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Interleaved adjoints of directed graphs

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ABSTRACT

For an integer $k \geq 1$, the *kth interleaved adjoint* of a digraph G is the digraph $\iota_k(G)$ with vertex-set $V(G)^k$, and arcs $((u_1, \dots, u_k), (v_1, \dots, v_k))$ such that $(u_i, v_i) \in A(G)$ for $i = 1, \dots, k$ and $(v_i, u_{i+1}) \in A(G)$ for $i = 1, \dots, k-1$. For every k, we derive upper and lower bounds for the chromatic number of $\iota_k(G)$ in terms of that of G. In the case where G is a transitive tournament, the exact value of the chromatic number of $\iota_k(G)$ has been determined by [H.G. Yeh, X. Zhu, Resource-sharing system scheduling and circular chromatic number, Theoret. Comput. Sci. 332 (2005) 447–460]. We use the latter result in conjunction with categorical properties of adjoint functors to derive the following consequence. For every integer ℓ, there exists a directed path Q_ℓ of algebraic length ℓ which admits homomorphisms into every directed graph of chromatic number at least 4. We discuss a possible impact of this approach on the multifactor version of the weak Hedetniemi conjecture.

1. Introduction

For an integer $k \geq 1$, the *kth interleaved adjoint* of a digraph G is the digraph $\iota_k(G)$ whose vertex-set is $V(\iota_k(G)) = V(G)^k$, the set of all k-sequences of vertices of G, and whose arc-set $A(\iota_k(G))$ is the set of couples (u, v) of sequences which "interleave" in the sense that for $u = (u_1, \dots, u_k)$, $v = (v_1, \dots, v_k)$, we have $(u_i, v_i) \in A(G)$ for $i = 1, \dots, k$ and $(v_i, u_{i+1}) \in A(G)$ for $i = 1, \dots, k-1$. Thus for every k, ι_k is a functor which makes a new digraph out of a given digraph G. We will show

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that interleaved adjoints share interesting properties with iterated "arc-digraph" constructions δ^k (see Section 2). Both classes of functors are categorical right adjoints; also from the graph-theoretic viewpoint, there are bounds for the chromatic number of $\delta^k(G)$ and $\iota_k(G)$ in terms of the chromatic number of G.

Here, the chromatic number $\chi(G)$ of a digraph G is the minimum number of colours needed to colour its vertices so that pairs of vertices joined by an arc have different colours. Alternatively, it is the usual chromatic number of its symmetrisation. We view undirected graphs as symmetric digraphs. Hence the chromatic number of a digraph G is the smallest integer n such that G admits a homomorphism (that is, an arc-preserving map) into K_n, the complete symmetric digraph on n vertices. Extending standard graph-theoretic concepts to the category of directed graphs sometimes allow us to use more elaborate categorical tools.

In Section 2, we find bounds for chromatic numbers of interleaved adjoints of general digraphs. In particular, the case of transitive tournaments has been dealt with in [21]. In Sections 3 and 4, we interpret the latter result in terms of category theory and finite duality. Our main result, Theorem 13, proves the existence of a rich class of paths admitting homomorphisms to all orientations of graphs with chromatic number at least 4.

In Section 5, we discuss a possible connection between interleaved adjoints and the conjecture of Hedetniemi on the chromatic number of a categorical product of graphs: either Theorem 13 can be refined to the existence of families of "steep" paths with similar properties, or the interleaved adjoints of transitive tournaments witness the boundedness of a multifactor version of the (directed) "Poljak–Rödl" function from [16,19].

2. Chromatic numbers of interleaved adjoints of digraphs

The *arc graph* $\delta(G)$ of a digraph G is defined by

$$V(\delta(G)) = A(G)$$
$$A(\delta(G)) = \{((u, v), (v, w)) : (u, v), (v, w) \in A(G)\}.$$

It is well known (see [7,16]) that $\log_2(\chi(G)) \le \chi(\delta(G)) \le 2\log_2(\chi(G))$. Therefore for $k \ge 1$, the iterated arc graph $\delta^k(G)$ has chromatic number in the order of $\log_2^{(k)}(\chi(G))$, where the exponential notation represents a composition. Since $\delta^k(G)$ has no (orientations of) odd cycles of length at most $2k+1$, the iterated arc graph construction provides a simple constructive proof of the existence of graphs with large odd girth and large chromatic number. The vertices of $\delta^k(G)$ correspond to chains $(u_0, u_1, \ldots, u_k)$ of vertices of G (with $(u_{i-1}, u_i) \in A(G)$, $1 \le i \le k$), and its arcs join consecutive chains $(u_0, u_1, \ldots, u_k)$, $(u_1, \ldots, u_k, u_{k+1})$.

