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Minimal Resolutions of Some Monomial Ideals

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Let $R = k[x_1, ..., x_n]$ be a polynomial ring in n variables over a field k. We view R as a graded algebra, where $\deg(x_i) = 1$.

A graded R-module V has an essentially unique minimal graded free resolution

$$0 \longrightarrow L_m \xrightarrow{d} L_{m-1} \xrightarrow{d} \cdots \xrightarrow{d} L_1 \xrightarrow{d} L_0 \xrightarrow{\alpha} V \longrightarrow 0$$

which is characterized, among free graded resolutions, by the condition

$$d(L_q) \subset M \cdot L_{q-1}$$

for all $q \geqslant 1$, where $M = (x_1, ..., x_n)$ is the augmentation ideal in $k[x_1, ..., x_n]$.

A particularly simple sort of graded modules is provided by the monomial ideals, i.e., the ideals generated by monomials. However, even in this case, it is still an open problem to describe explicitly the minimal resolution. (See [CEP, p. 29].)

In this paper, we give an explicit minimal resolution for a family of monomial ideals, which we call stable ideals, comprising in particular the "Borel fixed ideals" considered for instance by D. Bayer and M. Stillman in [BS, Proposition (2.7)]. See also [G].

We are grateful to L. Robbiano for drawing our attention to the class of Borel fixed ideals. The interest for these ideals motivated a revision of the paper. The preliminary version was concerned with a somewhat more restricted family.

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DEFINITION. If w is a monomial in a polynomial ring $R = k[x_1, ..., x_n]$, let $\max(w)$ denote the largest index of the variables dividing w. A monomial ideal I in R will be said to be *stable* if for every monomial $w \in I$ and index $i < m = \max(w)$, the monomial $x_i w/x_m$ again belongs to I.

Stable ideals are studied in Section 1. We also recall the definition of Borel fixed ideals at the end of Section 1. The resolution is constructed in Section 2 and provided with a structure of differential algebra. In Section 3 we have collected a few remarks on Betti numbers and Poincaré series.

1. Canonical Decompositions in Stable Ideals

We start with some remarks on monomial ideals in a polynomial ring $R = k[x_1, ..., x_n]$, k being now an arbitrary unitary ring.

Suppose that $I \subset R$ is the ideal generated by a set S of monomials. If $w \in I$ is any monomial in I, then w is a multiple of one of the generators in S. Assume moreover that none of the monomials $u \in S$ is a (proper) multiple of any other $v \in S$. Then, S is a *minimal* system of generators of I. This minimal system of generators is uniquely determined by I. It is the set of all monomials in I which are not proper multiples of any monomial in I. We shall denote this generator system by $G(I)$ and call it the *canonical generator system* of I. Of course, $G(I)$ is a finite set.

We will require a lemma stating that if a monomial w belongs to the stable ideal I, there is a canonical way of pinpointing a minimal generator $u \in G(I)$ of which w is a multiple.

First some notations. Recall that we denote by $\max(a)$ the largest index of the variables actually occuring in a. Thus, if $a = x_1^{a_1} \cdots x_n^{a_n}$, then $\max(a) = \max\{i \mid a_i > 0\}$. Similarly, $\min(a) = \min\{i \mid a_i > 0\}$. For convenience, we define $\min(1) = +\infty$, larger than any integer.

Recall that a monomial ideal I is stable if for every monomial $w \in I$ and index $i < m = \max(w)$, we have $x_i w/x_m \in I$.

LEMMA 1.1. *Let $I \subset R$ be a stable monomial ideal with canonical generating set $G(I)$. For every monomial $w \in I$, there is a unique decomposition*

$$w = u \cdot y$$

with $u \in G(I)$ and $\max(u) \leqslant \min(y)$.

Proof. If $w \in I$ is a monomial, then w is a multiple of some element $v \in G(I)$. Suppose that $w = v \cdot z$ and by misfortune $\max(v) > \min(z)$.

Then we can find an index $i < m = \max(v)$ such that x_i divides z. We can write

$$w = (x_i v/x_m) \cdot (x_m z/x_i).$$

By the stability hypothesis, $x_i v/x_m \in I$, and thus $x_i v/x_m$ is a multiple of some other generating monomial $v' \in G(I)$. Hence, $w = v' \cdot z'$ for some suitable monomial z'.

Clearly, after finitely may such mistreatments of the decomposition of w, we reach a situation $w = u \cdot y$, where $u \in G(I)$ and $\max(u) \leqslant \min(y)$. (On passage from $v = x_1^{b_1} \cdots x_n^{b_n}$ to v', the function $f(v) = \sum_{i=1}^n ib_i$, for instance, is strictly decreasing.)

What about uniqueness? If $w = u \cdot y = u' \cdot y'$, where u, $u' \in G(I)$, $\max(u) \leqslant \min(y)$ and $\max(u') \leqslant \min(y')$, then u, u' are both initial segments of w and therefore one of them must divide the other. But for u, $u' \in G(I)$, this is only possible if $u = u'$.

This unique decomposition $w = u \cdot y$, with $u \in G(I)$ and $\max(u) \leqslant \min(y)$, of a monomial $w \in I$ will be called the *canonical decomposition* of w, or the *canonical I-decomposition* if we wish to emphasize the dependence on the ideal I.

Remark. Note that conversely, if I is a monomial ideal in which every monomial has a canonical decomposition, then I must be stable. Indeed, if $w \in I$ is a monomial, $m = \max(w)$, and $i < m$, let

$$x_i w = u \cdot y, \qquad u \in G(I), \qquad \max(u) \leqslant \min(y)$$

be the canonical decomposition of $x_i w$. Then $y \neq 1$ because $u \in G(I)$ cannot be a multiple of w, and therefore x_m actually divides y. ($m = \max(w) = \max(x_i w)$.) Setting $y = y' x_m$, we have $x_i w/x_m = uy' \in I$ and therefore I is stable.

We shall need some properties of the canonical decomposition which are perhaps best expressed by introducing a decomposition function: If I is a stable ideal, let $M(I)$ be the set of all monomials in I. Define the *decomposition function*

$$g\colon M(I) \to G(I)$$

by $g(w) = u$ if $w = u \cdot y$ is the unique canonical I-decomposition of w, i.e. $u \in G(I)$, $\max(u) \leqslant \min(y)$.

LEMMA 1.2. *Let I be a stable ideal and let $g\colon M(I) \to G(I)$ be its decomposition function. Then for all $w \in M(I)$, and all monomials y, the equation $g(wy) = g(w)$ holds if and only if* $\max(g(w)) \leqslant \min(y)$.

Proof. Suppose $g(wy) = g(w)$. Then the canonical decomposition of wy reads

$$wy = g(wy) \cdot z = g(w) \cdot z,$$

where $\max(g(w)) = \max(g(wy)) \leqslant \min(z)$.

Since $g(w)$ divides w, the monomial y divides z and thus $\min(z) \leqslant \min(y)$. It follows that

$$\max(g(w)) \leqslant \min(y).$$

Conversely, if $\max(g(w)) \leqslant \min(y)$, consider the canonical decomposition of w: $w = g(w) \cdot z'$, $\max(g(w)) \leqslant \min(z')$. Then, $wy = g(w) \cdot z'y$ is the canonical decomposition of wy, i.e., $\max(g(w)) \leqslant \min(z'y)$. Therefore, $g(wy) = g(w)$.

Remark. It is interesting to note that Lemma 1.2 actually characterizes stable ideals. More precisely, let I be a monomial ideal (not known to be stable). Assume that I possesses a decomposition function

$$g: M(I) \to G(I)$$

such that for all monomials w, y with $w \in M(I)$, we have the axioms

DF1. $g(w)$ divides w,

DF2. $g(wy) = g(w)$ if and only if $\max(g(w)) \leqslant \min(y)$.

Claim. Then I is a stable ideal.

By the remark following Lemma 1.1 above, it suffices to prove that I admits canonical decompositions.

For this purpose, note first that we must have $g(u) = u$ for all $u \in G(I)$, because $g(u) \in G(I)$ divides $u \in G(I)$ by DF1.

