Characterisations of Classes of Multivalued Processes Using Riesz Approximations

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Let $E'$ be the separable dual of a Banach space $E$, and $\mathcal{K}$ the class of all non-empty, convex, weak*-compact subsets of $E'$. J. Neveu proved the convergence of $\mathcal{K}$-valued martingales called multivalued martingales. We prove Riesz approximations for some multivalued processes; i.e., for these processes, we show that they are close to some multivalued martingales. We also obtain Riesz decompositions of some single-valued processes; i.e., we show that they are the sums of a martingale and another process which goes to zero. The class of processes considered for Riesz approximation includes multivalued amarts. The Riesz decomposition of single-valued amarts was obtained by Edgar and Sucheston. Our proofs require some of their results in the multivalued form. Riesz decomposition for multivalued processes is not possible even in simple cases.

Multivalued random variables, also called set-valued random variables, have been studied by a number of authors. We refer to the works of Castaing and Valadier [2], Neveu [10], Hiai and Umegaki [6], Luu [7-9] and Bagchi [1] for details, although many more authors have contributed significantly in this field. As in [11], our definition of a multivalued random variables is the same as that in Neveu [10]. In this paper we study multivalued processes as an existension of vector-valued processes. The concept of martingales was generalized by amarts by Edgar–Sucheston [4] and subsequently to amarts of infinite order by Luu [7]. The notion of asymptotic martingales has been adapted to multivalued processes by Bagchi [1] and Luu [8, 9] among others.

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Neveu [10] proved the a.s. convergence of $L_1$-bounded multivalued martingales which take values in the class of non-empty, convex, weak*-compact subsets of the separable dual $E'$ of a Banach space $E$. Generalizations of Neveu's result to multivalued amarts and pramarts have been obtained by Bagchi [1]. In the present work we study another class of asymptotic martingales, namely amarts of infinite order, introduced by Luu [7].

Riesz decomposition of vector valued amarts was proved by Edgar and Sucheston [4]. Our purpose is to study whether or not similar results about multivalued processes can be obtained. First, a simple example shows that such a "decomposition" is not possible in the multivalued case. In that case can we "approximate" a given multivalued process by a multivalued martingale? If we can, i.e., if the given process is "close" to a multivalued martingale, then we can prove the convergence of the process using the convergence of multivalued martingales already established by Neveu [10]. Another question that can be asked is "how large is the class of multivalued processes that can be approximated in this way?" As it turns out, multivalued amarts of infinite order are precisely that class of processes. Thus while studying Riesz approximation, the amarts of infinite order appear somewhat naturally.

In the sequel, some definitions and basic results are given in Sections 1 and 2. Section 3 has some results on single (vector)-valued processes. The major results on multivalued processes are given in Section 4.

First, we give some known definitions and results. In this paper we assume that $E'$ (and hence $E$) is separable. Let $\mathcal{K} = \{K \subseteq E' : K \text{ is non-empty, convex, and weak*-compact} \}$. For a continuous sublinear map $\phi$ on $E$, define $\Delta(\phi) = \sup_{\|y\| \leq 1} |\phi(y)|$. A sublinear map $\phi: E \rightarrow \mathbb{R}$ is continuous iff $\Delta(\phi) < \infty$. For every $K \in \mathcal{K}$, define the map $K \mapsto \phi(K, \cdot)$ as follows:

$$\phi(K, y) = \sup_{x \in K} \langle y, x \rangle, \quad \text{for } y \in E.$$ 

The following lemma then is a consequence of the Hahn–Banach theorem:

**Lemma 1.1.** There is a one-to-one correspondence between the elements of $\mathcal{K}$ and the continuous sublinear functionals on $E$.

For $K_1, K_2$ in $\mathcal{K}$, $x_1$ in $K_1$, and $x_2$ in $K_2$, we define the Hausdorff's metric $\Delta$ as follows:

$$\Delta(K_1, K_2) = \max[ \sup_{K_1} \inf_{K_2} \|x_1 - x_2\|, \sup_{K_2} \inf_{K_1} \|x_1 - x_2\| ].$$
It can be checked that
\[ \Delta(K_1, K_2) = \sup_{\|y\| \leq 1} |\phi(K_1, y) - \phi(K_2, y)|. \]

\((\mathcal{X}, \Delta)\) is a complete metric space. In general, \(\mathcal{X}\) need not be separable. In view of Lemma 1.1, let us introduce the following notation: for \(K \in \mathcal{X}\),
\[ \Delta(K) = \sup_{x \in \mathcal{K}} \|x\| = \sup_{\|y\| \leq 1} |\phi(K, y)| = \Delta(\phi(K, \cdot)). \]

Let \((\Omega, \mathcal{F}, P)\) be a probability space. A map \(X\) from \((\Omega, \mathcal{F})\) to \(\mathcal{X}\) is called a multivalued random variable (mrv) if for every \(y \in E\), the map \(\omega \rightarrow \phi(X(\omega), y)\) is real-valued random variable (r.v.). An mrv \(X\) is called integrable if \(\Delta(X)\) is. It follows that if \(X\) is an integrable mrv, then for each \(y \in E\), \(\phi(X, y)\) is also integrable.

