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## Paperfolding morphisms, planefilling curves, and fractal tiles

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### ABSTRACT

An interesting class of automatic sequences emerges from iterated paperfolding. The sequences generate curves in the plane with an almost periodic structure. We generalize the results obtained by Davis and Knuth on the self-avoiding and planefilling properties of these curves, giving simple geometric criteria for a complete classification. Finally, we show how the automatic structure of the sequences leads to self-similarity of the curves, which turns the planefilling curves in a scaling limit into fractal tiles. For some of these tiles we give a particularly simple formula for the Hausdorff dimension of their boundary.

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### 1. Introduction

Folding starts with folding in two. In the early sixties John E. Heighway discovered that the pattern of creases that results from repeatedly folding a piece of paper in half generates an intriguing curve in the plane when the creases are unfolded to  $90^\circ$  angles. This curve is known as Heighway's dragon curve.

The abstract sequence of the symbols  $D$  and  $U$ , coding the 'down' folds and 'up' folds (cf. Fig. 1) is known as the paperfolding sequence,<sup>1</sup> and is extensively treated in Allouche and Shallit's monograph [1].

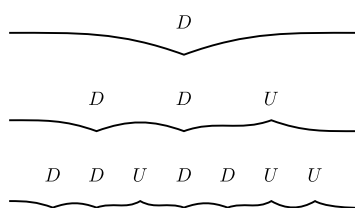


Fig. 1. Folding a strip of paper 1, 2, and 3 times, coding the folds.

On the monoid  $\{D, U\}^*$  we have the morphism  $\pi$  which permutes letters:

$$\pi(D) = U, \quad \pi(U) = D,$$

and the anti-morphism  $\tau$  which reverses words

$$\tau(w_1 w_2 \dots w_m) = w_m \dots w_2 w_1 \quad \text{for all } w_1 w_2 \dots w_m \in \{D, U\}^*.$$

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<sup>1</sup> We follow the convention of [1]: when  $A$  is an alphabet, elements from  $A^*$  will be called words or strings, elements from  $A^\infty$  will be called infinite words or sequences.

The process of iterated folding is allied to the composition of these two morphisms, the morphism  $\pi\tau$ . We therefore shortly write

$$\bar{S} := \pi\tau(S) \quad \text{for any word } S \text{ in } \{D, U\}^*.$$

We next define the *folding convolution*  $S * T$  of two words  $S = a_1 \dots a_{s-1}$  and  $T = b_1 \dots b_{t-1}$  by

$$S * T = \begin{cases} Sb_1\bar{S}b_2S \dots Sb_{t-1}\bar{S} & \text{if } t \text{ is even} \\ Sb_1\bar{S}b_2S \dots \bar{S}b_{t-1}S & \text{if } t \text{ is odd.} \end{cases} \quad (1)$$

It is quickly checked that this operation is associative:

$$(R * S) * T = R * (S * T) \quad \text{for all } R, S, T \in \{D, U\}^*.$$

Thus  $n$ -fold (sic) convolutions  $S^{*n}$  are unambiguously defined, and since  $S^{*(n+1)}$  starts with  $S^{*n}$  for all  $n$  we can also talk about the infinite folding sequence  $S^{*\infty}$ .

Note that the three strings in Fig. 1 are respectively  $D$ ,  $D * D$ , and  $D * D * D$ . It can be seen in general that folding first according to  $T$  and then according to  $S$  gives the pattern coded by  $S * T$ .

We end this section with a few historic remarks. The Heighway dragon was brought to a larger audience by Martin Gardner in 1967 [12]. Mathematical proofs of its properties were given by Davis and Knuth three years later in [3] (reprinted in 2010 in [13]). Around 1975 the present author generalized the notion of folding by introducing the folding convolution in Eq. (1). He announced this in two letters send to Martin Gardner. These were found by Donald Knuth last year in the Stanford archive. Knuth incited the author to write down the results with proofs. Some of these results have been announced in the report [5] and the paper [6] jointly written with Mendès France and van der Poorten.

## 2. Folding morphisms

Since the main interest in folding sequences is in the curves they generate in the (complex) plane, it is more convenient to describe these curves directly. We do this by passing from an alphabet of folding instructions  $\{U, D\}$  to an alphabet  $A = \{a, b, c, d\}$  of four letters, representing line segments in the positive real, positive imaginary, negative real, and negative imaginary direction.

It does, indeed, appear that the process of iterated folding and unfolding to  $90^\circ$  angles can be described by iterating a uniform morphism on the alphabet  $\{a, b, c, d\}$ . For 2-folding this morphism  $\theta$  is given by

$$\theta(a) = ab, \quad \theta(b) = cb, \quad \theta(c) = cd, \quad \theta(d) = ad.$$

How can this be seen?

Define the morphism  $\sigma$  on the monoid  $A^*$  which cyclically permutes the letters:

$$\sigma(a) = b, \quad \sigma(b) = c, \quad \sigma(c) = d, \quad \sigma(d) = a.$$

Also let the anti-morphism  $\tau$ , which reverses the words from  $A^*$ , be as before.

The crux of folding morphisms is that they commute with the morphism  $\sigma\tau$ . For the 2-folding morphism this holds for the letters:

$$\theta\sigma\tau(a) = \theta(b) = cb = \sigma\tau(ab) = \sigma\tau\theta(a),$$

similarly for the other three letters, and then for any word.

Suppose now that  $\theta^n(a)$  corresponds to the curve obtained by folding in two  $n$  times. Then if we unfold a strip folded  $n + 1$  times, the first  $2^n$  letters are equal to those we obtained before, and the last  $2^n$  letters are equal to the first  $2^n$ , but in reverse order and cyclically permuted (because of the  $90^\circ$  angle that we make half way). In other words, this word should be  $\theta^n(a)\sigma\tau\theta^n(a)$ .

But  $\theta^n$  also commutes with  $\sigma\tau$ , hence

$$\theta^n(a)\sigma\tau\theta^n(a) = \theta^n(a)\theta^n\sigma\tau(a) = \theta^n(a)\theta^n(b) = \theta^n(ab) = \theta^{n+1}(a),$$

so that indeed it follows by induction that the morphism  $\theta$  describes the iterated 2-folding.

In general we consider the following class of morphisms.

**Definition 1.** A *folding morphism* is any morphism  $\theta$  on  $A^*$  that commutes with  $\sigma\tau$ .

Such morphisms are of course determined by their value at the letter  $a$ , and we will require always that  $\theta(a)$  starts with the word  $ab$ .

We then have a one-to-one correspondence between strings  $S = D \dots$  of length  $m - 1$  and folding  $m$ -morphisms. To see this, define the homomorphism  $w : \{D, U\}^* \rightarrow \mathbb{Z}$  by

$$w(D) = +1, \quad w(U) = -1. \quad (2)$$

Then note that if  $S = a_1 \dots a_{m-1}$ , and  $\theta(a) = e_0 \dots e_{m-1}$ , then  $e_0 = a$  and

$$e_i = \sigma^{w(a_1 \dots a_i)}(a), \quad \text{for } i = 1, \dots, m - 1, \tag{3}$$

where  $\sigma$  is the cyclic permutation morphism on  $\{a, b, c, d\}^*$ .

Conversely, define the 2-block map  $\rho$  by

$$\rho(e \sigma(e)) = D, \quad \rho(\sigma(e) e) = U \quad \text{for } e \in A.$$

Then  $\rho(e_0 \dots e_{m-1}) = \rho(e_0 e_1) \rho(e_1 e_2) \dots \rho(e_{m-2} e_{m-1})$  gives the string back.

Our one-to-one correspondence is an operation-preserving isomorphism if we consider the convolution operation  $*$  on strings and composition of morphisms:

**Lemma 1.** *Let  $S = D \dots$  and  $T = D \dots$  be two strings with associated folding morphisms  $\theta_S$  and  $\theta_T$ . Then*

$$\theta_{S*T} = \theta_S \theta_T.$$

**Proof.** We give a purely geometrical argument:  $S*T$  describes the segments in the curve associated to first folding according to  $T$ , and then to  $S$ . But the same curve is described by replacing the segments in direction  $i^k$  ( $k = 0, 1, 2$  or  $3$ ) of the curve described by  $\theta_T(a)$  by the (scaled) curves associated to  $(\sigma \tau)^k \theta_S(a)$ .

The algebraically inclined reader may provide an alternative proof, using Eqs. (2) and (3).  $\square$

In the sequel all strings  $S$  from  $\{D, U\}^*$  will start with the letter  $D$ , and  $\theta$  will always be the morphism on  $A^* = \{a, b, c, d\}$  associated to  $S$ — and so  $\theta(a)$  will always start with the string  $ab$ .

### 3. Representations of morphisms

We now do a more elaborate analysis of the situation based on [4].

As usual we will write for a set  $B \subset \mathbb{C}$ , and  $r, t \in \mathbb{C}$ :

$$rB = \{rx : x \in B\} \quad \text{and} \quad B + t = \{x + t : x \in B\},$$

and we will write *segments* in the complex plane as

$$[y, y'] = \{x \in \mathbb{C} : x = \alpha y + (1 - \alpha)y', \quad 0 \leq \alpha \leq 1\}.$$

We introduce a homomorphism  $f : A^* \rightarrow \mathbb{C}$  by

$$f(a) = 1, \quad f(b) = i, \quad f(c) = -1, \quad f(d) = -i, \tag{4}$$

where the operation on  $A^*$  is concatenation, and that on  $\mathbb{C}$  addition. To associate curves to words we define a map  $K[\cdot] : A^* \rightarrow 2^{\mathbb{C}}$ , by first putting

$$K[a] = [0, 1], \quad K[b] = [0, i], \quad K[c] = [0, -1], \quad K[d] = [0, -i],$$

and then extending this by putting for any two words  $v, w \in A^*$

$$K[vw] = K[v] \cup (K[w] + f(v)). \tag{5}$$

So for example  $K[abcd] = K[a] \cup (K[b] + 1) \cup (K[c] + 1+i) \cup (K[d] + i)$  is nothing else but the unit square in  $\mathbb{C}$ .

