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Real and Complex Interpolation Methods for Finite and Infinite Families of Banach Spaces

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A comparative study is made of the various interpolation spaces generated with respect to n-tuples or infinite families of compatible Banach spaces by real and complex interpolation methods due to Sparr, Favini–Lions, Coifman–Cwikel–Rochberg–Sagher–Weiss, and Fernandez. Certain inclusions are established between these spaces and examples are given showing that in general they do not coincide. It is also shown that, in contrast to the case of couples of spaces, the spaces generated by the above methods may depend on the structure of the containing space in which the Banach spaces of the n-tuple ($n \geqslant 3$) or infinite family are embedded. Finally a construction is given which enables the spaces of Sparr and Favini-Lions, hitherto defined only with respect to n-tuples, to also be defined with respect to infinite families of Banach spaces.

INTRODUCTION

Most of the developments in the theory and applications of interpolation spaces in the past twenty years have occurred in the context of a couple of Banach spaces A_1 and A_2 both continuously embedded in a Hausdorff topological vector space $\mathscr{U}$, which in fact can also be taken to be a Banach space without loss of generality. There are several much studied constructions for obtaining interpolation spaces with respect to the couple (A_1, A_2), including in particular the real and complex methods [BL] which yield the spaces $(A_1, A_2)_{\theta,p}$ and $[A_1, A_2]_\theta$, respectively.

A more exotic variant of this theory has a different point of departure, namely an n-tuple $(A_1, A_2, \ldots, A_n)$ or even, more generally, an infinite family $\{A(\gamma)\}_{\gamma \in \Gamma}$ of Banach spaces all of which are required to be continuously

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embedded in a "containing" Hausdorff topological vector space or Banach space $\mathcal{U}$ as before. Using an appropriate generalization of the real or complex (or some other) method one can obtain interpolation spaces with respect to the n-tuple or infinite family, these being spaces $A \subset \mathcal{U}$ with the property that all linear operators which are continuous on $\mathcal{U}$ and on each A_j, or on each $A(\gamma)$, are also continuous on A.

We refer, e.g., to [Sp, F1] and also further references cited in [Sp] for more details concerning various types of real interpolation spaces defined with respect to n-tuples, including a more precise formulation of their interpolation properties. Analogous material concerning the complex methods, which have been defined for infinite families as well as n-tuples can be found, e.g., in [C1, C2, C3, Fa, F2, KN1, KN2, L, N, Sa]. Applications of one of the complex methods can be found, e.g., in [Sa] (spectral properties of convolution operators), [C1] (the Masani–Wiener theorem and estimates for Beckner's analytic semigroup of operators) and, at least implicitly, in [Pi] (K-convex Banach spaces). See also [HRW, R1, RW1, RW2]. We suggest that there are many further possible applications, for example, in the study of the resolvents of a given operator which may be considered as an analytic family of operators (cf. [Sa] and [C3, Theorem 4.2]).

The major part of this paper is devoted to a comparative study of the various types of interpolation spaces mentioned above. We obtain certain inclusions between them which generalize results already known in the "classical" setting of a couple (A_1, A_2). But we also show that several results in the setting of (A_1, A_2) do not extend to n-tuples or infinite families. The trouble usually begins already for a 3-tuple (cf. [C3, Appendix 1]) or even in one case, as we shall see, for a "2.19-tuple!" Our examples show that various methods which coincide in the context of couples yield different spaces in general, that these spaces also depend on the structure of the containing space $\mathcal{U}$, and furthermore, that in the interpolation theorems alluded to above, we cannot dispense with the requirement that the operator be well defined on $\mathcal{U}$.

These results will be explicitly formulated in Section 1, together with a recapitulation of the definitions of the various interpolation spaces to be studied. A diagram at the end of that section summarizes relations between these interpolation spaces.

The second purpose of this paper is to develop constructions which enable the real method spaces of Sparr and the complex method spaces of Favini–Lions, hitherto defined only with respect to n-tuples, to be obtained also for infinite families of Banach spaces. We establish various elementary properties of these new spaces, comparing them with those defined by the complex method of [C3].

These latter results are given in Section 2. The remaining Sections 3–6 contain proofs and detailed calculations related to results formulated in Section 1. We have found it convenient to subdivide the rather voluminous Section 4, which deals with various connections between real and complex methods for n-tuples, into six subsections 4A, 4B,..., 4F.

1. Definitions and Statements of Results for Interpolation Spaces Generated with Respect to n-tuples of Banach Spaces

1.1. Definition. For any positive integer n a *Banach n-tuple* (or *compatible Banach n-tuple*) is an ordered set of n Banach spaces $\bar{A} = (A_1, A_2, \ldots, A_n)$ all of which are linearly and continuously embedded in a Hausdorff topological vector space $\mathscr{U}$ which we shall call the *containing space.* We remark that the specification of $\mathscr{U}$ and of the embedding of each A_j into $\mathscr{U}$ are essential parts of the specification of $\bar{A}$, as will be clear below (see Theorem 1.32).

i. *The K and J Spaces of Sparr*

We begin by considering the real interpolation spaces generated with respect to the n-tuple $\bar{A}$ by the J and K methods of Sparr. A detailed study of these spaces can be found in [Sp]. They are of course generalizations of the spaces $(A_1, A_2)_{\theta,p}$. We recall their construction in the course of the following three definitions. (Similar spaces have also been studied by Yoshikawa [Y] and other authors cited in [Sp].)

1.2. Definition. (i) For any given Banach n-tuple $\bar{A}$ let $\Delta(\bar{A}) = A_1 \cap A_2 \cap \cdots \cap A_n$ with norm $\|a\|_{\Delta(\bar{A})} = \max_{j=1,2,\ldots,n} \|a\|_{A_j}$. More generally, for any n-tuple of positive numbers $\bar{t} = (t_1, t_2, \ldots, t_n)$, we may equivalently renorm $\Delta(\bar{A})$ by the *J-functional*

$$J(\bar{t}, a; \bar{A}) = \max_{j=1,\ldots,n} t_j \|a\|_{A_j} \qquad \text{for each } a \in \Delta(\bar{A}).$$

(ii) Let $\Sigma(\bar{A}) = A_1 + A_2 + \cdots + A_n$ with norm

$$\|a\|_{\Sigma(\bar{A})} = \inf \sum_{j=1}^{n} \|a_j\|_{A_j},$$

where the infimum is taken over all decompositions of a, $a = \sum_{j=1}^{n} a_j$ with $a_j \in A_j$, $j = 1, 2, \ldots, n$. $\Sigma(\bar{A})$ is of course contained in $\mathscr{U}$. More generally, for any n-tuple $\bar{t}$ as above, we may equivalently renorm $\Sigma(\bar{A})$ by the *K-functional*

$$K(\bar{t}, a; \bar{A}) = \inf \sum_{j=1}^{n} t_j \|a_j\|_{A_j}.$$

1.3. DEFINITION. Let H^n_+ be the set of all n-tuples $\bar{\theta} = (\theta_1, \theta_2, ..., \theta_n)$ of numbers in (0, 1) such that $\sum_{j=1}^n \theta_j = 1$. For each Banach n-tuple $\bar{A}$, each $\bar{\theta} \in H^n_+$ and each $p \in [1, \infty]$ we let $\bar{A}_{\bar{\theta}, p; K}$ be the space of all elements $a \in \Sigma(\bar{A})$ for which the norm

$$\|a\|_{\bar{A}_{\bar{\theta},p;K}} = \left(\int_E (\bar{t}^{-\bar{\theta}} K(\bar{t}, a; \bar{A}))^p \, d\mu(\bar{t}) \right)^{1/p} \tag{1.4}$$

is finite. Here the symbol $\bar{t}^{-\bar{\theta}}$ stands for the product $t_1^{-\theta_1} t_2^{-\theta_2} \cdots t_n^{-\theta_n}$ and μ is the measure $(dt_1 \, dt_2 \cdots dt_{n-1})/(t_1 t_2 \cdots t_{n-1})$ supported on the set

$$E = \{(t_1, t_2, ..., t_{n-1}, 1) \mid t_j > 0, j = 1, 2, ..., n-1\} \subset \mathbb{R}^n_+.$$

1.5. *Remark.* Since the K-functional has the homogeneity property $K(\lambda\bar{t}, a; \bar{A}) = \lambda K(\bar{t}, a; \bar{A})$ for each $\lambda > 0$ (where $\lambda\bar{t} = (\lambda t_1, \lambda t_2, ..., \lambda t_n)$) one can replace μ by other measures supported on other sets in $\mathbb{R}^n_+$ and still obtain the same norm. These matters are treated in detail in Section 3 of [Sp].

1.6. DEFINITION. Let $\bar{A}$, $\bar{\theta}$, p, μ, and E be as in Definition 1.3. Then the space $\bar{A}_{\bar{\theta}, p; J}$ is defined to consist of all those elements $a \in \Sigma(\bar{A})$ which have a representation of the form

$$a = \int_E u(\bar{t}) \, d\mu(\bar{t}), \tag{1.7}$$

where $u(\bar{t})$ is a strongly Borel measurable $\Delta(\bar{A})$ valued function on E which is absolutely (i.e., Bochner) integrable on all compact subsets of E and satisfies

$$\left(\int_E (\bar{t}^{-\bar{\theta}} J(\bar{t}, u(\bar{t}); \bar{A}))^p \, d\mu(\bar{t}) \right)^{1/p} < \infty. \tag{1.8}$$

The norm $\|a\|_{\bar{A}_{\bar{\theta},p;J}}$ is the infimum of the values of the integral (1.8) over all such representations (1.7) of a.

1.9. *Remark.* (i) The integral (1.7) can be conveniently interpreted in the weak sense, i.e., $\langle a, l \rangle = \int_E \langle u(\bar{t}), l \rangle \, d\mu(\bar{t})$ for all $l \in \Sigma(\bar{A})'$ (cf. [Sp, Remark 4.3]).

