraic equations of the form

$$(A + C)X = f, \tag{1.7}$$

where A is nonsingular. The method of factorization may fail for (1.7), but be perfectly satisfactory for the equivalent problem

$$(I + A^{-1}C)X = A^{-1}f.$$

An outline of the paper now follows.

In Section 2, we describe a general setting for linear boundary value problems of the type (1.1). We choose as our space of mappings the regulated functions as defined by Dieudonne [1]. This choice was made, first, because of the simplicity of defining definite and indefinite integrals of such functions with values in a Banach space, and, secondly, because for certain boundary operators B the coefficients of the generated initial value problems (e.g. (1.4) and (1.5)) will have discontinuities even when A and C are continuous. In fact, the differential equations appearing in this paper are only a symbolic representation for their integral equation counterparts. No more is demanded of the derivatives.

In Section 3, the Green's matrix $G_{A,B}$ is defined for the problem

$$y' = Ay + p, \quad By = \xi.$$

The standard properties of $G_{A,B}$ are then derived.

Section 4, contains the main factorization theorem.

THEOREM. *Let K be the integral operator on $[a, b]$ with kernel $G_{A,B}(t, s)C(s)$. Then*

$$(I - K) = (I - \Sigma^+)(I - \Sigma^-),$$

where Σ^+, Σ^- are upper and lower Volterra integral operators, if and only if the Riccati equation (1.4) *has a solution on* $[a, b]$.

Expressions for the Volterra factors and their resolvents, are obtained explicitly in terms of the solution of (1.4).

A comparison theorem for linear boundary value problems is given in Section 5. It relates the solvability of (1.1) when $C = 0$ to the case of general C. In addition another equivalent inital value algorithm is given for the solution of (1.1). This algorithm is of the two-pass variety but is of lower dimension than (1.5). At the end of this section a surprising property of Green's matrices is observed.

In Section 6 the connection between factorization and imbedding is made explicit. The Gohberg–Krein [8] theory of factorization shows that the Volterra factorization obtained in Section 4 is equivalent to the unique solvability of the family of problems (1.2b).

We then show that the solvability of (1.2b) is equivalent to the solvability of (1.2a) and derive the corresponding initial value algorithm (1.4) and (1.5).

Section 7 makes contact with work done in [2]. Here we focus our attention on factorization of the boundary value operators themselves. This leads to an alternate interpretation and derivation of the results given in Section 6.

Our final Section 8, considers the special case of the theory, mentioned earlier, in which the formalism of invariant imbedding applies. The fundemental kernel, introduced in [4], is shown to be a component of the Green's matrix $G_{A,B}$. In addition, the Volterra factors related to this fundamental kernel are shown to have the same relationship to the Volterra factors of $G_{A,B}C$.

2. Differential Equations in the Space of Regulated Maps

Following Dieudonne [1], we say a mapping f from an interval $[a, b]$ of reals into a Banach space E is regulated if f has one-sided limits at every point of $[a, b]$. Let $S([a, b], E)$ be the vector space of all such maps. Endowed with the norm $\|f\| = \sup_{t\in[a,b]} \|f(t)\|$, $S([a, b], E)$ is a Banach space. The interval $[a, b]$ will be fixed throughout this paper unless otherwise stated, and we abbreviate $S([a, b], E)$ to $S(E)$. The basic properties of regulated mappings concerning their definite and indefinite integrals, as derived in [1], will be assumed.

If E, F are two Banach spaces let $L(E, F)$ denote, as usual, the Banach space of all bounded linear mappings of E into F. We write $L(E)$ for $L(E, E)$.

Let $p \in S(E)$, $A \in S(L(E))$, then an element $y \in S(E)$ is said to be a solution of the equation

$$y' = Ay + p, \tag{2.1}$$

if

$$y(t) = y(a) + \int_a^t (A(\theta)\, y(\theta) + p(\theta))\, d\theta, \qquad t \in [a, b]. \tag{2.2}$$

We note that this definition implies that a solution to (2.1) is automatically continuous.

Let $S_A(E)$ be the linear space of all elements $y \in S(E)$ which satisfy the equation

$$y' = Ay. \tag{2.3}$$

An element $B \in L(S(E), E)$ is said to be nonsingular for A if $B \mid S_A(E)$, the restriction of B to $S_A(E)$, is a bijection. It is an elementary exercise (cf. [1, 10.5; 2]) to show that $S_A(E)$ is a closed subspace of $S(E)$. Hence, if B is nonsingular for A then F_A, the inverse of $B \mid S_A(E)$, is an element of $L(E, S_A(E))$.

Of special importance will be the point evaluation map $B^0(\tau) \in L(S(E), E)$ defined by $B^0(\tau)x = x(\tau)$, $x \in S(E)$, $\tau \in [a, b]$.

The global existence and uniqueness theorem for solutions of Eq. (2.3) satisfying given initial conditions shows that $B^0(\tau)$ is nonsingular for every $A \in S(L(E))$, see [1, 10.6]. Let $F_A{}^0(\tau)$ denote the inverse of $B^0(\tau)|\ S_A(E)$. Since $B^0(\tau)|\ S_A(E)$ is continuous in τ, so is $F_A{}^0(\tau)$.

An element $B \in L(S(E), E)$ will be called a boundary operator. A typical example of such an operator is given by

$$B = \sum_{i=1}^{\infty} \lambda_i B^0(\tau_i) + \int_a^b d\theta\, \lambda(\theta)\, B^0(\theta), \tag{2.4}$$

where

$$\lambda_i \in L(E), \sum_{i=1}^{\infty} \| \lambda_i \| < \infty, \qquad \lambda \in S(L(E)),$$

and

$$\int_a^b d\theta \, \| \lambda(\theta) \| < \infty.$$

We state and prove here certain elementary results concerning the properties of B^0, $F_A{}^0$, and B which will be needed in the following sections.

PROPOSITION 2.1. (a) *The restriction of B^0 to $S_A(E)$ satisfies*

$$B^0(t) = B^0(\tau) + \int_\tau^t A(\theta)\, B^0(\theta)\, d\theta. \tag{2.5}$$

(b) *Let $F_A{}^0(t) = \{B^0(t)|\ S_A(E)\}^{-1}$, then*

$$F_A{}^0(t) = F_A{}^0(\tau) - \int_\tau^t F_A{}^0(\theta)\, A(\theta)\, d\theta. \tag{2.6}$$

(c) *If boundary operator B is given by the representation* (2.4) *and if $D(s) = B\,\{h(\cdot - s)\, \Phi_A(\cdot, s)\}$, where*

$$\Phi_A(t, s) = B^0(t)\, F_A{}^0(s)$$

then D is a regulated mapping from $[a, b]$ into $L(E)$, i.e., $D \in S(L(E))$.

Proof. (a) If $u \in S_A(E)$, then by definition

$$u(t) = u(\tau) + \int_\tau^t A(\theta)\, u(\theta)\, d\theta, \qquad \tau, t \in [a, b].$$

Thus

$$B^0(t)u = B^0(\tau)u + \left(\int_\tau^t A(\theta)\, Bo(\theta)\, d\theta\right) u,$$

so that (2.5) holds.

(b) This is proved in more generality in Proposition 7.1b.

(c) Let $D^0(s, \tau) = B^0(\tau)\, \{h(\cdot - s)\, \Phi_A(\cdot, s)\}$.

Then $D^0(s, \tau) = h(\tau - s)\, \Phi_A(\tau, s) \in S(L(E))$ and

$$D = \sum_{i=1}^{\infty} \lambda_i D^0(\cdot, \tau_i) + \int_a^b d\theta\, \lambda(\theta)\, D^0(\cdot, \theta).$$

Since $S(L(E))$ is closed under uniform limits the absolute convergence of the sum and integral gives the desired results. Q.E.D.

3. Green's Matrix

We associate with the pair (A, B), where $A \in S(L(E))$, and $B \in L(S(E), E)$ is nonsingular for A, a kernel $G_{A,B}(t, s)$ defined by the following relationship:

$$y' = Ay + p, \qquad By = \xi, \qquad \xi \in E, \tag{3.1}$$

if and only if

$$y(t) = \int_a^b G_{A,B}(t, s)\, p(s)\, ds + B^0(t) F_A \xi, \qquad t \in [a, b], \tag{3.2}$$

where $F_A = \{B \mid S_A(E)\}^{-1}$.