Now, let $w \in I$ be a monomial in I and set $y = w/g(w)$. Applying g to both sides of the equation $w = g(w) \cdot y$, we get

$$g(w) = g(g(w) \cdot y).$$

Setting $u = g(w)$, this implies

$$g(uy) = g(g(w) \cdot y) = g(w) = u = g(u),$$

and thus, by DF2,

$$\max(g(w)) = \max(g(u)) \leqslant \min(y).$$

Hence, $w = g(w) \cdot y$ is a canonical decomposition and I is a stable ideal with g as a decomposition function.

We continue to list the properties of the decomposition function $g: M(I) \to G(I)$ of a stable ideal which will be needed in Section 2.

LEMMA 1.3. *Let I be a stable monomial ideal with decomposition function $g: M(I) \to G(I)$. Then, for any monomial a and any $w \in M(I)$,*

(1) $g(ag(w))=g(aw)$,

(2) $\max(g(aw))\leqslant\max(g(w))$.

Proof of (1). Assume first that $a=x_i$. We want to show that

$$g(x_ig(w))=g(x_iw).$$

Case 1. If $i\geqslant\max(g(w))$, then

$g(x_iw)=g(w)$, by Lemma 1.2, and

$g(x_i\,g(w))=g(g(w))=g(w)$, by Lemma 1.2 again, applied to $g(w)$.

Thus, $g(x_ig(w))=g(x_iw)$ as desired.

Case 2. If $i<\max(g(w))$, then start from the canonical decomposition

$$w=g(w)\cdot y, \qquad \text{with} \quad \max(g(w))\leqslant\min(y).$$

Multiplying this by x_i and applying g, we get

$$g(x_iw)=g(x_i\,g(w)\cdot y).$$

We have, since $g(x_i\,g(w))$ divides $x_i\,g(w)$,

$$\max g(x_ig(w))\leqslant\max(x_i\,g(w))=\max(g(w))\leqslant\min(y);$$

hence, by Lemma 1.2,

$$g((x_i\,g(w))\cdot y)=g(x_i\,g(w)),$$

and thus $g(x_iw)=g(x_i\,g(w))$.

The equation $g(aw)=g(ag(w))$ for an arbitrary monomial a follows by induction on $\deg(a)$.

Proof of (2). Again, as in the proof of (1), it suffices to prove that the assertion holds for $a=x_i$, i.e., $\max(g(x_iw))\leqslant\max(g(w))$.

If $i\geqslant\max(g(w))$, then $g(x_iw)=g(w)$ by Lemma 1.2, and

$$\max(g(x_iw))=\max(g(w)).$$

If $i\leqslant\max(g(w))$, then

$$\max g(x_i\,g(w))\leqslant\max(x_i\,g(w))\leqslant\max(g(w)),$$

and therefore $\max(g(x_iw))\leqslant\max(g(w))$ since $g(x_iw)=g(x_i\,g(w))$ by (1).

We still need one more lemma which involves an ordering on the set of monomials in $k[x_1, ..., x_n]$. We will use the graded reversed lexicographical order, i.e.

$$u=x_1^{a_1}\cdots x_n^{a_n}<v=x_1^{b_1}\cdots x_n^{b_n}$$

if and only if

$\deg(u) < \deg(v)$, or

$\deg(u) = \deg(v)$ and the last non-vanishing difference $a_1 - b_1, \ldots, a_n - b_n$ is negative.

LEMMA 1.4. *Let $w \in M(I)$ be a monomial in I and let $a \cdot w$ be some monomial multiple of w. Then*

$$g(a \cdot w) \leqslant g(w)$$

in the graded reversed lexicographical order.

Proof. Suppose first that $a = x_i$.

If $\max(g(w)) \leqslant i$, then $g(x_i w) = g(w)$ by Lemma 1.2 and there is nothing to prove.

If $i < \max(g(w))$, let

$$x_i\, g(w) = g(x_i\, g(w)) \cdot y = g(x_i w) \cdot y,$$

with $\max(g(x_i w)) \leqslant \min(y)$, be the canonical decomposition of $x_i\, g(w)$. (Here, the second equality is by Lemma 1.3 (1).)

Since $g(x_i w) \in G(I)$ is not a multiple of $g(w)$, we must have $\deg(y) > 0$, and thus $\deg(g(x_i w)) \leqslant \deg(g(w))$.

Again, if $\deg(g(x_i w)) < \deg(g(w))$, we are finished:

$$g(x_i w) < g(w).$$

If however $\deg(g(x_i w)) = \deg(g(w))$, then $\deg(y) = 1$ and y is a variable: $y = x_j$.

Since $i < \max(g(w))$, and $\max(g(x_i w)) \leqslant j$, it follows that $j = \max(g(w))$. The exponent of x_j in $g(w)$, or equivalently in $x_i\, g(w) = g(x_i w)\, x_j$, is strictly larger than the exponent of x_j in $g(x_i w)$, and thus $g(x_i w) < g(w)$ as desired.

The general case, $g(aw) \leqslant g(w)$, where a is arbitrary follows by induction on $\deg(a)$.

Remark. In their paper [BS], D. Bayer and M. Stillman consider what they call "Borel fixed ideals". A monomial ideal I is Borel fixed if whenever $w \in M(I)$ is a monomial in I and x_j is a variable dividing w, and $i < j$, then

$$x_i w / x_j \in I.$$

(It is not required here that $j = \max(w)$ as in the definition of a stable ideal.)

Of course, every Borel fixed ideal is stable. The converse is not true as seen from the example

$$I = (x_1^2, x_1 x_2, x_2^2, x_2 x_3)$$

in the polynomial ring $k[x_1, x_2, x_3]$. This ideal is stable but it is not Borel fixed since $x_1 x_3 = x_1 \cdot (x_2 x_3)/x_2$ does not belong to I.

As an exercise for the reader, we propose the following: Show that if I is Borel fixed, then $i \leqslant j$ implies the inequality $\max(g(x_i u)) \leqslant \max(g(x_j u))$ for all $u \in G(I)$. The example

$$I = (x_1^2, x_1 x_2, x_2^2, x_2 x_3, x_1 x_3^2, x_1 x_3 x_4, x_3^3, x_3^2 x_4, x_3 x_4^2, x_3 x_4 x_5)$$

shows that this fails in general for stable ideals. (Take $i = 1$, $j = 2$, $u = x_3 x_4 x_5$.)

More examples of stable ideals, some not Borel fixed, will be given at the end of Section 3.

2. Minimal Resolutions of Stable Ideals

We now proceed to describe the minimal graded free resolution $(L_*(I), d)$ of an arbitrary stable monomial ideal $I \subset R = k[x_1, ..., x_n]$.

We get a free graded resolution if k is a unitary ring. The resolution is minimal if k is a field.

Throughout this section, I is a stable monomial ideal and

$$g: M(I) \to G(I)$$

is its decomposition function as defined in Section 1, such that $g(w) = u$ if the canonical decomposition of w reads $w = u \cdot y$, with $u \in G(I)$ and $\max(u) \leqslant \min(y)$.

The properties of g that we need have been listed in Lemmas 1.2 to 1.4 in the previous section.

Here is the definition of the free R-modules $L_q(I)$, $q \geqslant 0$. First define a symbol $e(i_1, ..., i_q; u)$ to be *admissible* if the following three conditions are satisfied:

(1) $u \in G(I)$ is a monomial from the canonical, minimal set of monomial generators of I,

(2) $i_1, ..., i_q$ are integers such that

$$1 \leqslant i_1 < i_2 < \cdots < i_q \leqslant n,$$

(3) $i_q < m = \max(u)$.

In this definition, q may be 0. If $q = 0$, we consider that (2) and (3) are satisfied as void conditions.

Now, let $L_q = L_q(I)$ be the free R-module on the set of all admissible symbols $e(i_1, ..., i_q; u)$ for fixed $q \geqslant 0$.

In particular, $L_0(I)$ is the free R-module with set of generators $e(u)$ in bijection with $u \in G(I)$.