(ii) Using the homogeneity of the J-functional one can replace μ by other measures supported on other sets in $\mathbb{R}^n$ without changing the norm on $\bar{A}_{\bar{\theta}, p; J}$, exactly as for $\bar{A}_{\bar{\theta}, p; K}$ (cf. Remark 1.5).

In the case of Banach couples $\bar{A} = (A_1, A_2)$ much use is made of the important fact that the spaces $\bar{A}_{\bar{\theta}, p; J}$ and $\bar{A}_{\bar{\theta}, p; K}$ coincide to within equivalence of norms. For $n>2$ this result remains true for certain special Banach n-tuples. However, in general it fails, as is shown by an example of a Banach triple $\bar{A}=(A_1, A_2, A_3)$ due essentially to Yoshikawa. Instead we have only the inclusion $\bar{A}_{\theta,p;J}\subset\bar{A}_{\theta,p;K}$. For details of these matters we refer to [Sp], in particular Section 5 and p. 265. Yoshikawa's counterexample could be considered as somewhat "artificial" since it has the property that $\Delta(\bar{A})=\{0\}$ and consequently also $\bar{A}_{\theta,p;J}=\{0\}$. We are thus left with the question of whether the spaces $\bar{A}_{\theta,p;J}$ and $\bar{A}_{\theta,p;K}$ coincide when, as is the case in many "natural" examples, $\Delta(\bar{A})$ is nontrivial or even dense in each A_j, $j=1, 2,..., n$.

The following example and its corollary provide a negative answer to this question.

1.10. Example. Let $\bar{\theta}\in H^3_+$. For each $r>0$ there exists a triple $\bar{A}=\bar{A}^r$ of two-dimensional Hilbert spaces such that $V_K\geqslant rV_J$ where V_J and V_K denote the volumes (areas) of the unit balls of $\bar{A}_{\theta,2;J}$ and $\bar{A}_{\theta,2;K}$, respectively.

The details of the construction of $\bar{A}^r$ are given in Section 3.

1.11. Corollary. *There exists a Banach triple $\bar{A}$ such that $\Delta(\bar{A})$ is dense in A_j for $j=1, 2, 3$, but $\bar{A}_{\theta,2;J}$ is strictly smaller than $\bar{A}_{\theta,2;K}$.*

$\bar{A}$ is obtained by a simple construction using direct sums of spaces from the triples $\bar{A}^r$ for an unbounded sequence of values of r (see Remark 3.14).

(ii). *The Complex Interpolation Spaces of Favini–Lions*

We next consider a generalization of the complex interpolation spaces $[A_1, A_2]_\theta$ (see, e.g., [BL, Chap. 4]) for Banach n-tuples. We shall use essentially the same definition as suggested by Lions [L] which yields spaces which have been studied in detail by Favini [Fa].

1.12. Definition. Let $\bar{A}$ be a Banach n-tuple and let

$$\Omega=\Big\{z=(z_1, z_2,..., z_{n-1})\in\mathbb{C}^{n-1}\,|\,0<\operatorname{Re} z_j<1,$$

$$j=1, 2,..., n-1,\ 0<\sum_{j=1}^{n-1}\operatorname{Re} z_j<1\Big\}.$$

Let $\bar{\Omega}$ denote the closure (in $\mathbb{C}^{n-1}$) of Ω and let $\partial\Omega_j$, $j=1, 2,..., n$, denote the n components of the distinguished boundary of Ω. Thus, for $j=1, 2,..., n-1$,

$$\partial\Omega_j=\{z\in\bar{\Omega}\,|\,\operatorname{Re} z_j=1, \operatorname{Re} z_k=0, k\neq j\}$$

and

$$\partial\Omega_n = \{z \in \bar{\Omega} \mid \operatorname{Re} z_k = 0, k = 1, 2, \ldots, n-1\}.$$

Let $\mathscr{H}(\bar{A})$ denote the space of continuous bounded functions f: $\bar{\Omega} \to \Sigma(\bar{A})$ such that

(i) f is holomorphic in Ω,

(ii) for each $j = 1, 2, \ldots, n$ the restriction of f to $\partial\Omega_j$ is a continuous and bounded A_j valued function which vanishes at infinity.

$\mathscr{H}(\bar{A})$ is normed by

$$\|f\|_{\mathscr{H}(\bar{A})} = \sup\{\|f(z)\|_{A_j} \mid z \in \partial\Omega_j, j = 1, 2, \ldots, n\}.$$

For each $\bar{\theta} = (\theta_1, \theta_2, \ldots, \theta_n) \in H^n_+$ the space $[\bar{A}]_\theta$ is defined by

$$[\bar{A}]_\theta = \{f(\theta_1, \theta_2, \ldots, \theta_{n-1}) \mid f \in \mathscr{H}(\bar{A})\}$$

with norm

$$\|a\|_{[\bar{A}]_\theta} = \inf\{\|f\|_{\mathscr{H}(\bar{A})} \mid f(\theta_1, \theta_2, \ldots, \theta_{n-1}) = a\}.$$

1.13. *Remark.* A more symmetric formulation of this definition could be obtained by replacing the domain Ω by

$$\Omega_* = \left\{ z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \,\middle|\, \sum_{j=1}^{n} z_j = 1, \operatorname{Re} z_j \in (0, 1), \right.$$
$$\left. j = 1, 2, \ldots, n \right\}$$

whose distinguished boundary is the union of the sets

$$\partial\Omega_{*_j} = \{z \in \bar{\Omega}_* \mid \operatorname{Re} z_j = 1\}, \qquad j = 1, 2, \ldots, n.$$

Each $f \in \mathscr{H}(\bar{A})$ corresponds to a unique function g: $\bar{\Omega}_* \to \Sigma(\bar{A})$ with analogous properties, defined by $g(z_1, z_2, \ldots, z_n) = g(z_1, z_2, \ldots, z_{n-1}, 1 - \sum_{j=1}^{n-1} z_j) = f(z_1, z_2, \ldots, z_{n-1})$.

1.14. *Remark.* Favini in fact uses a slightly different space of analytic functions on Ω, which we shall denote here by $\mathscr{H}_1(\bar{A})$. It is defined and normed exactly like $\mathscr{H}(\bar{A})$ except that $\|f(z)\|_{A_j}$ is not required to vanish at infinity on $\partial\Omega_j$. However, it is easy to see that $[\bar{A}]_\theta = \{f(\theta_1, \theta_2, \ldots, \theta_{n-1}) \mid f \in \mathscr{H}_1(\bar{A})\}$ and $\|a\|_{[\bar{A}]_\theta} = \inf\{\|f\|_{\mathscr{H}_1(\bar{A})} \mid f \in \mathscr{H}_1(\bar{A}),\ f(\theta_1, \theta_2, \ldots, \theta_{n-1}) = a\}$ since, for each $f \in \mathscr{H}_1(\bar{A})$ and $\delta > 0$, the function $f_\delta \in \mathscr{H}(\bar{A})$, where $f_\delta(z) = e^{\delta \sum_{j=1}^{n-1} (z_j - \theta_j)^2} f(z)$ and $\|f_\delta\|_{\mathscr{H}(\bar{A})} \leq e^{(n-1)\delta} \|f\|_{\mathscr{H}_1(\bar{A})}$. It is often more convenient for us to use $\mathscr{H}(\bar{A})$ rather than $\mathscr{H}_1(\bar{A})$ in view of Lemma 1.16.

It should also be pointed out that both Favini's and our definitions differ from Lions' in that more stringent continuity conditions ((ii) above) are imposed on the boundary values of functions in $\mathscr{H}(\bar{A})$ and $\mathscr{H}_1(\bar{A})$. For a discussion of different boundary conditions encountered in complex interpolation methods and an example showing that they may lead to different interpolation spaces see [CJ].

Our first result concerning these spaces relates them to those of Sparr.

1.15. THEOREM. *For every Banach n-tuple $\bar{A}$ and every $\bar{\theta} \in H^n_+$*

$$\bar{A}_{\bar{\theta},1;J} \subset [\bar{A}]_{\bar{\theta}}$$

and the norm of the inclusion mapping is at most 1.

Proof. See Subsection 4A.

(For a related result for n-tuples of Hilbert spaces see Theorem 1.24.)

The following lemma generalizes a result of Calderón [Ca, Sect. 9.2] and can be proved via multiple Fourier series and a fairly straightforward adaptation of arguments used in [Ca, Sect. 23.2]. We have provided an alternative somewhat more direct proof in Subsection 4B.

1.16. LEMMA. *The set of all functions of the form*

$$g(z) = g(z_1, z_2, ..., z_{n-1}) = e^{\delta \sum_{j=1}^{n-1} z_j^2} \sum_{k-1}^{N} e^{(\lambda_k, z)} a_k$$

is dense in $\mathscr{H}(\bar{A})$, where $\delta > 0$, N is any positive integer, $\lambda_k = (\lambda_{k,1}, \lambda_{k,2}, ..., \lambda_{k,n-1}) \in \mathbb{R}^{n-1}$, $(\lambda_k, z) = \sum_{j=1}^{n-1} \lambda_{k,j} z_j$ and $a_k \in \Delta(\bar{A})$.