In analogy with the classical case, we call $G_{A,B}$ the Green's *matrix* for the pair (A, B). (A similar definition if given in [3].)

The Green's matrix $G_{A,B}$ is constructed as follows: Let $\Phi_A(t, s)$ be the fundamental matrix for A, i.e., the solution of the system

$$\frac{\partial \Phi_A}{\partial t}(\cdot, s) = A\Phi_A(\cdot, s), \qquad \Phi_A(s, s) = I$$

or equivalent by

$$\Phi_A(t, s) = I + \int_s^t A(\theta)\, \Phi_A(\theta, s)\, d\theta.$$

We note that

$$\Phi_A(t, s) = B^0(t) F_A{}^0(s).$$

The general solution to (3.1) can be written as

$$y(t) = \int_a^t \Phi_A(t, s)\, p(s)\, ds + \Phi_A(t, a)\, y(a). \tag{3.3}$$

Successively applying B and F_A to (3.3) yields (3.2) with

$$G_{A,B}(t, s) = h(t-s)\, \Phi_A(t, s) - B^0(t) F_A B\{h(\cdot - s)\Phi_A(\cdot, s)\}, \tag{3.4}$$

where $h(t)$ is the Heaviside step function,

$$\begin{aligned} h(t) &= 1,\ t \geqslant 0 \\ &= 0,\ t < 0, \end{aligned}$$

It is clear that $G_{A,B}$ is a regulated function of t for each s. To insure that $G_{A,B}$ is regulated in s for each t we must demand that the mapping D: $[a, b] \to L(E)$, defined by

$$s \to D(s) = B\{h(\cdot - s)\, \Phi_A(\cdot, s)\},$$

is regulated; i.e., $D \in S(L(E))$.

Henceforth it will be assumed that D is regulated. Proposition (2.1c) shows that this requirement is satisfied by all "reasonable" boundary operators.

The above construction shows that a solution to (3.1) is also a solution to (3.2). Conversely, if y satisfies (3.2) and $G_{A,B}$ is given by (3.4) then y satisfies (3.1).

$G_{A,B}$ has the usual continuity and differentiability properties associated with a Green's *matrix* (cf. [3]). The next proposition records these properties in their integrated form.

PROPOSITION 3.1. *The Green's matrix*

$$G_{A,B}(t, \tau) = h(t-\tau)\, \Phi_A(t, \tau) - B^0(t) F_A B\{h(\cdot - \tau)\, \Phi_A(\cdot, \tau)\}$$

has the following properties:

$$BG_{A,B}(\cdot, \tau) = 0, \qquad \textit{for all} \quad \tau \in [a, b], \tag{3.5a}$$

$$\begin{aligned} G_{A,B}(t, \tau) = G_{A,B}(\beta, \tau) - \int_t^\beta A(\theta)\, G_{A,B}(\theta, \tau)\, d\theta \\ - h(\beta - \tau)\, \Phi_A(t \vee \tau, \tau) + h(t-\tau)\, \Phi_A(t, \tau), \end{aligned} \tag{3.5b}$$

where $t \vee \tau = \max(t, \tau)$.

$$G_{A,B}(t,\tau) = B^0(t)F_AB\{[1-h(\cdot-\tau)]I\} + [h(t-\tau)-1]I - \int_a^\tau G_{A,B}(t,\theta)\,A(\theta)\,d\theta. \tag{3.5c}$$

Proof. (a) Note first that

$$G_{A,B}(\cdot,\tau) = h(\cdot-\tau)\,\Phi_A(\cdot,\tau) - F_AB\{h(\cdot-\tau)\,\Phi_A(\cdot,\tau)\}.$$

Thus, $BG_{A,B}(\cdot,\tau) = B\{h(\cdot-\tau)\,\Phi_A(\cdot,\tau)\} - B\{h(\cdot-\tau)\,\Phi_A(\cdot,\tau)\} = 0$, since $BF_A = I \in L(E)$.

(b)

$$\begin{aligned}
&\int_t^\beta A(\theta)\,G_{A,B}(\theta,\tau)\,d\theta\\
&\quad= \int_t^\beta A(\theta)\{h(\theta-\tau)\,\Phi_A(\theta,\tau) - B^0(\theta)F_AD(\tau)\}\,d\theta\\
&\quad= h(\beta-\tau)\int_{t\vee\tau}^\beta A(\theta)\,\Phi_A(\theta,\tau)\,d\tau - \{B^0(\beta)-B^0(t)\}F^A\,D(\tau)\\
&\quad= h(\beta-\tau)\{\Phi_A(\beta,\tau)-\Phi_A(t\vee\tau,\tau)\} - \{B^0(\beta)-B^0(t)\}F_AD(\tau)\\
&\quad= G_{A,B}(\beta,\tau) - G_{A,B}(t,\tau) - h(\beta-\tau)\,\Phi_A(t\vee\tau,\tau) + h(t-\tau)\,\Phi_A(t,\tau).
\end{aligned}$$

(c)

$$\begin{aligned}
&\int_a^\tau G_{A,B}(t,\theta)\,A(\theta)\,d\theta\\
&\quad= \int_a^\tau h(t-\theta)\,\Phi_A(t,\theta)\,A(\theta)\,d\theta - B^0(t)F_AB\int_a^\tau h(\cdot-\theta)\,\Phi_A(\cdot,\theta)\,A(\theta)\,d\theta\\
&\quad= \{\Phi_A(t,a)-\Phi_A(t,t\wedge\tau)\}\\
&\qquad - B^0(t)F_AB\{[\Phi_A(\cdot,a)-\Phi_A(\cdot,\cdot\wedge\tau)]\}\\
&\quad= -\Phi_A(t,t\wedge\tau) + B^0(t)F_AB\{h(\cdot-\tau)\,\Phi_A(\cdot,\cdot\wedge\tau)\\
&\qquad + (1-h(\cdot-\tau))\,\Phi_A(\cdot,\cdot\wedge\tau)\}\\
&\quad= -\Phi_A(t,t\wedge\tau) + B^0(t)F_AB\{h(\cdot-\tau)\,\Phi_A(\cdot,\tau)\}\\
&\qquad + B^0(t)F_AB\{I-h(\cdot-\tau)I\}\\
&\quad= -G_{A,B}(t,\tau) + B^0(t)F_AB\{[1-h(\cdot-\tau)]I\}\\
&\qquad + [h(t-\tau)-1]I.
\end{aligned}$$

Q.E.D.

4. Boundary Value Problems and Factorization

Our main interest is in solving the boundary value problem

$$y' = Ay + Cy + p, \qquad By = \xi, \tag{4.1}$$

where A, $C \in S(L(E))$, $p \in S(E)$, $\xi \in E$, and $B \in L(S(E), E)$ is nonsingular for A.

Using the construction of Section 3, problem (4.1) is equivalent to

$$\begin{aligned} y(t) = &\int_a^b G_{A,B}(t, s)\, C(s)\, y(s)\, ds \\ &+ \int_a^b G_{A,B}(t, s)\, p(s)\, ds + B^0(t) F_A \xi, \qquad t \in [a, b]. \end{aligned} \tag{4.2}$$

We write (4.2) as the Fredholm integral equation

$$y(t) = \int_a^b k(t, s)\, y(s)\, ds + f(t), \qquad t \in [a, b], \tag{4.3}$$

where

$$\begin{aligned} k(t, s) &= G_{A,B}(t, s)\, C(s), \qquad t, s \in [a, b], \\ f(t) &= \int_a^b G_{A,B}(t, s)\, p(s)\, ds + B^0(t) F_A \xi. \end{aligned} \tag{4.4}$$

Equation (3.2) shows that f satisfies the boundary value problem

$$f' = Af + p, \qquad Bf = \xi, \tag{4.5}$$

which is solvable by assumption.