For the definition of expectation or conditional expectation of an integrable mrv, see [10 or 11]. If \(X\) is an integrable mrv, then \(E(X)\) is an element of \(\mathcal{X}\) such that \(\phi(E(X), y) = E[\phi(X, y)]\) for all \(y \in E\). If \(\mathcal{G}\) is a sub-\(\sigma\)-field of \(\mathcal{F}\), then \(E(X \mid \mathcal{G})\) is an integrable mrv such that \(E(\phi(X, y) \mid \mathcal{G}) = \phi(E(X \mid \mathcal{G}), y)\) a.e., for all \(y \in E\). The existence of conditional expectation is a consequence of the following useful lemma proved in Neveu [10].

**Lemma 1.2.** Let \(y \rightarrow Z(\cdot, y)\) be a sublinear map from \(E\) to \(L^1(\Omega, \mathcal{F}, P)\), such that \(E[\sup_{\|y\| \leq 1} |Z(\cdot, y)|] < \infty\). Then there exists an integrable mrv \(X\) such that \(\phi(x, y) = Z(y)\), a.e., for every \(y \in E\).

Let \((\mathcal{F}_n)_{n=1}^{\infty}\) be a sequence of increasing sub-\(\sigma\)-fields of \(\mathcal{F}\). We shall assume that \(\mathcal{F}\) is generated by \(\bigcup_{n=1}^{\infty} \mathcal{F}_n\). Let \(T\) denote the class of all simple stopping times. \((T, \leq)\) is a directed set filtering to the right, where \(\leq\) is the usual order on \(T\). Let \(d \in \mathbb{N}\) and \(T^d\) be the class of all simple stopping times having at most \(d\) values a.s. Clearly, \(T^d\) is a directed set filtering to the right with the same order as in \(T\). Also, \(T^{d_1} \subseteq T^{d_2}\) whenever \(d_1 \leq d_2\) and \(T = \bigcup_{n=1}^{\infty} T^d\).

A sequence of mrv's \((X_n)\), such that each \((X_n)\) is \(\mathcal{F}_n\)-measurable, is called a multivalued process. For \(\tau \in T\), define \(X_{\tau} = \sum_{i=1}^{\tau} X_i 1_{\{\tau=i\}}\), where \(\tau \leq n\). A multivalued process \((X_n, \mathcal{F}_n)\) is called (i) integrable, if for every \(n \geq 1, E[A(X_n)] < \infty\), (ii) \(L^1\)-bounded, if \(\sup_{n \geq 1} E[A(X_n)] < \infty\), and (iii) of class (B) if \(\sup_{\tau \in T} E[A(X_{\tau})] < \infty\). We now define various kinds of processes.
**Definition 2.1.** An integrable multivalued process \((X_n, \mathcal{F}_n)\) is called:

(i) a martingale if \(E(X_{n+1} \mid \mathcal{F}_n) = X_n\), a.e., for every \(n \geq 1\),

(ii) an amart if the net \((EX_t)_{t \in T}\) converges in the \(\Delta\)-metric, i.e., if there is a \(K \in \mathcal{H}\) such that \(\lim_{T} \Delta(\text{EX}_t, K) = 0\),

(iii) an amart of order \(d\) if the net \((EX_t)_{t \in T'}\) converges in the \(\Delta\)-metric,

(iv) an amart of infinite order (i-amart) if \((X_n)\) is an amart of order \(d\) for every \(d \in \mathbb{N}\).

(v) a \(w^*\)-amart if there is \(K \in \mathcal{H}\) such that for every \(y \in E\),

\[
\lim_{T} \phi(\text{EX}_t, y) = \phi(K, y).
\]

Let \(A, A^d, \text{ and } A^\infty\) stand for the classes of multivalued amarts, multivalued amarts of order \(d\), and multivalued i-amarts, respectively. Clearly then, \(A \subseteq A^d\) for every \(d\) and hence \(A \subseteq A^\infty\). The inclusions are strict even when only real-valued processes are considered.

