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A Characterization of Minimizable Metrics in the Multifacility Location Problem

HANS-JÜRGEN BANDELT, VICTOR CHEPOI AND ALEXANDER V. KARZANOV

In the minimum 0-extension problem (a version of the multifacility location problem), one is given a metric m on a subset X of a finite set V and a non-negative function c on the unordered pairs of elements of V. The objective is to find a semimetric m' on V that minimizes the inner product $c \cdot m'$, provided that m' coincides with m within X and each element of V is at zero distance from X. For m fixed, this problem is solvable in strongly polynomial time if m is *minimizable*, which means that for any superset V and function c, the minimum objective value is equal to that in the corresponding linear relaxation.

In [9], Karzanov showed that the path metric of a graph H is minimizable if and only if all isometric cycles of H have length four and the edges of H can be oriented so that non-adjacent edges in each 4-cycle have opposite orientations along the cycle (such graphs are called *frames* in [9]). Extending this result to general metrics m, we show that m is minimizable if and only if m is modular and its underlying graph is a frame.

1. INTRODUCTION

A *semimetric* on a set X is a function $d : X \times X \to \mathbf{R}_+$ satisfying $d(x, x) = 0$, $d(x, y) = d(y, x)$, and $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$. If, in addition, $d(x, y) > 0$ for all $x \neq y$, then d is called a *metric*. A particular instance is the *path metric* d_G of a connected graph G: the distance $d_G(x, y)$ is the minimum number of edges in a path of G connecting the nodes x and y. A semimetric d' on a superset $V \supseteq X$ is called an *extension* of d if $d'(x, y) = d(x, y)$ for all $x, y \in X$, and a 0-*extension* if, in addition, for each $v \in V$, there exists some $x \in X$ such that $d'(v, x) = 0$.

Now, consider a metric m on a subset X of a finite set V and a non-negative integer-valued function c on the set $\binom{V}{2}$ of unordered pairs of elements (*points*) of V. The *minimum* 0-*extension problem* can be stated as follows:

$$\text{Find a 0-extension } m' \text{ of } m \text{ to } V \text{ minimizing } c \cdot m' := \sum\left(c(e)m'(e) : e \in \binom{V}{2}\right). \tag{1.1}$$

This problem is equivalent to a variant of the *multifacility location problem*, in which the existing facilities are located at points of X, the elements of $V - X$ are thought of as new facilities to be placed at points of X, and the numbers $c(x, y)$ represent a measure of mutual communication or supporting task between facilities x and y. (For a survey on location problems, see, e.g., [11].) When m is the path metric of the complete graph K_p with p nodes, (1.1) turns into the *minimum* p-*terminal* (or p-*way*) *cut problem*, which is known to be solvable in polynomial time if $p = 2$ (as being the classical minimum cut problem [7]), and strongly NP-hard if $p = 3$ [5].

Let $\tau(V, c, m)$ denote the minimum objective value $c \cdot m'$ in (1.1), and let $\tau^*(V, c, m)$ denote the minimum objective value in its relaxation:

$$\text{Find an extension } m' \text{ of } m \text{ to } V \text{ with } c \cdot m' \text{ minimum.} \tag{1.2}$$

Since every 0-extension is an extension, $\tau(V, c, m) \geq \tau^*(V, c, m)$. We call a metric m *minimizable* if $\tau(V, c, m) = \tau^*(V, c, m)$ holds for any choice of a finite superset V of X and non-negative function c. Since (1.2) is a linear program whose constraint matrix is of size

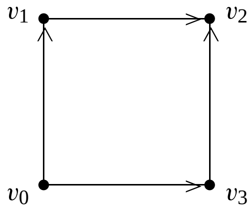


FIGURE 1. An orientation of a 4-cycle.

polynomial in $|V|$, this problem is solvable in strongly polynomial time using a version of the ellipsoid method [12]. This implies that for every minimizable metric m, (1.1) is solvable in strongly polynomial time as well. It turned out that the class of graphs whose path metrics are minimizable is rather large.

THEOREM A ([9]). *The metric d_H of a graph H is minimizable if and only if H is hereditary modular and orientable.*

Recall that a metric m on X is called *modular* if every three points $x_1, x_2, x_3 \in X$ have a *median*, that is, a point $x \in X$ satisfying $m(x_i, x) + m(x, x_j) = m(x_i, x_j)$ for all $1 \leq i < j \leq 3$. A graph H is called *modular* if its path metric d_H is modular, and *hereditary modular* if every isometric subgraph of H is modular. (A subgraph H' of H is *isometric* if $d_{H'}(u, v) = d_H(u, v)$ for all nodes u, v of H'; in other words, d_H is an extension of $d_{H'}$.) Every modular graph H is bipartite; moreover, one can easily show that the cycle space of H has a basis comprising only 4-cycles. We say that H is *orientable* if its edges can be oriented so that opposite (non-adjacent) edges in every 4-cycle have opposite orientations along the cycle; see Figure 1. For example, the complete bipartite graph $K_{p,r}$ is orientable if and only if $\min\{p, r\} \leq 2$; further the graph $K_{3,3}^-$, that is, $K_{3,3}$ minus one edge is not orientable (see Figure 2(b)). (In the orientable case, the orientation turns a modular graph into the Hasse diagram of an ordered set in which every order-interval constitutes a modular lattice. Indeed, every order-interval consists of the nodes on shortest paths between its end points, and therefore [3, Theorem 4.7] applies.) Following [9], we call an orientable hereditary modular graph a *frame*.