In this section, we obtain the factorization of the operator $(I - K)$, where K is the Fredholm integral operator with kernel $k(t, s)$. As discussed in earlier works [4–7], we say that a kernel $k(t, s)$ admits Volterra factorization if there exist kernels $\sigma+$, $\sigma-$ such that

$$k(t, s) = \sigma^+(t, s) + \sigma^-(t, s) - \int_a^{\min(t,s)} \sigma^+(t, \theta)\, \sigma^-(\theta, s)\, d\theta, \qquad t, s \in [a, b], \tag{4.6}$$

where

$$\sigma^+(t, s) \equiv 0, \qquad t < s, \qquad \sigma^-(t, s) \equiv 0, \qquad t > s.$$

The factorization equation (4.6) is equivalent to the operator equation

$$(I - K) = (I - \Sigma^+)(I - \Sigma^-).$$

Our first theorem gives necessary and sufficient conditions for Volterra factorization of k. This result generalizes the work of [7] which was concerned only with special two-point boundary value problems.

THEOREM 4.1. *Let k be given by* (4.4). *Then k admits the Volterra factorization* (4.6) *if and only if the Riccati initial-value problem*

$$R' = (A + C)R - RDCR, \qquad R(a) = B^0(a)F_A \tag{4.7}$$

has a solution in $S(L(E))$. In this case, the Volterra factors are given by

$$\begin{aligned} \sigma^+(t, s) &= \Phi_A(t, s)\{I - R(s)\,D(s)\}\,C(s), && t > s, \\ &= 0, && t < s, \quad t, s \in [a, b], \end{aligned} \tag{4.8a}$$

$$\begin{aligned} \sigma^-(t, s) &= -R(t)\,D(s)\,C(s), && t \leqslant s, \\ &= 0, && t > s, \quad t, s \in [a, b]. \end{aligned} \tag{4.8b}$$

Remarks. (a) In Section 2 we defined what was meant by a solution to a linear equation in the space of regulated mappings. The definition for nonlinear equations is the same, namely, $R \in S(L(E))$ satisfies (4.7) if

$$R(t) = B^0(a)F_A + \int_a^t [(A(\theta) + C(\theta))\,R(\theta) - R(\theta)\,D(\theta)\,C(\theta)\,R(\theta)]\,d\theta, \qquad t \in [a, b]. \tag{4.9}$$

Further, as before, if R satisfies (4.9), then R is automatically continuous.

(b) We adopt the convention that any function $S^+(t, s)$ of two variables t, s with a superscript $+$ vanishes for $t < s$. Similary a function $S^-(t, s)$ vanishes for $t > s$.

Proof. Suppose (4.6) holds. Note first

$$k^+(t, s) = (\Phi_A(t, s) - B^0(t)F_AD(s))\,C(s) = B^0(t)\{F_A{}^0(s) - F_AD(s)\}C(s),\ t > s,$$
$$k^-(t, s) = -B^0(t)F_AD(s)\,C(s), \quad t < s.$$

Thus, (4.6) is equivalent to the pair of equations

$$\sigma^+(t, s) = B^0(t)\{F_A{}^0(s) - F_AD(s)\}\,C(s) + \int_a^s \sigma^+(t, \theta)\,\sigma^-(\theta, s)\,d\theta, \qquad t > s,$$

$$\sigma^-(t, s) = -B^0(t)F_AD(s)\,C(s) + \int_a^t \sigma^+(t, \theta)\,\sigma^-(\theta, s)\,d\theta, \qquad t < s.$$

By the uniqueness of solutions of *linear* Volterra equations in the space of regulated maps (cf. [1, 11.7, Problem 8]), we can, therefore, write

$$\sigma^+(t, s) = B^0(t)\, T(s)\, C(s), \qquad t > s,$$
$$\sigma^-(t, s) = -R(t)\, D(s)\, C(s), \qquad t < s,$$

where the maps $T \in S(L(E, S(E)),\ R \in S(L(E))$ are given by

$$T(s) = F_A{}^0(s) - F_A D(s) - \int_a^s T(\theta)\, C(\theta)\, R(\theta)\, D(s)\, d\theta, \qquad s \in [a, b], \tag{4.10}$$

$$R(t) = B^0(t) F_A + \int_a^t B^0(t)\, T(\theta)\, C(\theta)\, R(\theta)\, d\theta, \qquad t \in [a, b]. \tag{4.11}$$

Two facts are important to note:

(i) $T(s)\,\xi \in S_A(E)$ for each $s \in [a, b]$, and each $\xi \in E$. This follows from the resolvent expansion of the solution to (4.10) on observing that the forcing function $(F_A{}^0(s) - F_A D(s))$ operating on ξ, belongs to $S_A(E)$, for each $s \in [a, b]$ and $\xi \in E$.

(ii) $B^0(t)\, T(t) + R(t)\, D(t) = I$.

This follows at once from (4.10) and (4.11).

We now show $R(t)$ satisfies (4.7).

If J is any element in $L(E, S_A(E))$, then by Proposition (2.1a),

$$B^0(t)\, J\xi = B^0(\tau)\, J\xi + \left(\int_\tau^t A(s)\, B^0(s) J\, ds\right) \xi, \qquad \text{for every} \quad \xi \in E,$$

so that

$$B^0(t) J = B^0(\tau) J + \int_\tau^t A(s)\, B^0(s) J\, ds, \qquad t, \tau \in [a, b]. \tag{4.12}$$

Identity (4.12) in (4.11) then given

$$\begin{aligned} R(t) = B^0(a) F_A &+ \int_a^t A(s)\, B^0(s) F_A\, ds \\ &+ \int_a^t \left\{B^0(\theta)\, T(\theta)\, C(\theta)\, R(\theta) + \int_\theta^t A(s)\, B^0(s)\, T(\theta)\, C(\theta)\, R(\theta)\, ds\right\} d\theta. \end{aligned} \tag{4.13}$$

Since

$$\begin{aligned} &\int_a^t d\theta \int_\theta^t A(s)\, B^0(s)\, T(\theta)\, C(\theta)\, R(\theta)\, ds \\ &\qquad = \int_a^t ds\, A(s)\, B^0(s) \left(\int_a^s T(\theta)\, C(\theta)\, R(\theta)\, d\theta\right), \end{aligned}$$

Eq. (4.13) becomes

$$R(t) = B^0(a) F_A + \int_a^t A(s) R(s)\, ds + \int_a^t (I - R(s) D(s)) C(s) R(s)\, ds$$

or, equivalently,

$$R' = (A + C)R - RDCR, \qquad R(a) = B^0(a)F_A .$$

This completes the proof of the first part of the theorem.

Now suppose problem (4.7) has a solution $R \in S(L(E))$. Defining $\sigma^{\pm}$ by Eq. (4.8a,b) gives, for $t > s$,

$$\begin{aligned} &\sigma^+(t, s) + \int_a^s \sigma^+(t, \theta)\, \sigma^-(\theta, s)\, d\theta \\ &\qquad = \Phi_A(t, s)\, C(s) - B^0(t)\, H(s)\, D(s)\, C(s), \end{aligned} \tag{4.14}$$

where

$$H(s) = F_A{}^0(s)\, R(s) - \int_a^s F_A{}^0(\theta)(I - R(\theta)\, D(\theta))\, C(\theta)\, R(\theta)\, d\theta. \tag{4.15}$$

We claim $H(s) = F_A$, so that the right side of (4.14) equals $k^+(t, s)$, $t > s$. This assertion is proved by applying the following lemma to the integral in (4.15) and using Proposition 2.1b.

LEMMA 4.2. *(Integration by parts* [1, 8.7]). *If* $U, V \in S(L(E, S(E))$, $X, Y \in S(L(E))$ *and*

$$U(s) = U(a) + \int_a^s V(\theta)\, d\theta,$$

$$X(s) = X(a) + \int_a^s Y(\theta)\, d\theta,$$

then

$$\int_a^s U(\theta)\, Y(\theta)\, d\theta = U(s)\, X(s) - U(a)\, X(a) - \int_a^s V(\theta)\, X(\theta)\, d\theta.$$

Analogously, Eq. (4.8 a,b) *give, for* $t < s$

$$\sigma^-(t, s) + \int_a^t \sigma^+(t, \theta)\, \sigma^-(\theta, s)\, d\theta = K(t)\, D(s)\, C(s), \tag{4.16}$$

where

$$K(t) = R(t) + \int_a^t \Phi_A(t, \theta)(I - R(\theta)\, D(\theta))\, C(\theta)\, R(\theta)\, d\theta.$$

A similar argument to the above then yields $K(t) = B^0(t)F_A$, so that the right side of (4.16) equals $k^-(t, s)$, $t < s$. Thus, the mappings $\sigma^\pm$ defined by Eq. (4.8 a,b) yield the Volterra factors of k. Q.E.D.