It follows of course from this lemma that $\Delta(\bar{A})$ is dense in $[\bar{A}]_{\bar{\theta}}$. This generalizes Teorema 9 of [Fa, p. 269].

iii. *The "St. Louis" Spaces*

We next relate the Favini-Lions spaces to a different kind of complex interpolation spaces introduced by Coifman, Cwikel, Rochberg, Sagher, and Weiss [C1, C2, C3] which we shall call "St. Louis" spaces for the sake of brevity. (Subsequent results concerning these spaces can be found in, e.g., [CF, H1, H2, J2, R1, R2, RW1, RW2, RW3]; cf. also the spaces introduced by Krein and Nikolova [KN1, KN2, N], i.e., "Voronež" spaces.) In fact the St. Louis spaces can be defined with respect to an infinite family of Banach spaces. (See the above references and Sect. 2.) However, at this stage we consider only the special case where they are generated by a Banach n-tuple $\bar{A}$. Thus we use a simply connected domain D in the complex plane whose boundary Γ is a rectifiable simple closed curve. We let $\bar{\Gamma} = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$ be an n-tuple of pairwise disjoint subsets of Γ, each measurable (with respect to harmonic measure), and whose union is Γ. $\mathscr{G} = \mathscr{G}(\bar{A}, \bar{\Gamma})$ shall denote the space of $\Delta(\bar{A})$ valued functions obtained by taking all finite sums of all functions of the form $\varphi(w)\, a$ where $a \in \Delta(\bar{A})$ and $\varphi(w)$ is a scalar valued bounded analytic function on D. Thus, for each $g \in \mathscr{G}$, the nontangential limit $\lim_{w \rhd \gamma} g(w) = g(\gamma)$ exists for a.e. $\gamma \in \Gamma$. Let $\mathscr{F} = \mathscr{F}(\bar{A}, \bar{\Gamma})$ denote the completion of $\mathscr{G}$ with respect to the norm $\|g\|_{\mathscr{F}(\bar{A}, \bar{\Gamma})} = \operatorname{ess\,sup}\{\|g(\gamma)\|_{A_j} \mid j = 1, 2, \ldots, n, \gamma \in \Gamma_j\}$ Clearly $\mathscr{F}$ is a space of analytic $\Sigma(\bar{A})$ valued functions on D whose boundary values are in A_j for a.e. $\gamma \in \Gamma_j$. Now we can define the St. Louis spaces $A[\zeta]$, or (using a notation more appropriate to the present context) $\bar{A}_{[\zeta], \bar{\Gamma}}$ for each fixed $\zeta \in D$ by

$$\bar{A}_{[\zeta], \bar{\Gamma}} = \{f(\zeta) \mid f \in \mathscr{F}(\bar{A}, \bar{\Gamma})\},$$

with norm $\|a\|_{\bar{A}_{[\zeta], \bar{\Gamma}}} = \inf\{\|f\|_{\mathscr{F}(\bar{A}, \bar{\Gamma})} \mid f \in \mathscr{F}(\bar{A}, \bar{\Gamma}), f(\zeta) = \mathrm{a}\}$.

We shall denote harmonic measure on Γ at a point $\zeta \in D$ by P_ζ, i.e., the Poisson integral of a function f on Γ is $u(\zeta) = \int_\Gamma f(\gamma)\, dP_\zeta(\gamma)$.

A relation between the St. Louis and Favini-Lions spaces is given by the following theorem.

1.17. THEOREM (Peetre). *Let $\bar{A}$ be a Banach n-tuple. Then for each $\zeta \in D$ and each partition $\bar{\Gamma} = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$ of Γ as above*

$$[\bar{A}]_{\bar{\theta}} \subset \bar{A}_{[\zeta], \bar{\Gamma}},$$

where $\bar{\theta} = (\theta_1, \ldots, \theta_n)$ is defined by $\theta_j = P_\zeta(\Gamma_j)$, $j = 1, 2, \ldots, n$. The norm of the inclusion mapping is at most 1.

The proof of this theorem is given in Subsection 4C.

For $n = 2$ the spaces $[\bar{A}]_{\bar{\theta}}$ and $\bar{A}_{[\zeta], \bar{\Gamma}}$ (with $\bar{\theta}$ and ζ related as above) coincide with inequality of norms [C3, Theorem 5.1] and therefore, in this case, the construction of St. Louis spaces is "rearrangement invariant" in the sense that if $\bar{\Gamma}^* = \{\Gamma_1^*, \ldots, \Gamma_n^*\}$ is a second partition of Γ into disjoint measurable subsets such that $P_\zeta(\Gamma_j^*) = P_\zeta(\Gamma_j)$ for $j = 1, 2, \ldots, n$, then $\bar{A}_{[\zeta], \bar{\Gamma}^*} = \bar{A}_{[\zeta], \bar{\Gamma}}$. Our next example shows that this "rearrangement invariance" fails when $n \geqslant 3$ (and even in a certain sense when $n \geqslant 2.1834$).

1.18. EXAMPLE. Let D be the unit disc and let its boundary Γ be divided into three arcs of equal length $\Gamma_1, \Gamma_2, \Gamma_3$. For each $m > 0$ there exists a triple of Banach spaces of analytic functions $\bar{A}^m = (A_1^m, A_2^m, A_3^m)$ and elements $x_m \in A_1^m \cap A_2^m \cap A_3^m$ for which $\|x_m\|_{\bar{A}^m_{[0], \bar{\Gamma}^*}} > m \|x_m\|_{\bar{A}^m_{[0], \bar{\Gamma}}}$, where $\bar{\Gamma} = \{\Gamma_1, \Gamma_2, \Gamma_3\}$ and $\bar{\Gamma}^* = \{\Gamma_3, \Gamma_2, \Gamma_1\}$. The spaces A_j^m may be taken to be isometric images of the disc algebra or of the Hardy class $H^p(D)$ for any $p \in [1, \infty)$.

The details of this construction are given in Subsection 4D where the reader may also find an example (Example 4.7) of an infinite interpolation family $\{A(\gamma)\}_{\gamma \in \Gamma}$ of two-dimensional spaces which has similar properties.

1.19. COROLLARY. *There exists a Banach triple $\bar{A}$ such that, for $\bar{\Gamma}$ and $\bar{\Gamma}^*$ as above, $\bar{A}_{[0],\bar{\Gamma}} \neq \bar{A}_{[0],\bar{\Gamma}^*}$.*

The construction uses Example 1.18 and a simple direct sum procedure, such as in Corollary 1.11 and Remark 3.14, In particular if we use l^2 direct sums of spaces isometric to $H^2(D)$ we may obtain a triple of Hilbert spaces with this property.

1.20. COROLLARY. The inclusion $[\bar{A}]_\theta \subset \bar{A}_{[\zeta],\bar{\Gamma}}$ *of Theorem* 1.17 *is strict in general.*

Proof. Suppose on the contrary that $[\bar{A}]_\theta = \bar{A}_{[\zeta],\bar{\Gamma}}$ whenever $\bar{\theta} = (\theta_1,..., \theta_n)$ and $\bar{\Gamma} = \{\Gamma_1, \Gamma_2,..., \Gamma_n\}$ satisfy $P_\zeta(\Gamma_j) = \theta_j$. Then for $\bar{\Gamma}$, $\bar{\Gamma}^*$ and $\bar{A}$ as in the preceding corollary and $\bar{\theta} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ we have $\bar{A}_{[0],\bar{\Gamma}} = [\bar{A}]_\theta = \bar{A}_{[0],\bar{\Gamma}^*}$ which is a contradiction.

1.21. *Remark.* The above examples also lead us to make some (rather discouraging) observations concerning the duals of Favini–Lions spaces. Favini showed [Fa, Teorema 10, p. 272] that $[A_1', A_2', A_3']_\theta \subset [A_1, A_2, A_3]_\theta'$ whenever $\Delta(\bar{A})$ is dense in A_1, A_2, A_3 and $[A_1, A_2, A_3]_\theta$. (In fact, density in $[A_1, A_2, A_3]_\theta$ is always assured by Lemma 1.16.) At first sight, it would seem reasonable to conjecture, by analogy with the description of the dual of $[A_1, A_2]_\theta$, that the above inclusion is in fact an equality, at least when the spaces A_j are reflexive. However, as we show in Subsection 4D, Remark 4.8, such a result is not true in general since for each $m > 0$ we can construct a triple of finite-dimensional spaces $\bar{Y} = (Y_1, Y_2, Y_3)$ and an element y such that $\|y\|_{[Y_1', Y_2', Y_3']_\theta} / \|y\|_{[Y_1, Y_2, Y_3]_\theta'} > m$. Despite this setback to characterizing such dual spaces (or maybe because of it) Peetre (see [P2]) has succeeded in obtaining a description of the dual spaces of the complex interpolation spaces of Fernandez (see below and [F2]) which have a definition roughly analogous to that of $[\bar{A}]_\theta$.

1.22. QUESTION. By Theorem 1.17 we in fact have $[\bar{A}]_\theta \subset \bigcap_{\bar{\Gamma}} \bar{A}_{[\zeta],\bar{\Gamma}}$ for every n-tuple $\bar{A}$ where the intersection is taken over the class of all decompositions of Γ, $\bar{\Gamma} = \{\Gamma_1, \Gamma_2,..., \Gamma_n\}$ for which $P_\zeta(\Gamma_j) = \theta_j$, $j = 1, 2,..., n$. Do these two spaces coincide? (By an obvious conformal map argument the above intersection of St. Louis spaces will be the same whether we consider ζ fixed or variable.)

The following theorem taken in combination with Theorems 1.15 and 1.17 provides an n-tuple analogue of the Lions–Peetre inclusions $(A_1, A_2)_{\theta,1} \subset [A_1, A_2]_\theta \subset (A_1, A_2)_{\theta,\infty}$ [BL, Theorem 4.7.1, p. 102].

1.23. THEOREM. (i) *Let $\bar{A} = (A_1, A_2, \ldots, A_n)$ be a Banach n-tuple and $\bar{\theta} \in H^n_+$. Then*

$$[\bar{A}]_{\bar{\theta}} \subset \bar{A}_{\bar{\theta},\infty;K}.$$

(ii) *If $\bar{\Gamma} = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$ is a partition of the rectifiable boundary Γ of the simply connected domain D such that, for some point $\zeta \in D$, $P_\zeta(\Gamma_j) = \theta_j$ for $j = 1, 2, \ldots, n$, then*

$$\bar{A}_{[\zeta],\bar{\Gamma}} \subset \bar{A}_{\bar{\theta},\infty;K}.$$

The norms of the inclusion mappings in (i) and (ii) are both at most 1.

Proof. See Subsection 4E.

iv. *The Case of n-tuples of Hilbert Spaces*

If (A_1, A_2) is a couple of Hilbert spaces then the above inclusions between real and complex interpolation spaces can be sharpened to yield $(A_1, A_2)_{\theta,2} = [A_1, A_2]_\theta$. (See, e.g. [P1]). Analogously, in the case where $\bar{A} = (A_1, A_2, \ldots, A_n)$ is an n-tuple of Hilbert spaces, it might be expected that we can obtain sharpened forms of Theorems 1.15 and 1.23. We shall present one partial result in this direction.