We define the map of R-modules

$$\alpha: L_0 \to I$$

by $\alpha(e(u)) = u$.

In order to define $d: L_q \to L_{q-1}$ for $q \geqslant 1$, we need some more notations as follows: Let $e(i_1, ..., i_q; u)$ be an admissible symbol. We shall often write σ for the sequence $(i_1, ..., i_q)$ and thus abbreviate $e(i_1, ..., i_q; u)$ to $e(\sigma; u)$. Occasionally, if $\sigma = (i_1, ..., i_q)$, we write σ_r for the sequence $\sigma_r = (i_1, ..., \hat{\imath}_r, ..., i_q)$ in which i_r has been deleted.

For every $r = 1, ..., q$, the monomial $x_{i_r}u \in I$ has a canonical decomposition

$$x_{i_r}u = u_r \cdot y_r,$$

where $u_r = g(x_{i_r}u) \in G(I)$ and $\max(u_r) \leqslant \min(y_r)$, which is unique by Lemma 1.1 above.

We write $m_r = \max(u_r)$ and denote by $A(\sigma; u) \subset \{1, ..., q\}$ the set of values of r for which $\max\{i_1, ..., \hat{\imath}_r, ..., i_q\} < m_r$.

Thus, $A(\sigma; u)$ is the set of $r \in \{1, ..., q\}$ such that

$$(i_1, ..., \hat{\imath}_r, ..., i_q, m_r)$$

is a strictly increasing sequence, or equivalently, such that $e(\sigma_r; u_r)$ is an admissible symbol.

We can now state the definition of $d: L_q \to L_{q-1}$. It is the R-module map determined by

$$de(\sigma; u) = \sum_{r=1}^{q} (-1)^r x_{i_r} e(\sigma_r; u) - \sum_{r \in A(\sigma; u)} (-1)^r y_r e(\sigma_r; u_r),$$

where $\sigma = (i_1, ..., i_q)$, $\sigma_r = (i_1, ..., \hat{\imath}_r, ..., i_q)$ and $u_r = g(x_{i_r}u)$, and $y_r = x_{i_r}u/u_r$ as above.

Remarks. Observe that since $x_{i_r}u$ is not a minimal generator of I, it follows that $\deg(y_r) \geqslant 1$, and thus $d(L_q) \subset M \cdot L_{q-1}$. In many examples, one actually has $\deg(y_r) > 1$. For instance, if $I = (x_1, x_2^2) \subset k[x_1, x_2]$, $de(1; x_2^2) = -x_1 e(x_2^2) + x_2^2 e(x_1)$.

Note also that the set $A(\sigma; u)$ always contains at least the index q. Indeed, $x_{i_q}u = u_q \cdot y_q$ and x_{i_q} cannot divide y_q, otherwise we would have $u = u_q$ and $i_q < \max(u) = \max(u_q) \leqslant \min(y_q) = i_q$, a contradiction. Hence, x_{i_q} divides u_q and so $i_q \leqslant m_q = \max(u_q)$. It follows that $(i_1, ..., i_{q-1}, m_q)$ is strictly increasing. In other words, $q \in A(\sigma; u)$. For $r \in \{1, ..., q-1\}$, the condition $r \in A(\sigma; u)$ boils down to $i_q < m_r$.

The module $L_q(I)$ is endowed with a natural (monomial valued) multigrading defined by

$$\mathrm{DEG}\{z \cdot e(i_1, ..., i_q; u)\} = zx_{i_1} \cdots x_{i_q} u,$$

for $q \geqslant 1$, and of course $\mathrm{DEG}\{z \cdot e(u)\} = zu$.

Note that the maps $d: L_q \to L_{q-1}$, as well as $\alpha: L_0 \to I$, preserve the multigrading.

THEOREM 2.1. *Let $R = k[x_1, ..., x_n]$ be the polynomial ring in n variables over the unitary ring k. Let I be a stable monomial ideal in R. Then, $(L_*(I), d)$ as described above is a free graded resolution of I over R. If k is a field, then $(L_*(I), d)$ is the minimal resolution.*

Proof. In two parts.

(1) We first have to check that $(L_*(I), d)$ is indeed a complex. The direct verification requires some painstaking calculations. So we prefer to exhibit $(L_*(I), d)$ as the quotient of another complex as follows.

Let C_q be the free R-module on all symbols $e(i_1, ..., i_q; u)$ satisfying only the two conditions

(1) $u \in G(I)$, and

(2) $1 \leqslant i_1 < \cdots < i_q \leqslant n$,

i.e., we drop the condition (3) in the definition of an admissible symbol and do not require $i_q < \max(u)$.

In $C_* = \oplus_q C_q$, define $D: C_q \to C_{q-1}$ to be the R-module map determined by

$$De(\sigma; u) = \sum_{r=1}^{q} (-1)^r x_{i_r} e(\sigma_r; u) - \sum_{r=1}^{q} (-1)^r y_r e(\sigma_r; u_r),$$

where, as in the formula for d, $u_r = g(x_{i_r} u)$ and $y_r = x_{i_r} u/u_r$.

We claim that (C_*, D) is a complex. (Not a resolution!)

For the proof, it is convenient to cut in two the operator D: Let $D = D_1 - D_2$, where

$$D_1 e(\sigma; u) = \sum_{r=1}^{q} (-1)^r x_{i_r} e(\sigma_r; u),$$

$$D_2 e(\sigma; u) = \sum_{r=1}^{q} (-1)^r y_r e(\sigma_r; u_r).$$

It is well-known and easy to verify that $D_1^2 = 0$.

Also, the formula $D_1 D_2 + D_2 D_1 = 0$ is a simple and straightforward calculation which does not require any of the lemmas in Section 1.

Finally, in order to calculate $D_2 D_2 e(\sigma; u)$, let

$$u_{r,s} = g(x_{i_r} x_{i_s} u).$$

By Lemma 1.3, (1), we have

$$u_{r,s} = g(x_{i_s} g(x_{i_r} u)),$$

and thus

$$D_2 D_2 e(\sigma; u) = \sum_{1 \leqslant s < r \leqslant n} (-1)^{r+s} y_r y_{s,r} e(\sigma_{s,r}; u_{s,r})$$
$$+ \sum_{1 \leqslant r < s \leqslant n} (-1)^{r+s-1} y_r y_{s,r} e(\sigma_{r,s}; u_{r,s}),$$

where $y_{s,r} = x_{i_s} g(x_{i_r} u)/g(x_{i_s} x_{i_r} u)$. Note that $u_{r,s} = u_{s,r}$.

Interchanging the summation indices r, s in the first sum, we see that the symbols $e(\sigma_{r,s}; u_{r,s})$ in the two sums are the same. By homogeneity, the monomial coefficients $y_s y_{r,s}$ and $y_r y_{s,r}$ must also agree: All terms have the same multidegree $x_\sigma u$, where $x_\sigma = x_{i_1} \cdots x_{i_q}$.

Of course, one can also verify directly that $y_r y_{s,r} = y_s y_{r,s}$.

It follows that $D_2^2 = 0$, and thus $D^2 = 0$.

Now, let $N_q \subset C_q$ be the submodule generated by the symbols $e(i_1, ..., i_q; u)$ with $\max(u) \leqslant i_q$. It is easily verified that $N_* \subset C_*$ is a subcomplex, i.e., $DN_q \subset N_{q-1}$.

Indeed, if $\max(u) \leqslant i_q$, then by Lemma 1.2, $g(x_{i_q} u) = u$, and thus $y_q = x_{i_q}$. So the last term $(-1)^q y_q e(\sigma_q; u_q)$ in $D_2 e(\sigma; u)$ coincides with the last term $(-1)^q x_{i_q} e(\sigma_q; u)$ of $D_1 e(\sigma; u)$.

It follows that if $\max(u) \leqslant i_q$, then

$$De(\sigma; u) = \sum_{r=1}^{q-1} (-1)^r x_{i_r} e(\sigma_r; u) - \sum_{r=1}^{q-1} (-1)^r y_r e(\sigma_r; u_r).$$

Since, by Lemma 1.3 (2),

$$\max(u_r) = \max g(x_{i_r} u) \leqslant \max(u) \leqslant i_q,$$

and i_q is the last index in σ_r for $r = 1, ..., q-1$, it follows that $De(\sigma; u) \in N_{q-1}$.