We use the iterated arc graph construction to find a lower bound for the chromatic number of interleaved adjoints of digraphs.

Theorem 1. *For an integer k and a digraph G, we have*

$$\chi(\delta^{2k-2}(G)) \le \chi(\iota_k(G)) \le \chi(G).$$

Proof. There exists a homomorphism ϕ of $\delta^{2k-2}(G)$ to $\iota_k(G)$ defined by

$$\phi(u_0, u_1, u_2, \ldots, u_{2k-2}) = (u_0, u_2, \ldots, u_{2k-2}).$$

Indeed for every arc $((u_0, \ldots, u_{2k-2}), (u_1, \ldots, u_{2k-1}))$ of $\delta^{2k-2}(G)$, $\phi(u_0, u_1, \ldots, u_{2k-2}) = (u_0, u_2, \ldots, u_{2k-2})$ interleaves $(u_1, u_3, \ldots, u_{2k-1}) = \phi(u_1, u_2, \ldots, u_{2k-1})$, whence ϕ preserves arcs. In particular, $\chi(\delta^{2(k-1)}(G)) \le \chi(\iota_k(G))$.

Also, there is an obvious homomorphism ψ of $\iota_k(G)$ to G defined by $\psi(u_1, \ldots, u_k) = u_1$. Therefore, $\chi(\iota_k(G)) \le \chi(G)$. □

Even though there is a large gap between the lower and upper bounds in Theorem 1, both bounds can be tight. The lower bound $\chi(\delta^{2k-2}(G))$ is tight when G is itself of the form $\delta(H)$. The non-isolated vertices $((u_0, u_1), (u_2, u_3), \ldots, (u_{2k-2}, u_{2k-1}))$ in $V(\iota_k(G))$ then correspond to chains

$(u_0, u_1, u_2, \ldots, u_{2k-2}, u_{2k-1})$ in $V(\delta^{2k-1}(H))$, and the arcs of $\iota_k(G)$ are interleaved sequences of the form

$$(((u_0, u_1), (u_2, u_3), \ldots, (u_{2k-2}, u_{2k-1})), ((u_1, u_2), (u_3, u_4), \ldots, (u_{2k-1}, u_{2k})))$$

which correspond to consecutive chains in H, that is, arcs of $\delta^{2k-1}(H)$. Hence the arcs of $\iota_k(G)$ span a copy of $\delta^{2k-1}(H) = \delta^{2k-2}(\delta(H)) = \delta^{2(k-1)}(G)$. The upper bound in $\chi(G)$ is tight in particular when G is undirected. Indeed the map $\psi : G \longrightarrow \iota_k(G)$ where $\psi(u) = (u, u, \ldots, u)$ is then a homomorphism, whence the inequality $\chi(G) \leq \chi(\iota_k(G))$ holds. The interested reader may find antisymmetric examples as well.

We now turn to the case of transitive tournaments, an important class of graphs where $\chi(\iota_k(G))$ is linear in $\chi(G)$. Let T_n denote the transitive n-tournament, that is, the vertex-set of T_n is $\{1, \ldots, n\}$ and its arc-set is $\{(i, j) : 1 \leq i < j \leq n\}$.

Remark 2. For every integer k,

(i) $\iota_k(T_{3k})$ contains a copy of T_3 induced by

$$\{(i, i + 3, \ldots, i + 3(k - 1)) : i \in \{1, 2, 3\}\},$$

(ii) $\iota_k(T_{3k})$ admits a proper 3-colouring $c : V(\iota_k(T_{3k})) \longrightarrow \{0, 1, 2\}$ defined by

$$f(u_1, \ldots, u_k) = \left\lfloor \sum_{i=1}^{k} u_i/k \right\rfloor \pmod 3.$$

Therefore $\chi(\iota_k(T_{3k})) = 3$.