For the mrv's \(X\) and \(Y\), we define the Pettis distance as

\[
H(X, Y) = \sup_{\|v\| \leq 1} \int |\phi(X, y) - \phi(Y, y)| dP.
\]

The following inequalities involving the Pettis distance which are similar to the well-known inequalities involving the Pettis norm, can be proved in a similar way.

**Proposition 2.2.** (a) Let \(X, Y\) be \(\mathcal{G}\)-measurable mrv's, where \(\mathcal{G}\) is a sub-\(\sigma\)-field of \(\mathcal{F}\). Then the following inequality holds:

\[
\sup_{A \in \mathcal{G}} \Delta(E_A X, E_A Y) \leq H(X, Y) \leq 4 \sup_{A \in \mathcal{G}} \Delta(E_A X, E_A Y).
\]

(b) If \(X, Y\) are mrv's and \(\mathcal{G}\) is a sub-\(\sigma\)-field of \(\mathcal{F}\), then

\[
H(\text{EX} \mid \mathcal{G}), E(Y \mid \mathcal{G})) \leq H(X, Y).
\]

(c) If \(X, Y, Z\) are mrv's, then \(H(X, Z) \leq H(X, Y) + H(Y, Z)\).

Furthermore, if \(H(X, Y) = 0\), then \(\int |\phi(X, y) - \phi(Y, y)| dP = 0\) for every \(y \in E\). Since \(E\) is separable, it follows that \(X = Y\) a.e.

We now proceed to prove Riesz decomposition theorems for single-valued i-amarts. In the next section Riesz approximation theorems will be proven for multivalued processes and we shall furthermore prove that this is a characterizing property of the multivalued i-amarts. First we prove a
lemma for multivalued amarts of order $d$ which is similar to the one proved by Chacon and Sucheston [3] for single-valued amarts. We shall make use of this lemma and its obvious single-valued counterpart in the sequel. The single valued case of this result has been proven by Luu [7] and similar techniques have been used in [3, 4].

**Lemma 3.1.** Let $d \in \mathbb{N}$ be arbitrary and fixed. Let $(X_n)$ be a multivalued amart of order $d + 1$. Fix $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ such that whenever $\sigma, \tau \in T^d$ and $N \leq \sigma \leq \tau$, we have $\sup_{A \in \mathcal{F}_\sigma} \Delta(E_A X_\sigma, E_A X_\tau) < \varepsilon$.

**Proof.** Given $\varepsilon > 0$, choose $N$ so large that whenever $\sigma_1, \tau_1 \in T^{d+1}$ and $\sigma_1 \geq N$, we have $\Delta(EX_{\sigma_1}, EX_{\tau_1}) < \varepsilon$. Let $\sigma, \tau \in T^d$ be such that $\tau \geq \sigma \geq N$. Fix $A \in \mathcal{F}_\sigma$ and $N_1 = \max(\sigma, \tau)$. Define $\sigma_1 = \sigma, \tau_1 = \tau$ on $A$ and $\sigma_1 = \tau_1 = N_1$ on $A'$. Clearly $\sigma_1$ and $\tau_1 \in T$ and, since they may have at most $d + 1$ values, they are in $T^{d+1}$. Since $\sigma_1, \tau_1 \geq N$, $\Delta(EX_{\sigma_1}, EX_{\tau_1}) < \varepsilon$. But

$$\begin{align*}
\Delta(EX_{\sigma_1}, EX_{\tau_1}) &= \Delta(E_A X_{\sigma_1} + E_A' X_{\sigma_1}, E_A X_{\tau_1} + E_A' X_{\tau_1}) \\
&= \Delta(E_A X_\sigma + E_A' X_{N_1}, E_A X_\tau + E_A' X_{N_1}) \\
&= \Delta(E_A X_\sigma, E_A X_\tau).
\end{align*}$$

Therefore $\Delta(E_A X_\sigma, E_A X_\tau) < \varepsilon$. Since $A \in \mathcal{F}_\sigma$ was arbitrarily chosen, we have that $\sup_{A \in \mathcal{F}_\sigma} \Delta(E_A X_\sigma, E_A X_\tau) < \varepsilon$.

**Corollary 3.2.** Let $(X_n)$ be a multivalued $i$-amart. Fix $\varepsilon > 0$ and $d \in \mathbb{N}$. Then there is $N \in \mathbb{N}$ such that whenever $\sigma, \tau \in T^d$ and $N \leq \sigma \leq \tau$, we have $\sup_{A \in \mathcal{F}_\sigma} \Delta(E_A X_\sigma, E_A X_\tau) < \varepsilon$.