Clearly, $L_* = C_*/N_*$ with L_q being the free R-module on admissible symbols $e(i_1, ..., i_q; u)$, such that $i_q < \max(u)$. The boundary operator $d\colon L_q \to L_{q-1}$ is induced by the boundary operator D on C_*. Hence, $d^2 = 0$.

The vanishing of the composition $\alpha \circ d\colon L_1 \to L_0 \to I$ is easily verified by direct computation.

Thus, $(L_*(I), d)$ is a complex.

(2) In order to prove

$$\operatorname{Ker}\{d_q: L_q \to L_{q-1}\} \subset \operatorname{Im}\{d_{q+1}: L_{q+1} \to L_q\},$$

we shall define a "normal form" to which every element of L_q may be reduced modulo $\operatorname{Im}(d_{q+1})$, and show that $\operatorname{Ker}(d_q)$, respectively $\operatorname{Ker}(\alpha)$, contains no normal element except 0.

This program will prove the required inclusion, since we already know by (1) that $\operatorname{Im}(d_{q+1}) \subset \operatorname{Ker}(d_q)$.

Consider the natural k-basis for L_q,

$$B = \{z \cdot e \mid z \text{ is a monomial in } x_1, ..., x_n \text{ and}$$
$$e = e(\sigma; u) \text{ is an admissible symbol in } L_q\},$$

whose elements will be called *terms*.

Definition. A term $z \cdot e(i_1, ..., i_q; u)$ will be called *normal* if $z = 1$, or if

$$\min(z) \geqslant i_1, \qquad \text{when } q \geqslant 1,$$
$$\min(z) \geqslant \max(u), \qquad \text{when } q = 0.$$

An element $f \in L_q$ is normal if it is a linear combination of normal terms. The element 0 is normal.

Let $<$ denote, as in Section 1, the graded reversed lexicographical order on the set of monomials.

Thus, $x_1^{a_1} \cdots x_n^{a_n} < x_1^{b_1} \cdots x_n^{b_n}$ if and only if either $\sum_i a_i < \sum_i b_i$, or $\sum_i a_i = \sum_i b_i$ and the last non-zero coordinate of $(a_1 - b_1, ..., a_n - b_n)$ is negative. Observe that the order relation is compatible with the product; that is, for all monomials, the inequalities $z < z'$ and $zx < z'x$ are equivalent.

By abuse of notation, we will let $<$ also denote an ordering on sequences and on terms defined as follows:

Given two sequences $\sigma = (i_1, ..., i_q)$, $\sigma' = (j_1, ..., j_q)$ of the same length q, define $\sigma < \sigma'$ if and only if $x_{i_1} \cdots x_{i_q} < x_{j_1} \cdots x_{j_q}$. Given two terms $z \cdot e(\sigma; u)$, $z' \cdot e(\sigma'; u')$ in L_q, define $z \cdot e(\sigma; u) < z' \cdot e(\sigma'; u')$ if and only if either $u < u'$, or $u = u'$ and $\sigma < \sigma'$, or $e(\sigma; u) = e(\sigma'; u')$ and $z < z'$.

Recall that on each L_q, we have the monomial valued multigrading defined by

$$\operatorname{DEG}\{z \cdot e(i_1, ..., i_q; u)\} = z \cdot x_{i_1} \cdots x_{i_q} \cdot u.$$

For the first part of the proof of $\operatorname{Ker}(d_q) \subset \operatorname{Im}(d_{q+1})$ we need two lemmas.

LEMMA 2.2. *Let* $a = e(i_0, i_1, ..., i_q; u)$ *be a term in* L_{q+1}, $q \geqslant 0$. *Then* $x_{i_0} e(i_1, ..., i_q; u)$ *is the biggest term in* $d(a)$.

Proof. Let $\sigma = (i_0, i_1, ..., i_q)$. We have

$$de(\sigma; u) = \sum_{r=0}^{q} (-1)^{r+1} x_{i_r} e(\sigma_r; u) - \sum_{r \in A(\sigma; u)} (-1)^{r+1} y_r e(\sigma_r; u_r),$$

where $u_r = g(x_{i_r} u)$. Since $i_r < \max(u)$ for all $r = 0, ..., q$, we have, by Lemma 1.2, $g(x_{i_r} u) \neq g(u) = u$. Hence, by Lemma 1.4. $g(x_{i_r} u) < u$. Thus all terms in the second sum are strictly smaller than $x_{i_0} e(\sigma_0; u)$. The terms of the first kind in $de(\sigma; u)$ satisfy $\sigma_r < \sigma_0$ for all $r \geqslant 1$.

It follows that $x_{i_0} e(\sigma_0; u)$ is indeed the biggest term in $d(a)$.

As an easy consequence, we get our next lemma.

LEMMA 2.3. *Let* $b = ze(i_1, ..., i_q; u)$ *be a non-normal term in* L_q, $q \geqslant 0$. *Then b is congruent* modulo $\mathrm{Im}(d_{q+1})$ *to an element whose terms are all strictly smaller than b.*

Proof. Let $i = \max(z)$. We have $i < i_1$ (respectively $i < \max(u)$ for $q = 0$), since b is assumed to be non-normal.

Consider the term $a = (z/x_i) \cdot e(i, i_1, ..., i_q; u)$ in L_{q+1}. By Lemma 2.2, b is the biggest term in $d(a)$. Thus all terms in $b + d(a)$, in which sum b cancels out, are strictly smaller than b.

We can now complete the first step in our program for proving that $\mathrm{Ker}(d_q) \subset \mathrm{Im}(d_{q+1})$.

PROPOSITION 2.4. *Any element in* L_q, $q \geqslant 0$, *is congruent to some normal element* modulo $\mathrm{Im}(d_{q+1})$.

Proof. Let $f \in L_q$. We may assume that f is multihomogeneous and non-normal. By Lemma 2.3, we can replace any non-normal term in f by a combination of strictly smaller terms, not changing the class of f modulo $\mathrm{Im}(d_{q+1})$. When iterated, this process must end up with a normal element in finitely many steps. Indeed, d preserves the multigrading, and there are only finitely many terms with a given multidegree.

Let us now turn to the second step of our program, namely showing that 0 is the only normal element in $\mathrm{Ker}(d_q)$ and $\mathrm{Ker}(\alpha)$.

The crucial point for this is contained in the following lemma.

LEMMA 2.5. *Let b be a normal term in* L_q, $q \geqslant 0$. *Let* b' *be any term in* L_q *and assume that the biggest term in* $d(b)$ *actually appears among the terms in* $d(b')$. *Then* $b \leqslant b'$.

Proof. We will treat separately the cases $q=0$ and $q \geqslant 1$.

Assume first $q=0$. Then b and b' have the form $b=y \cdot e(u)$, $b'=z \cdot e(v)$. Since $\alpha(b)=uy$, $\alpha(b')=vz$, the hypotheses amount to

$uy=vz$, and

$\max(u) \leqslant \min(y)$, by normality of b.

Thus, $u \cdot y$ is the canonical decomposition of vz, and so $u=g(vz)$. By Lemma 1.4, we conclude $u=g(vz) \leqslant g(v)=v$, and so $b \leqslant b'$.

Assume now $q \geqslant 1$. Let us write $b=y \cdot e(i_1, ..., i_q; u)$, $b'=z \cdot e(j_1, ..., j_q; v)$. By hypothesis, $c=x_{i_1} y \cdot e(i_2, ..., i_q; u)$, the biggest term in $d(b)$ according to Lemma 2.2, appears as a term in $d(b')$.

If it appears as a term of the second kind, i.e.,

$$x_{i_1} ye(i_2, ..., i_q; u)=z_r z \cdot e(j_1, ..., \hat{j}_r, ..., j_q; g(x_{j_r} v)),$$

where $z_r=x_{j_r} v/g(x_{j_r} v)$, then $u=g(x_{j_r} v)<v$ by Lemma 1.4, and so $b<b'$.