1.24. THEOREM. *Let $\bar{A} = (A_1, A_2, \ldots, A_n)$ be a compatible n-tuple of Hilbert spaces. Then for any $\bar{\theta} \in H^n_+$,*

$$[\bar{A}]_{\bar{\theta}} \subset \bar{A}_{\bar{\theta},2;J}$$

with continuous inclusion.

The proof, in Subsection 4F, implicitly contains the idea of identifying $\bar{A}_{\bar{\theta},2;J}$ with a variant of the Favini–Lions space which is constructed replacing $\mathscr{H}(A)$ by a similar space $\mathscr{H}^2(\bar{A})$ normed by $\|f\|_{\mathscr{H}^2(\bar{A})} = (\sum_{j=1}^n \int_{\partial\Omega_j} \|f(z)\|^2_{A_j}\, dm_j(z))^{1/2}$ (m_j is an $(n-1)$-dimensional Lebesgue measure on $\partial\Omega_j$). We shall not pursue this idea systematically here. One could also consider Favini–Lions type spaces corresponding to similar use of $\mathscr{H}^p(\bar{A})$ for other values of p. We are dealing here with vector valued analogues of spaces, H^p spaces on tubes, for which there is a well-developed theory (see [SW, Chap. III]; cf. also [DGV].)

In the case $n = 2$ all choices of p in the above construction yield the same complex interpolation spaces (see, e.g., [P1]). It is natural to ask whether this also happens if $n \geqslant 3$.

Concerning possible relations between $\bar{A}_{\theta,2;J}$ or $\bar{A}_{\theta,2;K}$ and the corresponding St. Louis spaces, let us note that in view of Remark 3.15 the space $\bar{A}_{[\zeta],\bar{\Gamma}}$ cannot coincide with either of its real method analogues $\bar{A}_{\theta,2;J}$, $\bar{A}_{\theta,2;K}$. This could also be shown using an argument similar to that used to prove Corollary 1.20. (Here again, as before, $\bar{\Gamma}=\{\Gamma_1, \Gamma_2,..., \Gamma_n\}$ and $\theta_j = P_\zeta(\Gamma_j)$.) The estimates of Remark 3.15 suggest that it is plausible to conjecture that $\bar{A}_{\theta,2;J} \subset \bar{A}_{[\zeta],\bar{\Gamma}} \subset \bar{A}_{\theta,2;K}$. Another natural question, related to this conjecture and the preceding question concerning Favini–Lions spaces and their L^p generalizations, is whether for an n-tuple of Banach spaces all of Fourier type p (see [M2, M3, P1]) one can obtain

$$\bar{A}_{\theta,p;J} \subset \bar{A}_{[\zeta],\bar{\Gamma}} \subset \bar{A}_{\theta,p',K}.$$

v. *The Real and Complex Interpolation Spaces of Fernandez*

Fernandez [F1, F2] has introduced versions of the real and complex methods for 2^n-tuples. His methods are similar to the Sparr and Favini–Lions methods defined above, but with the simplex replaced by an n-cube. For simplicity we will only treat the case of four spaces here, although the results easily extend to the general case. Following Fernandez' notation we will denote Banach 4-tuples by $\bar{A} = (A_{00}, A_{10}, A_{01}, A_{11})$. Then for $t_1, t_2 > 0$ and a in $\Delta(\bar{A})$ or $\Sigma(\bar{A})$ the J- and K-functionals of Fernandez are defined, respectively, by

$$J(t_1, t_2, a; \bar{A}) = \max(\|a\|_{A_{00}}, t_1\|a\|_{A_{10}}, t_2\|a\|_{A_{01}}, t_1 t_2\|a\|_{A_{11}})$$

and

$$K(t_1, t_2, a; \bar{A})$$
$$= \inf_{a = a_{00} + a_{10} + a_{01} + a_{11}} (\|a_{00}\|_{A_{00}} + t_1\| \|a_{10}\|_{A_{10}} + t_2\|a_{01}\|_{A_{01}} + t_1 t_2\|a_{11}\|_{A_{11}}).$$

Given $\theta_1, \theta_2 \in (0, 1)$ and $p_1, p_2 \in [1, \infty]$ Fernandez defines the space $\bar{A}_{\theta_1,\theta_2,p_1,p_2;J}$ to consist of all elements $a \in \Sigma(\bar{A})$ which can be represented in the form $a = \int_0^\infty \int_0^\infty u(t_1, t_2)(dt_1\, dt_2/t_1 t_2)$, where $u(t_1, t_2)$ is a strongly measurable $\Delta(\bar{A})$ valued function satisfying

$$\left(\int_0^\infty \left[t_2^{-\theta_2}\left(\int_0^\infty (t_1^{-\theta_1} J(t_1, t_2, u(t_1, t_2); \bar{A}))^{p_1} \frac{dt_1}{t_1}\right)^{1/p_1}\right]^{p_2} \frac{dt_2}{t_2}\right)^{1/p_2} < \infty.$$

Similarly $\bar{A}_{\theta_1,\theta_2,p_1,p_2;K}$ consists of all $a \in \Sigma(\bar{A})$ which satisfy

$$\left(\int_0^\infty \left[t_2^{-\theta_2}\left(\int_0^\infty (t_1^{-\theta_1} K(t_1, t_2, a; \bar{A}))^{p_1} \frac{dt_1}{t_1}\right)^{1/p_1}\right]^{p_2} \frac{dt_2}{t_2}\right)^{1/p_2} < \infty.$$

These spaces are normed in the obvious way.

For the definition of complex interpolation spaces Fernandez uses a space $H(\bar{A})$ of $\Sigma(\bar{A})$ valued continuous bounded functions $f(z_1, z_2)$ defined on the region in $\mathbb{C}^2$

$$S^2 = \{(x_1 + iy_1, x_2 + iy_2) \mid 0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 \leqslant 1\}$$

which are analytic in the interior and continuous bounded $A_{j_1 j_2}$ valued functions of (y_1, y_2) when restricted to the corresponding component $\{(j_1 + iy_1, j_2 + iy_2) \mid y_1, y_2 \in \mathbb{R}\}$ of the distinguished boundary of S^2, $(j_1 = 0, 1, j_2 = 0, 1)$. $H(\bar{A})$ is normed by

$$\|f\|_{H(\bar{A})} = \sup\{\|f(j_1 + iy_1, j_2 + iy_2)\|_{A_{j_1 j_2}} \mid j_1, j_2 = 0, 1, y_1, y_2 \in \mathbb{R}\}$$

and for $\theta_1, \theta_2 \in (0, 1)$ the space $[\bar{A}; \theta_1, \theta_2]$ consists of all elements of the form $a = f(\theta_1, \theta_2)$ with an obvious quotient norm.

As above (cf. Example 1.10) we wish to investigate whether the J and K spaces coincide in this context. An example has been given by Asekritova [A] where they do not, but in her case (cf. the example of Yoshikawa mentioned above) $\Delta(\bar{A}) = \{0\}$. The complex method spaces of Fernandez have also been studied by Dore, Guidetti, and Venni [DGV] who were led independently, and for different purposes, to consider a counterexample having some similarity with ours below.

We shall calculate the above spaces for certain values of the parameters when the 4-tuple $\bar{A}$ is "diagonally equal".

1.25. Examples. Let $\bar{A} = (A_{00}, A_{10}, A_{01}, A_{11})$ satisfy $A_{00} = A_{11} = B_1$, $A_{10} = A_{01} = B_2$ for any Banach couple (B_1, B_2). Then

$$[\bar{A}; \tfrac{1}{2}, \tfrac{1}{2}] = B_1 \cap B_2, \tag{1.26}$$

$$\bar{A}_{1/2, 1/2, 1, 1; J} = B_1 \cap B_2, \tag{1.27}$$

$$\int_0^\infty \int_0^\infty (t_1 t_2)^{-1/2} K(t_1, t_2, a; \bar{A})]^p \frac{dt_1\, dt_2}{t_1 t_2} = \frac{8}{p} \int_0^\infty \left(\frac{K(t, a; B_1, B_2)}{\max(1, t)} \right)^p \frac{dt}{t}, \tag{1.28}$$

$$\bar{A}_{1/2, 1/2, \infty, \infty; K} = B_1 + B_2, \tag{1.29}$$

$$\bar{A}_{1/2, 1/2, 1, 1; J} \neq \bar{A}_{1/2, 1/2, 1, 1; K} \quad \text{except in some trivial cases.} \tag{1.30}$$

For details of these calculations see Section 5.

1.31. *Remark.* Example (1.30) suggests that there is an inaccuracy in [F1, Theorem 3.4]. Equivalence of the J and K methods of Fernandez would imply, by an argument due to Milman [M1], that the real method spaces of Fernandez could be obtained by reiteration of the real method for couples.

vi. *Dependence on the Containing Space*

The final and perhaps, at first sight, the most surprising phenomenon which we shall discuss in this section exhibits the essential roles of the containing space $\mathscr{U}$ both in determining the interpolation spaces generated by a given n-tuple for $n \geqslant 3$ and also in the formulation of interpolation theorems.

For the sake of comparison we first remark that, if (A_1, A_2) and (B_1, B_2) are Banach couples such that A_j is isomorphic to B_j, $j = 1, 2$, and the isomorphism maps $T_1: A_1 \to B_1$, $T_2: A_2 \to B_2$ coincide on $A_1 \cap A_2$ and so define an isomorphism of $A_1 \cap A_2$ onto $B_1 \cap B_2$, then we can also extend T_1 and T_2 to define a (consistent) isomorphism T between $A_1 + A_2$ and $B_1 + B_2$ and deduce that $[A_1, A_2]_\theta$ and $(A_1, A_2)_{\theta,p}$ are isomorphic to $[B_1, B_2]_\theta$ and to $(B_1, B_2)_{\theta,p}$, respectively. This seems completely obvious (but we might begin to doubt it after reading what is to follow). We simply let $Ta = T_1 a_1 + T_2 a_2$ where $a = a_1 + a_2$, $a_j \in A_j$ and show that Ta is independent of the choice of decomposition $a = a_1 + a_2$.