In this paper we show that Theorem A can be extended to give a complete characterization of minimizable general metrics. Given a metric m on X, its *underlying graph $H(m)$* is obtained from the complete graph on X by deleting all edges xy such that there is a node z *between* x and y, i.e., $z \neq x, y$ and $m(x, z) + m(z, y) = m(x, y)$. In other words, $H(m)$ is the least connected graph on X in which any two nodes are connected by a path shortest for m.

We can now state the result of this paper.

THEOREM. *A metric m is minimizable if and only if m is modular and its underlying graph $H(m)$ is a frame.*

2. PRELIMINARIES

We begin with reformulating the property that a metric m on a set X is minimizable in polyhedral terms. We regard any semimetric on a finite superset $V \supseteq X$ as a vector of the euclidean space $\mathbf{R}^{\binom{V}{2}}$ whose coordinates are indexed by the edges of the complete graph on V. The set of extensions of m to V forms a polyhedron in $\mathbf{R}^{\binom{V}{2}}$, denoted by $\mathcal{P}_{V,m}$. For $m', m'' \in \mathcal{P}_{V,m}$, we say that m'' *decomposes* m' in $\mathcal{P}_{V,m}$ if $m' \geq \lambda m'' + (1 - \lambda)m''_0$ for some

$m_0'' \in \mathcal{P}_{V,m}$ and $0 < \lambda \leq 1$. If no extension $m'' \neq m'$ decomposes m', then m' is called *extreme*. The extreme extensions are precisely the vertices of the dominant $\mathcal{P}_{V,m} + \mathbf{R}_+^{\binom{V}{2}}$ of the polyhedron $\mathcal{P}_{V,m}$. In particular, every 0-extension of m is extreme.

It is easy to see that any extension that decomposes an optimal solution of (1.2) is an optimal solution as well. On the other hand, by linear programming arguments, every extreme extension is a unique optimal solution of (1.2) for some $c : \binom{V}{2} \to \mathbf{R}_+$. This implies the following characterization of minimizable metrics (cf. [9]):

$$\text{a metric } m \text{ on } X \text{ is minimizable if and only if for all finite supersets } V \text{ of } X, \text{ every extreme extension of } m \text{ to } V \text{ is a 0-extension.} \tag{2.1}$$

This property suggests the following approach to proving our theorem: in order to decide whether a given metric m is minimizable or not, it suffices to show that any extension of m is decomposable by a 0-extension or to find an extreme extension which is not a 0-extension. In order to verify that an extension is extreme we will use the fact that the extreme extensions have maximal sets of shortest paths. More precisely, let d be a semimetric on $V \supseteq X$. A *path on* V is a finite sequence $P = (v_0, v_1, \ldots, v_k)$ of points of V. The *d-length* $d(P)$ of P is $d(v_0, v_1) + \cdots + d(v_{k-1}, v_k)$, and P is called *d-shortest* if $d(P) = d(v_0, v_k)$. We say that P is an *X-path* on V if $v_0, v_k \in X$, and denote the set of d-shortest X-paths by $\mathcal{I}(d) = \mathcal{I}(X, d)$. It is not difficult to see that:

$$\text{for } m', m'' \in \mathcal{P}_{V,m},\ m'' \text{ decomposes } m' \text{ if and only if every } m'\text{-shortest } X\text{-path is } m''\text{-shortest, i.e., } \mathcal{I}(m') \subseteq \mathcal{I}(m''); \text{ this inclusion is strict when } m' \neq m''. \tag{2.2}$$

Next, in our proof we will use the fact that a modular metric and the path metric of its underlying graph have the same set of shortest paths. For a connected graph $H = (X, E)$ and a *length function* $\ell : E \to \mathbf{R}_+$, let $d_{H,\ell}$ denote the semimetric on X, where $d_{H,\ell}(x, y)$ is the minimum ℓ-length $\ell(P) = \ell(x_0x_1) + \cdots + \ell(x_{k-1}x_k)$ of a path $P = (x = x_0, x_1, \ldots, x_{k-1}, x_k = y)$ between x and y in H. If H is the underlying graph of a metric m and ℓ is the restriction of m to E, then $d_{H,\ell}$ is just m. For an edge xy of $H(m)$, we therefore refer to $m(x, y)$ as the length of xy. We say that two edges e, e' of H are *projective* if there is a sequence $e = e_0, e_1, \ldots, e_k = e'$ of edges such that every two consecutive edges e_i, e_{i+1} are opposite in some 4-cycle of H. A maximal set of mutually projective edges is called an *orbit*. Each bridge e of H constitutes an orbit consisting only of e (recall that a *bridge* is an edge whose removal disconnects H).

PROPOSITION 1 ([1]).