For some morphisms  $\theta$  we will have a commuting diagram

$$\begin{array}{ccc} A^* & \xrightarrow{\theta} & A^* \\ \downarrow f & & \downarrow f \\ \mathbb{C} & \xrightarrow{L_\theta} & \mathbb{C} \end{array}$$

where  $L_\theta$  is multiplication by a complex number. It is not hard to see that  $L_\theta$  exists for any folding morphism and is multiplication by  $z = f(\theta(a))$ . For example for the 2-folding morphism determined by  $\theta(a) = ab$ ,  $L_\theta$  is multiplication by  $1 + i$ . As another example:  $L_\sigma$  is multiplication by  $i$  for the cyclic permutation morphism  $\sigma$ .

Folding convolutions behave nicely with respect to the embedding in  $\mathbb{C}$ :

**Lemma 2.** *Let  $S$  and  $T$  be two folding strings, whose curves end in  $z_S$  and  $z_T$ . Then the curve of  $S * T$  ends in  $z_S z_T$ .*

**Proof.** We have that  $L_{\theta_S}$  is multiplication by  $z_S = f(\theta_S(a))$ , and  $L_{\theta_T}$  is multiplication by  $z_T = f(\theta_T(a))$ . Lemma 1 implies that the curve of  $S * T$  ends in the point  $f(\theta_{S*T}(a))$ . The result thus follows from a look at the following commutative diagram, recalling that  $f(a) = 1$ .

$$\begin{array}{ccccc} A^* & \xrightarrow{\theta_T} & A^* & \xrightarrow{\theta_S} & A^* \\ \downarrow f & & \downarrow f & & \downarrow f \quad \square \\ \mathbb{C} & \xrightarrow{L_{\theta_T}} & \mathbb{C} & \xrightarrow{L_{\theta_S}} & \mathbb{C} \end{array}$$

In Sections 6 and 8 we will need a map from unions of Gaussian integer segments to itself that describes the action of  $\theta$ . We consider neighboring Gaussian integers  $z$  and  $z'$ , defining the  $A$ -code  $e_{z,z'}$  of the segment  $[z, z']$  by

$$e_{z,z'} = e \text{ if } z' - z = f(e), \text{ where } e = a, b, c, \text{ or } d.$$

If  $[z, z']$  is such a segment then we define the  $K$ - $\theta$ -image of  $[z, z']$  by

$$K[\theta]([z, z']) = K[\theta(e_{z,z'})] + L_\theta(z). \tag{6}$$

For example, if  $\theta$  is the 2-folding morphism, then

$$\begin{aligned} K[\theta]([1 + i, i]) &= K[\theta(c)] + L_\theta(1 + i) = K[cd] + 2i \\ &= (K[c] \cup (K[d] - 1)) + 2i = [2i, 2i - 1] \cup [2i - 1, i - 1]. \end{aligned}$$

The map  $K[\theta](\cdot)$  is extended to unions of  $k$  segments, in the natural way:

$$K[\theta]([z_1, z'_1] \cup \dots \cup [z_k, z'_k]) = K[\theta]([z_1, z'_1]) \cup \dots \cup K[\theta]([z_k, z'_k]). \tag{7}$$

Note that if the union  $Q$  of the segments can be written as  $Q = K[w]$  for some word  $w$  from  $A^k$ , then by (5) simply  $K[\theta](Q) = K[\theta(w)]$ .

#### 4. Symmetric folds

We call a string of  $U$ 's and  $D$ 's *symmetric* if  $S = \bar{S}$ . For example the string  $DDUU$  coding a 5-fold is symmetric. Obviously only strings of folds of odd length can be symmetric. We call a morphism  $\theta$  *palindromic* if it commutes with the reversal map, i.e., if  $\tau\theta = \theta\tau$ .

**Lemma 3.** *A string is symmetric if and only if the associated folding morphism is palindromic.*

**Proof.** Suppose  $S = a_1 \dots a_{m-1}$  is symmetric. It is easy to see that we only have to check that the associated  $\theta(a) = e_0 \dots e_{m-1}$  is a palindrome word. First note that the symmetry of  $S$  implies that the number of  $U$ 's in  $S$  equals the number of  $D$ 's, and hence that  $w(S) = 0$ , where  $w$  is the homomorphism from (2). But then we have for  $j = 1, \dots, m - 1$

$$w(a_1 \dots a_{m-j-1}) = -w(a_{m-j} \dots a_{m-1}) = w(a_1 \dots a_j),$$

since symmetry is equivalent to  $w(a_{m-i}) = -w(a_i)$  for  $i = 1, \dots, m - 1$ . Then we have by Eq. (3) that

$$e_{m-j-1} = \sigma^{w(a_1 \dots a_{m-j-1})}(a) = \sigma^{w(a_1 \dots a_j)}(a) = e_j \text{ for } j = 1, \dots, m - 1,$$

which exactly means that  $\theta(a) = e_0 \dots e_{m-1}$  is a palindrome.

Conversely, if  $\theta(a)$  is a palindrome, then for  $i = 1, \dots, m - 1$

$$a_i = \rho(e_{i-1}e_i) = \rho(e_{m-i}e_{m-i-1}) = \overline{\rho(e_{m-i-1}e_{m-i})} = \overline{a_{m-i}},$$

and thus  $S$  is symmetric.  $\square$

#### 5. Self-avoiding folding sequences

We want to consider folds that can be physically unfolded to  $90^\circ$  degree angles.

**Definition 2.** We call a folding string  $S$  *self-avoiding* if the curve associated to the sequence  $S^{*\infty}$  does not traverse a segment twice.

It will appear that all information that we need to decide whether this infinite sequence satisfies this property is in the finite word  $\theta(abcd)$ , if  $\theta$  is the morphism associated to  $S$ .

It is important to observe that all words generated by a folding morphism have the property that letters from  $\{a, c\}$  alternate with letters from  $\{b, d\}$ . We will call such words  $\mathbb{D}_2$ -words, for reasons that will become clear soon (see also Fig. 5).

For the following development it is convenient to consider curves  $\tilde{K}[w]$ , obtained from  $K[w]$  by rounding of the corners (see Fig. 2).

The point of this is that the curves  $\tilde{K}[w]$  do not pass through the Gaussian integers  $u + iv$ , with  $u, v \in \mathbb{Z}$ . In this way, if  $w$  is a  $\mathbb{D}_2$ -word, and  $K[w]$  does not traverse a segment twice, then  $\tilde{K}[w]$  is a simple curve in the topological sense (i.e., the image of an injective map from an interval to the plane).

The central notion in the sequel will be the following.

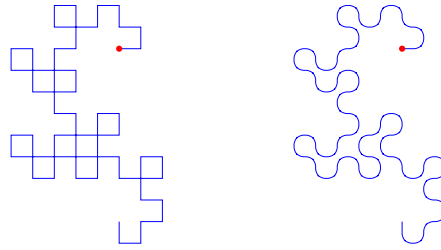


Fig. 2. The curves  $K[w]$  and  $\tilde{K}[w]$  with  $w = \theta^6(a)$ , where  $\theta(a) = ab$ .

**Definition 3.** The  $\theta$ -loop of a morphism  $\theta$  is the set  $\tilde{K}[\theta(abcd)]$ .

See Figs. 3 and 4 for examples of  $\theta$ -loops.

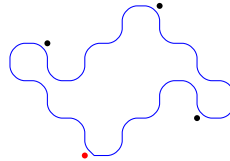


Fig. 3. The  $\theta$ -loop of  $\theta(a) = ababad$ .

**Lemma 4.** Let  $\theta$  be a folding morphism. Then its  $\theta$ -loop is a closed curve.

**Proof.** The endpoint of  $K[\theta(abcd)]$  is given by

$$f(\theta(abcd)) = L_\theta(f(abcd)) = L_\theta(f(a) + f(b) + f(c) + f(d)) = 0. \quad \square$$

**Theorem 1.** Let  $S$  be a folding string. Then  $S$  is a self-avoiding string if and only if the  $\theta$ -loop of  $S$  is a simple curve.

**Proof. Part 1:** Suppose that the  $\theta$ -loop is not simple, i.e., there is a segment that is traced twice. We have to show that there exist  $n \geq 1$  such that  $\tilde{K}[\theta^n(a)]$  is not simple.

Let  $z = f(\theta(a))$ . The  $\theta$ -loop (ignoring the rounding off) consists of the four curves  $K_a := K[\theta(a)]$ ,  $K_b := K[\theta(b)] + z$ ,  $K_c := K[\theta(c)] + (1+i)z$ , and  $K_d := K[\theta(d)] + iz$ , which can be mapped to each other by a rotation and a translation (see Lemma 8). Therefore, if one  $K_e$  has a segment that is traced by another segment of the same  $K_e$ , then all four curves have such a segment. So the curve  $\tilde{K}[\theta(a)]$  is already not simple.

The next possibility is that a segment of a  $K_e$  is traced by  $K_{\sigma(e)}$ , and again, if this happens for one  $e$ , then it happens for all  $e \in A$ . Thus, since  $\theta(a)$  starts with  $ab = a\sigma(a)$ , the curve  $\tilde{K}[\theta^2(a)]$  will not be simple.

The last (note that  $K_e$  has a segment in common with  $K_{\sigma^2(e)}$  iff  $K_{\sigma(e)}$  has a segment in common with  $K_e$ ) possibility is that a segment of a  $K_e$  is traced by  $K_{\sigma^2(e)}$ , and now the situation is different, since this might happen for  $K_a$  and  $K_c$ , but not for  $K_b$  and  $K_d$  (cf. Fig. 4), or vice versa. In the first case this overlap will occur in  $K[\theta(abc)]$ , in the second case in  $K[\theta(bad)]$ . So our goal is to show that both  $abc$  and  $bad$  will occur in some  $\theta^n(a)$ .

Now, since  $\theta(a)$  starts with  $ab$ , there exists a  $k \geq 1$  such that either  $\theta(a) = (ab)^k$  or  $(ab)^k a$ , or  $\theta(a)$  starts with  $(ab)^k c$  or with  $(ab)^k ad$ . The former case cannot occur, since clearly the  $\theta$ -loops of  $(ab)^k$  or  $(ab)^k a$  are simple for any  $k \geq 1$ . In the second case  $\theta(a)$  contains the word  $abc$ , or the word  $bad$ . But since  $\tau\sigma(bad) = abc$  and  $(\tau\sigma)^3(abc) = bad$ ,  $\theta^3(a)$  will contain

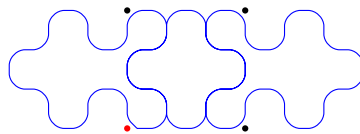


Fig. 4. The  $\theta$ -loop of  $\theta(a) = abcbabadadcd$ .

both  $abc$  and  $bad$ . It follows that  $\tilde{K}[\theta^3(a)]$  will not be simple.