This remark means, in other words, that we can embed A_1 and A_2 in different containing spaces $\mathscr{U}$ or $\mathscr{V}$ and, provided A_1 and A_2 always intersect in the same way, these different embeddings will not change the interpolation spaces generated by (A_1, A_2).

For three or more spaces the situation is drastically different:

1.32. THEOREM. *Let $\bar{A} = (A_1, A_2, A_3)$ be a triple of Banach spaces continuously embedded in a Banach space $\mathscr{U}$ and let $\bar{A}_\Phi$ denote an interpolation space generated by $\bar{A}$ using any of the methods discussed above and containing $\Delta(\bar{A})$ densely. Suppose further that $\bar{A}_\Phi$ contains an element a_* which is not in any of the spaces $A_1 + A_2$, $A_2 + A_3$, $A_3 + A_1$. Then there exists a triple $\bar{B} = (B_1, B_2, B_3)$ of Banach spaces, all embedded continuously in a Banach space $\mathscr{V}$, and a linear map S from $\Delta(\bar{B})$ to $(\Delta\bar{A})$ such that*

$$\sup_{b \in \Delta(\bar{B})} \|Sb\|_{\bar{A}_\Phi} / \|b\|_{\bar{B}_\Phi} = \infty$$

despite the fact that for $j = 1, 2, 3$,

$$\sup_{b \in \Delta(\bar{B})} \|Sb\|_{A_j} / \|b\|_{B_j} = 1$$

and indeed S is the common restriction to $\Delta(\bar{B})$ of three linear maps S_j: $B_j \to A_j$ which also agree on pairwise intersections and define isometries between A_j and B_j and also between $A_j \cap A_k$ and $B_j \cap B_k$ for $j, k = 1, 2, 3$.

Proof. See Section 6, which also contains some further related remarks.

$$\begin{tikzcd}
 & \bar{A}_{\theta,p;K} \arrow[rr] & & \bar{A}_{\theta,q;K} \arrow[dr] & \\
\bar{A}_{\theta,1;J} \arrow[r] \arrow[ur] \arrow[dr] & \bar{A}_{\theta,p;J} \arrow[u] \arrow[rr] & & \bar{A}_{\theta,q;J} \arrow[u] \arrow[r] & \bar{A}_{\theta,\infty;K} \\
 & {[\bar{A}]_{\theta}} \arrow[u, dashed] \arrow[rr] \arrow[urr, phantom, "\text{(For Hilbert n-tuples when $p \geqslant 2$)}"] & & \bar{A}_{[z];\Gamma} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Favini–Lions Space   St. Louis Space

FIG. 1. Inclusions between various interpolation spaces generated with respect to an n-tuple $\bar{A} = (A_1, \ldots, A_n)$ of compatible Banach spaces. The spaces in the first two lines are generated by the J and K methods of Sparr, with $1 < p < q < \infty$. Each solid line arrow points from a given space towards another which contains it. All these inclusions are known to be strict in general, and the inclusion maps are continuous. The dotted line arrow represents an inclusion which holds when A_j, $j = 1, 2, \ldots, n$, are all Hilbert spaces and $p \geqslant 2$, but may fail to hold otherwise. $\bar{\Gamma} = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$ is a partition of Γ, the boundary curve of a domain containing z such that $P_z(\Gamma_j) = \theta_j$, $j = 1, 2, \ldots, n$.

2. EXTENSIONS OF THE METHODS OF SPARR AND FAVINI–LIONS TO THE CASE OF INFINITE FAMILIES OF BANACH SPACES.

As pointed out in [C1, p. 274], the construction of the St. Louis spaces $A[z]$ from a given interpolation family $\{A(\gamma)\}_{\gamma \in \Gamma}$ may be likened to solving a Dirichlet problem where the values of the "boundary function" $A(\gamma)$ and its "Poisson integral" $A[z]$ are Banach spaces rather than numbers or elements of some vector space. Developing this analogy further we could say that, for a given n-tuple $\bar{A}$, the space $\bar{A}_{[z];\bar{\Gamma}}$ can be considered as a sort of "Poisson integral" at z of the "simple function" $A(\gamma) = \sum_{j=1}^{n} A_j \chi_{\Gamma_j}(\gamma)$. We shall start by thinking of the processes of calculating the spaces $[\bar{A}]_\theta$, $\bar{A}_{\theta,p;J}$ and $\bar{A}_{\theta,p;K}$ also as processes akin to "integration" of the same simple function $A(\gamma) = \sum_{j=1}^{n} A_j \chi_{\Gamma_j}(\gamma)$. The mechanism which will enable us to make the transition from these spaces to their new counterparts, defined for infinite interpolation families $\{A(\gamma)\}_{\gamma \in \Gamma}$, will be reminiscent of the transition from integration of simple functions to integration of more general functions.

In this context the curve Γ used for constructing the St. Louis spaces can just as well be replaced by an arbitrary measure space $(\Gamma, \mathscr{S}, Z)$ where Z is a probability measure (corresponding to harmonic measure on Γ at z in the case where Γ is a curve).

We shall begin by restating some obvious things about integration and (real valued) simple and measurable functions. The notions to be discussed and the notation for them have been chosen to permit an easy generalization to a parallel "integration/interpolation" theory for "functions" whose values are Banach spaces. This theory will yield the constructions we require.

A. *A Scalar or "One-Dimensional" Model of the Theory*

Let Z be a probability measure on a σ-algebra of subsets of an abstract set Γ. Let A denote an arbitrary real valued function on Γ which satisfies the inequalities

$$0 \leqslant A(\gamma) \leqslant \mathscr{U} \tag{2.1}$$

for all $\gamma \in \Gamma$ where $\mathscr{U}$ is a fixed positive number.

By a *partition* of Γ we shall mean a finite collection $\bar{\Gamma} = \{\Gamma_1, \Gamma_2, ..., \Gamma_n\}$ of disjoint measurable subsets of Γ, each of positive Z-measure, such that $Z(\Gamma \setminus \bigcup_{j=1}^n \Gamma_j) = 0$. (The seemingly more natural requirement that $\Gamma = \bigcup_{j=1}^n \Gamma_j$ would lead to a technical problem as we shall see in a moment.) Let $\mathscr{P}$ denote the set of all such partitions. For each $\bar{\Gamma} \in \mathscr{P}$ let $A^{\bar{\Gamma}}$ denote the simple function which assumes the value $\sup_{\gamma \in \Gamma_j} A(\gamma)$ on Γ_j for each Γ_j in $\bar{\Gamma}$. Similarly let $A_{\bar{\Gamma}}$ denote the simple function which assumes the value $\inf_{\gamma \in \Gamma_j} A(\gamma)$ on Γ_j. Define the "upper and lower exponentiated sums" of $\log A$ on $\bar{\Gamma}$ by

$$\mathrm{U}(A, \bar{\Gamma}, Z) = \exp \int_\Gamma \log A^{\bar{\Gamma}}(\gamma)\, dZ(\gamma)$$

and

$$\mathrm{L}(A, \bar{\Gamma}, Z) = \exp \int_\Gamma \log A_{\bar{\Gamma}}(\gamma)\, dZ(\gamma).$$

Clearly $0 \leqslant \mathrm{L}(A, \bar{\Gamma}, Z) \leqslant \mathrm{U}(A, \bar{\Gamma}, Z) \leqslant \mathscr{U}$.

Now let $\bar{\Omega}$, $\bar{\Gamma}$ be partitions in $\mathscr{P}$ such that $\bar{\Omega}$ is a refinement of $\bar{\Gamma}$ (meaning of course that each Ω_j of $\bar{\Omega}$ is a subset of some Γ_k of $\bar{\Gamma}$). We shall denote this by the notation $\bar{\Omega} \succ \bar{\Gamma}$. Obviously

$$\mathrm{L}(A, \bar{\Omega}, Z) \geqslant \mathrm{L}(A, \bar{\Gamma}, Z) \tag{2.2}$$

and

$$\mathrm{U}(A, \bar{\Omega}, Z) \leqslant \mathrm{U}(A, \bar{\Gamma}, Z). \tag{2.3}$$

Clearly $\mathscr{P}$ is a directed set with respect to the partial ordering $\succ$. (This would not be the case if we required $\Gamma = \bigcup_{j=1}^n \Gamma_j$.) The estimates (2.2) and (2.3) imply the existence of the generalized limits:

$$\lim_{\bar{\Gamma} \in \mathscr{P}} \mathrm{L}(A, \bar{\Gamma}, Z) = \sup_{\bar{\Gamma} \in \mathscr{P}} \mathrm{L}(A, \bar{\Gamma}, Z) \tag{2.4}$$

and

$$\lim_{\bar{\Gamma}\in\mathscr{P}} \mathrm{U}(A, \bar{\Gamma}, Z) = \inf_{\bar{\Gamma}\in\mathscr{P}} \mathrm{U}(A, \bar{\Gamma}, Z). \tag{2.5}$$

These are the exponential lower and upper integrals of $\log A$ with respect to Z and it is convenient to denote them here by $\mathrm{L}(A, Z)$ and $\mathrm{U}(A, Z)$, respectively. Clearly $\mathrm{L}(A, Z) = \mathrm{U}(A, Z)$ if and only if A is a measurable function on (Γ, Z) in which case

$$\mathrm{L}(A, Z) = \mathrm{U}(A, Z) = \exp \int_\Gamma \log A(\gamma)\, dZ(\gamma).$$

A trivial instance in which this occurs is when A is a simple function assuming constant values on each set Γ_j of some partition $\bar{\Gamma} \in \mathscr{P}$. In this case we also have $\mathrm{L}(A, \bar{\Omega}, Z) = \mathrm{U}(A, \bar{\Omega}, Z) = \mathrm{L}(A, Z) = \mathrm{U}(A, Z)$ for all $\bar{\Omega} \in \mathscr{P}$ with $\bar{\Omega} \succ \bar{\Gamma}$, since of course $A = A^{\bar{\Gamma}} = A_{\bar{\Gamma}} = A^{\bar{\Omega}} = A_{\bar{\Omega}}$.