(i) *If m is a modular metric, then the graph $H(m)$ is modular and m is constant on the edges of each orbit of $H(m)$.*
(ii) *Conversely, if $H = (X, E)$ is a modular graph and ℓ is a positive length function on E which is constant within each orbit of H, then the metric $d_{H,\ell}$ is modular, and the metrics d_H and $d_{H,\ell}$ have the same sets of shortest paths.*

Finally, we will use the following properties of hereditary modular graphs.

PROPOSITION 2 ([2]).

(i) *A graph is hereditary modular if and only if it is bipartite and contains no isometric cycles of length six or more.*
(ii) *A modular but not hereditary modular graph contains an isometric 6-cycle.*

3. Proof of the 'only if' Part

Our method of proof is close to that for the corresponding part of Theorem 1.1 in [9]. Although the objects we deal with are more general, the constructions we apply in subsequent proofs of this section are relatively simpler than those used in [9].

Let m be a metric on X. We will rely on the following simple fact.

LEMMA 1 ([9]). *Let m_0 be the restriction of m to a set $X_0 \subseteq X$. Let m'_0 be an extreme extension of m_0 to a set V_0 with $V_0 \cap X = X_0$. Then there exists an extreme extension m' of m to $V = V_0 \cup X$ which coincides with m'_0 on V_0.*

Indeed, define $d(x, y)$ to be $m'_0(x, y)$ for $x, y \in V_0$, $m(x, y)$ for $x, y \in X$, and $\min\{m'_0(x, z) + m(z, y) : z \in X_0\}$ for $x \in V_0$ and $y \in X$. One can easily check that d is a metric on V and, therefore, d is an extension of m. Take any extreme extension m' of m that decomposes d. Then the restriction of m' to V_0 decomposes m'_0 in $\mathcal{P}_{V_0,m_0}$. Since m'_0 is extreme in $\mathcal{P}_{V_0,m_0}$, the semimetric m' coincides with m'_0 on V_0, as required.

Next we will show that if the graph $H(m)$ is not a frame, then m has an extreme extension to some $V \supset X$ which is not a 0-extension. By (2.1), this would imply that m is not minimizable.

LEMMA 2. *Let $H(m)$ be non-modular. Then m is not minimizable.*

PROOF. Since $H(m)$ is non-modular, by Proposition 1(i), m is not modular either. So there exist points $x_1, x_2, x_3 \in X$ which do not have a median for m. Let m_0 denote the restriction of m to $X_0 = \{x_1, x_2, x_3\}$. Define the numbers $r_1, r_2, r_3 \geq 0$ so that $r_i + r_j = m(x_i, x_j)$ for all $1 \leq i < j \leq 3$; such numbers exist because m is a metric, and they are unique. Add a new point x and define the distance from x to x_i to be r_i for $i = 1, 2, 3$. This gives an extension m'_0 of m_0 to the set $V' = X_0 \cup \{x\}$. Evidently, m'_0 is an extreme extension. By Lemma 1, m has an extreme extension m' to the set $X \cup \{x\}$ that coincides on V' with m'_0. Since the triplet x_1, x_2, x_3 does not have a median, $m'(x, y) > 0$ for all $y \in X$, i.e., m' is not a 0-extension. Hence, m is not minimizable. □

LEMMA 3. *Let $H(m)$ be modular but not orientable. Then m is not minimizable.*

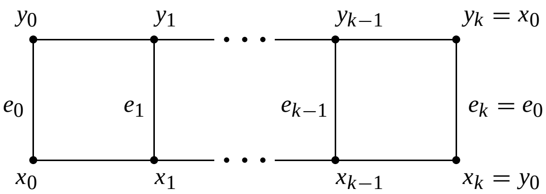
PROOF. Since $H(m)$ is not orientable, it contains a *Möbius sequence* ('orientation-reversing dual cycle'), i.e., a circular sequence $(e_0 = x_0y_0, e_1 = x_1y_1, \ldots, e_k = x_ky_k = e_0)$ of edges such that:

(i) the edges $e_0, \ldots, e_{k-1}$ are distinct;
(ii) x_ix_{i+1} and y_iy_{i+1} are edges of $H(m)$ for $i = 0, \ldots, k-1$;
(iii) $x_0 = y_k$ and $y_0 = x_k$ (yielding the 'twist');

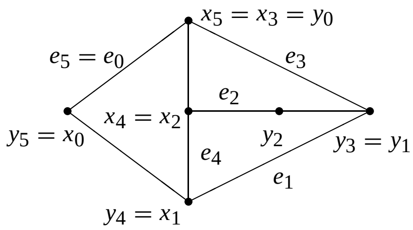
see Figure 2. Let X_0 be the set of (different) nodes occurring among $x_0, y_0, \ldots, x_{k-1}, y_{k-1}$, and let m_0 be the restriction of m to X_0. We extend the complete graph on X_0 to the graph $G' = (V', E')$ by adding k new nodes $z_1, \ldots, z_k = z_0$ and $3k$ new edges x_iz_i, y_iz_i and z_iz_{i+1} for $i = 0, \ldots, k-1$.