Remarks. The above formalism also gives a representation for the resolvents $V^\pm$ of the Volterra factors $\sigma^\pm$. Thus, if the initial value problem (4.7) has a solution $R \in S(L(E))$ and if the kernels $V^\pm(t, s)$ are defined by

$$V^+(t, s) = \sigma^+(t, s) + \int_s^t \sigma^+(t, \theta)\, V^+(\theta, s)\, d\theta, \qquad t > s,$$

$$V^-(t, s) = \sigma^-(t, s) + \int_t^s \sigma^-(t, \theta)\, V^-(\theta, s)\, d\theta, \qquad t < s,$$

then the following equations hold:

$$\frac{\partial V^+}{\partial t}(t, s) = (A + C)\, V^+(t, s) - RDCV^+(t, s),$$

$$V^+(s, s) = (I - R(s)\, D(s))\, C(s),$$

$$\frac{\partial V^-}{\partial t}(t, s) = (A + C)\, V^-(t, s),$$

$$V^-(s, s) = R(s)\, D(s)\, C(s).$$

These in turn lead to the representations,

$$V^+(t, s) = R(t)\, R^{-1}(s)\, (I - R(s)\, D(s))\, C(s),$$
$$V^-(t, s) = -\Phi_{A+C}(t, s)\, R(s)\, D(s)\, C(s),$$

where $R^{-1} \in S(L(E))$ and satisfies the linear equation

$$(R^{-1})' = -R^{-1}(A + C) + DC,\ R^{-1}(a) = BF_A{}^0(a).$$

5. A Comparison Theorem for Boundary Value Problems

As in Section 4, we are still interested in solving the boundary value problem (4.1). The theme of this section is, however, the following. Under the assumption of the solvability of (4.1) when $C = 0$, we show via the preceeding factorization theory that this can lead to an initial value algorithm for the solution of (4.1) with general C. Our main result has the flavor of a comparison theorem.

THEOREM 5.1. *If B is nonsingular for A and if the Riccati initial value* *problem* (4.7) *has a solution* $R \in S(L(E))$ *then* B *is nonsingular for* $A + C$. *Further the boundary value problem* (4.1)

$$y' = Ay + Cy + p, \qquad By = \xi$$

is equivalent to the initial value problem

$$w' = p + (A + C)w - RDCW, \qquad w(a) = f(a), \tag{5.1}$$

$$y' = (A + C)y + p, \qquad y(b) = w(b), \tag{5.2}$$

where f *is the solution of the boundary value problem*

$$f' = Af + p, \qquad Bf = \xi. \tag{5.3}$$

Proof. The boundary value problem (4.1) is equivalent to the Fredholm integral equation

$$(I - K)y = f.$$

By virtue of Theorem 4.1 this in turn is equivalent to the pair of Volterra integral equations

$$(I - \Sigma^{+})w = f,$$
$$(I - \Sigma^{-})y = w,$$

or explicity

$$w(t) = f(t) + \int_a^t B^0(t) F_A{}^0(\theta)(I - R(\theta)\, D(\theta))\, C(\theta)\, w(\theta)\, d(\theta), \tag{5.4}$$

$$y(t) = w(t) - \int_t^b R(t)\, D(\theta)\, C(\theta)\, y(\theta)\, d\theta, \tag{5.5}$$

$$f(t) = f(a) + \int_a^t A(\theta) f(\theta)\, d\theta. \tag{5.6}$$

Applying the operator identity (4.12) to (5.4) gives

$$w(t) = f(t) + \int_a^t (I - R(\theta)\, D(\theta))\, C(\theta)\, w(\theta)\, d\theta$$
$$+ \int_a^t ds\, A(s)\, B^0(s) \int_a^s (I - R(\theta)\, D(\theta))\, C(\theta)\, w(\theta)\, d\theta.$$

On using (5.4) and (5.6) we have

$$w(t) = f(a) + \int_a^t \{(A(s) + C(s))\, w(s) - R(s)\, D(s)\, C(s)\, w(s)\}\, ds$$

or, equivalently,

$$w' = (A + C)w - RDCw, \qquad w(a) = f(a).$$

A similar argument using Eq. (4.7) and (5.1) applied to (5.5) yields

$$y' = (A + C)y + p, \qquad y(b) = w(b).$$

Thus, the initial value formulation (5.1) and (5.2) shows that a unique solution y of (4.1) exists for every $\xi \in E$. Thus, B is nonsingular for $A + C$. Q.E.D.

Remarks. (a) The full force of factorization theory is not needed to prove that B nonsingular for A implies B nonsingular for $A + C$. The following result is, in fact, trivial: If B is nonsingular for A and if $(I - G_{A,B}C)$ is invertible then B is nonsingular for $A + C$. This result, however, does not give the corresponding initial value algorithm. On the other hand, it does lead to what may be a surprising property Green's *matrices*. Namely, *The Fredholm resolvent of $G_{A,B}C$ is equal to $G_{A+C,B}C$*; or in operator notation

$$(I - G_{A,B}C)^{-1} - I = G_{A+C,B}C.$$

(b) A "converse" to Theorem 5.1 is given in Section 7 (cf. Remarks after Theorem 7.2).

6. Factorization and Imbedding

The general theory of Gohberg and Krein [8] for factorization of integral operators leads, in the setting of Section 4, to an imbedding of the corresponding boundary value problem. The meaning of this imbedding for special two-point boundary value problems was discussed in detail in [7]. The question arises as to the meaning of the imbedding in the general case.

The Gohberg–Krein theory states: The kernel $k(t, s)$ has a Volterra factorization for $t, s \in [a, b]$ if and only if the operators $I - P_\tau K P_\tau$ are invertible for all $\tau \in [a, b]$, where $P_\tau : S(E) \to S(E)$ is defined by

$$\begin{aligned}(P_\tau x)(t) &= x(t), \quad a \leqslant t \leqslant \tau,\\ &= 0, \qquad \tau < t \leqslant b.\end{aligned}$$

This result applied in the setting of Section 4 is equivalent to the statement: The kernel $G_{A\cdot B}(t, s)\, C(s)$ has a Volterra factorization if and only if the boundary operator B is nonsingular for all the operators $A + C_\tau$, where

$$C_\tau(t) = C(t) \quad a \leqslant t \leqslant \tau,$$
$$= 0, \qquad \tau < t \leqslant b.$$

This last condition is, in other words, that the boundary value problems

$$y' = Ay + C_\tau y, \quad By = \xi, \quad \tau \in [a, b], \tag{6.1}$$

all have unique solutions.

We now give another interpretation of this condition by relating it to an imbedding of the boundary operator itself. This, then, is a generalization of invariant imbedding [7].