B. *Basic Definitions for the General Theory: "Functions" Taking Values in a Class of Banach Spaces*

Let us now consider the analogue of the above in the context of a "function" A on Γ whose "values" $A(\gamma)$ are each Banach spaces rather than numbers. Here we shall define the "inequality" $E \leqslant F$ between two Banach spaces to mean that $E \subset F$ and $\|x\|_F \leqslant \|x\|_E$ for each $x \in E$. Thus we require the existence of a fixed Banach space $\mathscr{U}$ such that $A(\gamma) \leqslant \mathscr{U}$ for all $\gamma \in \Gamma$. (This is in fact the analogue of (2.1) since we of course have $\{0\} \leqslant A(\gamma) \leqslant \mathscr{U}$.) We shall call a family of Banach spaces $\{A(\gamma) \mid \gamma \in \Gamma\}$ which has this property a *bounded family* on Γ. (This is of course reminiscent of the notion of interpolation family as defined in [C3, Definition 2.1, p. 206].)

We next define "simple functions" $A_{\bar{\Gamma}}$ and $A^{\bar{\Gamma}}$ for any $\bar{\Gamma} = \{\Gamma_1, \Gamma_2, ..., \Gamma_n\} \in \mathscr{P}$ by $A_{\bar{\Gamma}}(\delta) = \inf_{\gamma \in \Gamma_j} A(\gamma)$ and $A^{\bar{\Gamma}}(\delta) = \sup_{\gamma \in \Gamma_j} A(\gamma)$ for all $\delta \in \Gamma_j$ and all $j = 1, 2, ..., n$. Here $\inf_{\gamma \in \Gamma_j} A(\gamma)$ is the Banach space consisting of all elements $a \in \bigcap_{\gamma \in \Gamma_j} A(\gamma)$ for which $\|a\|_{\inf_{\gamma\in\Gamma_j} A(\gamma)} = \sup_{\gamma \in \Gamma_j} \|a\|_{A(\gamma)}$ is finite (it may in some cases be the trivial space $\{0\}$) and $\sup_{\gamma \in \Gamma_j} A(\gamma)$ is the Banach space consisting of elements $a \in \mathscr{U}$ of the form $a = \sum_{\gamma \in \Gamma_j} u(\gamma)$ (convergence in $\mathscr{U}$) where $u(\gamma) \in A(\gamma)$ for all $\gamma \in \Gamma_j$ and $\sum_{\gamma \in \Gamma_j} \|u(\gamma)\|_{A(\gamma)} < \infty$. This latter space is normed by $\|a\|_{\sup_{\gamma\in\Gamma_j} A(\gamma)} = \inf \sum_{\gamma \in \Gamma_j} \|u(\gamma)\|_{A(\gamma)}$, where the infimum is taken over all representations $\sum_{\gamma \in \Gamma_j} u(\gamma)$ as above for a.

The "ranges" of each of the "functions" $A_{\bar{\Gamma}}$ and $A^{\bar{\Gamma}}$ are compatible n-tuples of Banach spaces, each with containing space $\mathscr{U}$:

$$A_{\bar{\Gamma}}(\Gamma) = (\inf_{\gamma \in \Gamma_1} A(\gamma), \inf_{\gamma \in \Gamma_2} A(\gamma), ..., \inf_{\gamma \in \Gamma_n} A(\gamma)),$$

$$A^{\bar{\Gamma}}(\Gamma) = (\sup_{\gamma \in \Gamma_1} A(\gamma), \sup_{\gamma \in \Gamma_2} A(\gamma), ..., \sup_{\gamma \in \Gamma_n} A(\gamma)).$$

We can now define three analogues of the exponentiated upper and lower sums $\mathrm{U}(A, \bar{\Gamma}, Z)$ and $\mathrm{L}(A, \bar{\Gamma}, Z)$ corresponding, respectively, to the Sparr K spaces, Sparr J spaces and Favini–Lions spaces. Thus, for each $p \in [1, \infty]$ and each $\bar{\Gamma} = \{\Gamma_1, \Gamma_2, ..., \Gamma_n\} \in \mathscr{P}$, define

$$\bar{\theta} = \bar{\theta}(\bar{\Gamma}) = (\theta_1, \theta_2, ..., \theta_n) \in H_+^n$$

by setting

$$\theta_j = Z(\Gamma_j), \qquad j = 1, 2, ..., n$$

and let

$$L_{K,p}(A, \bar{\Gamma}, Z) = (A_{\bar{\Gamma}}(\Gamma))_{\bar{\theta}, p; K}$$

and

$$L_{J,p}(A, \bar{\Gamma}, Z) = (A_{\bar{\Gamma}}(\Gamma))_{\bar{\theta}, p; J}$$

up to equivalence of norms (see below). We also take

$$L_{\mathrm{FL}}(A, \bar{\Gamma}, Z) = [A_{\bar{\Gamma}}(\Gamma)]_{\bar{\theta}}$$

with equality of norms. The spaces $U_{K,p}$, $U_{J,p}$, and U_{FL} are defined analogously using the n-tuple $A^{\bar{\Gamma}}(\Gamma)$ in place of $A_{\bar{\Gamma}}(\Gamma)$. There is also a fourth analogue corresponding to St. Louis spaces. In this case we must of course assume that Γ is a rectifiable simple closed curve constituting the boundary of a domain $D \subset \mathbb{C}$. We shall take $Z = P_z$ to be harmonic measure on Γ at some fixed point $z \in D$. Then, as in the notation used above in Theorem 1.17, we take

$$\mathrm{L}_{\mathrm{St.L}}(A, \bar{\Gamma}, P_z) = (A_{\bar{\Gamma}}(\Gamma))_{[z], \Gamma}$$

and

$$\mathrm{U}_{\mathrm{St.L}}(A, \bar{\Gamma}, P_z) = (A^{\bar{\Gamma}}(\Gamma))_{[z], \Gamma}$$

with equality of norms. It will be convenient to collectively denote these various spaces by the notation $\mathrm{L}_{\mathrm{M}}(A, \bar{\Gamma}, Z)$ and $\mathrm{U}_{\mathrm{M}}(A, \bar{\Gamma}, Z)$ where M stands for any of the "methods" J,p, K,p, FL, or St.L and where it will always be understood that if $\mathrm{M} = \mathrm{St.L}$ then Γ and $Z = P_z$ are necessarily of the form specified above.

For completeness we shall also define all these spaces for the case $n = 1$, ($\bar{\Gamma} = \{\Gamma_1\}$) by adopting the convention that for a "1-tuple" $\bar{A} = (A_1)$ each of the spaces $[\bar{A}]_{\theta}$, $(\bar{A})_{\theta, p; K}$, and $(\bar{A})_{\theta, p; J}$ coincide with A_1 with equality of norms. (This is also automatically true for $(\bar{A})_{[z], \Gamma}$.)

C. *Monotonicity of* $\mathrm{L_M}$ *and* $\mathrm{U_M}$ *Spaces with Respect to Refinement of Partitions of* Γ

The next and relatively lengthy step is to show that each of the above spaces $\mathrm{L_M}$ and $\mathrm{U_M}$ satisfies the analogues of (2.2) and (2.3),

$$\mathrm{L_M}(A, \bar{\Omega}, Z) \geqslant \mathrm{L_M}(A, \bar{\Gamma}, Z), \tag{2.6}$$

$$\mathrm{U_M}(A, \bar{\Omega}, Z) \leqslant \mathrm{U_M}(A, \bar{\Gamma}, Z), \tag{2.7}$$

for any bounded family $\{A(\gamma) \mid \gamma \in \Gamma\}$ and any partitions $\bar{\Omega}, \bar{\Gamma} \in \mathscr{P}$ satisfying $\bar{\Omega} \succ \bar{\Gamma}$. It suffices to do this in the case where $\bar{\Gamma} = \{\Gamma_1, \Gamma_2, ..., \Gamma_n\}$ and $\bar{\Omega} = \{\Omega_1, \Omega_2, ..., \Omega_{n+1}\}$ with $\Gamma_n = \Omega_n \cup \Omega_{n+1}$ and $\Gamma_j = \Omega_j$ for $j = 1, 2, ..., n-1$. We can then obtain the general result by successive applications of this case, since the definitions of each of the spaces $\mathrm{L_M}(A, \bar{\Gamma}, Z)$ and $\mathrm{U_M}(A, \bar{\Gamma}. Z)$ and their norms are independent of the order in which we label the sets in $\bar{\Gamma}$.

Since $\mathrm{L_{St.L}}(A, \bar{\Gamma}, P_z)$ is obtained by applying the construction in [C3, Definition 2.3, p. 209] to the interpolation family $\{A_{\bar{\Gamma}}(\gamma) \mid \gamma \in \Gamma\}$ and since for all $\gamma \in \Gamma$, $A_{\bar{\Gamma}}(\gamma) \leqslant A_{\bar{\Omega}}(\gamma)$ we immediately obtain that $\mathrm{L_{St.L}}(A, \bar{\Gamma}, P_z) \leqslant \mathrm{L_{St.L}}(A, \bar{\Omega}, P_z)$. Similarly, since $A^{\bar{\Gamma}}(\gamma) \geqslant A^{\bar{\Omega}}(\gamma)$ for all $\gamma \in \Gamma$, we deduce that $\mathrm{U_{St.L}}(A, \bar{\Omega}, P_z) \leqslant \mathrm{U_{St.L}}(A, \bar{\Gamma}, P_z)$.

To obtain (2.6) and (2.7) for Favini–Lions spaces we need the following "reducibility" property of these spaces (see [Fa, p. 263] for a special case of this result).