**Proof.** This follows from the last lemma and the fact that $A^\infty = \bigcap_{d=1}^{\infty} A^d$.

**Lemma 3.3.** Let $(X_n)$ be a multivalued amart. For a fixed $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that whenever $\sigma, \tau \in T$ and $N \leq \sigma \leq \tau$, we have $\sup_{A \in \mathcal{F}_\sigma} \Delta(E_A X_\sigma, E_A X_\tau) < \varepsilon$.

**Proof.** The proof is similar to that of Lemma 3.1.

Next, we proceed to prove a Riesz decomposition theorem for single-valued $i$-amarts. Riesz decomposition for amarts was proven by Edgar and Sucheston [5].

**Theorem 3.4.** Let $(X_n, \mathcal{F}_n)$ be an $E'$-valued $i$-amart (i.e., for every $n \geq 1$ and every $\omega \in \Omega$, $X_n(\omega) \in E'$ and $(X_n)$ is an $i$-amart) such that
lim inf_{n \to \infty} E \|X_n\| < \infty. Then \( X_n = Y_n + Z_n \), where \((Y_n)\) is an \( L_1 \)-bounded martingale and \((Z_n)\) is an \( i \)-mart such that for every \( d \in \mathbb{N} \),

\[
\lim \sup_{\sigma \in T^d} \int_{\{y \mid \|y\| < 1\}} |\langle y, Z_\sigma \rangle| \, dP = 0. 
\]

**Proof.** Fix \( d \in \mathbb{N} \), \( \sigma \) in \( T \), and \( \varepsilon > 0 \). By the single-valued version of Corollary 3.2, there is \( N > \sigma \) such that whenever \( \tau, \rho \in T^d \) and \( \tau, \rho > N \), we have \( \sup_{A \in \mathcal{F}_\sigma} \|E_A X_\rho - E_A X_\tau\| < \varepsilon \). Consequently, \( \sup_{A \in \mathcal{F}_\sigma} \|E_A X_\rho - E_A X_\tau\| \) is a uniform Cauchy net and hence for every \( A \in \mathcal{F}_\sigma \), there is \( \mu(A) \in \mathbb{E}' \) such that

\[
\lim \sup_{\tau \in T^d, \sigma \in \mathcal{F}_\sigma} \|E_A X_\tau - \mu(A)\| = 0. 
\]

Clearly, \( \mu \) is a finitely additive measure on \( \mathcal{F}_\sigma \) and since in particular \( \mu(A) = \lim_{n \to \infty} \int_A X_n \, dP \), by Vitali–Hahn–Saks theorem it follows that \( \mu \) is in fact countably additive on \( \mathcal{F}_\sigma \). \( \mu \) is also of bounded variation on \( \mathcal{F}_\sigma \) since variation of \( \mu \leq \lim_{n \to \infty} \inf E \|X_n\| < \infty \). Thus, for each \( n \), one can define a measure \( \mu_n \) on \( \mathcal{F}_\sigma \) as

\[
\mu_n(A) = \lim_{m \to \infty} \int_A X_m \, dP \quad \text{for} \quad A \in \mathcal{F}_n.
\]

Since \( \mathbb{E}' \) is separable, it possesses the Radon–Nikodym property. Clearly \( \mu_n \) coincides with \( \mu_{n+1} \) on \( \mathcal{F}_n \) and \( \mu_n \ll P \). Define the sequence of random variables \((Y_n)\), defined by \( Y_n = d\mu_n/dP \). It follows that \((Y_n, \mathcal{F}_n)\) is an \( L_1 \)-bounded martingale. From (2) it follows that for a fixed \( n \),

\[
\lim \sup_{\tau \in T^d, A \in \mathcal{F}_\sigma} \|E_A X_\tau - \mu_n(A)\| = 0 \quad \text{for every} \quad d \in \mathbb{N}. 
\]