If c is equal to a term in $d(b')$ of the first kind, i.e.,

$$x_{i_1} ye(i_2, ..., i_q; u)=x_{j_r} ze(j_1, ..., \hat{j}_r, ..., j_q; v),$$

then $u=v$. We compare the sequences: If $r>1$, then we have $i_s=j_s$ for $r+1 \leqslant s \leqslant q$ and $i_r=j_{r-1}<j_r$. Hence,

$$(i_1, ..., i_q)<(j_1, ..., j_q)$$

and thus $b<b'$. If $r=1$, we have $(i_2, ..., i_q)=(j_2, ..., j_q)$ and $x_{i_1} y=x_{j_1} z$.

By normality of b, we have $i_1=\min(x_{i_1} y)$, so $i_1=\min(x_{j_1} z)$. Hence, $i_1 \leqslant j_1$. If $i_1=j_1$, this implies $b=b'$. If $i_1<j_1$, then we conclude again that $b<b'$.

It is now easy to complete the second and last step of our program.

Proposition 2.6. *Let f be a non-zero, normal element in L_q, $q \geqslant 0$. Then $d(f)$, respectively $\alpha(f)$, is non-zero.*

Proof. Let b be the biggest term in f and c the biggest term in $d(b)$, resp. $\alpha(b)$. By Lemma 2.5, c cannot cancel against any other term in $d(f)$, resp. $\alpha(f)$. Hence $d(f)$, resp. $\alpha(f)$, is non-zero.

This finishes the proof of the theorem. If k is a field $(L_*(I), d)$ is the minimal free graded resolution of I over R since $d(L_q) \subset M \cdot L_{q-1}$ for all $q \geqslant 1$.

Remark. 1. The requirement that k be a field in Theorem 2.1 is only needed to make sense of the assertion that $(L_*(I), d)$ is the minimal resolution of I over R.

For $k = \mathbb{Z}$, $(L_*(I), d)$ is a "generic" resolution in the sense of the following theorem due to I. Kaplansky and to J. Eagon and M. Hochster (see [EH, p. 70]).

THEOREM. *Let J be a monomial ideal in $A = \mathbb{Z}[X_1, ..., X_n]$, and let $L_* \to J$ be a resolution of J over A. Let R be a unitary commutative ring, let $x_1, ..., x_n \in R$, and let $\Phi: A \to R$ be the map determined by $\Phi(X_i) = x_i$, $i = 1, ..., n$. Denote by $I \subset R$ the ideal generated by $\Phi(J)$.*

If $x_1, ..., x_n$ is an R-sequence in every order, then $L_* \otimes_A R \to I$ is a resolution of I over R.

Hence, we obtain the following corollary:

Let R be a unitary commutative ring and let $x_1, ..., x_n \in R$ be an R-sequence in every order. Let $A = \mathbb{Z}[X_1, ..., X_n]$ and let $\Phi: A \to R$ be the obvious map determined by $\Phi(X_i) = x_i$ for $i = 1, ..., n$. Let $I \subset R$ be the ideal generated by the Φ-image of a stable monomial ideal $J \subset A$.

Let $L_* \to J$ be the $\mathbb{Z}[X_1, ..., X_n]$-resolution described in our Theorem 2.1 for $k = \mathbb{Z}$.

Then $L_* \otimes_{\mathbb{Z}} R \to I$ is a resolution for I over R. Its localization at any prime P of R containing I is the minimal resolution over R_P of the localized ideal I_P.

Remark 2. The reader may perhaps prefer the following alternate notation for our resolution.

Still with $R = k[x_1, ..., x_n]$, let V be the free R-module on the set $e_1, ..., e_n$.

If I is a stable ideal in R, define the complex $(C_*(I), D)$ by $C_q = \Lambda^q V \otimes_R L_0$, where $L_0 = L_0(I)$, as above, is the free R-module on the set of symbols $e(u)$ in bijection with $G(I)$, and where $\Lambda^q V$ is the qth exterior power of V over R. The boundary operator $D: C_q \to C_{q-1}$ is defined by

$$D(\xi \otimes e(u)) = \sum_{r=1}^{q} (-1)^r \xi_r \otimes \{x_{i_r} e(u) - y_r e(g(x_{i_r} u))\},$$

where $y_r = x_{i_r} u / g(x_{i_r} u)$, $\xi = e_{i_1} \wedge \cdots \wedge e_{i_q}$, and $\xi_r = e_{i_1} \wedge \cdots \wedge \hat{e}_{i_r} \wedge \cdots \wedge e_{i_q}$.

Let $N_q \subset C_q$ be the submodule generated by the elements $e_{i_1} \wedge \cdots \wedge e_{i_q} \otimes e(u)$, with $\max(u) \leqslant \max\{i_1, ..., i_q\}$. Then $N_* \subset C_*$ is a subcomplex and our resolution is isomorphic to $L_*(I) = C_*/N_*$.

As a byproduct of this description, observe that the multiplication

$$L_0 \times L_0 \to L_0$$

given by

$$e(u) \cdot e(v) = (uv/g(uv)) \cdot e(g(uv))$$

turns C_* into a differential algebra with the product

$$(\xi \otimes e(u)) \cdot (\eta \otimes e(v)) = (\xi \wedge \eta) \otimes e(u) \cdot e(v).$$

The associativity of the product is guaranteed by Lemma 1.3, clause (1), which implies that

$$g(g(uv)\, w) = g(ug(vw)).$$

Also, the formula $d(a \cdot b) = da \cdot b + (-1)^p\, a \cdot db$, for $a \in C_p$, relies again on Lemma 1.3, (1).

The subcomplex N_* is easily seen to be an ideal in C_*. (Use Lemma 1.3, clause (2).) Hence, $L_*(I)$ inherits the structure of a (graded-commutative) graded differential algebra.

We do not pursue this here.

The referee has suggested that our admissible symbols $e(i_1, ..., i_q; u)$ with $u \in G(I)$, $1 \leqslant i_1 < \cdots < i_q \leqslant n$, and $i_q < \max(u)$ could be pictured by a "hook", a notion arising in the representation theory of permutation groups.

The hook corresponding to $e(i_1, ..., i_q; u)$ with $m = \max(u)$ would be

$$\begin{array}{|c|ccccccc}
\cline{1-1}
i_1 \\
\cline{1-1}
i_2 \\
\cline{1-1}
\vdots \\
\cline{1-1}
i_q \\
\hline
m & \multicolumn{1}{|c|}{m} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{m-1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\
\hline
\end{array},$$

where $m \cdot m \cdots m-1 \cdots 1 \cdot 1$ represents the sequence determined by the monomial $u = x_1^{a_1} \cdots x_m^{a_m}$ as

$$\underbrace{m \cdot \cdots \cdot m}_{a_m\text{-times}} \cdot \underbrace{(m-1) \cdot \cdots \cdot (m-1)}_{a_{m-1}\text{-times}} \cdot \cdots \underbrace{1 \cdot \cdots \cdot 1}_{a_1\text{-times}}.$$

The above hook is a column-strict, row-non-strict decreasing standard Young tableau.

For more details on Young tableaux in this connection, the reader is referred to the thesis of Hema Srinivasan [Sr, Chap. 2].

3. Some remarks on Betti Numbers and Poincaré Series

Let I be a stable ideal in the polynomial ring $R = k[x_1, ..., x_n]$ over a field k and let $G(I)$ be its canonical generator system. The above description of the minimal resolution $(L_*(I); d)$ of I over R provides, of course, a formula for the Betti numbers

$$\beta_q(I) = \dim_k \operatorname{Tor}_q^R(I, k) = \operatorname{rank} L_q(I)$$

in terms of the elements of $G(I)$.