In [21], Yeh and Zhu give a general version of this result. Let $B(n, k)$ be the symmetrisation of $\iota_k(T_n)$, and $K_{n/k}$ be the "n/k circular complete graph" in the sense of [23].

Theorem 3 ([21, Lemma 9]). *For integers $n \geq 2k$, there exist homomorphisms both ways between $B(n, k)$ and $K_{n/k}$.*

Corollary 4. *For integers $n \geq 2k$, $\chi(\iota_k(T_n)) = \lceil n/k \rceil$.*

In fact, Yeh and Zhu proved that the circular chromatic number of $B(n, k)$ is n/k, linking the "interleaved multicolourings of resource-sharing systems" devised by Barbosa and Gafni [1] to the circular chromatic number. In the next section, we examine the interleaved adjoints of transitive tournaments from the point of view of category theory and finite duality.

3. Right and left adjoints

The interleaved adjoints share with the arc graph construction the property of being categorical right adjoints. Following [17,3], they each have a corresponding left adjoint which acts as a kind of inverse in the sense detailed in Theorem 5. For an integer $k \geq 1$, we define the *kth inverse interleaved adjoint* of a digraph G as the digraph $\iota_k^*(G)$ constructed as follows. For every vertex u of G, we put k vertices $u_1, u_2, \ldots, u_k$ in $\iota_k^*(G)$, and for every arc (u, v) of G, we put the arcs (u_i, v_i), $i = 1, \ldots, k$ and (v_i, u_{i+1}), $i = 1, \ldots, k - 1$ in $\iota_k^*(G)$.

Theorem 5 ([17,3]). *For two digraphs G and H, there exists a homomorphism of G to $\iota_k(H)$ if and only if there exists a homomorphism of $\iota_k^*(G)$ to H.*

Proof. If $\phi : G \longrightarrow \iota_k(H)$ is a homomorphism, we can define a homomorphism $\psi : \iota_k^*(G) \longrightarrow H$ by $\psi(u_i) = x_i$, where $\phi(u) = (x_1, \ldots, x_k)$. Conversely, if $\psi : \iota_k^*(G) \longrightarrow H$ is a homomorphism, we can define a homomorphism $\phi : G \longrightarrow \iota_k(H)$ by $\phi(u) = (\psi(u_1), \ldots, \psi(u_k))$. □

In light of this property, we interpret the interleaved adjoints of transitive tournaments in terms of finite path duality. For an integer $n \geq 1$, let P_n be the path with vertex-set $\{0, 1, \ldots, n\}$ and arc-set $A(P_n) = \{(0, 1), (1, 2), \ldots, (n - 1, n)\}$. We use the following classic result.

Theorem 6 ([4,5,18,20]). *A digraph G admits a homomorphism to T_n if and only if there is no homomorphism of P_n to G.*

For an integer k, let $\mathcal{P}_{n,k}$ denote the family of paths obtained from P_n by reversing at most k arcs.

Theorem 7. *Let G be a digraph. Then the following are equivalent.*

(i) *There is no homomorphism of G to $\iota_k(T_n)$.*
(ii) *Some path in $\mathcal{P}_{n,k-1}$ admits a homomorphism to G.*

Proof. If there is no homomorphism of G to $\iota_k(T_n)$, then there is no homomorphism of $\iota_k^*(G)$ to T_n, whence there exists a homomorphism $\phi : P_n \longrightarrow \iota_k^*(G)$. Let P_ϕ be the path obtained from P_n by reversing the arc $(i, i+1)$ if $\phi(i) = v_j$ and $\phi(i+1) = u_{j+1}$ for some $(u, v) \in A(G)$, and leaving it as is otherwise (that is, if $\phi(i) = u_j$ and $\phi(i+1) = v_j$ for some $(u, v) \in A(G)$). Then $P_\phi \in \mathcal{P}_{n,k-1}$, and ϕ composed with the natural projection of $\iota_k^*(G)$ to G is a homomorphism of P_ϕ to G.