Since the edges $e_0, e_1, \ldots, e_{k-1}$ are projective, by Proposition 1(i) they have the same length, say α. Any two edges x_ix_{i+1} and y_iy_{i+1} are opposite in a 4-cycle and, therefore, they have the same length

$$\beta_i := m(x_i, x_{i+1}) = m(y_i, y_{i+1}) \qquad \text{for} \quad i = 0, \ldots, k-1.$$



(*a*) Generic instance.



(*b*) $K_{3,3}^-$.

FIGURE 2. Möbius sequences.

We define a length function ℓ on the edge set E' by letting

$$\ell(vw) = m(v, w) \qquad \text{for} \quad v, w \in X_0;$$
$$\ell(x_i z_i) = \ell(y_i z_i) = \alpha/2 \qquad \text{for} \quad i = 1, \ldots, k;$$
$$\ell(z_i z_{i+1}) = \beta_i \qquad \text{for} \quad i = 1, \ldots, k.$$

We assert that $d = d_{G',\ell}$ is an extension of m_0. To see this, it suffices to verify $\ell(P) \geq m(v, w)$ for any simple path P in G' with end nodes v, w from X_0 and all intermediate nodes in $V' - X_0$. We may assume without loss of generality that $P = (v, z_i, \ldots, z_j, w)$ with $v = x_i$ and $w \in \{x_j, y_j\}$ for some $0 \leq i \leq j \leq k - 1$. Then

$$\begin{aligned}\ell(P) &= \alpha/2 + \sum_{p=i}^{j-1} \beta_p + \alpha/2 \\ &= \ell(x_j y_j) + \sum_{p=i}^{j-1} \ell(x_p x_{p+1}) \\ &\geq m(x_j, w) + m(v, x_j) \geq m(v, w).\end{aligned}$$

Take an extreme extension m'_0 of m_0 to V' that decomposes d. By Lemma 1, there exists an extreme extension m' of m to $V = V' \cup X$ that coincides with m'_0 on V'. We assert that m' cannot be a 0-extension. Indeed, consider the paths $P_i = (x_i, z_i, y_i)$, $Q_i = (x_i, z_i, z_{i+1}, y_{i+1})$, and $R_i = (y_i, z_i, z_{i+1}, x_{i+1})$ for $i = 1, \ldots, k$. By the definition of ℓ and taking into account that the paths (x_i, y_i, y_{i+1}) and (y_i, x_i, x_{i+1}) are m-shortest by Proposition 1, we conclude that

$$\begin{aligned}\ell(P_i) &= \alpha = m(x_i, y_i), \\ \ell(Q_i) &= \alpha + \beta_i = m(x_i, y_i) + m(y_i, y_{i+1}) = m(x_i, y_{i+1}), \\ \ell(R_i) &= \alpha + \beta_i = m(y_i, x_i) + m(x_i, x_{i+1}) = m(y_i, x_{i+1}).\end{aligned}$$

Hence, $P_i, Q_i, R_i \in \mathcal{I}(V, m')$ by (2.2). Suppose that m' is a 0-extension of m. Then for each new node z_i there exists a node $w_i \in X$ such that $m'(z_i, w_i) = 0$. The only m-shortest path on X between x_i and y_i is $x_i y_i$, whence $w_i \in \{x_i, y_i\}$. Assume $w_0 = x_0$; the case $w_0 = y_0$ is analogous. Then $w_1 = x_1$; otherwise, $w_1 = y_1$ would imply that R_0 is not m'-shortest because $\alpha + \beta_0 + \alpha > m'(y_0, x_1)$. Similarly, $w_2 = x_2$, and so on, until one arrives at $w_0 = x_0 = y_k$, obtaining a contradiction. Thus, m is not minimizable. □

LEMMA 4. *Let $H(m)$ be modular and orientable but not hereditary modular. Then m is not minimizable.*

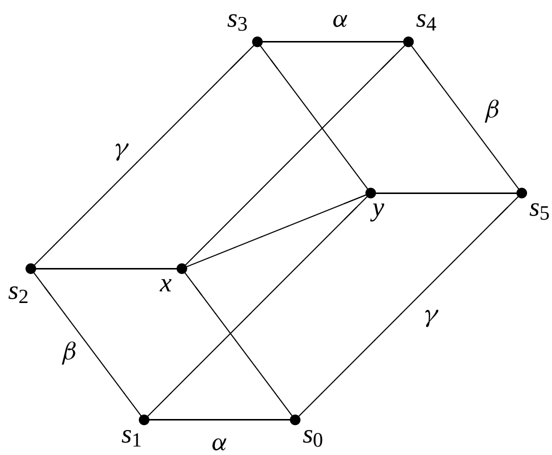


FIGURE 3. Graph G' in the proof of Lemma 4.