2.8. Lemma. *Let* $\bar{E} = (E_1, E_2, ..., E_n)$ *be a compatible n-tuple of Banach spaces. Define the compatible* $(n+1)$*-tuple* $\bar{F} = (F_1, F_2, ..., F_{n+1})$ *by* $E_j = F_j$ *for* $j = 1, 2, ..., n-1$ *and* $F_n = F_{n+1} = E_n$ *with equality of norms. Then, for each* $\bar{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_{n+1}) \in H_+^{n+1}$ *and each corresponding* $\bar{\beta} = (\alpha_1, \alpha_2, ..., \alpha_{n-1}, \alpha_n + \alpha_{n+1}) \in H_+^n$,

$$[\bar{E}]_{\bar{\beta}} = [\bar{F}]_{\bar{\alpha}} \tag{2.9}$$

with equality of norms.

Proof. Suppose first that $a \in [\bar{F}]_{\bar{\alpha}}$ and let $f = f(z_1, z_2, ..., z_n) \in \mathscr{H}(\bar{F})$ with $f(\alpha_1, \alpha_2, ..., \alpha_n) = a$. For any fixed $z_1, z_2, ..., z_{n-1}$ all having zero real part, f is a continuous bounded E_n valued function of z_n vanishing at infinity on the line $z_n = 1 + it$, $-\infty < t < \infty$, corresponding to points in $\partial\Omega_n$, and also on the line $z_n = it$, $-\infty < t < \infty$, corresponding to points in $\partial\Omega_{n+1}$. Since f is also a continuously bounded $\Sigma(\bar{E})$ valued function of z_n on the strip $\bar{S} = \{z_n \mid 0 \leqslant \operatorname{re} z_n \leqslant 1\}$ and analytic in $\{z_n \mid 0 < \operatorname{re} z_n < 1\}$, f must equal the Poisson integral of its boundary values and thus be a continuous E_n valued function of z_n on all of $\bar{S}$, which vanishes at infinity. Furthermore,

$$\sup_{z_n \in \bar{S}} \|f(it_1, it_2, ..., it_{n-1}, z_n)\|_{E_n}$$
$$= \sup_{t \in \mathbb{R}, j=0,1} \|f(it_1, it_2, ..., it_{n-1}, j+it)\|_{E_n} \leqslant \|f\|_{\mathscr{H}(\bar{F})}.$$

Therefore the function $g = g(z_1, z_2, ..., z_{n-1})$ defined by

$$g(z_1, z_2, ..., z_{n-1}) = f\left(z_1, z_2, z_{n-1}, \alpha_n \left(1 - \sum_{j=1}^{n-1} z_j\right) \Big/ (\alpha_n + \alpha_{n+1})\right)$$

is in the space $\mathscr{H}(\bar{E})$ and $\|g\|_{\mathscr{H}(\bar{E})} \leqslant \|f\|_{\mathscr{H}(\bar{F})}$. Since $g(\alpha_1, \alpha_2, ..., \alpha_{n-1}) = f(\alpha_1, \alpha_2, ..., \alpha_n) = a$ it follows that $[\bar{F}]_{\bar{\alpha}} \subset [\bar{E}]_{\beta}$ and indeed, by taking the infimum over all functions f as above, we obtain that $[\bar{F}]_{\bar{\alpha}} \leqslant [\bar{E}]_{\beta}$.

Conversely, if $a \in [\bar{E}]_{\beta}$ and $a = g(\alpha_1, \alpha_2, ..., \alpha_{n-1})$ where $g \in \mathscr{H}(\bar{E})$, let f be defined simply by $f(z_1, z_2, ..., z_n) = g(z_1, z_2, ..., z_{n-1})$. It is obvious that $f \in \mathscr{H}(\bar{F})$ and, furthermore, that $[\bar{E}]_{\beta} \leqslant [\bar{F}]_{\bar{\alpha}}$. ∎

We now apply Lemma 2.8 to the n-tuple $\bar{E} = A_{\bar{\Gamma}}(\Gamma)$ where we also choose $\bar{\alpha} = \bar{\theta}(\bar{\Omega}) = (Z(\Gamma_1), Z(\Gamma_2), ..., Z(\Gamma_{n-1}), Z(\Omega_n), Z(\Omega_{n+1}))$ so that $\bar{\beta} = \bar{\theta}(\bar{\Gamma}) = (Z(\Gamma_1), Z(\Gamma_2), ..., Z(\Gamma_n))$. Since $F_n = F_{n+1} = E_n = \inf_{\gamma \in \Gamma_n} A(\gamma) \leqslant \inf_{\gamma \in \Omega_j} A(\gamma)$ for $j = n, n+1$ it follows that $[\bar{E}]_{\beta} = [\bar{F}]_{\bar{\alpha}} \leqslant [A_{\bar{\Omega}}]_{\bar{\alpha}}$ which is precisely (2.6) for $\mathrm{M} = \mathrm{FL}$. A very similar argument yields (2.7) for $\mathrm{M} = \mathrm{FL}$.

Before establishing (2.6) and (2.7) for spaces generated by the Sparr K and J methods we first have to choose suitable norms for these spaces as follows:

For any $\bar{\Gamma} = \{\Gamma_1, \Gamma_2, ..., \Gamma_n\}$ with $\bar{\theta} = (\theta_1, \theta_2, ..., \theta_n)$ where $\theta_j = Z(\Gamma_j)$, $j = 1, 2, ..., n$, define

$$\|a\|_{L_{K,p}(A,\bar{\Gamma},Z)} = \left(p^{n-1} \prod_{j=1}^{n} \theta_j\right)^{1/p} \|a\|_{(A_{\bar{\Gamma}}(\Gamma))_{\bar{\theta},p;K}}$$

and

$$\|a\|_{U_{K,p}(A,\bar{\Gamma},Z)} = \left(p^{n-1} \prod_{j=1}^{n} \theta_j\right)^{1/p} \|a\|_{(A^{\bar{\Gamma}}(\Gamma))_{\bar{\theta},p;K}}$$

for $1 \leqslant p \leqslant \infty$. Similarly we take

$$\|a\|_{L_{J,p}(A,\bar{\Gamma},Z)} = \left(p'^{n-1} \prod_{j=1}^{n} \theta_j\right)^{-1/p'} \|a\|_{(A_{\bar{\Gamma}}(\Gamma))_{\bar{\theta},p;J}}$$

and

$$\|a\|_{U_{J,p}(A,\bar{\Gamma},Z)} = \left(p'^{n-1} \prod_{j=1}^{n} \theta_j\right)^{-1/p'} \|a\|_{(A^{\bar{\Gamma}}(\Gamma))_{\bar{\theta},p;J}},$$

where $1/p+1/p'=1$. (The above expressions $(p^{n-1}\prod_{j=1}^{n}\theta_j)^{1/p}$ and $(p'^{n-1}\prod_{j=1}^{n}\theta_j)^{-1/p'}$ are taken to be 1 if $p=\infty$ or $p=1$, respectively. Note that they also equal 1 for all values of p in the special case $n=1$.)

As before, the key step is to establish a "reducibility" property, namely the following "quantitative" version of Proposition 6.3 of [Sp, p. 270] (cf. also [Y, Propositions 4.4 and 4.5]).

2.10. LEMMA. *Let $\bar{E}$, $\bar{F}$, $\bar{\alpha}$, and $\bar{\beta}$ be as in the statement of Lemma 2.8 Let $1\leqslant p\leqslant\infty$. Then the spaces $\bar{E}_{\bar{\beta},p;K}$ and $\bar{F}_{\bar{\alpha},p;K}$ coincide and, for all elements a in these spaces,*

$$\left(p^{n-1}\left(\prod_{j=1}^{n-1}\alpha_j\right)(\alpha_n+\alpha_{n+1})\right)^{1/p}\|a\|_{\bar{E}_{\bar{\beta},p;K}}=\left(p^{n}\prod_{j=1}^{n+1}\alpha_j\right)^{1/p}\|a\|_{\bar{F}_{\bar{\alpha},p;K}}. \tag{2.11}$$

Similarly the spaces $\bar{E}_{\bar{\beta},p;J}$ and $\bar{F}_{\bar{\alpha},p;J}$ coincide and

$$\left(p'^{n-1}\left(\prod_{j=1}^{n-1}\alpha_j\right)(\alpha_n+\alpha_{n+1})\right)^{-1/p'}\|b\|_{\bar{E}_{\bar{\beta},p;J}}=\left(p'^{n}\prod_{j=1}^{n+1}\alpha_j\right)^{-1/p'}\|b\|_{\bar{F}_{\bar{\alpha},p;J}} \tag{2.12}$$

for all elements b of these spaces.

Proof. If $n=1$ then $\bar{E}=(E)$ and $\bar{F}=(E,E)$ and, in accordance with the convention we have adopted above (see Subsect. B), the left-hand sides of (2.11) and (2.12) equal $\|a\|_E$ and $\|b\|_E$, respectively. Since $K(1,t,a;\bar{F})=\min(1,t)\|a\|_E$, (2.11) follows by a straightforward integration. Note that for calculating $\|b\|_{\bar{F}_{\bar{\alpha},p;J}}$, the optimal choice of decomposition $b=\int_0^\infty u(t)\,dt/t$ is of the form $u(t)=\varphi(t)\,b$ where $\varphi(t)$ is a nonnegative scalar function. Via Hölder's inequality, we see that

$$\begin{aligned}\inf_{\varphi}\left(\int_0^\infty (t^{-\alpha_2}\max(1,t)\,\varphi(t)^p\,dt/t)\right)^{1/p}&\Big/\int_0^\infty\varphi(t)\,dt/t\\ =1\Big/\left(\int_0^\infty [t^{\alpha_2}/\max(1,t)]^{p'}\,dt/t\right)^{1/p'}&=(p'\alpha_1\alpha_2)^{1/p'}\end{aligned}$$

and the infimum is attained for suitable $\varphi(t)$. Since $J(1,t,u(t),\bar{F})=\max(1,t)\|u(t)\|_E$, we obtain (2.12). Thus from here on we can assume that $n\geqslant 2$.