Namely, if $u \in G(I)$, let $m(u) = \max(u)$. Then the basis elements $e(i_1, ..., i_q; u) \in L_q(I)$ with fixed u are in bijection with the strictly increasing sequences

$$1 \leqslant i_1 < i_2 < \cdots < i_q \leqslant m(u) - 1,$$

and therefore are $\binom{m(u)-1}{q}$ in number. Thus,

$$\beta_q(I) = \sum_{u \in G(I)} \binom{m(u)-1}{q}.$$

In particular, we can read off directly the projective dimension of a stable ideal from its minimal generator system as

$$\text{proj. dim } I = \max\{m(u) - 1 \mid u \in G(I)\}.$$

We can also easily read off the Poincaré series of a stable monomial ideal from the minimal resolution.

By exactness of the resolution, we have

$$P(I, t) = \sum_{q=0}^{\infty} (-1)^q P(L_q, t),$$

where P denotes the Poincaré series.

We have

$$P(L_q, t) = \sum_{u \in G(I)} \binom{m(u)-1}{q} \frac{t^{q+\deg(u)}}{(1-t)^n},$$

since $\deg(e(i_1, ..., i_q; u)) = \deg(x_{i_1} \cdots x_{i_q} u) = q + \deg(u)$, and $\beta_q(I) = \sum_{u \in G(I)} \binom{m(u)-1}{q}$ is the rank of $L_q(I)$ as an R-module. Thus,

$$P(I, t) = \sum_{u \in G(I)} \left(\sum_{q=0}^{\infty} (-1)^q \binom{m(u)-1}{q} t^q \right) \frac{t^{\deg(u)}}{(1-t)^n},$$

and therefore

$$P(I, t) = \sum_{u \in G(I)} \frac{t^{\deg(u)}}{(1-t)^{n-m(u)+1}}$$

In [BS], it is shown (as a corollary of a more general result) that every homogeneous ideal has the same Poincaré series as some Borel fixed ideal. Since Borel fixed ideals are stable, the above formula gives the outlook of the Poincaré series of an arbitrary homogeneous ideal.

The above formula for the Betti numbers of a stable ideal yields similarly

$$\begin{aligned} P(\operatorname{Tor}_*^R(I, k), t) &= \sum_{q=0}^{\infty} \beta_q(I)\, t^q \\ &= \sum_{u \in G(I)} (1+t)^{m(u)-1}. \end{aligned}$$

There also exists a formula due to M. Hochster and R. Stanley that expresses the Betti numbers of an arbitrary monomial ideal in terms of the homology of some finite simplicial complexes associated with the ideal. (See [St, p. 49] and [H, Section 5].)

We now describe a situation in which the Betti numbers of a monomial ideal are expressible in terms of those of simpler components.

Let U, V be monomial ideals, and let $W = U \cap V$. Then W is again a monomial ideal and its canonical system of generators $G(W)$ satisfies

$$G(W) \subset lcm(G(U), G(V)) = \{lcm(u, v) \mid u \in G(U), v \in G(V)\}.$$

Of course, the inclusion can be strict, and a given w in $G(W)$ may in general be written in several ways as $w = lcm(u, v)$, with $u \in G(U)$ and $v \in G(V)$.

Now, let I be a monomial ideal, not necessarily stable.

Definition. We say that I is *splittable* if I is the sum of two non-zero monomial ideals U and V such that

(1) $G(I)$ is the disjoint union of $G(U)$ and $G(V)$,

(2) There is a *splitting function*

$$\begin{aligned} G(U \cap V) &\to G(U) \times G(V) \\ w &\to (\phi(w), \psi(w)) \end{aligned}$$

satisfying the following properties:

(S1) $w = lcm(\phi(w), \psi(w))$ for all $w \in G(W) = G(U \cap V)$,

(S2) For every subset $G' \subset G(W)$, both $lcm\, \phi(G')$ and $lcm\, \psi(G')$ strictly divide $lcm\, G'$.

A pair U, V satisfying the above condition will be called a splitting of I.

The condition (1), i.e., $G(I) \amalg G(V)$, is in fact a consequence of the axioms (S1), (S2) for the splitting function. The argument is easy and left to the reader. Alternatively, observe that the proof of Proposition 3.1 below never uses (1). Thus (1) also follows from the formula $\beta_0(I) = \beta_0(U) + \beta_0(V)$.

PROPOSITION 3.1. *Let I be a splittable monomial ideal with splitting U, V. Then, for all $q \geqslant 0$,*

$$\beta_q(I) = \beta_q(U) + \beta_q(V) + \beta_{q-1}(U \cap V),$$

where $\beta_{-1} = 0$ as usual.

Let us recall D. Taylor's resolution of monomial ideals [T], which will be used in the proof of the proposition.

Let $U \subset R = k[x_1, ..., x_n]$ be a monomial ideal with $G(U) = \{u_1, ..., u_r\}$. Let L be the free R-module of rank r with basis $e_1, ..., e_r$. Set $T_q = \Lambda^{q+1}L$, the $(q+1)$-*st* exterior power of L over R, and define $d\colon T_q \to T_{q-1}$ by the formula

$$de(i_0, ..., i_q) = \sum_{s=0}^{q} (-1)^s \frac{p(i_0, ..., i_q)}{p(i_0, ..., \hat{\imath}_s, ..., i_q)} \cdot e(i_0, ..., \hat{\imath}_s, ..., i_q)$$

where $e(i_0, ..., i_q)$ stands for $e_{i_0} \wedge e_{i_1} \wedge \cdots \wedge e_{i_q}$ and $p(i_0, ..., i_q) = lcm(u_{i_0}, u_{i_1}, ..., u_{i_q})$.

In low degree, $T_0 = L$ and let $\alpha\colon T_0 \to U$ be given by $\alpha(e_i) = u_i$.

A simple calculation shows that

$$0 \to T_{r-1} \to T_{r-2} \to \cdots \to T_1 \to T_0 \to U \to 0$$

is a complex, and it is easy to produce a k-linear contracting homotopy $h\colon T_q \to T_{q+1}$ (i.e., $dh + hd = 1$), showing that (T_*, d) is in fact a resolution of U as an R-module.

One possible choice for the (k-linear) map h is provided by the formula

$$h\{u \cdot e(i_1, ..., i_q)\} = \frac{u \cdot p(i_1, ..., i_q)}{p(i_0, ..., i_q)} \cdot e(i_0, i_1, ..., i_q),$$

where u is a monomial in R, and i_0 is the smallest index $i \in \{1, ..., r\}$ such that u_i divides $u \cdot p(i_1, ..., i_q)$.

Of course $i_0 \leqslant i_1$, and if $i_0 = i_1$, then simply $e(i_0, i_1, ..., i_q) = 0$.

Digression. Recently G. Lyubeznik [L] has exhibited a subcomplex $L_* \subset T_*$ which provides a "smaller" resolution of U (if still not minimal, in general). Lyubeznik's L_q is the R-subcomplex of T_q generated by the vectors $e(i_0, i_1, ..., i_q)$ with $i_0 < i_1 < \cdots < i_q$ such that for all j $(0 \leqslant j \leqslant q)$ and all $i < i_j$, the monomial u_i does not divide $lcm(u_{i_j}, u_{i_{j+1}}, ..., u_{i_q}) =$

$p(i_j, i_{j+1}, ..., i_q)$. Now, if $e(i_1, ..., i_q)$ satisfies Lyubeznik's condition and u is some monomial, then $e(i_0, i_1, ..., i_q)$, where $i_0 = \min\{i \mid u_i$ divides $u \cdot p(i_1, ..., i_q)\}$, also satisfies Lyubeznik's condition. Indeed, if u_i divides $p(i_0, i_1, ..., i_q)$, then it divides $u \cdot p(i_0, ..., i_q)$ and thus $i \geqslant i_0$. It follows that the above homotopy contraction $h: T_q \to T_{q+1}$ maps L_q into L_{q+1} and hence is a contracting homotopy for Lyubeznik's resolution as well.

Proof of the Proposition. Consider the exact sequence

$$O \longrightarrow W \xrightarrow{\Phi} U \oplus V \xrightarrow{\pi} I \longrightarrow O,$$

where $\pi(u, v) = u - v$ and $\Phi(w) = (w, w)$.