**Part 2:** Suppose the  $\theta$ -loop of  $S$  is a simple curve. The crucial observation here is that in whatever contorted way a curve  $K[\theta(w)]$  (with  $w$  a  $\mathbb{D}_2$ -word, and  $\theta$  a folding morphism) lies in the plane, it will always be a subset of a very regular checkerboard pattern of  $\theta$ -loops and anti- $\theta$ -loops (ignoring rounding off). Here we define the anti- $\theta$ -loop as the curve  $\tilde{K}[\theta(dcba)]$ .

The regular pattern is created by the structure of the directed graph  $\mathbb{D}_2$ , which as vertices has the Gaussian integers, and directed edges between neighboring vertices as in Fig. 5.

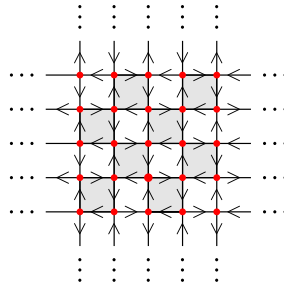


Fig. 5. The directed graph  $\mathbb{D}_2$ .

To avoid cumbersome notation we shall often identify the vertices with the corresponding Gaussian integers (which we will also call *grid points*), and the directed edges with the corresponding segments in the plane.

We call a Gaussian integer  $z = u + iv$  *even* if  $u + v$  is even, and *odd* if  $u + v$  is odd. The regions bounded by the curves  $K[abcd]$  and  $K[dcb a]$  and their translations over even Gaussian integers correspond to the white, respectively, gray squares in Fig. 5.

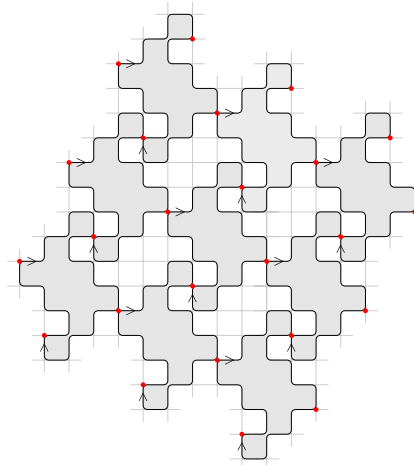


Fig. 6.  $\theta$ -loops and anti- $\theta$ -loops when  $\theta(a) = ababad$ .

It follows that the curves  $\tilde{K}[\theta(abcd)]$  and  $\tilde{K}[\theta(dcb a)]$  are  $\theta$ -loops (white interior) and anti- $\theta$ -loops (gray interior) as in the example in Fig. 6 (we see in fact, in this figure part of the  $K$ - $\theta$ -image  $K[\theta](\mathbb{D}_2)$  of  $\mathbb{D}_2$ ). Let  $w$  be a  $\mathbb{D}_2$ -word such that  $K[w]$  is self-avoiding. Since the  $\theta$ -loops and anti- $\theta$ -loops are self-avoiding, the same will be true for the curve  $K[\theta(w)]$ , which will be a subset of the full  $\theta$ -loops and anti- $\theta$ -loops pattern (after rounding off in the appropriate way). It thus follows by iteration of this observation that all  $\tilde{K}[\theta^n(a)]$  will be self-avoiding, and so by definition  $S$  is self-avoiding.  $\square$

### 6. Plane-filling folding sequences

A surprising property of the curve generated by 2-folding is that it is not only self-avoiding, but very compactly traverses large parts of the set of Gaussian integers ([12,3]).

**Definition 4.** A folding string  $S$  is *plane-filling* if the curve associated to the infinite sequence  $S^{*\infty}$  passes twice through every grid point lying in some arbitrary large circle.

It will be convenient to reformulate this to the equivalent property that this curve passes through all the segments connecting Gaussian integers forming a  $(2k + 1)$ -cross in  $\mathbb{D}_2$  for every  $k = 1, 2, \dots$ . Here the  $(2k + 1)$ -crosses are translations of  $(2k + 1)$ -crosses centered at  $(1 + i)/2$  defined as the set of edges in  $\mathbb{D}_2$  lying within the square through the points  $k + 1 + q$ ,  $(k + 1)i + q$ ,  $-k - 1 + q$  and  $-(k + 1)i + q$ , where  $q = (1 + i)/2$  (Fig. 7). Note that as directed graphs there are actually two types of  $(2k + 1)$ -crosses: those where the center loop runs clockwise, and those where the center loop runs anti-clockwise. In the sequel we will not make this distinction explicitly.

The goal of this section is to give a characterization of self-avoiding plane-filling folding strings. In the next section we will see that there exist many self-avoiding plane-filling folding strings.

**Definition 5.** A folding string  $S$  is *maximally simple*, if its  $\theta$ -loop is simple and if the region the  $\theta$ -loop bounds only contains grid points that are passed twice by the  $\theta$ -loop.

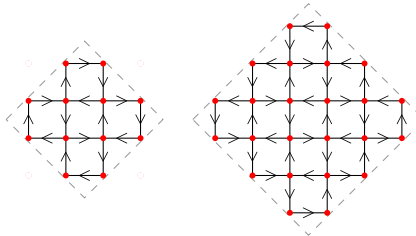


Fig. 7. The 3-cross and 5-cross graphs.

Note that we used Jordan’s Curve Theorem here.

**Lemma 5.** *Let  $S$  and  $T$  be two maximally simple folding strings. Then  $S * T$  is also maximally simple.*

**Proof.** This is similar to the proof of Lemma 1.

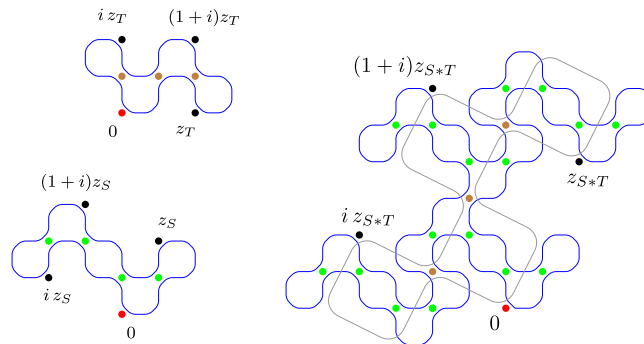


Fig. 8. The  $\theta$ -loops of the folds  $S = DUDD$ ,  $T = DUU$ , and  $S * T$ .

We obtain the  $\theta_{S*T}$ -loop by replacing scaled unit-loops in the  $\theta_T$ -loop by  $\theta_S$ -loops. Grid points in the interior of the  $\theta_{S*T}$ -loop will either lie in these  $\theta_S$ -loops, or will be in the interior of the scaled  $\theta_T$ -loop, cf. Fig. 8. In both cases these points are visited twice, and so the  $\theta_{S*T}$ -loop is maximally simple.  $\square$

Lemma 5 forms the basis for the planefilling property. It implies that for a given string  $S$  with maximally simple  $\theta$ -loop all its  $\theta^n$ -loops will also be maximally simple. However, there are some subtleties in actually proving that this implies that  $S$  is planefilling.

We call a word  $w$  from  $A^*$  a *loop word* if it is a  $\mathbb{D}_2$ -word, and  $f(w) = 0$ , where  $f$  is the homomorphism from Eq. (4).

We call a loop word *maximally simple* if  $\tilde{K}[w]$  is a self-avoiding curve that visits each grid point in its interior twice. Examples are the 8 words

$$\sigma^k(abcd), \text{ and } \sigma^k(adcb) \text{ for } k = 0, 1, 2, 3, \tag{8}$$

which we call *square words*.

**Lemma 6.** *Let  $w$  be a maximally simple loop word. Then  $w$  contains at least one of the square words.*

**Proof.** A point  $u + iv$  with  $u = N + 1/2$  and  $v = M + 1/2$ , where  $N$  and  $M$  are integers will be called a *mid grid point*. Maximally simple loop words  $w$  of length  $m$  are associated to particularly nice curves  $\tilde{K}[w]$ . The curve  $\tilde{K}[w]$  contains exactly  $m/4$  mid grid points in its interior. Consider the graph  $\mathcal{G}$  whose vertices are these mid grid points, and where there is an edge if two vertices are neighbors (i.e., the corresponding mid grid points are at distance  $\sqrt{2}$ ), see also Fig. 9. If  $m = 4$ , then  $w$  is a square word, so consider  $m \geq 8$ . Since  $\tilde{K}[w]$  is simple,  $\mathcal{G}$  is connected, and since  $\tilde{K}[w]$  is maximally simple,  $\mathcal{G}$  has no circuits,

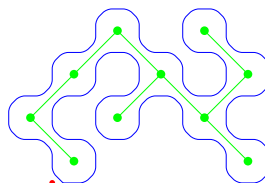


Fig. 9. The curve  $\tilde{K}[w]$  of a maximally simple loop word of length  $m = 40$ .

and so  $\mathcal{G}$  is a tree. A mid grid point corresponding to any leaf of this tree is the center of a  $\tilde{K}[\sigma^k(abcd)]$  or a  $\tilde{K}[\sigma^k(adcb)]$  (ignoring rounding of). Since  $m \geq 8$  the tree has at least two leaves. At least one of the corresponding two square words will occur in  $w$  (one has to take into account the possibility that one of the two corresponding mid grid points is the point  $(1+i)/2$ ).  $\square$

**Lemma 7.** *Let a folding morphism  $\theta$  be maximally simple. Then a square word will occur in  $\theta^4(a)$ .*

**Proof.** If  $m = 2$ , then  $\theta(a) = ab$ , and it is easily checked that  $w_{sq} = bcda$  occurs in  $\theta^4(a)$ , so suppose  $m \geq 3$ . From Lemma 6 we know that  $w = \theta(abcd)$  contains a square word  $w_{sq}$ . As  $m \geq 3$ ,  $w_{sq}$  has to occur in at least one of  $\theta(ab)$  or  $\theta(bc)$ , or  $\theta(cd)$ .