Let \( d \in \mathbb{N} \) and \( \varepsilon > 0 \) be arbitrary, but fixed. Using Corollary 3.2 we choose \( N \geq 1 \) so large that whenever \( \sigma, \tau \in T^d \) are such that \( N \leq \sigma \leq \tau \), we have \( \sup_{A \in \mathcal{F}_\sigma} \|E_A X_\sigma - E_A X_\tau\| < \varepsilon \). Fix \( \sigma \in T^d \) such that \( \sigma \geq N \). Let \( m > \sigma \). Using (3) we choose \( \tau \geq m, \tau \in T^d \) such that \( \sup_{A \in \mathcal{F}_n} \|E_A X_\tau - E_A Y_m\| < \varepsilon \). Thus,

\[
\sup_{A \in \mathcal{F}_n} \|E_A X_\sigma - E_A Y_n\| \\
\leq \sup_{A \in \mathcal{F}_\sigma} \|E_A X_\sigma - E_A Y_\tau\| + \sup_{A \in \mathcal{F}_\tau} \|E_A X_\tau - E_A Y_m\| \\
\leq \sup_{A \in \mathcal{F}_\sigma} \|E_A X_\sigma - E_A Y_\tau\| + \sup_{A \in \mathcal{F}_m} \|E_A X_\sigma - E_A Y_m\| \\
\leq 2\varepsilon.
\]
Hence, defining $Z_n = X_n - Y_n$ we have that
\[
\lim_{n \to \infty} \sup_{\sigma \in T^d} \|E_A Z_n\| = 0.
\tag{4}
\]
This shows that \(\lim_{\sigma \in T^d} \sup_{\|y\| < 1} \int |\langle y, Z_n \rangle| \, dP \leq \lim_{\sigma \in T^d} 4 \sup_{A \in \mathcal{A}} \|E_A Z_\sigma\| = 0\). This proves (1). Clearly (4) shows that \(\lim_{\sigma \in T^d} \|EZ_\sigma\| = 0\). Since this is true for any \(d \in \mathbb{N}\), \((Z_n)\) is an i-amart.

The following is an amart version of Theorem 3.4.

**Theorem 3.5.** Let \((X_n)\) be an \(\mathcal{E}'\)-valued amart such that
\[
\lim_{n \to \infty} \inf_{E} \|X_n\| < \infty.
\]
Then \(X_n = Y_n + Z_n\), where \((Y_n)\) is an \(L_1\)-bounded martingale and \((Z_n)\) is an amart such that
\[
\lim_{\sigma \in T^d} \sup_{\|y\| < 1} \int |\langle y, Z_\sigma \rangle| \, dP = 0.
\]

**Proof.** The proof is similar to that of the preceding theorem and uses Lemma 3.3 instead of Lemma 3.1.

As a straightforward application of Theorem 3.4 we have the following result:

**Corollary 3.6.** Every real valued i-amart \((X_n)\) such that
\[
\lim_{n \to \infty} \inf_{E} |X_n| < \infty,
\]
converges in probability.

**Proof.** If \((X_n)\) is a real-valued i-amart, Theorem 3.4 implies that \(Z_n \to 0\) in \(L_1\) and hence in probability. Since \((Y_n)\) is \(L_1\)-bounded, \(Y_n \to Y\) a.e. for some integrable r.v. \(Y\). This shows that \(X_n \to Y\) in probability.

Even \(L_1\)-bounded real-valued i-amarts need not converge a.s.

Next we prove some results concerning multivalued i-amarts. The first one characterizes a class of i-amarts in terms of convergence in Pettis distance.

**Theorem 4.1.** Let \((X_n, \mathcal{F}_n)\) be an integrable process. Then the following are equivalent:

(i) there is an integrable mrv \(X_\infty\) such that \(\lim_{n \to \infty} H(X_n, X_\infty) = 0\)

(ii) \((X_n)\) is an i-amart and there is a martingale \((Y_n, \mathcal{F}_n)\) and an integrable mrv \(Y\) such that \(\lim_{n \to \infty} H(X_n, Y_n) = 0\) and \(\lim_{n \to \infty} H(Y_n, Y) = 0\).

**Proof.** (i) \(\to\) (ii) Since \(\lim_{n \to \infty} H(X_n, X_\infty) = 0\), from Proposition 2.2(b) it follows that
\[
\lim_{n \to \infty} H(E(X_n \mid \mathcal{F}_n), E(X_\infty \mid \mathcal{F}_\infty)) \leq \lim_{n \to \infty} H(X_n, X_\infty) = 0.
\]
Thus,

$$\lim_{n \to \infty} H(X_n, E(X_n \mid F_n)) = 0, \quad \lim_{n \to \infty} H(E(X_{n+1} \mid F_n), X_{n+1}) = 0.$$ 

Set $Y_n = E(X_{n+1} \mid F_n)$ and $Y = X_{n+1}$. Clearly $(Y_n, F_n)$ is a martingale and we have $\lim_{n \to \infty} H(X_n, Y_n) = 0$ and $\lim_{n \to \infty} H(Y_n, Y) = 0$.