We shall prove that the map

$$\Phi_* : \operatorname{Tor}_q^R(W, F) \to \operatorname{Tor}_q^R(U, F) \oplus \operatorname{Tor}_q^R(V, F)$$

induced by Φ is O for all $q \geqslant 0$. The proposition then follows from the long homology sequence

$$\cdots \xrightarrow{\Phi_*} \operatorname{Tor}_q^R(U, k) \oplus \operatorname{Tor}_q^R(V, k) \to \operatorname{Tor}_q^R(I, k) \to \operatorname{Tor}_{q-1}^R(W, k) \xrightarrow{\Phi_*} \cdots.$$

Let A_*, B_*, C_* be the Taylor resolutions for U, V, W respectively. Denote by $G(A_0)$, similarly $G(B_0)$, $G(C_0)$ mutatis mutandis, the R-basis of A_0 mapping bijectively to $G(U)$ under $\alpha: A_0 \to U$.

Our splitting function $G(W) \to G(U) \times G(V)$ can be lifted in an obvious way to a function

$$G(C_0) \to G(A_0) \times G(B_0)$$

$$c \to (\phi(c), \psi(c))$$

A lifting of Φ: $W \to U \oplus V$ to a map of resolutions

$$\Phi: C_* \to A_* \oplus B_*$$

is then determined by the formula

$$\Phi(c_1 \wedge \cdots \wedge c_q) = \left(\frac{lcm\ \alpha c}{lcm\ \alpha\phi(c)} \cdot \phi(c), \frac{lcm\ \alpha c}{lcm\ \alpha\psi(c)} \cdot \psi(c)\right),$$

where $c_1, ..., c_q \in G(C_0)$, $c = c_1 \wedge \cdots \wedge c_q$, and

$$\phi(c) = \phi(c_1) \wedge \cdots \wedge \phi(c_q),\ lcm\ \alpha c = lcm(\alpha c_1, ..., \alpha c_q), \text{ etc.}$$

An easy calculation shows that Φ commutes with Taylor's boundary operators. The fact that Φ induces O in homology follows from axiom (S2) of splitting functions, implying here that $(lcm\ \alpha c)/(lcm\ \alpha\phi(c))$ and $(lcm\ \alpha c)/(lcm\ \alpha\psi(c))$ both belong to the augmentation ideal M.

Remark 1. In the case where I is stable, there is a natural splitting U, V of I, where U is generated by the subset $G(U) \subset G(I)$ comprising the generators divisible by x_1, and V is the ideal generated by $G(V) = G(I) - G(U)$.

Obviously, U is stable. If $R' = k[x_2, ..., x_n]$ and V' is the R'-ideal generated by $G(V)$, then V' is stable as an R'-ideal. (Equivalently, one could shift $G(V)$ to $\tau G(V) \subset k[x_1, ..., x_{n-1}]$ by $\tau(x_i) = x_{i-1}$ for $i = 2, ..., n$ and take the ideal V_1 in $k[x_1, ..., x_{n-1}]$ generated by $G(V_1) = \tau G(V)$. Then V_1 is stable.)

It is easy to see that $W = U \cap V$ is equal to $x_1 V$.

Clearly, $W \subset x_1 V$. Conversely, if $v \in G(V)$, let $x_1 v = uy$ be the canonical decomposition of $x_1 v$ relative to I. Since $\max(u) \leqslant \min(y)$, x_1 must divide u, i.e., $u \in G(U)$. Therefore $x_1 v \in W$. Note that this argument actually shows that $G(W) = x_1 G(V)$.

One can then define

$$G(U \cap V) \to G(U) \times G(V),$$
$$w \to (u, w/x_1),$$

where u is determined by the canonical I-decomposition $w = u \cdot y$, $u \in G(I)$, $\max(u) \leqslant \min(y)$. (Again, $u \in G(U)$, since w is divisible by x_1.)

An easy argument shows that this map is a splitting function for I.

Now, note that the ideal V in $k[x_1, ..., x_n]$ has the same Betti numbers as the ideal V' in $R' = k[x_2, ..., x_n]$. In fact, if L'_* is a minimal resolution of V' over R', then $R \otimes_{R'} L'_*$ is a minimal resolution of $V = R \otimes_{R'} V'$ over R.

It follows that $\beta_q(V) = \beta_q(V')$ for all $q \geqslant 0$. Also, $\beta_{q-1}(U \cap V) = \beta_{q-1}(V)$. Hence,

$$\beta_q(I) = \beta_q(U) + \beta_q(V) + \beta_{q-1}(V).$$

Of course, this formula could be obtained, more simply actually, from the explicit formula $\beta_q(I) = \sum_{u \in G(I)} \binom{m(u)-1}{q}$.

Remark 2. Not every (non-principal) monomial ideal is splittable. For instance, in $k[x_1, ..., x_5]$ there is no splitting of the ideal

$$I = (x_1 x_2 x_3, x_1 x_3 x_5, x_1 x_4 x_5, x_2 x_3 x_4, x_2 x_4 x_5),$$

as can easily be verified by inspection.

This ideal is associated with the triangulation of the Möbius strip shown in the Fig. 1. (See [H] and [R].)

Note also that the stable monomial ideals have a minimal resolution which is completely determined by the divisibility properties of the

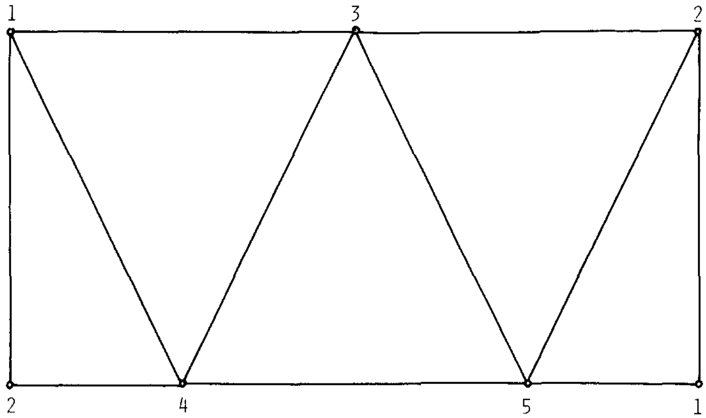

FIGURE 1

monomials in I and $G(I)$, and in particular is independent of the characteristic of the ground field k.

This is not the case in general, as shown by the example

$$I=(x_1x_2x_3, x_1x_3x_5, x_1x_4x_5, x_2x_3x_4, x_2x_4x_5, x_1x_2x_6,$$
$$x_1x_4x_6, x_2x_5x_6, x_3x_4x_6, x_3x_5x_6)$$

in $k[x_1, ..., x_6]$, associated with the triangulation of the projective plane shown in Fig. 2.

The Betti numbers of this ideal are

$$\beta_0=10, \beta_1=15, \beta_2=6, \beta_q=0 \quad \text{for} \quad q \geqslant 3, \quad \text{if} \quad \text{char}(k) \neq 2$$

and

$$\beta_0=10, \beta_1=15, \beta_2=7, \beta_3=1, \beta_q=0 \quad \text{for} \quad q \geqslant 4, \quad \text{if} \quad \text{char}(k)=2.$$

We conclude with some examples.

EXAMPLE 1. Let M_n^d be the dth power of the augmentation ideal

$$M_n=(x_1, ..., x_n) \text{ in } k[x_1, ..., x_n].$$

Clearly, this ideal is stable and thus our Theorem 2.1 provides the minimal resolution for M_n^d.