Obviously  $ab$  occurs in  $\theta^3(a) = ab\dots$ . We also claim that  $bc$  occurs in  $\theta^3(a)$ . To see this, note that if  $bc$  occurs in  $\theta(a)$ , then it occurs in  $\theta^2(a)$  and in  $\theta^3(a)$ , since  $\theta(b)$  ends in  $b$  and  $\theta(c)$  starts with  $c$ . Now either  $\theta(a) = abc\dots$ , and then  $bc$  will also appear in  $\theta^3(a)$ , or  $\theta(a) = aba\dots$ , and then  $bc$  will appear in  $\theta^2(a)$  (and as argued before, in  $\theta^3(a)$ ), since in that case  $\theta(b) = \sigma\tau\theta(a)$  ends in  $bc$ . Finally,  $cd$  occurs in  $\theta^3(a)$ , since  $c$  occurs in  $\theta(b)$ , and  $\theta(c) = (\sigma\tau)^2\theta(a) = cd\dots$

Since  $ab, bc$  and  $cd$  all occur in  $\theta^3(a)$ ,  $w_{sq}$  will occur in  $\theta^4(a)$ .  $\square$

**Theorem 2.** *Let  $S$  be a self-avoiding folding string. Then  $S$  is a planefilling string if and only if the  $\theta$ -loop of  $S$  is maximally simple.*

**Proof.** Part 1: If the  $\theta$ -loop is not maximally simple, then at least one of the grid points in its interior is visited only once. It follows that a  $(2k+1)$ -cross can nowhere occur if  $k$  is large enough.

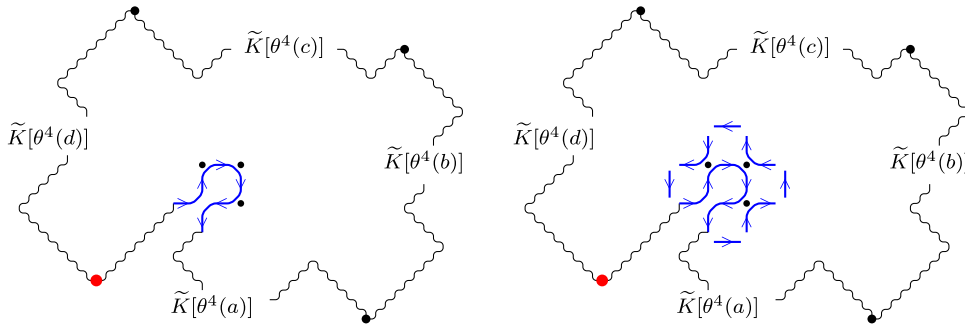
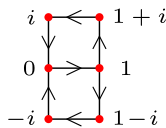


Fig. 10. Left:  $\tilde{K}[w_{sq}]$  in  $\theta^4$ -loop. Right: forced segments in the  $\theta^4$ -loop.

Part 2: Note that the 8 square words have two different types, those associated to a  $\theta$ -loop:  $abcd, bcda, cdab$ , and  $dabc$ , and the other four, which are associated to an anti- $\theta$ -loop (cf. the white, respectively gray squares in Fig. 5). In terms of curves this means that the  $\theta$ -loop runs counter clockwise, and the anti- $\theta$ -loop clockwise.

Let  $w_{sq}$  be the square word occurring in  $\theta^4(a)$  according to Lemma 7. Suppose that  $w_{sq}$  is of the second (clockwise) type (otherwise we replace the  $\theta^4$ -loop by the anti- $\theta^4$ -loop everywhere below). Now consider the curve  $\tilde{K}[\theta^4(a)]$  in the  $\theta^4$ -loop. Since the type of this loop is opposite to the type of  $w_{sq}$ , the 3 grid points passed by  $\tilde{K}[w_{sq}]$  must be interior points of this loop, cf. Fig. 10, left. This implies that the 10 segments indicated in Fig. 10, right must all be part of the  $\theta^4$ -loop. In other words: the  $\theta^4$ -loop contains a 3-cross. Since  $w_{sq}$  occurs in  $\theta^4(a)$  this implies that  $\tilde{K}[\theta^8(a)]$  contains a 3-cross.



With a figure-8 configuration we denote the subgraph of  $\mathbb{D}_2$  with 6 vertices and 7 edges picture right above this sentence. Let  $Q_{\square} = [0, 1] \cup [1, 1+i] \cup \dots \cup [-i, 0]$  be the corresponding set of segments.

Now consider  $K[\theta^8]([0, 1]) = K[\theta^8(a)]$  in  $K[\theta^8](Q_{\square})$ , see Fig. 11. Since all the segments of  $K[\theta^8]([0, 1])$  lie in the interior of the (rounded off) closed curve corresponding to  $K[\theta^8](Q_{\square}) \setminus K[\theta^8]([0, 1])$ , the eight boundary points of the 3-cross are interior points of this curve, and so it must contain a 5-cross.

Now note that  $K[\theta^8](Q_{\square})$  (rounded off) appears in  $\tilde{K}[\theta^{16}(a)]$ , since  $\tilde{K}[\theta^8(a)]$  contains a figure-8 configuration (as a subgraph of the 3-cross, both for the clockwise and for the anti-clockwise type). Thus  $\tilde{K}[\theta^{16}(a)]$  has to contain a 5-cross. The same type of argument yields that  $K[\theta^{24}(a)]$  contains a 7-cross, and more generally that  $\tilde{K}[\theta^{8k}(a)]$  will contain a  $(2k+1)$ -cross, for each natural number  $k$ .  $\square$



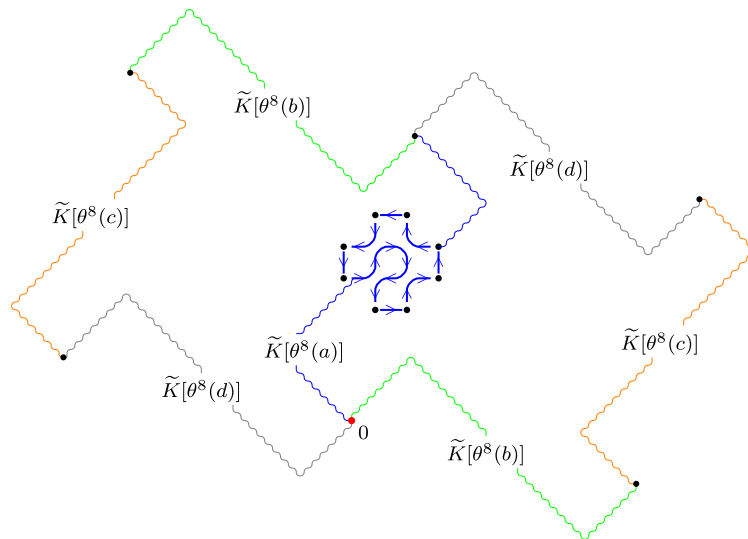


Fig. 11. A 3-cross in a figure eight configuration at level 8.

**Theorem 3.** Let  $S$  and  $T$  be two self-avoiding plane-filling folding strings. Then  $S * T$  is a self-avoiding plane-filling folding string.

**Proof.** This follows directly from Lemma 5 and Theorem 2.  $\square$

**7. Existence of plane-filling folds**

Let  $z = f(\theta(a))$  be the endpoint of the curve associated to a  $m$ -fold. By Lemma 8 in Section 8 the sets  $\tilde{K}[\theta(a)]$  and  $\tilde{K}[\theta(c)] + (1 + i)z$  are congruent, and so are  $\tilde{K}[\theta(b)] + z$  and  $\tilde{K}[\theta(d)] + iz$ . This implies that the interior of the  $\theta$ -loop and the square through  $0, z, (1 + i)z$  and  $iz$  have the same area. For a maximally simple  $\theta$ -loop this area is equal to  $m$  (ignoring second order effects caused by rounding off), since all  $4m$  segments of such a  $\theta$ -loop contribute one side to a unit square. It thus follows from Theorem 2 that self-avoiding plane-filling  $m$ -folds can only exist if  $m = |z|^2$  is a sum of two squares.

**Theorem 4.** For any integer  $m > 1$  which is a sum of two squares there exists a self-avoiding and plane-filling folding string of length  $m$ .

**Proof.** We will prove in fact more, namely that for every Gaussian integer  $z = u + iv$  with  $u^2 + v^2 > 1$  there is a self-avoiding plane-filling folding string whose curve ends in  $z$ . By Theorem 3 and Lemma 2, it suffices to prove this for those  $z = u + iv$  which are Gaussian prime numbers. Moreover, we can assume, by rotating if necessary, that  $u > 0, v \geq 0$ . We already know that there exists a plane-filling folding string to the Gaussian prime number  $z = 1 + i$ . Moreover,  $z = u + iv$  is certainly not prime if both  $u$  and  $v$  are even, and since

$$u + iv = (1 + i)\left(\frac{u + v}{2} + i\frac{v - u}{2}\right),$$

$z$  is also not prime if  $u$  and  $v$  are both odd (and not both equal to 1). So we may assume in the sequel that  $z = u + iv$  is an odd Gaussian integer, i.e.,  $u + v$  is odd. Moreover, by exchanging  $u$  and  $v$  if necessary, we can assume that  $u$  is odd and  $v$  even.

First consider the case  $v \neq 0$ .

We will define an octagon  $\Omega$ , based on the mid grid point  $s = \frac{1}{2}u(1 + i)$  and the grid point  $t = \frac{1}{2}v(1 + i)$  (see Fig. 12). The octagon is given by its vertices:

$$\Omega \approx (0, s, s + it, s + (1 + i)t, z, -is + (1 + i)t, -is + t, -is).$$

The regular way in which  $\Omega$  is constructed ensures that it is simply connected. Now consider the finite directed subgraph of  $\mathbb{D}_2$  whose edges lie in  $\Omega$ . It is easily checked that all its vertices have in-degree 1 and out-degree 1 or in-degree 2 and out-degree 2, except the vertices  $0$  and  $z$ , which have out-degree 1, respectively in-degree 1 (here again we use that  $u + v$  is odd).