PROOF. By Proposition 2(ii) the graph $H(m)$ contains an isometric 6-cycle $C = (s_0, s_1, \ldots, s_5, s_0)$. Let m_0 be the restriction of m to $X_0 = \{s_0, \ldots, s_5\}$. Since C is isometric, each path $(s_i, s_{i+1}, s_{i+2}, s_{i+3})$ is shortest in $H(m)$ and, therefore, it is m-shortest (taking indices modulo 6). As s_{i+2} and s_{i+5} are between s_i and s_{i+3}, and vice versa, it follows that the m-lengths of opposite edges of C are equal:

$$m(s_0, s_1) = m(s_3, s_4) =: \alpha,$$
$$m(s_1, s_2) = m(s_4, s_5) =: \beta,$$
$$m(s_2, s_3) = m(s_5, s_0) =: \gamma.$$

We construct an extension m'_0 of m_0 as follows. Assume $\alpha \leq \beta, \gamma$. Consider the graph $G' = (V', E')$ with $V' = X_0 \cup \{x, y\}$ shown in Figure 3 (G' is the skeleton of the cube with one diagonal added). Note that G' is not orientable (as it includes $K^-_{3,3}$). Assign the following lengths $\ell(e)$ to its edges $e \in E'$:

$$\ell(s_i s_{i+1}) = m(s_i, s_{i+1}) \qquad \text{for} \quad i = 0, \ldots, 5,$$
$$\ell(s_2 x) = \ell(s_5 y) = \alpha,$$
$$\ell(s_0 x) = \ell(s_3 y) = \beta,$$
$$\ell(s_4 x) = \ell(s_1 y) = \gamma,$$
$$\ell(xy) = \beta + \gamma - \alpha.$$

Then x and y are medians of the triplets $\{s_0, s_2, s_4\}$ and $\{s_1, s_3, s_5\}$, respectively. Note also that $\ell(xy)$ is as small as possible subject to the requirement that the ℓ-length of each path from s_i to s_{i+3} passing through the edge xy be at least $m(s_i, s_{i+3}) = \alpha + \beta + \gamma$, taking into account that $\alpha \leq \beta, \gamma$. Then $m'_0 = d_{G',\ell}$ is an extension of m_0.

We first prove that m'_0 is an extreme extension, by showing that m'_0 is uniquely determined by the set of ℓ-shortest X_0-paths in G' cf. (2.2). To see the latter, note that the distances from x to s_0, s_2, s_4 are determined by the ℓ-lengths of the paths (s_0, x, s_2), (s_2, x, s_4), (s_4, x, s_0), and then the distances from x to s_1, s_3, s_5 are determined by the paths (s_1, s_2, x, s_4), (s_3, s_4, x, s_0),

(s_5, s_0, x, s_2). Similarly, one can uniquely characterize the distance from y to each s_i. Finally, the distance between x and y is determined by the ℓ-shortest path (s_2, x, y, s_5) because $m'_0(s_2, x)$ and $m'_0(s_5, y)$ have already been determined.

Let m' be an extreme extension of m to $X \cup \{x, y\}$ that coincides with m'_0 on V'. Suppose that m' is a 0-extension, and let u and v be the points of X obeying $m'(u, x) = m'(v, y) = 0$. We assert that G' is isomorphic to the subgraph of $H(m)$ induced by $X \cup \{u, v\}$. Indeed, since x is a median of the triplet $S = \{s_0, s_2, s_4\}$ for m', the node u is a median of S for m, and thus, by Proposition 1, u is a median of S in $H(m)$ as well. Since $d_{H(m)}(s_i, s_{i+2}) = 2$ for each i, the node u is adjacent in $H(m)$ to each of the nodes s_0, s_2, s_4. Similarly, v is adjacent with each of the nodes s_1, s_3, s_5. The fact that C is isometric implies that $u \neq v$ and that $u, v \notin C$. Finally, the path $P = (s_2, u, v, s_5)$ on X is m-shortest because the path (s_2, x, y, s_5) is m'-shortest. Therefore, u and v belong to a shortest path from s_2 to s_5 in $H(m)$. Since $d_{H(m)}(s_2, s_5) = 3$ and s_2, u, v, s_5 are distinct, u and v are adjacent in $H(m)$. Thus, $H(m)$ contains a subgraph isomorphic to G' (which is non-orientable). This contradicts the orientability of $H(m)$, and hence we conclude that m' is not a 0-extension. $\square$

Lemmas 2–4 cover all cases when $H(m)$ is not a frame, completing the proof of the 'only if' part of the theorem.

4. Proof of the 'if' Part

The proof is based on the explicit construction of the tight span of the path metric of a frame given in [9]. We review that construction, starting with necessary definitions.

An extension d' of a metric d on X to a (possibly infinite) set $V \supseteq X$ is called *tight* if no other extension of d to V is coordinatewise less than or equal to d'. This is equivalent to the property that for any $x, y \in V$, there are $s, t \in X$ such that $d'(s, x) + d'(x, y) + d'(y, t) = d(s, t)$.

It is shown in [8] (and independently in [6]) that for every metric space (X, d), there exists a unique metric space $\mathcal{T}(d) = (\mathcal{X}, \delta)$ such that δ is a tight extension of d and any tight extension (V, d') of d is isometrically embeddable in $\mathcal{T}(d)$, in the sense that there exists a mapping $\gamma : V \to \mathcal{X}$ with the identity on X satisfying $d'(x, y) = \delta(\gamma(x), \gamma(y))$ for all $x, y \in V$. The space $\mathcal{T}(d)$ is called the *tight span* (or injective envelope, or T_X-space) of (X, d). When X is finite, $\mathcal{X}$ can be represented as a polyhedral complex of dimension at most $|X|/2$; see [6].