Conversely, suppose that some path P in $\mathcal{P}_{n,k-1}$ admits a homomorphism $\phi : P \longrightarrow G$. Let $f : V(P) \longrightarrow \mathbb{Z}$ be the function defined recursively by $f(0) = 1$ and

$$f(i+1) = \begin{cases} f(i) & \text{if } (i, i+1) \in A(P), \\ f(i) + 1 & \text{if } (i+1, i) \in A(P). \end{cases}$$

Since P is in $\mathcal{P}_{n,k-1}$, we have $1 \le f(i) \le k$ for all i. If $(i, i+1) \in A(P)$, then $(\phi(i), \phi(i+1)) \in A(G)$ and $f(i+1) = f(i)$, hence $(\phi(i)_{f(i)}, \phi(i+1)_{f(i+1)}) \in A(\iota_k^*(G))$. If $(i+1, i) \in A(P)$, then $(\phi(i+1), \phi(i)) \in A(G)$ and $f(i+1) = f(i)+1$, hence $(\phi(i)_{f(i)}, \phi(i+1)_{f(i+1)}) \in A(\iota_k^*(G))$. Therefore, the map $\psi : P_n \longrightarrow \iota_k^*(G)$ defined by $\psi(i) = \phi(i)_{f(i)}$ is a homomorphism. This implies that there is no homomorphism of $\iota_k^*(G)$ to T_n, and no homomorphism of G to $\iota_k(T_n)$. □

Both [1] and [21] provide polynomial algorithms to decide whether an input digraph G admits a homomorphism to $\iota_k(T_n)$. From the point of view of descriptive complexity, Theorem 7 shows that the problem has "width 1" in the sense of [2], and is solvable polynomially through the "arc consistency check" algorithm. More precisely, $\iota_k(T_n)$ has "finite duality". We will discuss the structural consequences of this fact in the next section. We end this section by noting the following consequence of Theorem 7 and Corollary 4.

Corollary 8. *For every integers c, k and for every digraph G such that $\chi(G) > c$, there exists a path $P \in \mathcal{P}_{ck,k-1}$ which admits a homomorphism to G.* □

Actually, this is also an easy consequence of "Minty's painting lemma" [10]. We include it here for reference; we will discuss hypothetical strengthening of it in Section 5.

4. Finite duality

A digraph H has *finite duality* if there exists a finite family $\mathcal{F}$ of digraphs such that a graph G admits a homomorphism to H if and only if there is no homomorphism of a member of $\mathcal{F}$ to G. The family $\mathcal{F}$ is then called a *complete set of obstructions* for H. For instance, by Theorem 6, T_n has finite duality and admits $\{P_n\}$ as a complete set of obstructions. By Theorem 7, $\iota_k(T_n)$ also has finite duality and admits $\mathcal{P}_{n,k-1}$ as a complete set of obstructions.

The finite dualities were characterised in [12], in terms of homomorphic equivalence with categorical products of structures with singleton duality. Two digraphs G, H are called *homomorphically equivalent* if there exist homomorphisms of G to H and of H to G. The *categorical product* of a family $\{G_1, \dots, G_n\}$ of digraph is the digraph $\prod_{i=1}^n G_i$ defined by

$$V\left(\prod_{i=1}^n G_i\right) = \prod_{i=1}^n V(G_i),$$

$$A\left(\prod_{i=1}^n G_i\right) = \{((u_1, \dots, u_n), (v_1, \dots, v_n)) : (u_i, v_i) \in A(G_i) \text{ for } 1 \le i \le n\}.$$

We use mostly categorical products of sets of digraphs. For $\mathcal{F} = \{G_1, \ldots, G_n\}$, we write $\prod \mathcal{F}$ for $\prod_{i=1}^{n} G_i$. This allows one to simplify the notation without loss of generality, since the categorical product is commutative and associative (up to isomorphism).

Theorem 9 *([12]). For every directed tree T, there exists a directed graph $D(T)$ (called the dual of T) which admits $\{T\}$ as a complete set of obstructions. A digraph H has finite duality if and only if there exists a finite family $\mathcal{F}$ of trees such that H is homomorphically equivalent to $\prod\{D(T) : T \in \mathcal{F}\}$. $\mathcal{F}$ is then a complete set of obstructions for H.*

Corollary 10. *$\iota_k(T_n)$ is homomorphically equivalent to $\prod\{D(P) : P \in \mathcal{P}_{n,k-1}\}$.*

The dual $D(T)$ of a tree T is not unique, but all duals of a given tree are homomorphically equivalent. According to [13], we get a possible construction for $D(T)$ by taking for $V(D(T))$ the set of all functions $f : V(T) \longrightarrow A(T)$ such that $f(u)$ is incident to u for every $u \in V(T)$, and putting an arc from f to g if for every $(u, v) \in A(T), f(u) \neq g(v)$. This is the simplest general construction known, and yet is far from transparent. This combined with the fact that the family $\mathcal{P}_{n,k-1}$ is large makes it difficult to describe explicit homomorphisms between $\iota_k(T_n)$ and $\prod\{D(P) : P \in \mathcal{P}_{n,k-1}\}$. However, combined with the concept of algebraic length, we use this structural insight to exhibit some common features of digraphs with a large chromatic number.