We now show that $(X_n, F_n)$ is an i-amart. Let $d \in \mathbb{N}$ be arbitrary, but fixed, and $\varepsilon > 0$. There is $N \geq 1$ such that whenever $m, n \geq N$, we have $H(X_m, X_n) < \varepsilon/d^2$. Let $\sigma, \tau \in T^d$ be such that $\sigma, \tau \geq N$. Then

$$\Delta(\mathcal{E}X_{\sigma}, \mathcal{E}X_{\tau}) \leq \sum_{m, n \geq N} \Delta(E(X_m 1_{\{\sigma = m\} \cap \{\tau = n\}}, E(X_n 1_{\{\sigma = m\} \cap \{\tau = n\}}))$$

Since there are $d^2$ terms in the summation in the right-hand side and, since each term is majorised by $\sup_{A \in \mathcal{E}_n} \sup_{m, n \geq N} A(E X_m, E X_n)$, we have $\Delta(\mathcal{E}X_{\sigma}, \mathcal{E}X_{\tau}) \leq d^2 \sup_{m, n \geq N} H(X_m, X_n) < \varepsilon$. This shows that $X_n$ is an amart of order $d$. But since $d$ was arbitrary, $(X_n)$ is an i-amart.

(ii) $\Rightarrow$ (i) This implication is obvious if we set $X_{n+1} = Y$ and observe that $\lim_{n \to \infty} H(X_n, X_{n+1}) \leq \lim_{n \to \infty} H(X_n, Y_n) + \lim_{n \to \infty} H(Y_n, Y)$. 

We now proceed to prove the main result which characterises i-amarts in terms of Riesz approximation.

**Theorem 4.2.** Let $(X_n, F_n)$ be a multivalued process such that $\lim_{n \to \infty} \inf E(\Delta(X_n)) < \infty$. Then $(X_n)$ is an i-amart if and only if there is a unique $L_1$-bounded martingale $(Y_n, F_n)$ such that for every fixed $d \in \mathbb{N}$ and $\sigma \in T$, we have

$$\limsup_{n \to \infty} d(E X_n, E Y_n) = 0.$$ 

Proof. (only if) Let us prove the uniqueness first. Suppose that there is a martingale $(Y_n)$ satisfying the same properties as stated in the theorem. Then for a fixed $m$,

$$\limsup_{n \to \infty} \Delta(E_A Y_n, E_A Y'_n) = 0.$$ 

However, the left-hand side is $\sup_{A \in \mathcal{F}_m} \Delta(E_A Y_n, E_A Y'_m)$ and hence $E_A Y_m = E_A Y'_m$ for every $A \in \mathcal{F}_m$. Hence $Y_m = Y'_m$ a.e. This proves uniqueness.

Consider the real-valued i-amart $(\phi(X_n, y), F_n)$ for a fixed $y \in E$. By Theorem 3.4 we know that $\phi(X_n, y) = f_n(y) + g_n(y)$, where $(f_n(y), F_n)$ is a martingale. From (4) we know that

$$\limsup_{\tau \in T^d} \left| \sum_{A \in \mathcal{F}_n} \int_A (\phi(X_\tau, y) - f_\tau(y)) dP \right| = 0. \quad (5)$$
It follows that for a fixed $m$,
\[
\lim_{n \to \infty} \sup_{A \in \mathcal{F}_m} \left| \int_A (\phi(X_n, y) - f_n(y)) \, dP \right| = 0
\]
and hence
\[
\lim_{n \to \infty} \sup_{A \in \mathcal{F}_m} \left| \int_A E[\phi(X_n, y) | \mathcal{F}_m] - \int_A f_m(y) \right| = 0.
\]
Therefore,
\[
\lim_{n \to \infty} E[\phi(X_n, y) | \mathcal{F}_m] = f_m(y).
\]
(6)

It therefore follows that $y \to f_m(y)$ is a sublinear functional on $E$. Due to (6),
\[
E \left[ \sup_{\|y\| \leq 1} |f_m(y)| \right] \leq E \left[ \lim_{n \to \infty} \inf E(A(X_n) | \mathcal{F}_m) \right] \\
\leq \lim_{n \to \infty} \inf E(E(A(X_n) | \mathcal{F}_m)) \\
= \lim_{n \to \infty} \inf E(A(X_n)) \\
< \infty.
\]

Therefore, by Lemma 1.2, there is an integrable mrw $Y_m$ such that $\phi(Y_m, y) = f_m(y)$. Clearly $(Y_n, \mathcal{F}_n)$ is an $L_1$-bounded martingale. From (5) it follows that
\[
\lim_{\tau \in T^d} \sup_{A \in \mathcal{F}_\tau} \left| \int_A \phi(X_\tau, y) - \phi(Y_\tau, y) \right| = 0.
\]
(7)