Define $B_A(\tau) \in L(E), E), \tau \in [a, b]$ by

$$B_A(\tau)x = B[(1 - h(\cdot - \tau))x(\cdot)] + B[h(\cdot - \tau)\,\Phi_A(\cdot, \tau)]x(\tau), x \in S(E). \tag{6.2}$$

PROPOSITION 6.1. *$B_A(\tau)$ is nonsingular for $A + C$ if and only if B is nonsingular for $A + C_\tau$. Moreover, if y_τ is the solution to* (6.1) *and if z_τ is the solution to the boundary value problem*

$$z' = (A + C)z, \qquad B_A(\tau) = \xi, \tag{6.3}$$

then

$$P_\tau y_\tau = P_\tau z_\tau .$$

Proof. First observe that the general solution to (6.1) is

$$y_\tau(t) = (1 - h(t - \tau))\,\Phi_{A+C}(t, \tau)\,\alpha + h(t - \tau)\,\Phi_A(t, \tau)\alpha$$

for some $\alpha \in E$, ($\alpha = y_\tau(\tau)$). B is nonsingular for $A + C_\tau$, if and only if the operator $M(\tau) \in L(E)$ given by

$$M(\tau) = B\{(1 - h(\cdot - \tau))\,\Phi_{A+C}(\cdot, \tau) + h(t - \tau)\,\Phi_A(\cdot, \tau)\}$$

is invertible. Next observe that the general solution to (6.3) is

$$z_\tau(t) = \Phi_{A+C}(t, \tau)\beta,$$

for $\beta \in E$, ($\beta = z_\tau(\tau)$). $B_A(\tau)$ is nonsingular for $A + C$, if and only if the operator $N(\tau) \in L(E)$ given by

$$N(\tau) = B\{(1 - h(\cdot, \tau))\,\Phi_{A+C}(\cdot, \tau)\} + B\{h(\cdot, \tau)\,\Phi_A(\cdot, \tau)\}\,\Phi_{A+C}(\tau, \tau)$$

is invertible. Since $\Phi_{A+C}(\tau, \tau) = I$, we have $M(\tau) = N(\tau)$ and the first part of the proposition is proved.

The second part of the proposition is obtained by noting that $\alpha = \beta$ and therefore from the above expressions for y_τ and z_τ,

$$y_\tau(t) = z_\tau(t), \qquad a \leqslant t \leqslant \tau. \qquad \text{Q.E.D.}$$

The main results of this section now follows from the Proposition 6.1 and Theorem 4.1. The first of these is a direct corollary.

THEOREM 6.2. *$B_A(\tau)$ is nonsingular for $A + C$ for all $\tau \in [a, b]$ if and only if the Riccati equation* (4.7):

$$R' = (A + C)R - RDCR, \qquad R(a) = B^0(a)F_A$$

has a solution $R \in S(L(E))$.

Next, the factorization formalism provides an algorithm for the solution of the imbedded boundary value problems (6.3).

THEOREM 6.3. *If $B_A(\tau)$ is nonsingular for $A + C$ for all $\tau \in [a, b]$, then the solution to the imbedded boundary value problem*

$$y' = (A + C)y + p, \qquad B_A(\tau)y = \xi$$

is given by

$$y(t, \tau) = R(t)\, U(t, \tau)\xi + R(t)\, w(t, \tau) + v(t), \tag{6.4}$$

where the maps U, w, v, u are determined by either of the two initial value systems

$$\frac{\partial u}{\partial t}(t, \tau) = DCRu(t, \tau), \qquad u(\tau, \tau) = \xi, \tag{6.5a}$$

$$\frac{\partial w}{\partial t}(t, \tau) = DCRw + D(Cv + p), \qquad w(\tau, \tau) = 0, \tag{6.5b}$$

$$v' = (A + C)v + p - RD(Cv + p), \qquad v(a) = 0, \tag{6.5c}$$

$$u(t, \tau) = U(t, \tau)\xi; \tag{6.5d}$$

and, equivalently,

$$\frac{\partial U}{\partial \tau}(t, \tau) = -U(t, \tau)\, DCR, \qquad U(t, t) = I, \tag{6.6a}$$

$$\frac{\partial w}{\partial \tau}(t, \tau) = -U(t, \tau)\, D(Cv + p), \qquad w(t, t) = 0, \tag{6.6b}$$

$$v' = (A + C)v + p - RD(Cv + p), \qquad v(a) = 0. \tag{6.6c}$$

Remarks. (a) This result, along with the initial value system (5.1)–(5.3), gives three methods of solution to the original boundary value problem (4.1). The system (5.1)–(5.3) is a two-pass algorithm and in some cases (e.g. real time problems) would not be appropriate. It does, however, have the advantage that only one "operator" equation, namely (4.7) need be solved. System (6.5) is also a two pass algorithm with the same advantage. However, (6.5) may be "stable" in cases when (5.1)–(5.3) is not. Finally, system (6.6) is a one-pass algorithm. It has the same stability advantage as (6.5) but requires an additional operator equation in (6.6a).

(b) The algorithm implied by (6.6) is as follows: To determine the solution of (4.1) $y(t_0, b)$, for a fixed $t = t_0$, and given b, integrate (4.7) and (6.6c) up to the point t_0; then adjoin (6.6a,b) and integrate (4.7) and (6.6) to the end point b.

(c) The proof of Theorem 6.3 follows the same lines as that given in Theorem 5.1. We will rederive the same result in Section 7 by alternate means, and, thus, we omit the details here.

7. Factorization of the Differential Equations

Up to this point we have viewed the boundary value problem (4.1) in terms of its equivalent Fredholm integral equation and derived the preceeding results via Volterra factorization. An alternate approach involves factorizing the operators associated with the boundary value problem itself. This idea was initiated in [2] for the case $A = 0$. Here we extend the theory of [2] and also cover the case of general A.

Let $B_A(\tau)$ be defined by (6.2), i.e.,

$$B_A(\tau)x = B[(1 - h(\cdot - \tau))\, x(\cdot)] + B[h(\cdot - \tau)\, \Phi_A(\cdot, \tau)]\, x(\tau). \tag{7.1}$$

The elements A, B, C and D are as before. We write, however, S_{A+C} for $S_{A+C}(E)$.

Proposition 7.1. (a) *The restriction of $B_A(\tau)$ to S_{A+C} satisfies*

$$\text{(a)}\quad B_A(\tau) = B_A(\alpha) + \int_\alpha^\tau D(\theta)\, C(\theta)\, B^0(\theta)\, d\theta, \qquad \alpha, \tau \in [a, b] \tag{7.2a}$$

or, equivalently,

$$(B_A)' = DCB^0.$$

(b) *If* $B_A(\tau)| S_{A+C}$ *is invertible for* $\tau \in [\alpha, \beta] \subset [a, b]$, *and if* $F_{A,C}(\tau)$ *denotes the inverse of* $B_A(\tau)| S_{A+C}$ *then*

$$F_{A,C}(\tau) = F_{A,C}(\alpha) - \int_\alpha^\tau F_{A,C}(\theta)\, C(\theta)\, B(\theta) F_{A,C}(\theta)\, d\theta, \qquad \tau \in [\alpha, \beta] \quad (7.2b)$$

or, equivalently,

$$(F_{A,C})' = -F_{A,C}\, DCB^0 F_{A,C}\,.$$

Proof. (a) If $y \in S_{A+C}(E)$, we see from Eq. (7.1) that

$$B_A(\tau)y = By + B\{h(\cdot - \tau)[\Phi_A(\cdot, \tau) - \Phi_{A+C}(\cdot, \tau)]\, B^0(\tau)\}y,$$

now define $Z(\tau)$ by

$$Z(\tau) = B - \int_\tau^b D(\theta)\, C(\theta)\, B^0(\theta)\, d\theta.$$

Using the definition of $D(\theta)$ in Section 3, we have that

$$\begin{aligned} Z(\tau) &= B - B \int_\tau^b h(\cdot - \theta)\, \Phi_A(\cdot, \theta)\, C(\phi)\, B^0(\theta)\, d\theta \\ &= B - B \left\{ h(\cdot - \tau) \int_\tau^{\cdot} \Phi_A(\cdot, \theta)\, C(\theta)\, B^0(\theta)\, d\theta \right\}. \end{aligned}$$

By Proposition (2.1a), $B^0(t)| S_{A+C}$ satisfies

$$B^0(t) = B^0(\tau) + \int_\tau^t (A(\theta) + C(\theta))\, B^0(\theta)\, d\theta.$$

From the definition of $\Phi_A(t, s)$ we obtain the adjoint equation

$$\Phi_A(s, t) = \Phi_A(s, \tau) - \int_\tau^t \Phi_A(s, \theta)\, A(\theta)\, d\theta. \qquad (7.3)$$

Further,

$$\begin{aligned} &\int_\tau^t \Phi_A(t, \theta)\, C(\theta)\, B^0(\theta)\, d\theta \\ &\quad = \int_\tau^t \Phi_A(t, \theta)[(A(\theta) + C(\theta))\, B^0(\theta)]\, d\theta + \int_\tau^t [-\Phi_A(t, \theta)\, A(\theta)]\, B^0(\theta)\, d\theta. \end{aligned} \qquad (7.4)$$

Using Lemma 4.2, the left side of (7.4) restricted to S_{A+C} becomes

$$B^0(t) - \Phi_A(t, \tau)\, B^0(\tau) = \{\Phi_{A+C}(t, \tau) - \Phi_A(t, \tau)\}\, B^0(\tau).$$

Thus, $Z(\tau)| S_{A+C}$ is given by

$$\begin{aligned} Z(\tau) &= B + B\{h(\cdot - \tau)\, [\Phi_A(t, \tau) - \Phi_{A+C}(t, \tau)]\}\, B^0(\tau) \\ &= B_A(\tau), \end{aligned}$$

and, hence, the definition of $Z(\tau)$ implies that $B_A(\tau)$ satisfies (7.2).