We can use Proposition 3.1 to calculate the Betti numbers $\beta_q(M_n^d)$. The result is

$$\beta_q(M_n^d)=\binom{d+n-1}{d+q}\binom{d+q-1}{q}.$$

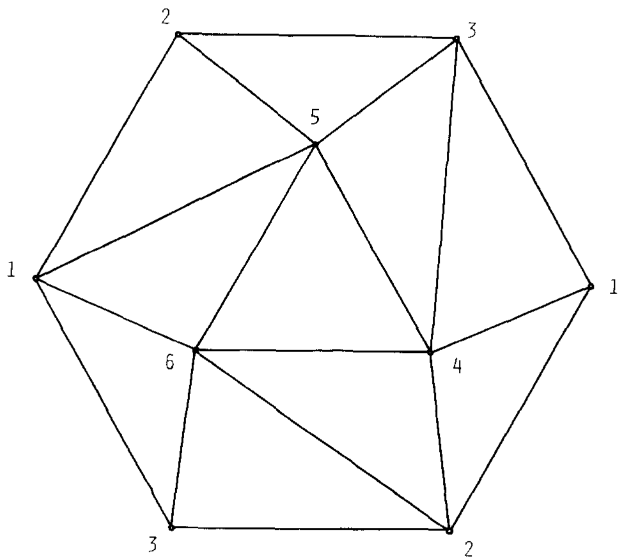

FIGURE 2

Since M_n^d is stable, the splitting described in Remark 1 above applies. Here, $U = x_1 M_n^{d-1} \cong M_n^{d-1}$, and V has the same Betti numbers as M_{n-1}^d (by shifting). Hence

$$\beta_q(M_n^d) = \beta_q(M_n^{d-1}) + \beta_q(M_{n-1}^d) + \beta_{q-1}(M_{n-1}^d).$$

An easy calculation then shows that the desired expression yields the correct values of $\beta_q(M_n^d)$ for small d or n and satisfies the induction formula.

Of course, $\beta_q(M_n^d) = \binom{d+n-1}{d+q}\binom{d+q-1}{q}$ can also be proved by counting the number of monomials u in $G(M_n^d)$ with a given $m = \max(u)$. This number is $\binom{d+m-2}{m-1}$, because division by x_m provides a bijection between this set and the set of monomials in $x_1, ..., x_m$ of degree $d-1$.

The result follows then from the easily verified identity

$$\sum_{m=1}^{n} \binom{d+m-2}{m-1}\binom{m-1}{q} = \binom{d+n-1}{d+q}\binom{d+q-1}{q}.$$

We are grateful to M. Hochster for pointing out to us that a minimal resolution for the powers M_n^d of the augmentation ideal has been known for some time: It can be extracted from the Eagon–Northcott complex [EN, p. 201], as noted by D. Buchsbaum and D. S. Rim in [BR, Lemma 3.9, p. 215].

EXAMPLE 2. Let $S_n \subset k[x_1, ..., x_n]$ be the ideal generated by the set of monomials w with $\deg(w) = \max(w)$, i.e.,

$$\{x_1^{a_1} \cdots x_m^{a_m} \mid a_1 + \cdots + a_m = m \text{ for } m = 1, ..., n\}.$$

With a mild abuse of notation,

$$S_n = M_1 + M_2^2 + \cdots + M_n^n.$$

This ideal is Borel fixed, hence stable.

It is not difficult to count the number g_m of generators $u \in G(S_n)$ with prescribed $m = \max(u)$.

Note that g_m is the cardinality of the set of monomials $x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}$ such that $a_1 + a_2 + \cdots + a_m = m$, and such that moreover, for all $j = 1, ..., m-1$, the exponents satisfy the inequality

$$\sum_{i=1}^{j} a_i \leqslant j - 1.$$

(For instance, this implies $a_1 = 0$, a_2 is at most 1, $a_2 + a_3$ is at most 2, etc.. Also, $a_m \geqslant 2$.)

This number is equal to the number of weakly increasing paths joining $(1, 0)$ to $(m, m-1)$ in the (t, s) plane, not running above the diagonal $\Delta = \{s = t - 1\}$, and progressing along the sides of unit squares with integral coordinates. (See Fig. 3.)

Such a path can perhaps best be described by a monotone step function

$$f\colon [1, m] \to \mathbb{Z} \subset R$$

with jumps at integral values of $t \in [1, m]$, such that $f(1) = 0$, $f(m) = m - 1$, and $f(t) \leqslant t - 1$ for all t.

The step function associated with a monomial $x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}$ in $G(S_n)$ is $f(t) = \sum_{i=1}^{[t]} a_i$, where $[t]$ denotes the integral part of t.

Thus g_m is the well-known Catalan number

$$g_m = \frac{1}{m} \binom{2m-2}{m-1}$$

and we have

$$\beta_q(S_n) = \sum_{m=1}^{n} \frac{1}{m} \binom{2m-2}{m-1} \binom{m-1}{q}.$$

EXAMPLE 3. If $w = x_1^{a_1} \cdots x_n^{a_n}$ is a monomial, there is a well defined

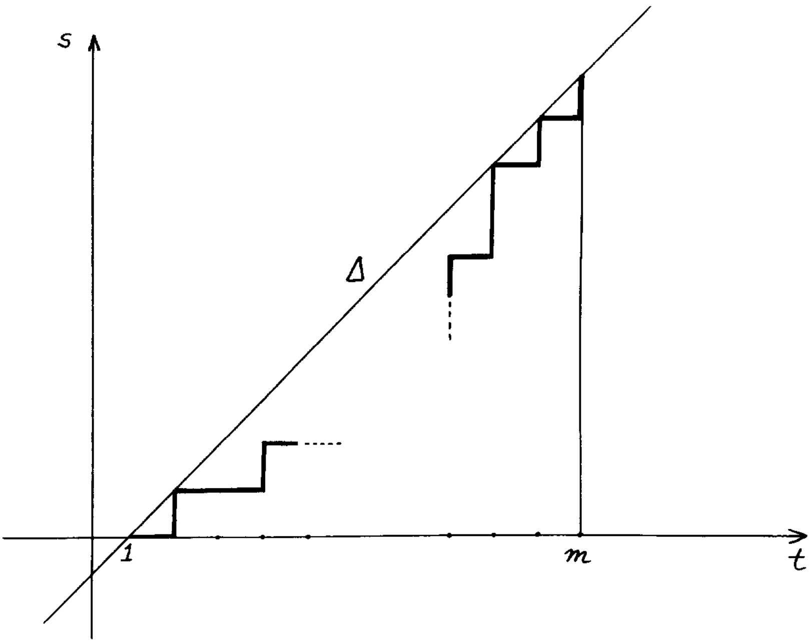

FIGURE 3

stable ideal $\langle w \rangle$ whose canonical generator system consists of the monomials u obtained from w by a finite sequence

$$w = w_0, w_1, \ldots, w_t = u$$

of transformations of the kind

$$w_{i+1} = x_{k_i} w_i / x_{m_i},$$

where $m_i = \max(w_i)$ and $k_i < m_i$.

If $T_m(a_1, \ldots, a_n)$ is the number of such transforms u of the given monomial w, with $\max(u) = m$, then

$$T_m(a_1, \ldots, a_n) = \binom{a_n + \cdots + a_m + m - 2}{m - 1}.$$

To prove this, observe that each transform u of w with $\max(u) = m$ is in fact a transform of $w_m = x_1^{a_1} \cdots x_{m-1}^{a_{m-1}} x_m^{s_m}$, where $s_m = a_m + \cdots + a_n$. The number of transforms $x_1^{b_1} \cdots x_m^{b_m}$ of w_m with $b_m = s_m - i$ is equal to the number $\binom{m+i-2}{i}$ of monomials $x_1^{i_1} \cdots x_{m-1}^{i_{m-1}}$ of degree $i_1 + \cdots + i_{m-1} = i$.

Therefore, if $w = x_1^{a_1} \cdots x_n^{a_n}$,

$$\beta_q(\langle w \rangle) = \sum_{m=1}^{n} \binom{a_n + \cdots + a_m + m - 2}{m - 1} \binom{m - 1}{q}.$$

We do not know whether this can in general be written without summation sign.

If $a_1 = \cdots = a_n = 1$, we get

$$\beta_q(\langle x_1 \cdots x_n \rangle) = \binom{n-1}{q} \cdot 2^{n-q-1},$$

as follows from the calculation

$$\sum_{q=0}^{\infty} \beta_q(\langle x_1 \cdots x_n \rangle)\, t^q = \sum_{m=1}^{n} \binom{n-1}{m-1} (1+t)^{m-1} = (2+t)^{n-1}.$$

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