Let $\sigma \in T$. Fix $\varepsilon > 0$. Using Corollary 3.2 choose $N \geq \sigma$ such that whenever $\rho, \tau \in T^d$ and $N \leq \rho \leq \tau$, we have
\[
\sup_{A \in \mathcal{F}_\rho} \Delta(E_A X_\rho, E_A X_\tau) < \varepsilon.
\]
Since $\sigma \leq \rho$, it follows that the net $(E_A X_\tau, \tau \in T^d)$ is uniformly Cauchy in $A \in \mathcal{F}_\sigma$ and hence for some $M(A) \in \mathcal{K}$,
\[
\lim_{\tau \in T^d} \sup_{A \in \mathcal{F}_\tau} \Delta(E_A X_\tau, M(A)) = 0.
\]
(8)

Since $(Y_n)$ is a martingale, from (7) we can say that
\[
\lim_{\tau \in T^d} \sup_{A \in \mathcal{F}_\tau} |\phi(E_A X_\tau, y) - \phi(E_A Y_\sigma, y)| = 0.
\]
(9)
Combining (8) and (9) one obtains that \( M(A) = E_A Y_\tau = E_A Y_\sigma \) for \( \tau \geq \sigma \). This concludes the proof.

Conversely, suppose that there is an \( L_1 \)-bounded martingale with the given property. Let \( K = EY_1 \) and fix \( d \in \mathbb{N} \):

\[
\lim_{\sigma \downarrow \tau} A(EX_\sigma, K) \leq \lim_{\sigma \downarrow \tau} A(EX_\sigma, EY_\sigma) \leq \lim_{\sigma \downarrow \tau} H(X_\sigma, Y_\sigma) = 0.
\]

Since \( d \in \mathbb{N} \) is arbitrary, \( (X_n) \) is an i-amart.

**Corollary 4.3.** Let \( (X_n, \mathcal{F}_n) \) be a multivalued process such that \( \lim_{n \to \infty} \inf EX_\Delta(X_n) < \infty \). Then \( (X_n) \) is an i-amart if and only if there is a unique \( L_1 \)-bounded martingale \( (Y_n, \mathcal{F}_n) \) such that for every fixed \( d \in \mathbb{N} \),

\[
\lim_{n \to \infty} H(X_n, Y_n) = 0.
\]

**Proof.** This follows from Theorem 4.2 and Proposition 2.2(a).

**Remark.** Let \( \mathcal{E} = \mathcal{E}' = \mathbb{R}^2 \). Let \( S \) be the square with vertices at \((0,0)\), \((0,1)\), \((1,0)\), and \((1,1)\) and let \( A_n = \) the triangle with vertices at \((0,1)\), \((\frac{1}{2}, 1 + 1/n)\) and \((1,1)\). Define the process \( X_n = S \cup A_n \) over any probability space. The process \( Y_n = S \) for all \( n \geq 1 \) is obviously a martingale. Since \( H(X_n, Y_n) = 1/n \), it follows from Theorem 4.1 that \( (X_n) \) is an i-amart. It is obvious that \( X_n \) cannot be expressed as the sum of \( Y_n \) and another process.

Some results on the convergence of multivalued processes are given below. Since the martingale \( (Y_n) \) in the Riesz approximation of \( (X_n) \) is \( L_1 \)-bounded, the following theorem of Neveu [10] applies to it.

**Theorem 4.4.** Let \( (X_n, \mathcal{F}_n) \) be an \( L_1 \)-bounded multivalued martingale. Then there is an integrable mrv \( X_\infty \) such that \( \lim_{n \to \infty} \phi(X_n, y) = \phi(X_\infty, y) \), a.s., for every \( y \in \mathcal{E} \). Moreover, if \( X_\infty \) takes its values in a separable subset of \( \mathcal{H} \), then we have \( \lim_{n \to \infty} A(X_n, X_\infty) = 0 \) a.s.