(b) Now assume $B_A(\tau)| S_{A+C}$ invertible for $\tau \in [\alpha, \beta]$. Since $B_A(\tau)| S_{A+C}$ is continuous in τ, the same is true of $F_{A,C}(\tau)$. For convenience, we write $F = F_{A,C}$ in the remainder of this proof.

Define $X(\tau)$ by

$$X(\tau) = F(\alpha) - \int_\alpha^\tau F(\theta)\, D(\theta)\, C(\theta)\, B^0(\theta) F(\theta)\, d\theta.$$

Using (7.2a) and Lemma 4.2 we then have that

$$\begin{aligned} B_A(\tau)\, X(\tau) - B_A(\alpha)\, X(\alpha) &= \int_\alpha^\tau D(\theta)\, C(\theta)\, B^0(\theta)\, X(\theta)\, d\theta \\ &\quad - \int_\alpha^\tau B_A(\theta) F(\theta)\, D(\theta)\, C(\theta)\, B_0(\theta) F(\theta)\, d\theta \\ &= \int_\alpha^\tau D(\theta)\, C(\theta)\, B_0(\theta)\{X(\theta) - F(\theta)\}\, d\theta. \qquad (7.5) \end{aligned}$$

But $X(\alpha) = F(\alpha)$ so that $B_A(\alpha)\, X(\alpha) = I = B_A(\tau) F(\tau)$. Thus, (7.5) can be written as

$$\{X(\tau) - F(\tau)\} = F(\tau) \int_\alpha^\tau D(\theta)\, C(\theta)\, B^0(\theta)\{X(\theta) - F(\theta)\}\, d\theta, \qquad (7.6)$$

where we have used the fact that $X(\tau) \in L(E, S_{A+C})$. By the uniqueness of solutions to Volterra integral equations (cf. [1, 11.6]) it follows that $X(\tau) = F(\tau)$ and consequently (7.2b) is verified. Q.E.D.

We now utilize the idea of factorization of the operators associated with the imbedded boundary value problems.

THEOREM 7.2. *$B_A(\tau)| S_{A+C}$ is invertible for $\tau \in [\alpha, \beta]$ if and only if the Riccati equation*

$$R' = (A + C)R - RDCR, \qquad R(\alpha) = B^0(\alpha) F_{A,C}(\alpha) \qquad (7.7)$$

has a solution in $S([\alpha, \beta], E)$. In this case

$$R(\tau) = B^0(\tau) F_{A,C}(\tau).$$

Proof. (a) Assume $B_A(\tau)|\, S_{A+C}$ is invertible for $\tau \in [\alpha, \beta]$. Define

$$S(\tau) = B^0(\tau)\, F_{A,C}(\tau).$$

From Eqs. (7.2b), (7.3) and Lemma 4.2

$$S(\tau) - S(\alpha) = \int_\alpha^\tau \{[A(\theta) + C(\theta)]\, S(\theta) - S(\theta)\, D(\theta)\, C(\theta)\, S(\theta)\}\, d\theta,$$

so that $S(\tau)$ satisfies (7.7) on $[\alpha, \beta]$.

(b) Assume (7.7) has a solution $R \in S([\alpha, \beta], E)$. Since $R(\alpha) = B^0(\alpha) F_{A,C}(\alpha)$, we have $B_A(\alpha)|\, S_{A+C}$ is invertible, by assumption. By the continuity of $B_A(\tau)|\, S_{A+C}$ as a function of τ, we have that $B_A(\tau)|\, S_{A+C}$ is invertible for $\tau \in [\alpha, \sigma)$ for some $\sigma > 0$. Let Ω be the set of singular points of $B_A(\tau)|\, S_{A+C}$ for $\tau \in [\alpha, \beta]$. If Ω is not empty let $\beta_0 = \inf \Omega$. Clearly $\beta_0 \geqslant \sigma$. By a standard argument $\| F_{A,C}(\tau)\| \to \infty$ as $\tau \to \beta_0$, $\tau \in [\alpha, \beta_0)$. On the other hand, $R(\tau) = B^0(\tau)\, F_{A,C}(\tau)$, $\tau \in [\alpha, \beta_0)$ by part (a) of this theorem and the uniqueness property of Volterra equations. Since $B^0(\tau)\, \|\, S_{A+C}$ is invertible for $\tau \in [\alpha, \beta_0)$ this implies $\| R(\tau)\| \to \infty$ as $\tau \to \beta_0$, $\tau \in [\alpha, \beta_0)$. This contradicts the assumption that $R \in S([\alpha, \beta], E)$. Thus Ω is empty. Q.E.D.

Remark. The representation $R(t) = B^0(t)\, F_{A,C}(t)$ gives a partial converse to Theorem 5.1. Namely, *If B is nonsingular for $A + C$ and the initial value problem $R' = (A + C)\, R - RDCR$, $R(b) = B^0(b)\, F_{A+C}$, has a solution $R \in S(L(E))$ then B is nonsingular for A.* The proof follows from the observations

$$R(b) = B^0(b)\, F_{A,C}(b) = B^0(b)\, F_{A+C}$$

and

$$R(a) = B^0(a)\, F_{A,C}(a) = B^0(a)\, F_A\, .$$

The ingredients are now at hand to give a proof of Theorem 6.3. The main idea of the proof is the factorization of $B^0(t)\, F_{A,C}(\tau)$.

Proof of Theorem 6.3. (a) We first assume $p \equiv 0$. The solution $y(t, \tau)$ of the imbedded boundary value problem

$$y' = (A + C)y, \qquad B_A(\tau)y = \xi$$

can be represented as

$$y(t, \tau) = B^0(t)\, F_{A,C}(\tau)\xi.$$

Since $F_{A,C}(t)\, B_A(t)$ is the identity of S_{A+C}, we have

$$y(t, \tau) = B^0(t) F_{A,C}(t)\, B_A(t) F_{A,C}(\tau)\xi \tag{7.8}$$
$$= R(t)\, U(t, \tau)\xi,$$

where

$$U(t, \tau) = B_A(t) F_{A,C}(\tau).$$

Evidently, $U(\tau, \tau) = I \in L(E)$, and from Eqs. (7.2a,b) we immediately obtain

$$U(t, \tau) = \left\{B_A(\tau) + \int_\tau^t D(\theta)\, C(\theta)\, B^0(\theta)\, d\theta\right\} F_{A,C}(\tau)$$

$$= I + \int_\tau^t D(\theta)\, C(\theta)\, B_0(\theta) F_{A,C}(\theta)\, B_A(\theta) F_{A,C}(\tau)\, d\theta$$

$$= I + \int_\tau^t D(\theta)\, C(\theta)\, R(\theta)\, U(\theta, \tau)\, d\theta \tag{7.9a}$$

and

$$U(t, \tau) = B_A(t) \left\{F_{A,C}(t) + \int_\tau^t F_{A,C}(\theta)\, D(\theta)\, C(\theta)\, B_0(\theta) F_{A,C}(\theta)\, d\theta\right\}$$
$$= I + \int_\tau^t U(t, \theta)\, D(\theta)\, C(\theta)\, R(\theta)\, d\theta. \tag{7.9b}$$

Equations (6.5a,d) and (6.6a) follow from the above and the remaining Eqs. (6.5b,c) and (6.6b,c) all have trivial solutions when $p \equiv 0$.