Now remember that a directed Euler path is a path along a connected directed graph that connects all the vertices and that traverses every edge of the graph only once, and that the degree conditions above yield the existence of a directed Euler path from  $0$  to  $z$  [10]. But such a path is nothing else than the curve given by a self-avoiding folding string  $S$  of length  $u^2 + v^2$ , since the octagon  $\Omega$  has half the area of the square  $\mathcal{E}$  with vertices  $0, z, (1 + i)z$ , and  $iz$ . Moreover, the curve obtained by joining the four curves produced by rotating thrice over  $90^\circ$  around the center  $s + it = (1 + i)z/2$  of  $\mathcal{E}$  is the  $\theta$ -loop of

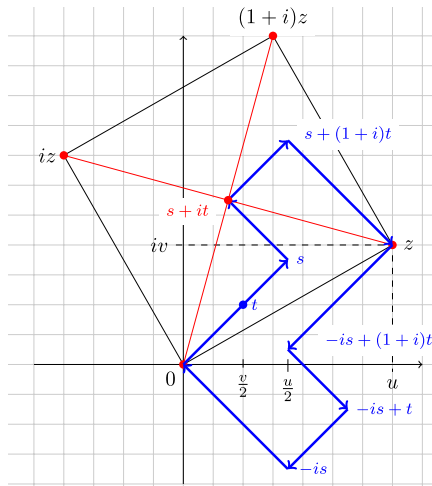


Fig. 12. The octagon  $\Omega$  associated to  $z = u + iv$  when  $u = 7$  and  $v = 4$ .

$S$ , and it visits each of its internal points exactly twice (cf. Fig. 13). It follows from Theorem 2 that the string  $S$  generates a plane-filling curve.

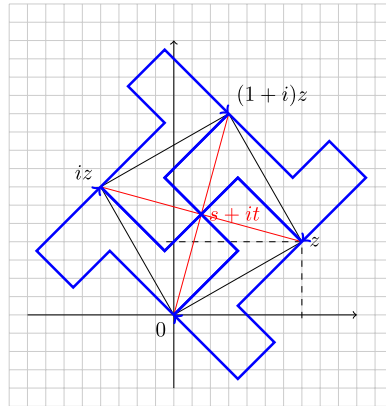


Fig. 13. Four octagons forming a  $\theta$ -loop region.

When  $v = 0$  the octagon  $\Omega$  degenerates into a square, but the arguments will remain the same.  $\square$

It can be seen that actually one can find a symmetric fold to  $u + iv$  for any odd Gaussian integer. The idea here is that Euler's theorem will guarantee a  $\mathbb{D}_2$ -word  $w$  of length  $(m - 1)/2$  with its curve filling half of  $\Omega$  after the central mid-grid square has been removed. The symmetric  $m$ -fold will then be given by  $w a \tau(w)$  if  $u + v - 1$  is divisible by 4, or else by  $w c \tau(w)$ .

### 8. Perfect sequences: the carousel theorem

Davis and Knuth prove in [3] that the 2-fold  $S = D$  has the following property, stating that this is “probably the most interesting property of the dragon curve”.

**Definition 6.** A string  $S$  is called *perfect* if it is self-avoiding, and the four curves  $K[S^{*\infty}]$ ,  $iK[S^{*\infty}]$ ,  $-K[S^{*\infty}]$ , and  $-iK[S^{*\infty}]$  together cover every edge of  $\mathbb{D}_2$  exactly once.

See Fig. 14 for perfectness of the 2-fold with string  $S = D$ .

Actually it is remarkable that a fold  $S$  can be perfect, since the set above, which we call the *carousel* associated to  $S$ , is incompatible with the directed graph structure of  $\mathbb{D}_2$ —note, e.g., that the out-degree of the origin is four for the carousel. Nevertheless, perfect folding sequences abound, and the following characterization holds.

**Theorem 5.** Let  $S$  be a folding string that generates a self-avoiding plane-filling curve. Then  $S$  is perfect if and only if the curve  $K[\theta(a)]$  of  $S$  does not traverse the segment from  $(1 + i)z - i - 1$  to  $(1 + i)z - i$ , where  $z = f(\theta(a))$  is the endpoint of the curve of  $S$ .

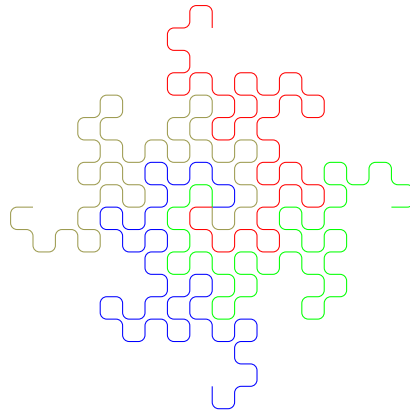


Fig. 14. The carousel of order 6 for  $S = D$ .

We first present a lemma which resolves the incompatibility observed above.

**Lemma 8.** *Let  $S$  be a folding string with  $z = f(\theta(a))$ . Then*

$$iK[\theta(a)] = K[\theta(d)] + iz, \quad i^2K[\theta(a)] = K[\theta(c)], \quad i^3K[\theta(a)] = K[\theta(b)] + i^3z.$$

**Proof.** Note that these are equalities of sets, which we prove by going through the segments making up the sets in a certain order. To see that the first one holds, traverse  $iK[\theta(a)] = K[\sigma\theta(a)]$  from  $iz$  to 0. This amounts to applying  $\sigma^2\tau$  to the word  $\sigma\theta(a)$  (since we reversed the direction the segments in the curve are rotated over  $180^\circ$ ). But since  $\sigma$  and  $\tau$  commute, and by definition of the folding morphism we have

$$\sigma^2\tau(\sigma\theta(a)) = (\sigma\tau)^3\theta(a) = \theta(d).$$

For the second equality, note that  $i^2K[\theta(a)] = K[\sigma^2\theta(a)]$ , and that

$$\sigma^2\theta(a) = (\sigma\tau)^2\theta(a) = \theta(c).$$

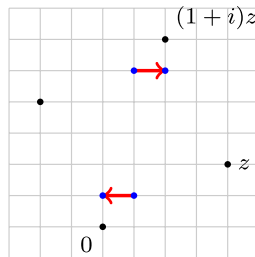
The third equality follows as the first one.  $\square$

The segment  $[1 + i, i]$  will play a crucial role in the proof of Theorem 5.

**Lemma 9.** *Let  $S$  be a self-avoiding plane-filling string. Then the  $\theta$ -loop of  $S$  contains the segment  $[1 + i, i]$ .*

**Proof.** The statement is true when  $m = 2$ , and for  $m \geq 3$  we either have  $\theta(a) = abc\dots$ , and then  $K[\theta(a)]$  already contains  $[1 + i, i]$ , or  $\theta(a) = aba\dots$ , and then the point  $1 + i$  is in the interior of the  $\theta$ -loop, what implies that the segment  $[1 + i, i]$  must be traversed by it.  $\square$

**Proof of Theorem 5.** Let  $\theta$  be the morphism associated to  $S$ , with  $\theta(a) = ab\dots$ . First we will translate the mysterious condition on the curve  $K[\theta(a)]$  of  $S$  to a slightly less mysterious condition on the  $\theta$ -loop of  $S$ . Note that (by Lemma 8) the  $K[\theta(c)]$  part of the  $\theta$ -loop (ignoring the rounding off) can be written as  $K[\theta(c)] + (1 + i)z = -K[\theta(a)] + (1 + i)z$ , where  $z = f(\theta(a))$  is the endpoint of  $K[\theta(a)]$ .



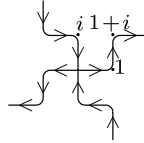
It follows that  $K[\theta(a)]$  does go through the segment from  $(1 + i)z - i - 1$  to  $(1 + i)z - i$  if and only if  $K[\theta(c)] + (1 + i)z$  goes through the segment from  $1 + i$  to  $i$  (see also figure right above).

In the sequel, let

$$Q_n := K[\theta^n(a)] \cup iK[\theta^n(a)] \cup -K[\theta^n(a)] \cup -iK[\theta^n(a)]$$

be the carousel of order  $n$ . Since we can replace the 2-fold  $D$  by the 4-fold  $D * D = DDU$ , we will assume henceforth that  $m \geq 3$ .

Part 1: Suppose  $[1 + i, i]$  is traversed by  $K[\theta(c)] + (1 + i)z$  in the  $\theta$ -loop.



Then  $\theta(a)$  must start with  $aba$ , so  $\theta(d)$  must end in  $dad$ , and in  $Q_1$  the segment  $[1 + i, i]$  is absent (cf. figure right above). In  $Q_2$ , the segment  $[1 + i, i]$  is still missing, since it could only be traversed by  $K[\theta(c)] + (1 + i)z$ , which is missing from the incomplete  $\theta$ -loop with  $i$  as one of its internal points. Continuing in this fashion we see that the segment  $[1 + i, i]$  will be missing in all  $Q_n$ , and so not all edges of  $\mathbb{D}_2$  are visited.

Part 2: We now suppose  $[1 + i, i]$  is not traversed by  $K[\theta(c)] + (1 + i)z$  in the  $\theta$ -loop. Note that either  $\theta(a)$  starts with  $abc$ , or with  $aba$ . The latter case will be considered at the end of the proof. If  $\theta(a)$  starts with  $abc$ , then the 3rd segment of  $K[\theta(a)]$  traverses  $[1 + i, i]$ . Since  $K[\theta(a)]$  does not end in the point  $i$ , it will then also traverse the segment  $[i, 2i]$ . It thus follows from rotational symmetry that  $Q_1$  will contain a 1-star, see Fig. 15.

In general the  $k$ -star graph is the subgraph  $S_k$  of  $\mathbb{D}_2$  consisting of all the vertices and edges lying in the square through  $k + 1, (k + 1)i, -(k - 1)$  and  $-(k + 1)i$ . It will be important later that we have the following property of the star graphs: one obtains  $S_{k+1}$  from  $S_k$  by adding all edges (and corresponding vertices) to the vertices that are not yet of degree 4 in  $S_k$ . The

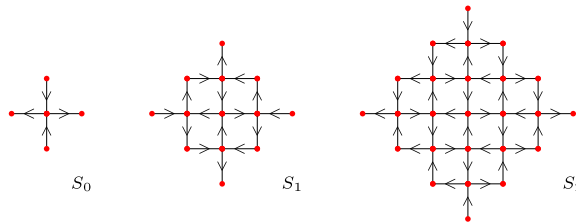


Fig. 15. The 0-star, 1-star and 2-star graphs.

edges and vertices that are added, except for the outer 4, will be called the boundary  $B_k$  of  $S_k$  (see also Fig. 16).