When $H = (X, E)$ is a frame, the tight span $\mathcal{T}(d_H) = (\mathcal{X}, \delta)$ of its path metric is a two-dimensional complex obtained in the following way. Let $K(A; B)$ denote the complete bipartite graph with parts A and B. We call a maximal subgraph $K(A; B)$ of H a *bi-clique* if $|A|, |B| \geq 2$. Since H is orientable, any bi-clique $K(A; B)$ satisfies $\min\{|A|, |B|\} = 2$. As H is bipartite and does not include $K^-_{3,3}$ as an induced subgraph, it easily follows that:

$$\text{the intersection of two bi-cliques of } H \text{ is either empty, or a single node, or a single edge.} \tag{4.1}$$

Therefore, every 4-cycle of H is contained in precisely one bi-clique. Note that every edge e of H which is not a bridge is contained in a 4-cycle: e belongs to an isometric cycle, and by Proposition 2(i) all isometric cycles of H have length 4.

To construct the ground set $\mathcal{X} = \mathcal{X}_H$, we turn each edge into a homeomorphic copy of the segment $[0, 1] \subset \mathbf{R}^1$. Each 4-cycle $C = (v_0, v_1, v_2, v_3, v_0)$ (considered as a graph) is extended to a two-dimensional disc D^C. Formally, D^C is a homeomorphic copy of the square

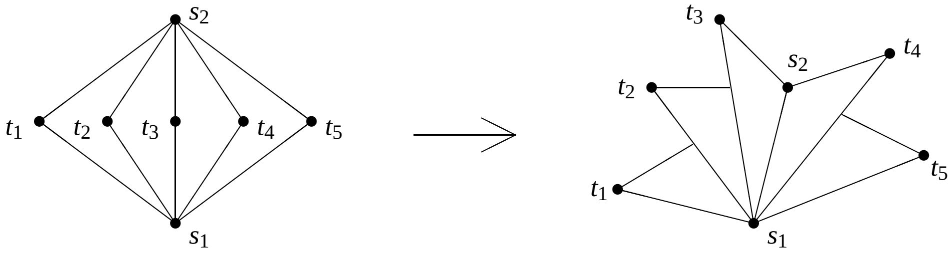


FIGURE 4. Creation of a folder.

$[0,1] \times [0,1] \subset \mathbf{R}^2$, the nodes v_0, v_1, v_2, v_3 are identified with the points (0,0), (0,1), (1,1), (1,0), respectively, and the edges with the corresponding segments. If C and another 4-cycle $C' = (u_0, u_1, u_2, u_3, u_0)$ have three nodes in common, say, $v_i = u_i$ for $i = 0, 1, 2$, we identify the corresponding triangular halves in D^C and $D^{C'}$. More precisely, assuming that v_0, v_1, v_2 are represented in both discs by (0,0), (0,1), (1,1), respectively, we identify each point in D^C coordinatized by (ξ, η) for $0 \le \xi \le \eta \le 1$ with the corresponding point (ξ, η) in $D^{C'}$. As a result, every bi-clique $K = K(A; B)$ with $A = \{s_1, s_2\}$ and $B = \{t_1, \ldots, t_k\}$ is turned into the space $F(K)$, called the *folder* of K, homeomorphic to the space obtained by gluing together k copies of the triangle $\{(\xi, \eta) : 0 \le \xi \le \eta \le 1\}$ along the side $\{(\alpha, \alpha) : 0 \le \alpha \le 1\}$; see Figure 4 for $k = 5$. By 4.1, each D^C lies in one folder, and two overlapping folders intersect in a node or an edge. This gives the desired set $\mathcal{X}$.

The segment $F = F(e)$ (of length 1) associated with a bridge e of H carries its natural metric; for convenience, F is also referred to as a bridge of $\mathcal{X}$. Each folder $F = F(K)$ obtained from a bi-clique K of H is endowed with the metric δ_F inherited from the participating (overlapping) squares. More precisely, any two points x, y of F belong to at least one disc D^C (for some 4-cycle C in K) with coordinates $x = (\xi, \eta)$ and $y = (\xi', \eta')$. Then $\delta_F(x, y)$ is defined to be the l_1-distance $|\xi - \xi'| + |\eta - \eta'|$; this number is the same for all discs D^C containing x, y. So each δ_F is well defined, and moreover, any two points (on a segment) shared by different folders F and F' are at the same distance with respect to δ_F and $\delta_{F'}$. The desired intrinsic metric $\delta = \delta_H$ on $\mathcal{X}$ is defined in a natural way: for $x, y \in \mathcal{X}$, $\delta(x, y)$ is the infimum of the values $\delta(P) = \delta_{F_1}(x_0, x_1) + \cdots + \delta_{F_r}(x_{r-1}, x_r)$ over all finite sequences $P = (x = x_0, x_1, \ldots, x_r = y)$ in which each pair x_{i-1}, x_i belongs to the same folder or the same bridge F_i. One can show that δ coincides with δ_F within each folder or bridge F.