(b) We now consider general p. First observe that the nonhomogeneous problem

$$y' = (A + C)y + p, \qquad B_A(\tau)y = \xi$$

may be written as a homogeneous problem in the extended space $\bar{E} = E \times R'$, where R' = real line. Thus,

$$\bar{y} = \begin{pmatrix} y \\ z \end{pmatrix}, \qquad \bar{\xi} = \begin{pmatrix} \xi \\ 1 \end{pmatrix}, \qquad \bar{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

$$\bar{C} = \begin{pmatrix} C & p \\ 0 & 0 \end{pmatrix}, \qquad \bar{B}(\tau) = \begin{pmatrix} B_A(\tau) & 0 \\ 0 & B^0(a) \end{pmatrix},$$

gives rise to the equivalent problem

$$\bar{y}' = (\bar{A} + \bar{C})\bar{y}, \qquad \bar{B}(\tau)\bar{y} = \bar{\xi}.$$

It is readily shown, as in [2], that

$$\bar{R}(t) = \begin{pmatrix} R(t) & v(t) \\ 0 & 1 \end{pmatrix}, \qquad \bar{U}(t, \tau) = \begin{pmatrix} U(t, \tau) & w(t, \tau) \\ 0 & 1 \end{pmatrix},$$

$$\bar{D}(t) = \begin{pmatrix} D(t) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \bar{F}_{\bar{A},\bar{C}}(\tau) = \begin{pmatrix} F_{A,C}(\tau) & f_{A,C}(\tau) \\ 0 & 1 \end{pmatrix},$$

where $f_{A,C}(\tau)$ satisfies

$$B_A(t) f_{A,C}(\tau) = w(t, \tau).$$

The natural extensions of Eqs. (7.8), (4.7), and (7.9a,b) lead, respectively, to (6.4), (6.5c), and (6.5b), (6.6b) and also regenerate Eqs. (4.7), (6.5a,d), (6.6a).

Q.E.D.

8. Fundamental Kernels

The concept of a fundamental kernel was defined in [4]. Such kernels are closely related to special Green's matrices but this connection was not made explicit. In this section we precisely define this relationship.

Let $E = E_1 \times E_2$, where E_1 and E_2 are two Banach spaces; write $y = \binom{u}{v} \in E$, $u \in E_1$ $v \in E_2$; and let α, $\gamma \in S(L(E_2))$, β, $\delta \in S(L(E_1))$, $\omega \in L(E_2)$. A continuous map $\Gamma : [a, b] \times [a, b] \to L(E_1)$ is said to be fundamental relative to $(\alpha, \beta, \gamma, \omega)$ if the following equations hold:

$$\Gamma(t, s) = \Gamma(s, s) + \int_s^t \beta(\theta)\, \Gamma(\theta, s)\, d\theta, \qquad t \geqslant s, \tag{8.1a}$$

$$\Gamma(t, s) = \Gamma(t, t) + \int_t^s \Gamma(t, \theta)\, \gamma(\theta)\, d\theta, \qquad s \geqslant t, \tag{8.1b}$$

$$\Gamma(t, t) = \omega + \int_a^t \{\alpha(\theta) + \beta(\theta)\, \Gamma(\theta, \theta) + \Gamma(\theta, \theta)\, \gamma(\theta)\}\, d\theta, \qquad t \geqslant a. \tag{8.1c}$$

Matrix kernels of the type $\Gamma(t, s)\, \delta(s)$ are a generalization of those which arise in the theory of linear filtering and control (cf. [4, 9]). These kernels are intimately connected with special two-point boundary value problems. In particular, the Green's matrix formulation for these problems contains the fundamental kernel as a component. The special problem is as follows:

$$\text{(8.2a)} \quad y' = Ay + Cy + p, \qquad By = \xi, \tag{8.2a}$$

$$A = \begin{pmatrix} \beta & \alpha \\ 0 & -\gamma \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 \\ -\delta & 0 \end{pmatrix}, \tag{8.2b}$$

$$B = \begin{pmatrix} I & -\omega \\ 0 & 0 \end{pmatrix} B^0(a) + \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} B^0(b). \tag{8.2c}$$

THEOREM 8.1. *Let $G_{A,B}$ be the Green's matrix associated with the problem (7.2) and assume B is nonsingular for A. Then the following representation holds:*

$$G_{A,B}(t, s) = \begin{pmatrix} \Phi_\beta{}^+(t, s) & -\Gamma(t, s) \\ 0 & -\Phi_{-\gamma}^-(t, s) \end{pmatrix},$$

where Γ is a fundamental kernel relative to $(\alpha, \beta, \gamma, \omega)$.

Proof. Write

$$G_{A,B} = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}.$$

Proposition 3.1, Eq. (3.5a) gives

$$\begin{pmatrix} G_1(a, \tau) - \omega G_3(a, \tau) & G_2(a, \tau) - \omega G_4(a, \tau) \\ G_3(b, \tau) & G_4(b, \tau) \end{pmatrix} = 0. \tag{8.3}$$

Also

$$A(t)\, G_{A,B}(t, \tau) = \begin{pmatrix} \beta G_1(t, \tau) + \alpha G_3(t, \tau) & \beta G_2(t, \tau) + \alpha G_4(t, \tau) \\ -\gamma G_3(t, \tau) & -\gamma G_4(t, \tau) \end{pmatrix}. \tag{8.4}$$

Using (3.5b) with $\beta = b$, we have

$$G_{A,B}(t, \tau) = G_{A\,B}(b, \tau) - \{1 - h(t - \tau)\} I \int_t^b A(\theta)\, G_{A,B}(\theta, \tau)\, d\theta. \tag{8.5}$$

Considering $G_3(t, \tau)$: Eqs. (8.3), (8.4), and (8.5) give

$$G_3(t, \tau) = 0 + \int_t^b \gamma(\theta)\, G_3(\theta, \tau)\, d\theta;$$

hence, $G_3(t, \tau) \equiv 0$.
Considering $G_4(t, \tau)$: Eqs. (8.3), (8.4), and (8.5) give

$$G_4(t, \tau) = -\{1 - h(t - \tau)\} I + \int_t^b \gamma(\theta)\, G_4(\theta, \tau)\, d\theta;$$

so that

$$\begin{aligned} G_4(t, \tau) &\equiv 0, \qquad t > \tau \\ &= -\Phi_{-\gamma}(t, \tau) \qquad t \leqslant \tau, \end{aligned}$$

i.e.,

$$G_4(t, \tau) = -\Phi_{-\gamma}^-(t, \tau).$$

Using (3.5b), with $\beta = t$, $t = a$, $\tau > a$, we have

$$G_{A,B}(t, \tau) = G_{A,B}(a, \tau) + h(t - \tau) I + \int_a^t A(\theta)\, G_{A,B}(\theta, \tau)\, d\theta. \tag{8.6}$$

Considering $G_1(t, \tau)$: Eqs. (8.3), (8.4), and (8.6) give

$$G_1(t, \tau) = h(t - \tau)I + \int_a^t \beta(\theta)\, G_1(\theta, \tau)\, d\theta$$
$$= \Phi_\beta{}^+(t, \tau).$$

Considering $G_2(t, \tau)$: Eqs. (8.3), (8.4), and (8.6) give

$$G_2(t, \tau) = -\, \omega\Phi_{-\gamma}(a, \tau) + \int_a^t [\beta(\theta)\, G_2(\theta, \tau) - \alpha(\theta)\, \Phi_{-\gamma}^-(\theta, \tau)]\, d\theta$$

$$= -\, \omega\Phi_{-\gamma}(a, \tau) - \int_a^\tau \alpha(\theta)\, \Phi_{-\gamma}(\theta, \tau)\, d\theta + \int_a^\tau \beta(\theta)\, G_2(\theta, \tau)\, d\theta$$
$$+ \int_\tau^t \beta(\theta)\, G_2(\theta, \tau)\, d\theta$$

or

$$G_2(t, \tau) = G_2(\tau, \tau) + \int_\tau^t \beta(\theta)\, G_2(\theta, \tau)\, d\theta, \tag{8.7}$$

$$G_2(\tau, \tau) = -\omega\Phi_{-\gamma}(a, \tau) + \int_a^\tau \{-\alpha(\theta)\, \Phi_{-\gamma}(\theta, \tau) + \beta(\theta)\, G_2(\theta, \tau)\}\, d\theta. \tag{8.8}$$

In particular, (8.7) hold for $t > \tau$.