The *algebraic length* of a path P is the value

$$\mathrm{al}(P) = \min\{n : \text{there exists a homomorphism of } P \text{ to } P_n\}.$$

If we picture a path drawn from left to right, then its algebraic length is the (absolute value of) the difference between the number of its forward arcs and the number of its backward arcs. In particular, for $n \geq k \geq 1$, the paths in $\mathcal{P}_{n,k-1}$ all have algebraic length at least $n - 2k + 2$.

We use the following results.

Theorem 11 *([8]). A digraph G admits a homomorphism to P_n if and only if no path of algebraic length $n + 1$ admits a homomorphism to G.*

Theorem 12 *([11]). A categorical product $\prod_{i=1}^{n} G_i$ of digraphs admits a homomorphism to P_n if and only if at least one of the factors does.*

Our main result is the following.

Theorem 13. *For every length ℓ, there exists a path Q_ℓ such that $al(Q_\ell) = \ell$ and for every digraph G with chromatic number at least 4, there exists a homomorphism of Q_ℓ to G.*

Proof. For $\ell \leq 3$, we can take $Q_\ell = P_\ell$ by Theorem 6. For $\ell \geq 4$, we put $k = \ell - 2$. The paths in $\mathcal{P}_{3k,k-1}$ all have algebraic length at least ℓ, hence none of them admits a homomorphism to $P_{\ell-1}$. Thus by Theorem 12, their categorical product $\prod \mathcal{P}_{3k,k-1}$ does not admit a homomorphism to $P_{\ell-1}$. Therefore by Theorem 11, there exists a path Q_ℓ of algebraic length ℓ which admits a homomorphism to $\prod \mathcal{P}_{3k,k-1}$. We will show that Q_ℓ has the required property.

Since Q_ℓ admits a homomorphism to $\prod \mathcal{P}_{3k,k-1}$, it admits a homomorphism to every path P in $\mathcal{P}_{3k,k-1}$. By Theorem 9, this implies that there is no homomorphism of P to $D(Q_\ell)$. Since this holds for every P in $\mathcal{P}_{3k,k-1}$, by Theorem 7 there exists a homomorphism of $D(Q_\ell)$ to $\iota_k(T_{3k})$. Therefore by Corollary 4, we have $\chi(D(Q_\ell)) \leq \chi(\iota_k(T_{3k})) = 3$.

Now let G be a digraph such that there is no homomorphism of Q_ℓ to G. Then by Theorem 9 there exists a homomorphism of G to $D(Q_\ell)$ whence $\chi(G) \leq \chi(D(Q_\ell)) \leq 3$. Therefore, Q_ℓ admits homomorphisms to all digraphs with chromatic number at least 4. □

The study of the relation between the algebraic length of paths and the chromatic number of their duals was initiated in [13], where it was shown that the bound $\chi(D(P)) < \mathrm{al}(P)$ can hold for paths with arbitrarily large algebraic lengths. Paper [14] provides examples of paths P such that $\chi(D(P)) \in O(\log(\mathrm{al}(P)))$, and raises the question of the existence of paths with arbitrarily large algebraic lengths whose duals have bounded chromatic number. Theorem 13 settles this question in the affirmative. The focus now shifts to how steep can such paths be, that is, how many arcs must there be in a path with large algebraic length whose dual has a small chromatic number. As we shall see, this question is connected to a long-standing conjecture in graph theory.

5. The multifactor Poljak–Rödl function

The Poljak–Rödl function $f : \mathbb{N} \longrightarrow \mathbb{N}$ for directed graphs is defined by

$$f(c) = \min\{\chi(G_1 \times G_2) : G_1 \text{ and } G_2 \text{ are } c\text{-chromatic digraphs}\}.$$

Its undirected version $g : \mathbb{N} \longrightarrow \mathbb{N}$ is defined by

$$g(c) = \min\{\chi(G_1 \times G_2) : G_1 \text{ and } G_2 \text{ are } c\text{-chromatic undirected graphs}\}.$$

These functions are related to the long-standing conjecture of Hedetniemi, first formulated in 1966.