THEOREM B ([9]). *For a frame H, the metric space $(\mathcal{X}_H, \delta_H)$ is precisely the tight span $\mathcal{T}(d_H)$.*

We will use a generalization of this theorem given in [10] where the class of finite metrics with two-dimensional tight spans is completely characterized. More precisely, consider a modular metric m on X such that $H(m)$ is a frame. Let $O_1, \ldots, O_k$ be the orbits of $H(m)$. We know that m is constant within each orbit O_i, say $m(e) = h_i$ for all $e \in O_i$. Note that all edges of a bi-clique $K = K(A; B)$ with $|A| + |B| \ge 5$ are projective and, therefore, they belong to a common orbit. On the other hand, if K is a 4-cycle, it may happen that the two pairs of opposite edges belong to distinct orbits. Accordingly, we introduce a metric δ_F^m on the folder $F = F(K)$ of a bi-clique K or on a bridge F as follows.

(i) If K is a bi-clique whose edges belong to one and the same orbit O_i, then for each 4-cycle C in K and points $x, y \in D^C$, define $\delta_F^m(x, y) = h_i \delta_F(x, y)$ (i.e., δ_F^m is obtained by uniformly stretching the metric δ_F by a factor of h_i in 'all directions').

(ii) If K is a bi-clique given by a 4-cycle $(v_0, v_1, v_2, v_3, v_0)$ with $v_0v_1 \in O_i$ and $v_1v_2 \in O_j$ $(i \neq j)$, then for points $x = (\xi, \eta)$, $y = (\xi', \eta')$ of F, define $\delta^m_F(x, y) = h_j|\xi - \xi'| + h_i|\eta - \eta'|$ (i.e., δ^m_F is obtained by stretching δ_F by a factor of h_i in 'vertical direction' and by a factor of h_j in 'horizontal direction').

(iii) If $e = uv$ is a bridge with $\{e\} = O_i$, say, then the segment $F = F(e)$ of length 1 with its metric δ_F is stretched by the factor of h_i, that is, $\delta^m_F = h_i\delta_F$.

These local metrics determine the intrinsic metric δ^m on the complex $\mathcal{X}_{H(m)}$ in an analogous fashion as for δ_H above.

THEOREM C ([10]). *If m is a modular metric such that $H(m)$ is a frame, then $\mathcal{T}(m)$ is $(\mathcal{X}_{H(m)}, \delta^m)$.*

(See also [4] for another proof.) Note that in [10] this theorem was proved for rational-valued metrics; however, it remains valid for real-valued metrics by standard rational approximation and compactness arguments. Indeed, take an infinite sequence $\ell_1, \ell_2, \ldots$ of positive rational-valued functions on the edges of $H = H(m)$ such that each ℓ_i is constant within each orbit of H and the sequence of metrics $m_i = d_{H,\ell_i}$, $i = 1, 2, \ldots$, converges to m. Since the tight spans $\mathcal{T}(m_i)$ have the same ground set $\mathcal{X}_H$, one can see that the metrics δ^{m_i} converge to some metric δ on $\mathcal{X}_H$. That $(\mathcal{X}_H, \delta)$ is indeed the tight span of m follows from the obvious fact that for every tight extension m' of m, there are tight extensions of m_i's which converge to m'.

We are now ready to show that m is minimizable, arguing in a way similar to [9]. Consider any extension m' of m to a finite superset V of X. We wish to show that there exists a 0-extension m'' of m to V such that every m'-shortest X-path on V is m''-shortest. Then m'' decomposes m' (by (2.2)), implying that every extreme extension of m is a 0-extension, i.e., m is minimizable (by (2.1)). Clearly we may assume that m' is coordinatewise minimal, i.e., m' is a tight extension of m. Therefore, we may regard V as a subset of the ground set $\mathcal{X} = \mathcal{X}_{H(m)}$ of the tight span $\mathcal{T}(m)$ and m' as the restriction of δ^m to V.

We construct a mapping $\phi : \mathcal{X} \to X$ which is identical on X and brings every δ^m-shortest X-path on $\mathcal{X}$ to an m-shortest path on X. Choose a feasible orientation of $H = H(m)$. Then every bi-clique K of H has a unique node $v = v_K$ such that all edges of K incident to v are oriented towards v (if $K = K(\{s_1, s_2\}; \{t_1, \ldots, t_r\})$ with $r \geq 3$, then v is either s_1 or s_2). For $x \in \mathcal{X}$, define ϕ as follows:

(i) if $x \in X$, then $\phi(x) = x$;

(ii) if x is an *interior* point on an edge $e = yz \in E$ (i.e., $x \neq y, z$) and e is oriented from y to z, then $\phi(x) = z$;

(iii) if x is an *interior* point of the folder $F(K)$ for a bi-clique K of H (i.e., x is not in the boundary K of $F(K)$), then $\phi(x) = v_K$.