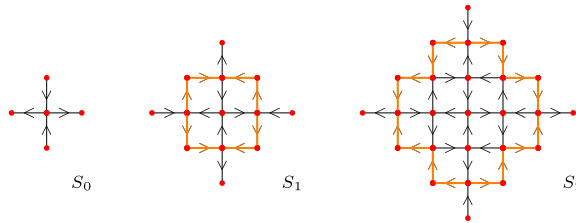


Fig. 16. The boundaries in the 1-star and 2-star graphs.

Now note that  $B_1$  is a closed simple loop in  $S_1$ , which has all the segments of  $S_0$  in its interior, and that we know that  $Q_1$  contains  $S_1$ . It follows (recall Eqs. (6) and (7)) that  $Q_2$  contains the  $K$ - $\theta$ -image of  $B_1$ , which will be a closed (simple) loop, that contains  $S_1$  in its interior (except for the 8 extremal points), since the  $K$ - $\theta$ -image of  $S_0$  is  $Q_1$ . Now we get to the main idea of the argument: since the  $\theta$ -loop is maximally simple, all grid points within the  $K$ - $\theta$ -image of  $B_1$  will be visited twice by  $Q_2$ . But then  $S_1$  has to grow<sup>2</sup> till  $S_2$ , and so  $Q_2$  will contain  $S_2$ . Continuing in this fashion we find that  $Q_k$  contains  $S_k$  for all  $k$ , and so the  $Q_k$  increase to cover all of  $\mathbb{Z}_2$ . See Fig. 17 for an example with a self-avoiding and planefilling 8-fold.

We still have to consider the possibility that  $\theta(a)$  starts with  $aba$ . By Lemma 9 one of the appropriate translates of  $K[\theta(e)]$ ,  $e = a, b$  or  $d$  traverses  $[1 + i, i]$ . First, suppose that  $[1 + i, i]$  is in  $K[\theta(a)]$  or  $K[\theta(d)] + iz$ . Then obviously  $Q_1$  contains the 1-star graph.

Second, if  $[1 + i, i]$  is in  $K[\theta(b)] + z$ , then the segment  $[1 + i, i]$  will not be in  $Q_1$ . However, since  $\theta(a)$  starts with  $aba$ ,  $\theta^2(a)$  starts with  $\theta(a)\theta(b)$ , and thus the segment  $[1 + i, i]$  will be contained in  $Q_2$ . In all cases  $Q_2$  will contain the 1-star graph. Thus for the string  $S * S$  the carousel  $Q_1$  will contain the 1-star graph, and we can copy the proof of the  $abc$  case.  $\square$

Genetically speaking, non-perfectness is recessive, and perfectness is dominant.

<sup>2</sup> There is one special case ( $m = 4$  and  $\theta(a) = abc$ ) where this is not true, due to the fact that  $B_1$  does not correspond to a  $\mathbb{D}_2$ -word and because for this particular  $\theta$  the  $K$ - $\theta$ -image of  $S_0$  equals  $S_1$ .

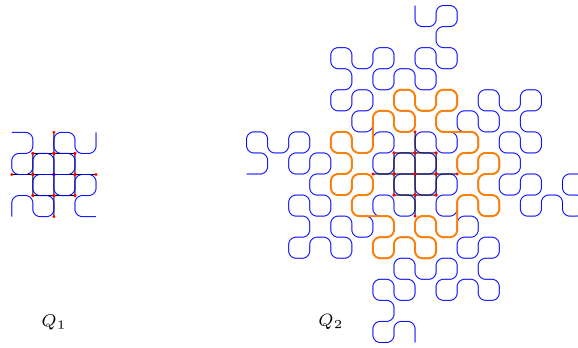


Fig. 17. The case  $\theta(a) = abcabad$ ; the carousel  $Q_1$  and  $Q_2$  with the 1-star. On the right the  $K$ - $\theta$ -image of  $B_1$  is indicated in orange.

**Theorem 6.** Let  $S$  and  $T$  be self-avoiding plane-filling folding strings. If neither  $S$  nor  $T$  is perfect, then  $S * T$  is not perfect. If either  $S$  or  $T$  is perfect, then  $S * T$  is perfect.

**Proof.** We use the characterization of perfectness based on the segment  $[1 + i, i]$ . We write  $z_S = f(\theta_S(a))$ ,  $z_T = f(\theta_T(a))$ , and  $z_{S*T} = z_S z_T$ .

*Part 1:* Suppose neither  $S$  nor  $T$  is perfect. Then  $K[\theta_T(c)] + (1 + i)z_T$  traverses  $[1 + i, i]$ , and so part of the curve  $K[\theta_S(\theta_T(c))] + (1 + i)z_{S*T}$  will go from the point  $(1 + i)z_S$  to the point  $iz_S$ . But this part must be  $K[\theta_S(c)] + (1 + i)z_S$  in the  $\theta_S$ -loop, and since  $\theta_S$  is not perfect, this part traverses  $[1 + i, i]$ . Conclusion:  $K[\theta_S\theta_T(c)] + iz_{S*T}$  traverses  $[1 + i, i]$ , and thus  $S * T$  is not perfect.

*Part 2:* (a) First suppose  $S$  is perfect,  $T$  perfect or not. There is the special case that  $S = D$ , the 2-fold. Then  $\theta_S(a) = ab$  and  $\theta_S(b) = cb$ , so

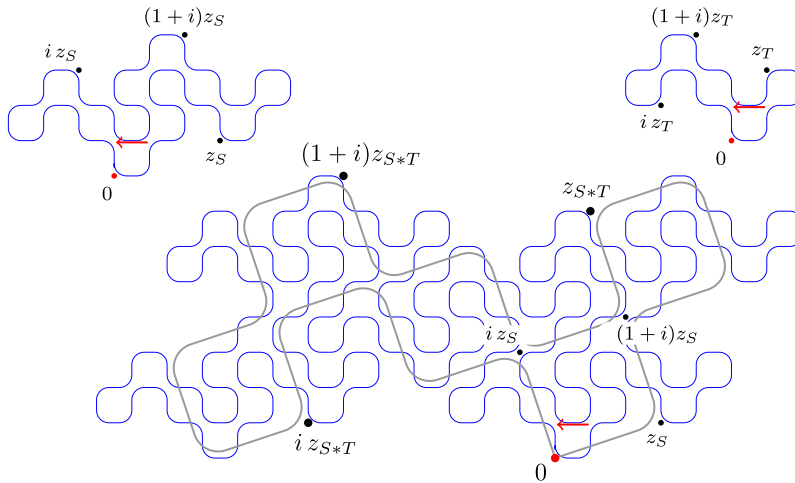


Fig. 18. The  $\theta$ -loops of the non-perfect fold  $S = DUDDUUUDU$ , the perfect fold  $T = DUDD$ , and the perfect 50-fold  $S * T$ .

$$\theta_S\theta_T(a) = \theta_S(ab\dots) = abc\dots$$

starts with  $abc$ , and hence  $S * T$  is perfect. The following argument is illustrated by Fig. 8. If  $m_S \geq 3$ , then either  $\theta_S(a) = abc\dots$ , and then  $[1 + i, i]$  is traversed by  $K[\theta_S\theta_T(a)]$ , or  $\theta_S(a) = aba\dots$ , in which case  $1 + i$  is in the interior of the  $\theta_S\theta_T$ -loop, but also in the interior of the  $\theta_S$ -loop (actually the loop corresponding to the  $K$ - $\theta_S$ -image of the unit square, since the segments will not be traversed in the direct order). By perfectness of  $S$ ,  $[1 + i, i]$  is traversed by either  $K[\theta_S(ab)]$ , which is the beginning part of  $K[\theta_S\theta_T(a)]$ , or by  $K[\theta_S(d)] + iz_S$  which is the last part of  $K[\theta_S\theta_T(d)] + iz_{S*T}$ . Therefore  $[1 + i, i]$  can not be traversed by  $K[\theta_S\theta_T(c)] + (1 + i)z_{S*T}$ , and so  $S * T$  is perfect.

(b) This argument is illustrated by Fig. 18. If  $T$  is perfect, then (by Lemma 9) part of one of the curves  $K[\theta_S(\theta_T(a))]$ ,  $K[\theta_S(\theta_T(b))]$  +  $z_S z_T$  or  $K[\theta_S(\theta_T(d))]$  +  $iz_S z_T$  will go from the point  $(1 + i)z_S$  to the point  $iz_S$ , and this part must be  $K[\theta_S(c)] + (1 + i)z_S$  in the  $\theta_S$ -loop (or rather the  $K$ - $\theta_S$ -image of the unit square). Then if  $S$  is not perfect, this  $K[\theta_S(c)] + (1 + i)z_S$  traverses  $[1 + i, i]$ . So if  $T$  is perfect, and  $S$  is not, then  $[1 + i, i]$  is not traversed by  $K[\theta_S\theta_T(c)] + (1 + i)z_{S*T}$ , and so  $S * T$  is perfect.  $\square$

The simplest example of a non-perfect fold, already given by Davis and Knuth [3], is the 4-fold  $S = DUU$ . Here  $K[\theta^n(a)]$  fills a  $45^\circ - 45^\circ - 90^\circ$  triangle for all  $n$ , and the carousel fills only half of the plane.

Remarkably, the non-perfect 10-fold  $S = DUDDUUUDU$  from Fig. 18 can be written as  $S = DUDD * U$ , and  $DUDD$  and  $U$  are both perfect folds. This does not contradict Theorem 6, since we always require  $S$  and  $T$  to start with  $D$ . The  $Q_2$  of this perfect 50-fold is depicted in Fig. 19.

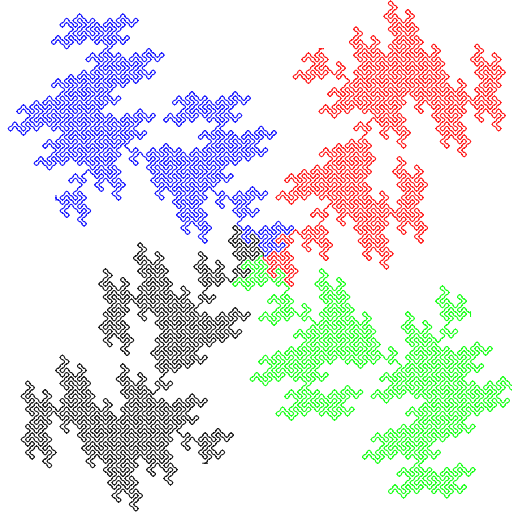


Fig. 19. The carousel  $Q_2$  of the 50-fold  $S * T$  of Fig. 18.