To obtain the adjoint to (8.7), we use (3.5c). First note, for $a < \tau < b$, (8.2c) shows that $B\{[1 - h(\cdot - \tau)]I\}$ is independent of τ. Thus, Eq. (3.5c) gives, for $\tau \geqslant t \geqslant a$,

$$G_{A,B}(t, \tau) = G_{A,B}(t, t) - \int_t^\tau G_{A,B}(t, \theta)\, A(\theta)\, d\theta.$$

Since,

$$G_{A,B}(t, s)\, A(s) = \begin{pmatrix} G_1(t, s)\, \beta(s) & G_1(t, s)\, \alpha(s) - G_2(t, s)\, \gamma(s) \\ 0 & G_4(t, s)\, \gamma(s) \end{pmatrix}$$

$$G_2(t, \tau) = G_2(t, t) + \int_t^\tau G_2(t, \theta)\, \gamma(\theta)\, d\theta, \qquad \tau \geqslant t. \tag{8.9}$$

We now consider $G_2(\tau, \tau)$. Using Eqs. (8.8), (8.9), and (7.3), we have

$$G_2(\tau, \tau) = -\, \omega \left\{I + \int_a^\tau \Phi_{-\gamma}(a, \theta)\, \gamma(\theta)\, d\theta\right\} \tag{8.10}$$
$$- \int_a^\tau \alpha(\theta) \left\{I + \int_\theta^\tau \Phi_{-\gamma}(\theta, s)\, \gamma(s)\, ds\right\}$$
$$+ \int_a^\tau \beta(\theta) \left\{G_2(\theta, \theta) + \int_\theta^\tau G_2(\theta, s)\, \gamma(s)\, ds\right\}$$
$$= -\, \omega + \int_a^\tau \{-\alpha(\theta) + \beta(\theta)\, G_2(\theta, \theta) + G_2(\theta, \theta)\, \gamma(\theta)\}\, d\theta.$$

Equations (8.7), (8.9), and (8.10) show that $-G_2(t, \tau)$ is a fundamental kernel relative to $(\alpha, \beta, \gamma, \omega)$. Q.E.D.

As an example of the earlier theory we apply the results of Sections 4, 5, and 6 to the case at hand.

The boundary operator $B_A(\tau)$ is given by

$$B_A(\tau) = \begin{pmatrix} I & -\omega \\ 0 & 0 \end{pmatrix} B^0(a) + \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} B^0(\tau).$$

The imbedded boundary value problems (6.4) are equivalent to

$$\begin{aligned} u' &= \beta u + \alpha v + p, \\ -v' &= \delta u + \gamma v + q, \\ u(a) - \omega v(a) &= \lambda,\ v(\tau) = \eta. \end{aligned} \tag{8.11}$$

The boundary operator B is always nonsingular for A since the boundary value problem (8.11), with $\delta = 0$, separates into two initial value problems. Thus, $G_{A,B}$ always exists. Moreover, if we write

$$B^0(t) F_A(\tau) = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix} (t, \tau),$$

then it is found that

$$\begin{aligned} F_1(t, \tau) &= \Phi_\beta(t, a), \\ \frac{\partial}{\partial t} F_2(t, \tau) &= \beta F_2(t, \tau) + \alpha \Phi_{-\gamma}(t, \tau), \\ F_2(a, \tau) &= \omega \Phi_{-\gamma}(a, \tau). \\ F_3(t, \tau) &\equiv 0, \\ F_4(t, \tau) &= \Phi_{-\gamma}(t, \tau). \end{aligned}$$

The Riccati initial value problem (4.7) for $R = \left(\begin{smallmatrix} R_1 & R_2 \\ R_3 & R_4 \end{smallmatrix}\right)$ becomes

$$\begin{aligned} R_1' &= \beta R_1 + \alpha R_3 + R_2 \Phi_{-\gamma}(b, \cdot)\, \delta R_1, & R_1(a) &= I, \\ R_2' &= \alpha R_4 + \beta R_2 + R_2 \Phi_{-\gamma}(b, \cdot)\, \delta R_2, & R_2(a) &= \omega \Phi_{-\gamma}(a, b), \\ R_3' &= -\delta R_1 - \gamma R_3 + R_4 \Phi_{-\gamma}(b, \cdot)\, \delta R_1, & R_3(a) &= 0, \\ R_4' &= -\delta R_2 - \gamma R_4 + R_4 \Phi_{-\gamma}(b, \cdot)\, \delta R_2, & R_4(a) &= \Phi_{-\gamma}(a, b), \end{aligned} \tag{8.12}$$

where we have used the fact that

$$D(\tau) = B\{h(\cdot - \tau)\, \Phi_A(\cdot, \tau)\} = \begin{pmatrix} 0 & 0 \\ 0 & \Phi_{-\gamma}(b, \tau) \end{pmatrix}.$$

We observe

$$R_4 = \Phi_{-\gamma}(\cdot, b),$$

so that $R_3 \equiv 0$.

Next note that the mapping

$$\rho = R_2 \Phi_{-\gamma}(b, \cdot)$$

satisfies the Riccati initial value problem,

$$\rho' = \alpha + \beta\rho + \rho\gamma + \rho\delta\rho, \qquad \rho(a) = \omega. \tag{8.13}$$

Hence, $R_1' = (\beta + \rho\delta)R_1$.

Further a solution $R \in S(L(E))$ to (8.12) exists if and only if a solution $\rho \in S(L(E_2))$ to (8.13) exists. This fact gives the standard result on the solvability of the imbedded boundary value problem (8.11) (cf. [4, 7]).

Finally, the Volterra factors of $G_{A,B}C$ are related to the Volterra factors of $\Gamma\delta$ in the following simple way: If

$$(I - G_{A,B}C) = (I - \sigma^+)(I - \sigma^-)$$

and

$$(I - \Gamma\delta) = (I - S^+)(I - S^-)$$

then

$$\sigma^+ = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}, \qquad \sigma^- = \begin{pmatrix} S^- & 0 \\ -\Phi_{-\gamma}^-\delta & 0 \end{pmatrix}.$$

and, consequently,

$$S^+(t, s) = \Phi_\beta{}^+(t, s)\, \rho(s)\, \delta(s),$$

$$S^-(t, s) = \rho(t)\, \Phi_{-\gamma}^-(t, s)\, \delta(s).$$

References

1. J. Dieudonné, "Foundations of Modern Analysis," Academic Press, New York, 1960.
2. A. McNabb, Initial value method for linear boundary value problems, *J. Math. Anal. Appl.* **39** (1972), 495–526.
3. R. N. Bryan, A linear differential system with general linear boundary conditions, *J. Differential Equations* **5** (1969), 34–38.
4. A. Schumitzky, On the equivalence between matrix Riccati equations and Fredholm resolvents, *J. Comput. System Sci.* **2** (1968), 76–87.
5. A. McNabb and A. Schumitzky, Factorization of operators. I. Algebraic theory and examples, *J. Functional Analysis* **9** (1972), 262–295.

6. A. McNabb and A. Schumitzky, Factorization of operators. II. A nonlinear Volterra method for numerical solution of linear Fredholm equations, *J. Comput. System Sci.* **4** (1970), 103–128.
7. A. McNabb and A. Schumitzky, Factorization of operators. III. Initial value method for linear two-point boundary value problems, *J. Math. Anal. Appl.* **31** (1970), 391–405.
8. I. C. Goldberg and M. G. Krein, On the factorization of operators in Hilbert space, *Acta. Sci. Math. Szeged.* **25** (1964), 90–123; English Translation: *Amer. Math. Soc. Transl.* **51** (1966), 151–188.
9. H. H. Kagiwada, R. Kalaba, A. Schumitzky, and R. Sridhar, Invariant imbedding and sequential interpolating filters for nonlinear processes, *Trans. ASME, J. Basic Eng.* **91** (1969), 195–199.
10. A. Schumitzky and T. Wenska, Imbedding a class of linear integral equations through the first critical point, *SIAM J. Math. Anal.*, to appear.