Conjecture 14 (*[6]*). *If G and H are undirected graphs, then*

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

The Hedetniemi conjecture states that $g(c) = c$ for all c, but for the moment it is not even known whether g is bounded or unbounded. In [19], it is shown that f is unbounded if and only if g is unbounded, and in [15,22], it is shown that f is bounded if and only if $f(c) \leq 3$ for all c.

We will now consider an application of this topic to a variation of Corollary 8.

Conjecture 15. *There exists a number C such that for every k and every partition of $\mathcal{P}_{3k,k-1}$ in two sets $\mathcal{Q}_1, \mathcal{Q}_2$, we can select a set $\mathcal{Q}_i$ with the following property.*

For every digraph G such that $\chi(G) \geq C$, there exists a path $P \in \mathcal{Q}_i$ which admits a homomorphism to G.

Theorem 16. *If f is unbounded, then Conjecture 15 is true.*

Proof. Suppose that f is unbounded, and let C be the first value such that $f(C) > 3$. When $\mathcal{P}_{3k,k-1}$ is partitioned into two sets $\mathcal{Q}_1, \mathcal{Q}_2$, we have

$$\prod\{D(P) : P \in \mathcal{P}_{3k,k-1}\} = \left(\prod\{D(P) : P \in \mathcal{Q}_1\}\right) \times \left(\prod\{D(P) : P \in \mathcal{Q}_2\}\right).$$

We have $\chi(\prod\{D(P) : P \in \mathcal{P}_{3k,k-1}\}) = \chi(\iota_k(T_{3k})) = 3$ by Remark 2 and Corollary 10. Since $f(C) > 3$, one of the factors $\prod\{D(P) : P \in \mathcal{Q}_i\}$ has chromatic number less than C. Therefore, for every digraph G such that $\chi(G) \geq C$, there exists no homomorphism of G to $\prod\{D(P) : P \in \mathcal{Q}_i\}$, whence there exists a path $P \in \mathcal{Q}_i$ which admits a homomorphism to G. □

We now turn to multifactor versions of the Poljak–Rödl functions. Define $f_m, g_m : \mathbb{N} \longrightarrow \mathbb{N}$ by

$$f_m(c) = \min\left\{\chi\left(\prod \mathcal{D}\right) : \mathcal{D} \text{ is a finite family of } c\text{-chromatic digraphs}\right\}$$

$$g_m(c) = \min\left\{\chi\left(\prod \mathcal{G}\right) : \mathcal{G} \text{ is a finite family of } c\text{-chromatic graphs}\right\}.$$

The Hedetniemi conjecture is obviously equivalent to its multifactor version, that is, the identity $\chi(\prod \mathcal{G}) = \min\{\chi(G) : G \in \mathcal{G}\}$ for every finite family $\mathcal{G}$ of undirected graphs. (Miller [9] notes that the identity can fail if $\mathcal{G}$ is infinite.) However, it is not known whether the unboundedness of g_m would imply that of f_m. These hypotheses involving various versions of the Poljak–Rödl functions are summarised in the following table.

$$\begin{array}{ccc} g(c) \equiv c & \Leftrightarrow & g_m(c) \equiv c \\ \Downarrow & & \Downarrow \\ g(c) \text{ is unbounded} & \Leftarrow & g_m(c) \text{ is unbounded} \\ \Updownarrow & & \Uparrow \\ f(c) \text{ is unbounded} & \Leftarrow & f_m(c) \text{ is unbounded} \end{array}$$

At the top of the diagram is the Hedetniemi conjecture, which has withstood scrutiny for more than forty years. At the bottom right is the hypothesis of the unboundedness of f_m, which is very much in doubt. Consider the function $h : \mathbb{N} \longrightarrow \mathbb{N}$ defined by

$$h(k) = \min\{\chi(D(P)) : P \in \mathcal{P}_{3k,k-1}\}.$$

If h is unbounded, then the products $\prod\{D(P) : P \in \mathcal{P}_{3k,k-1}\}$ witness the boundedness of f_m. On the other hand, if $h(k) < C$ for all k, then every set $\mathcal{P}_{3k,k-1}$ contains a path P such that $\chi(D(P)) < C$. This would imply Conjecture 15, as well as a variation of Theorem 13 for chromatic number C, using steep paths rather than the family $\{\mathcal{Q}_\ell\}_{\ell\in\mathbb{N}}$. All in all, it seems to be too strong a certificate for $(C-1)$-colourability.