### 9. Boundaries

Let  $w$  be a  $\mathbb{D}_2$ -word. The curve  $K[w]$  starts in 0, and ends in  $f(w)$ . The smallest simple closed polygonal curve that contains  $K[w]$  and has vertices  $0, f(w)$  and further only mid grid points will be called the *mid grid border* of the curve  $K[w]$ , see Fig. 20, Left. We have already encountered such borders: the octagon  $\Omega$  from Section 7 is the mid grid border of the curve  $K[\theta(a)]$  generated by any of the morphisms  $\theta$  that are constructed in the existence proof.

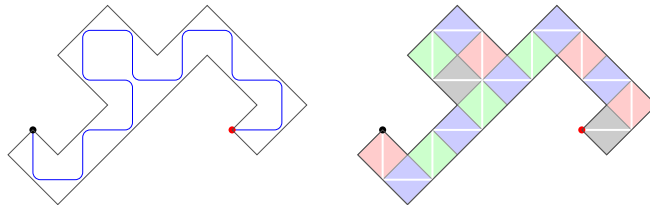


Fig. 20. Left: mid grid border of the curve  $\tilde{K}[abc bcdcbcdadcdcb]$ . Right: the polyomino  $K_\diamond[abc bcdcbcdadcdcb]$ .

There is an alternative and attractive way to define this border, given by Knuth in the Addendum of [13] (see also the recent [14]): the mid grid border of  $K[w]$  is the topological border of the polyomino generated by  $w$ . In the context of Section 3 this polyomino is obtained via the diamond map  $K_\diamond[\cdot]$  mapping words to compact subsets of  $\mathbb{C}$ , given by

$$K_\diamond[a] = B\left(\frac{1}{2}, \frac{1}{2}\right), \quad K_\diamond[b] = B\left(\frac{i}{2}, \frac{1}{2}\right), \quad K_\diamond[c] = B\left(-\frac{1}{2}, \frac{1}{2}\right), \quad K_\diamond[d] = B\left(-\frac{i}{2}, \frac{1}{2}\right),$$

where  $B(z, r)$  is the ball with center  $x$  and radius  $r$  in the  $L_1$ -metric. Eq. (5) then defines the polyomino  $K_\diamond[w]$  associated to any word  $w$ .

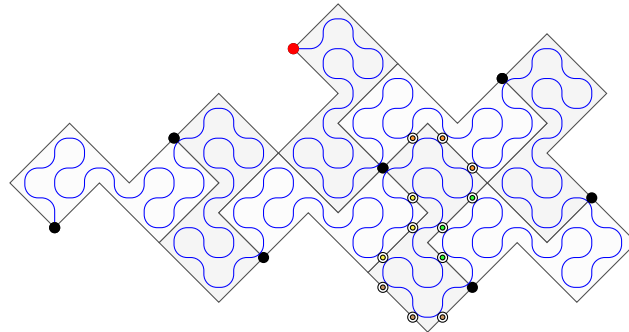
Let  $S$  be a folding string with associated morphism  $\theta$ . Gaussian integers that are visited twice by the curve  $K[\theta(a)]$  will be called *internal points*, and those visited once *boundary points*. Obviously, if  $S$  codes an  $m$ -fold, and we denote by  $i(S)$  and  $r(S)$  the number of internal and boundary points of  $S$ , then

$$m = 2i(S) + r(S) - 1.$$

**Proposition 1.** Let  $r(S)$  be the number of Gaussian integer boundary points of the curve associated to a fold  $S$ . Let  $S$  and  $T$  be the strings of two self-avoiding folds. Suppose that  $S$  is plane-filling and symmetric ( $S = \bar{S}$ ). Then

$$r(S * T) = \frac{1}{2} r(S)r(T).$$

**Proof.** Let  $\theta_S$  and  $\theta_T$  of length  $m_S$  and  $m_T$  be the folding morphisms of  $S$  and  $T$ . It can be deduced from the symmetry that the mid grid border of  $K[\theta_S(e)]$ ,  $e = a, b, c, d$  consists of four congruent quarter-borders, each bordering say,  $k_S$  boundary points (cf. Fig. 21). Since  $S$  is plane-filling, there are no other boundary points, except for the first and the last point of  $K[\theta_S(e)]$ . Hence  $r(S) = 4k_S + 2$ . The curve  $K[\theta_S\theta_T(a)]$  generated by  $S * T$ , is obtained by replacing the  $m_T$  scaled segments of  $L_S(K[\theta_T(a)])$  by translations of curves  $K[\theta_S(e)]$ ,  $e = a, b, c$  or  $d$ , where  $L_S$  is multiplication by  $z_S = f(\theta_S(a))$ .



**Fig. 21.** The symmetric self-avoiding plane-filling 25-fold  $\theta_S(a) = abadadcbcdadcdadcbcdadaba$ : its quarter-borders in  $\tilde{K}[\theta_S(c)] + 2z_S$  in  $\tilde{K}[\theta_S\theta_T(a)]$ , where  $T$  is the 8-fold with  $\theta_T(a) = abadcdcd$ . In this example  $k_S = 3$ .

In this process we pass  $i(T)$  (scaled) internal points twice, where we lose eight times  $k_S$  points, and four endpoints. Further we pass  $r(T)$  (scaled) boundary points once, where we lose (except at the begin and endpoint of  $L_S(K[\theta_T(a)])$ ) one endpoint and two times  $k_S$  points. We thus obtain that

$$\begin{aligned} r(S * T) &= m_T r(S) - i(T)(8k_S + 4) - (r(T) - 2)(2k_S + 1) \\ &= (2i(T) + r(T) - 1)r(S) - 2i(T)r(S) - (r(T) - 2)r(S)/2 \\ &= r(S)r(T)/2. \quad \square \end{aligned}$$

A slightly surprising consequence of Proposition 1 is that if  $S$  is any symmetric plane-filling string with no internal points, then the fold  $S * S$  has  $(m - 1)^2/4$  internal points.

### 10. Tiles

By Theorem 2.4 of [4] the sequence of sets  $L_\theta^{-n}(K[\theta^n(abcd)])$  converges to a compact set  $\mathcal{T}_S$ , which we call the *tile* generated by  $S$ . Now if  $S$  is a self-avoiding and plane-filling  $m$ -fold, then  $K[\theta(abcd)]$  is maximally simple, and this implies that there exists a set of  $m$  Gaussian integers  $\mathcal{D}$  such that

$$K[\theta(abcd)] = \bigcup_{\vartheta \in \mathcal{D}} (K[abcd] + \vartheta).$$

But then also for all  $n = 1, 2, \dots$

$$K[\theta^{n+1}(abcd)] = \bigcup_{\vartheta \in \mathcal{D}} (K[\theta^n(abcd)] + L_\theta^n(\vartheta)),$$

and so for all  $n = 1, 2, \dots$

$$L_\theta L_\theta^{-n-1} K[\theta^{n+1}(abcd)] = \bigcup_{\vartheta \in \mathcal{D}} (L_\theta^{-n} K[\theta^n(abcd)] + \vartheta).$$

Taking the limit as  $n \rightarrow \infty$  we obtain the *self-similarity equation*

$$L_\theta(\mathcal{T}_S) = \bigcup_{\vartheta \in \mathcal{D}} (\mathcal{T}_S + \vartheta). \tag{9}$$

Applying  $L_\theta^{-1}$  on both sides we see that  $\mathcal{T}_S$  is the unique attractor of a contracting iterated function system<sup>3</sup>.

A set  $\mathcal{T}$  satisfying (9) is called a *self-affine tile*. Such tiles have been extensively studied, mostly under the condition that  $L_\theta(\mathbb{G}) \subset \mathbb{G}$  and that the digit set  $\mathcal{D}$  is a complete residue system for  $\mathbb{G}/L_\theta(\mathbb{G})$ , where we write  $\mathbb{G}$  for the Gaussian integers. In this case  $\mathcal{D}$  is called a *standard digit set*. The authors of [8,16] determine under the standard digit set condition the Hausdorff dimension of the boundary  $\partial\mathcal{T}_S$  of  $\mathcal{T}_S$ , with a rather involved algorithm in both papers (see [7] for an even more involved algorithm that does not require the standard digit set assumption).

<sup>3</sup> This connection between recurrent sets as in [4], and language directed IFS's, also called recurrent IFS's, or graph directed IFS's has been known for a long time, see [15].

For our folding tiles  $\mathcal{T}_5$  it is obvious that they will not have a standard digit set when  $m$  is even (all digits will be even Gaussian integers). For the case  $m = 2$  we will still obtain a tiling of the plane by a scaling, translation and rotation of the well known twin-dragon tile (it has been observed by Bandt ([2], p. 550) that this holds for any 2 element digit set  $\{0, \vartheta\}$ ).

For symmetric self-avoiding plane-filling folds  $S$  we present a particularly simple formula for  $\dim_{\text{H}}(\partial\mathcal{T}_S)$ , where  $\dim_{\text{H}}(\cdot)$  denotes the Hausdorff dimension of a set. (Actually [8,16] do apply in this case, as it is easy to show that for symmetric  $S$  the digit set will be standard.)