(This mapping can be interpreted as follows. The orientation of H induces a partial order $\leq$ on X. The restriction of this order to any bi-clique K is extended to the folder $F(K)$ in a natural way, by assuming that the smallest node of K is coordinatized as (0,0) in the discs of all 4-cycles of K. This turns $F(K)$ into a complete modular lattice. Also the order $\leq$ is extended in a natural way within each bridge of $\mathcal{X}$. Then ϕ maps any point x of a folder or bridge F to the smallest point from $X \cap F$ which is greater than or equal to x and, therefore, the interior of F is mapped to the unique top point of F.)

LEMMA 5. *Let $P = (x_0, x_1, \ldots, x_k)$ be a δ^m-shortest X-path on $\mathcal{X}$. Then $\phi(P) = (\phi(x_0), \phi(x_1), \ldots, \phi(x_k))$ is an m-shortest path on X.*

PROOF. We may assume that each pair x_{i-1}, x_i belongs to a common folder or bridge, because we can always connect x_{i-1} and x_i by a δ^m-shortest path in which each consecutive pair satisfies this property.

We use induction on the distance between the ends x_0, x_k of P in the graph H. The assertion is trivial if $d_H(x_0, x_k) \leq 1$ (in case $d_H(x_0, x_k) = 1$ all intermediate points $x_1, \ldots x_{k-1}$ of P lie on the edge $x_0 x_k$). Also the assertion easily follows by induction if P is *splittable*, which means that some intermediate point of P is in $X - \{x_0, x_k\}$. So assume that $d_H(x_0, x_k) \geq 2$ and that P is not splittable. Then none of the intermediate points lies on a bridge.

Consider the maximal initial subpath $P_0 = (x_0, x_1, \ldots, x_q)$ of P which is entirely contained in some folder $F(K)$. Then x_q lies on the part of the boundary of $F(K)$ formed by the edges of K not incident to x_0. Moreover, since the path P_0 is δ^m-shortest, it is contained in the disc D^C for some 4-cycle $C = (v_0, v_1, v_2, v_3, v_0)$ in K. We may assume that $x_0 = v_0$, that x_q lies on the edge (segment) $v_1 v_2$ of D^C, and that v_0, v_1 have the coordinates (0,0) and (0,1), respectively. Then $0 = \xi(x_0) \leq \xi(x_1) \leq \cdots \leq \xi(x_q)$ and $0 = \eta(x_0) \leq \eta(x_1) \leq \cdots \leq \eta(x_q) = 1$, where $(\xi(x), \eta(x))$ are the coordinates of a point x in D^C. By the construction of ϕ, any point of D^C is mapped by ϕ to some node of C. Considering the possible orientations of C, one can see that in all cases for any $x, y \in D^C$, if $\xi(x) \leq \xi(y)$ and $\eta(x) \leq \eta(y)$, then $\xi(\phi(x)) \leq \xi(\phi(y))$ and $\eta(\phi(x)) \leq \eta(\phi(y))$. Therefore, $\xi(\phi(x_0)) \leq \cdots \leq \xi(\phi(x_q))$ and $\eta(\phi(x_0)) \leq \cdots \leq \eta(\phi(x_q))$, i.e., the path $\phi(P_0)$ is δ^m-shortest.

So we can delete the elements $x_1, \ldots, x_{q-1}$ from P, obtaining the path $P' = (x_0, x_q, \ldots, x_k)$ such that $\delta^m(P') = \delta^m(P)$ and $\delta^m(\phi(P')) = \delta^m(\phi(P))$. Recall that $\phi(x_0) = v_0$ and $\phi(x_q) \in \{v_1, v_2\}$. Insert v_1 between x_0 and x_q in P', which results in the path $R = (x_0, v_1, x_q, \ldots, x_k)$ satisfying $\delta^m(R) = \delta^m(P')$ and $\delta^m(\phi(R)) = \delta^m(\phi(P'))$. Since $d_H(x_0, v_1) = 1$ and $d_H(x_0, x_k) \geq 2$, the point v_1 of R is different from both x_0, x_k. Hence, R is a splittable δ^m-shortest path. By the above argument, we have $\delta^m(R) = m(x_0, x_k)$, and the result follows. □

By this lemma, the metric m' (being the restriction of δ^m to V) is decomposed by the 0-extension m'' of m, defined by $m''(x, y) = m(\phi(x), \phi(y))$ for $x, y \in V$. Thus, m is minimizable. This completes the proof of the theorem.

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HANS-JÜRGEN BANDELT
Mathematisches Seminar,
Universität Hamburg,
Bundesstrasse 55,
D-20146 Hamburg,
Germany

VICTOR CHEPOI
Laboratoire d'Informatique de Marseille,
Université de la Méditerranée, Faculté des Sciences de Luminy,
F-13288 Marseille Cedex 8,
France
E-mail: chepoi@lim.univ-mrs.fr
AND

ALEXANDER V. KARZANOV
Institute for System Analysis,
9, Prospect 60 Let Oktyabrya,
117312 Moscow,
Russia
E-mail: sasha@cs.isa.ac.ru