It would be interesting to know whether a family $\{Q_\ell\}_{\ell\in\mathbb{N}}$ of paths which respects the conclusion of Theorem 13 needs an exponential number of arcs, and also which of the missing implications can be added to the table above.

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References

[1] V.C. Barbosa, E. Gafni, Concurrency in heavily loaded neighborhood-constrained systems, ACM Trans. Program. Lang. Syst. 11 (1989) 562–584.
[2] T. Feder, M. Vardi, The computational structure of monotone monadic SNP and constraint satisfaction: a study through datalog and group theory, SIAM J. Comput. 28 (1998) 57–104.
[3] J. Foniok, C. Tardif, Adjoint functors and tree duality, 2008, 14 pages MS (preprint) arXiv:0805.2978v1.
[4] T. Gallai, On directed paths and circuits, in: Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, pp. 115–118.
[5] M. Hasse, Zur algebraischen Begründung der Graphentheorie. I, Math Nachr. 28 (1964/1965) 275–290.
[6] S.H. Hedetniemi, Homomorphisms of graphs and automata, University of Michigan Technical Report 03105-44-T, 1966.
[7] P. Hell, J. Nešetřil, Graphs and Homomorphisms, in: Oxford Lecture Series in Mathematics and its Applications, vol. 28, Oxford University Press, Oxford, 2004, xii+244 pp.
[8] P. Hell, X. Zhu, Homomorphisms to oriented paths, Discrete Math. 132 (1994) 107–114.
[9] D.J. Miller, The categorical product of graphs, Canad. J. Math. 20 (1968) 1511–1521.
[10] G.J. Minty, A theorem on n-colouring the points of a linear graph, Amer. Math. Monthly 67 (1962) 623–624.
[11] J. Nešetřil, A. Pultr, On classes of relations and graphs determined by subobjects and factorobjects, Discrete Math. 22 (1978) 287–300.
[12] J. Nešetřil, C. Tardif, Duality theorems for finite structures (Characterising gaps and good characterisations), J. Combin. Theory Ser. B 80 (2000) 80–97.
[13] J. Nešetřil, C. Tardif, A dualistic approach to bounding the chromatic number of a graph, European J. Combin. 29 (2008) 254–260.
[14] J. Nešetřil, C. Tardif, Path homomorphisms, graph colourings and boolean matrices, J. Graph Theory 63 (3) (2010) 198–209.
[15] S. Poljak, Coloring digraphs by iterated antichains, Comment. Math. Univ. Carolin. 32 (1991) 209–212.
[16] S. Poljak, V. Rödl, On the arc-chromatic number of a digraph, J. Combin. Theory Ser. B 31 (1981) 190–198.
[17] A. Pultr, The right adjoints into the categories of relational systems, in: Reports of the Midwest Category Seminar, IV, in: Lecture Notes in Mathematics, vol. 137, Springer, Berlin, 1970, pp. 100–113.
[18] B. Roy, Nombre chromatique et plus longs chemins d'un graphe, Rev. Francaise Informat. Recherche Opérationelle 1 (1967) 129–132.
[19] C. Tardif, D. Wehlau, Chromatic numbers of products of graphs: the directed and undirected versions of the Poljak–Rödl function, J. Graph Theory 51 (2006) 33–36.
[20] L.M. Vitaver, Determination of minimal colouring of vertices of a graph by means of Boolean powers of the incidence matrix, Dokl. Akad. Nauk SSSR 147 (1962) 758–759 (in Russian).
[21] H.G. Yeh, X. Zhu, Resource-sharing system scheduling and circular chromatic number, Theoret. Comput. Sci. 332 (2005) 447–460.
[22] X. Zhu, A survey on Hedetniemi's conjecture, Taiwanese J. Math. 2 (1998) 1–24.
[23] X. Zhu, Circular chromatic number: a survey, Discrete Math. 229 (2001) 371–410.