**Corollary 4.5.** (i) Let \( (X_n) \) be a multivalued i-amart such that \( \lim_{n \to \infty} \inf E[\Delta(X_n)] < \infty \), \( (Y_n) \) be the approximating martingale (as in Theorem 4.2), and \( X_\infty \) the limiting mrv of \( (Y_n) \) (as in Theorem 4.5). Then for every \( y \in \mathcal{E} \),

\[
\lim_{n \to \infty} \phi(X_n, y) = \phi(X_\infty, y) \quad \text{in probability}.
\]

(ii) If, moreover, \( (Y_n) \) converges in the Pettis distance, then

\[
\lim_{n \to \infty} H(X_n, X_\infty) = 0.
\]
(iii) If $X_\infty$ takes values in a separable subset of $\mathcal{H}$ and
\[
\lim_{n \to \infty} \Delta(X_n, Y_n) = 0 \text{ in probability, then}
\]
\[
\lim_{n \to \infty} \Delta(X_n, X_\infty) = 0 \quad \text{in probability.}
\]

Proof. (i) First, for the i-amart $(X_n)$, the existence of the martingale
$(Y_n)$ and the limit $X_\infty$ of the martingale $(Y_n)$ are guaranteed by
Theorem 4.2 and Theorem 4.4, respectively. Fix $y \in E$. From Theorem 4.4 it
follows that $\phi(Y_n, y) \to \phi(X_\infty, y)$ a.s., and hence in probability. By
Corollary 4.3, it follows that $|\phi(X_n, y) - \phi(Y_n, y)| \to 0$ in $L_1$ and hence also
in probability. This proves (i).

(ii) Suppose that there is an mrv $Y$ such that $\lim_{n \to \infty} H(Y_n, Y) = 0$.
From Corollary 4.3, $\lim_{n \to \infty} H(X_n, Y_n) = 0$. Hence from Theorem 4.1, it
follows that $\lim_{n \to \infty} H(X_n, Y) = 0$ and that $X_\infty = Y$. This proves (ii).

(iii) This follows from Theorem 4.5.

We now prove a Riesz approximation theorem for multivalued amarts.

**Theorem 4.6.** (i) Let $(X_n, \mathcal{F}_n)$ be a multivalued weak*-amart such that
\[
\lim_{n \to \infty} \inf E(\Delta(X_n)) < \infty.
\]
Then there is a unique $L_1$-bounded martingale
$(Y_n)$ such that if we define $Z_n(y) = \phi(X_n, y) - \phi(Y_n, y)$, then for every $y \in E$,
$(Z_n(y))$ is an amart such that $Z_n(y) \to 0$ a.s. and in $L_1$.

(ii) If, moreover, $(X_n)$ is an amart then
\[
\sup_{\|y\| \leq 1} \int |Z_n(y)| \, dP \to 0.
\]

**Proof.** (i) This can be proved using the same line of argument as that
in Theorem 4.2.

(ii) Suppose, moreover, that $(X_n)$ is an amart. Fix $\varepsilon > 0$. Using
Lemma 3.3, choose $N$ such that if $\sigma, \tau \in T$ and $N \leq \sigma \leq \tau$, then
\[
\sup_{A \in \mathcal{F}_\sigma} d(E_A X_\sigma, E_A X_\tau) < \varepsilon/2.
\]
Fix $\sigma \in T$ such that $\sigma \geq N$. Using the argument of Theorem 4.2, choose $\tau$ in $T$, $\tau \geq \sigma$ such that
\[
\sup_{A \in \mathcal{F}_\sigma} d(E_A X_\tau, E_A Y_\tau) < \varepsilon/2.
\]
Therefore,
\[
\sup_{\|y\| \leq 1} \int |Z_\sigma(y)| \, dP \leq 4 \sup_{A \in \mathcal{F}_\sigma} (E_A X_\sigma, E_A Y_\sigma) < 4\varepsilon.
\]
Thus $\sup_{\|y\| \leq 1} \int |Z_\tau(y)| \, dP \to 0$.

**Corollary 4.7.** ([1].) Let $(X_n)$ be a weak*-amart of class (B). Then
there is a null set $N$ and an mrv $X_\infty$ such that, outside of $N$,
\[
\lim_{n \to \infty} \phi(X_n, y) = \phi(X_\infty, y) \quad \text{for every } y \in E.
\]
Proof. Since \((X_n)\) is of class (B), we may assume (a proof is given in [1]) that \(\sup_n \mathcal{A}(X_n) \in L_1\). From Theorem 4.5 and Theorem 4.7, we have an integrable mrv \(X_\infty\) such that for every \(y \in \mathcal{E}\), \(\phi(X_n, y) \rightarrow \phi(X_\infty, y)\) a.e. Since \(\mathcal{E}\) is separable and \(\sup \mathcal{A}(X_n) < \infty\) a.e., one can have a single null set independent of \(y\) such that outside of that null set, for every \(y \in \mathcal{E}\), \(\phi(X_n, y) \rightarrow \phi(X_\infty, y)\).

REFERENCES