**Theorem 7.** *Let  $S$  be a self-avoiding plane-filling symmetric  $m$ -fold with  $r$  boundary points. Let  $\mathcal{T}_S$  be the tile generated by  $S$  with topological boundary  $\partial\mathcal{T}_S$ . Then*

$$\dim_{\text{H}}(\partial\mathcal{T}_S) = \frac{2 \log(r) - \log(4)}{\log(m)}.$$

**Proof.** The tile  $\mathcal{T}_S$  is a union of 4 sets: the limits of  $L_\theta^{-n}(K[\theta^n(e)] + f(e))$ , where  $e = a, b, c, d$ . By Lemma 8, the boundaries of these sets all have the same Hausdorff dimension. So it follows with the usual counting and scaling argument (cf. [11]), that

$$\dim_{\text{H}}(\partial\mathcal{T}_S) = \lim_{n \rightarrow \infty} \frac{\log r(S^{*n})}{\log(|z|^n)}.$$

where  $z$  is the endpoint of  $K[\theta(a)]$ . Since  $S$  is self-avoiding and plane-filling, we have  $|z| = \sqrt{m}$ , and from Proposition 1 we obtain

$$r(S^{*n}) = \left(\frac{1}{2}\right)^{n-1} r^n = 2\left(\frac{r}{2}\right)^n.$$

This leads to the formula in the theorem.  $\square$

We give an example: for  $m = 5$  we have the symmetric fold  $S = DDUU$  with  $\theta(a) = abcba$ . Since there are no internal points, the boundary of the tile  $\mathcal{T}_S$  has Hausdorff dimension

$$\frac{2 \log(6) - \log(4)}{\log(5)} = \frac{\log(9)}{\log(5)}.$$

### 11. Other angles, more morphisms

The strip of paper can be opened at other angles than  $90^\circ$ . In their paper [3] Davis and Knuth observe that the fold  $S = DU$ , when opened at  $60^\circ$  angles yields an interesting curve, which they coined the *terdragon*. They prove, using a ternary number system, that 6 infinite terdragons rotated over successive multiples of  $60^\circ$  pass exactly once through every edge of the triangular lattice. Actually, the theory developed in this paper can give a completely different proof of this property. We should now use a 3 letter alphabet, say  $\{a, c, e\}$ . The letters correspond to segments in the complex plane by the map  $f$ , now defined by

$$f(a) = 1, \quad f(c) = \omega, \quad f(e) = \omega^2, \quad \text{with } \omega := e^{2\pi i/3}.$$

The anti-morphism  $\tau$  is as before, and the morphism  $\sigma$  is now defined by  $\sigma(a) = c, \sigma(c) = e, \sigma(e) = a$ . Folding morphisms again are those that commute with  $\sigma\tau$ , and, moreover, do not contain  $aa, cc$  or  $ee$ . The basic example is  $\theta$  with  $\theta(a) = aca$ , which generates the terdragon.

Defining the  $\theta$ -loop by the curve  $\tilde{K}[\theta(ace)]$ , and the anti- $\theta$ -loop by the curve  $\tilde{K}[\theta(aec)]$ , then Theorem 1 remains true provided we replace ‘the  $\theta$ -loop’ by ‘the  $\theta$ -loop and the anti- $\theta$ -loop’. The difference is caused by the different symmetries of the triangular lattice.

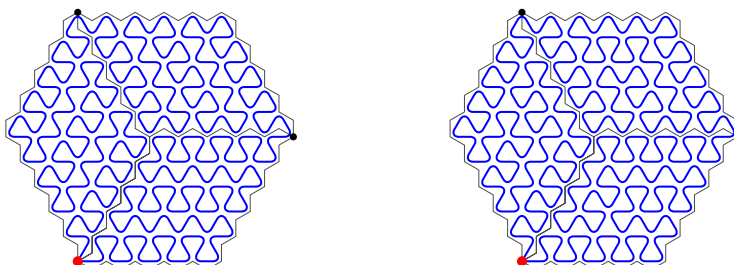


Fig. 22. The  $\theta$ -loop of a symmetric and an asymmetric plane-filling 75-fold.



Theorems 2 and 3 will remain true with the obvious definitions of self-avoiding and plane-filling, but the proof of the analogue of the existence Theorem 4 fails miserably. The analogue of Theorem 4 would be that there is a self-avoiding plane-filling curve on the triangular lattice generated by a  $m$ -fold for every  $m > 1$  that can be written as

$$m = r^2 + s^2 - rs, \quad r, s \in \mathbb{Z}^+,$$

since  $|z| = \sqrt{r^2 + s^2 - rs}$  is the norm of an Eisenstein integer  $z = r + s\omega$ , and we should have  $|z|^2 = m$ . This set consists of the so called L schian numbers larger than 1

$$\mathcal{L} := \{3, 4, 7, 9, 12, 13, 16, 19, 21, \dots\}.$$

The reason that the proof fails is that not all paths in the directed triangular lattice correspond to allowed folding words (as, e.g., the word  $aa$ ). In fact not only does the proof fail, but the result is also not true, since one can show that there is no self-avoiding plane-filling  $m$ -fold on the triangular lattice when  $m$  is even. (The proof considers the union of the  $\theta$ -loop and the anti- $\theta$ -loop: if we remove from this set the set  $\tilde{K}[\theta(a)]$  then one obtains a simple curve that (as an application of Pick’s Theorem shows) has  $m - 1$  Eisenstein integers in its interior. By symmetry of the midgrid borders of the  $\theta$ -loop and the anti- $\theta$ -loop, these integers have to be distributed evenly over the two loops, which is only possible if  $m - 1$  is even.)

Nevertheless, self-avoiding plane-filling curves generated by folds opened at  $60^\circ$  abound. For small  $m$  these all have to be symmetric, but for large  $m$  they can be asymmetric (see e.g., Fig. 22, right). An infinite family is given, for example, by all  $m$ -folds with  $m = 3k^2$ ,  $k = 1, 3, 5, 7, \dots$ . The folds here end in  $2k + k\omega$ , and trace a parallelogram-shaped midgrid border in a zig-zag way (see Fig. 22, left for the  $\theta$ -loop of this fold when  $k = 5$ ). Here the midgrid border is defined as in the  $90^\circ$  case, by a diamond map  $K_\diamond$ , where now, for example,  $K_\diamond[a]$  is the quadrangle with vertices  $0, \frac{1}{2} + \frac{1}{6}\sqrt{3}i, \frac{1}{2} - \frac{1}{6}\sqrt{3}i$ , and  $1$ .

Curves on the hexagonal lattice are obtained when we unfold at  $120^\circ$  degree angles. We need a 6 letter alphabet, say  $\{a, b, c, d, e, f\}$ , which corresponds to the 6th roots of unity. Theorem 1 remains true, when we define the  $\theta$ -loop in the usual way. See Fig. 23 for how the 3-fold  $S = DD$  generates a self-avoiding curve on the hexagonal lattice. Actually the limiting fractal is an old friend: it is easily shown that the curves  $L_\theta^{-n}K[\theta^n(a)]$  are exponentially close in the Hausdorff metric to the usual approximants of the Sierpinski triangle, also known as the Sierpinski gasket. The notion of plane-filling is problematic in the hexagonal case, since the grid points all have degree 3.

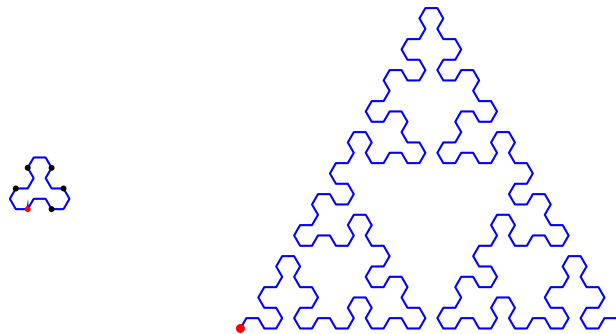


Fig. 23. Folding in three according to  $S = DD$ , and opening at  $120^\circ$  degree angles. Shown are the  $\theta$ -loop and the 5th order curve.

The concept of self-avoiding and plane-filling curves generated by morphisms on the 4 letter alphabet can be generalized in another direction. In fact, a slight generalization of Theorems 2 and 3 will hold for morphisms that preserve  $\mathbb{D}_2$ -words (see [6]). From that paper we have the following example of a non-uniform morphism that generates a self-avoiding and plane-filling curve:

$$\theta(a) = abad, \quad \theta(b) = ab, \quad \theta(c) = cb, \quad \theta(d) = cdcd.$$

Finally, as a response to a query of one of the referees I would like to mention a three dimensional version of folding studied in [9]. Regretfully, the ideas of Section 3 do not seem to apply in this case.

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**References**

[1] Jean-Paul Allouche, Jeffrey Shallit, Automatic Sequences, Theory, Applications, Generalizations, Cambridge University Press, Cambridge, 2003, xvi+571.  
 [2] Christoph Bandt, Self-similar sets. V. Integer matrices and fractal tilings of  $\mathbb{R}^n$ , Proc. Amer. Math. Soc. 112 (1991) 549–562.

- [3] C. Davis, D. Knuth, Number representations and dragon curves, *J. Recreat. Math.* 3 (1970) 66–81. 133–149.
- [4] F.M. Dekking, Recurrent sets, *Adv. Math.* 44 (1) (1982) 78–104.
- [5] F.M. Dekking, Iterated paperfolding and plane-filling curves, Report 8126 of the Mathematical Institute, Katholieke Universiteit, Nijmegen, The Netherlands, 1981, pp. 1–16.
- [6] Michel Dekking, Michel Mendès France, Alf van der Poorten, Folds III, More morphisms, *Math. Intelligencer* 4 (4) (1982) 190–195.
- [7] F.M. Dekking, P. van der Wal, The boundary of the attractor of a recurrent iterated function system, *Fractals* 10 (2002) 77–89.
- [8] P. Duvall, J. Keesling, A. Vince, The Hausdorff dimension of the boundary of a self-similar tile, *J. Lond. Math. Soc.* (2) 61 (2000) 748–760.
- [9] Michel Mendès France, J.O. Shallit, Wire bending, *J. Combin. Theory Ser. A* 50 (1989) 1–23.
- [10] L. Euler, *Solutio problematis ad geometriam situs pertinentis*, *Comment. Academiae Sci. I. Petropolitanae* 8 (1736) 128–140.
- [11] K. Falconer, *Fractal Geometry*, in: *Mathematical Foundations and Applications*, Wiley, Chichester, 1990, xxii+288.
- [12] M. Gardner, *Mathematical games*, *Sci. Am.* 216 (1967) 124–125.
- [13] D. Knuth, Addendum to Number representations and dragon curves, in: *Selected Papers on Fun and Games*, in: *CSLI Lecture Notes*, vol. 192, Center for the Study of Language and Information, Stanford, California, 2010, pp. 603–614.
- [14] D. Knuth, *Diamonds and dragons: notes on Dekking’s generalized dragon curves*, unpublished manuscript at <http://www-cs-faculty.stanford.edu/~uno/preprints.html#unpub> (2010) 1–5.
- [15] M. Morcrette, Sur l’équivalence de descriptions de figures itérées (Equivalence of descriptions of iterated pictures), *Theoret. Comput. Sci.* 165 (2) (1996) 325–354 (in French).
- [16] Robert S. Strichartz, Wang, Yang, *Geometry of self-affine tiles. I*, *Indiana Univ. Math. J.* 48 (1999) 1–23.