Let us first deal with K spaces (cf. [Sp, pp. 270, 271]). For any $a\in\Sigma(\bar{E})=\Sigma(\bar{F})$ and $(t_1,t_2,\ldots,t_{n+1})\in\mathbb{R}_+^{n+1}$ we clearly have

$$K(t_1,t_2,t_{n+1},a;\bar{F})=K(t_1,t_2,\ldots,t_{n-1},\min(t_n,t_{n+1}),a;\bar{E})$$

and also

$$\|a\|^p_{\bar{F}_{\bar{\alpha},p;K}} = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \left[\prod_{j=2}^{n+1} t_j^{-\alpha_j} K(1, t_2, \ldots, t_{n+1}, a; \bar{F}) \right]^p \frac{dt_2}{t_2} \cdots \frac{dt_{n+1}}{t_{n+1}}$$

for $1 \leqslant p < \infty$ (cf. Definition 1.3 and Remark 1.5). We can now obtain (2.11) by replacing the integration with respect to the variables t_n and t_{n+1} in the above multiple integral by integration with respect to a single real variable s in accordance with the identity:

$$\int_0^\infty \int_0^\infty [t_n^{-\alpha_n} t_{n+1}^{-\alpha_{n+1}} \varphi(\min(t_n, t_{n+1}))]^p \frac{dt_n}{t_n} \frac{dt_{n+1}}{t_{n+1}}$$

$$= \frac{(\alpha_n + \alpha_{n+1})}{p\alpha_n \alpha_{n+1}} \int_0^\infty [s^{-(\alpha_n + \alpha_{n+1})} \varphi(s)]^p \frac{ds}{s} \tag{2.13}$$

which holds for any positive measurable function $\varphi(s)$. (To obtain (2.13) simply split the double integration into separate calculations on the two subsets where $t_n \leqslant t_{n+1}$ and where $t_n > t_{n+1}$, respectively.) We leave the details of the easy case $p = \infty$ to the reader.

Now we return to the case of J spaces. This is essentially the dual of the result for K spaces but we choose to give a direct proof. For any $b \in \Delta(\bar{E}) = \Delta(\bar{F})$ we clearly have

$$J(t_1, t_2, \ldots, t_{n+1}, b; \bar{F}) = J(t_1, t_2, \ldots, t_{n-1}, \max(t_n, t_{n+1}), b; \bar{E}).$$

Suppose then that $b \in \bar{F}_{\bar{\alpha},p;J}$. Then there exists a strongly measurable $\Delta(\bar{F})$ valued function u on the set $\{(1, t_2, t_3, \ldots, t_{n+1}) \mid t_j > 0, j = 2, 3, \ldots, n+1\} \subset \mathbb{R}_+^{n+1}$ such that

$$b = \int_0^\infty \cdots \int_0^\infty u(t_2, t_3, \ldots, t_{n+1}) \frac{dt_2}{t_2} \cdots \frac{dt_{n+1}}{t_{n+1}}$$

and the expression

$$\left(\int_0^\infty \cdots \int_0^\infty [t_2^{-\alpha_2} \cdots t_{n+1}^{-\alpha_{n+1}} J(1, t_2, \ldots, t_{n+1}, u(t_2, t_3, \ldots, t_{n+1}); \bar{F})]^p \right.$$
$$\left. \times \frac{dt_2}{t_2} \cdots \frac{dt_{n+1}}{t_{n+1}} \right)^{1/p}$$

is arbitrarily close to $\|b\|_{\bar{F}_{\bar{\alpha},p;J}}$. Define the function v on

$$\{(1, t_2, \ldots, t_n) \mid t_j > 0, j = 2, 3, \ldots, n\} \subset \mathbb{R}_+^n$$

by

$$v(t_2,\dots,t_n)=\int_0^{t_n}[u(t_2,\dots,t_{n-1},t_n,s)+u(t_2,\dots,t_{n-1},s,t_n)]\,\frac{ds}{s}.$$

Clearly $\int_0^\infty\cdots\int_0^\infty v(t_2,\dots,t_n)(dt_2/t_2\cdots dt_n/t_n)=b$ and also

$$\begin{aligned}
&J(1,t_2,\dots,t_n,v(t_2,\dots,t_n);\bar{E})\\
&\quad\leqslant\int_0^{t_n}[J(1,t_2,\dots,t_n,u(t_2,\dots,t_{n-1},t_n,s);\bar{E})\\
&\qquad+J(1,t_2,\dots,t_n,u(t_2,\dots,t_{n-1},s,t_n);\bar{E})]\,\frac{ds}{s}\\
&\quad=\int_0^{t_n}[J(1,t_2,\dots,t_n,s,u(t_2,\dots,t_{n-1},t_n,s);\bar{F})\\
&\qquad+J(1,t_2,\dots,t_{n-1},s,t_n,u(t_2,\dots,t_{n-1},s,t_n);\bar{F})]\,\frac{ds}{s}.
\end{aligned}$$

An application of Hölder's inequality on the measure space consisting of two copies of $[0,t_n]$ shows that the preceding integral is dominated by

$$\begin{aligned}
&\left[\int_0^{t_n}[t_n^{-\alpha_n}s^{-\alpha_{n+1}}J(1,t_2,\dots,t_n,s,u(t_2,\dots,t_n,s);\bar{F})]^p\right.\\
&\qquad\left.+[s^{-\alpha_n}t_n^{-\alpha_{n+1}}J(1,t_2,\dots,t_{n-1},s,t_n,u(t_2,\dots,t_{n-1},s,t_n);\bar{F})]^p\,\frac{ds}{s}\right]^{1/p}\\
&\quad\times\left[\int_0^{t_n}[t^{\alpha_n p'}s^{\alpha_{n+1}p'}+s^{\alpha_n p'}t_n^{\alpha_{n+1}p'}]\,\frac{ds}{s}\right]^{1/p'}
\end{aligned}$$

for all $p\in[1,\infty)$. The second factor equals $t_n^{\alpha_n+\alpha_{n+1}}[(\alpha_n+\alpha_{n+1})/\alpha_n\alpha_{n+1}p']^{1/p'}$, so an appropriate integration of the above estimates show that $b\in\bar{E}_{\beta,p;J}$. By passing to an appropriate infimum we also have

$$(\alpha_n+\alpha_{n+1})^{-1/p'}\|b\|_{E_{\beta,p;J}}\leqslant(p'\alpha_n\alpha_{n+1})^{-1/p'}\|b\|_{\bar{F}_{\bar{\alpha},p;J}}.\tag{2.14}$$

The preceding argument is essentially the same for $p=\infty$.

Conversely, if $b\in\bar{E}_{\beta,p;J}$ and $b=\int_0^\infty\cdots\int_0^\infty u(t_2,\dots,t_n)(dt_2/t_2)\cdots(dt_n/t_n)$ with

$$\left(\int_0^\infty\int_0^\infty\cdots\int_0^\infty[t_2^{-\alpha_2},\dots,t_{n-1}^{-\alpha_{n-1}}t_n^{-(\alpha_n+\alpha_{n+1})}J(1,t_2,\dots,t_n,u(t_2,\dots,t_n);\bar{E})]^p\right.$$
$$\left.\times\frac{dt_2}{t_2}\cdots\frac{dt_n}{t_n}\right)^{1/p}$$

arbitrarily close to $\|b\|_{\bar{E}_{\beta,p;J}}$, then we define v by

$$v(t_2,..., t_{n+1}) = \left(\frac{t_n^{\alpha_n} t_{n+1}^{\alpha_{n+1}}}{\max(t_n, t_{n+1})^{\alpha_n+\alpha_{n+1}}}\right)^{p'} u(t_2,..., t_{n-1}, \max(t_n, t_{n+1})) \quad (2.15)$$

for $p \in (1, \infty]$. Using the identity

$$\int_0^\infty \int_0^\infty \left(\frac{t_n^{\alpha_n} t_{n+1}^{\alpha_{n+1}}}{\max(t_n, t_{n+1})^{\alpha_n+\alpha_{n+1}}}\right)^{p'} \varphi(\max(t_n, t_{n+1})) \frac{dt_n\, dt_{n+1}}{t_n t_{n+1}}$$
$$= \frac{(\alpha_n + \alpha_{n+1})}{p'\alpha_n \alpha_{n+1}} \int_0^\infty \varphi(s) \frac{ds}{s} \quad (2.16)$$

which is in fact a variant of (2.13) and which holds for any scalar or vector valued absolutely integrable function φ, we deduce that

$$\int_0^\infty \cdots \int_0^\infty v(t_2,..., t_{n+1}) \frac{dt_2}{t_2} \cdots \frac{dt_{n+1}}{t_{n+1}} = \frac{(\alpha_n + \alpha_{n+1})}{p'\alpha_n \alpha_{n+1}} b.$$

Furthermore, if $p > \infty$,

$$[t_n^{-\alpha_n} t_{n+1}^{-\alpha_{n+1}} J(1, t_2,..., t_{n+1}, v(t_2,..., t_{n+1}); \bar{F})]^p$$
$$= [\max(t_n, t_{n+1})^{-(\alpha_n+\alpha_{n+1})} J(1, t_2,..., t_{n-1}, \max(t_n, t_{n+1}),$$
$$u(t_2,..., t_{n-1}, \max(t_n, t_{n+1})); \bar{E})]^p$$
$$\times \left(\frac{t_n^{\alpha_n} t_{n+1}^{\alpha_{n+1}}}{\max(t_n, t_{n+1})^{\alpha_n+\alpha_{n+1}}}\right)^{(p'-1)p}. \quad (2.17)$$

Since $(p'-1)p = p'$ we can invoke (2.16) again, taking the first factor on the "right-hand side" of (2.17) to be $\varphi(\max(t_n, t_{n+1}))$. After also multiplying by $(t_2^{-\alpha_2} \cdots t_{n-1}^{-\alpha_{n-1}})^p$ and integrating with respect to the remaining variables, we obtain that $b \in \bar{F}_{\bar{\alpha}, p;J}$ and

$$\left\| \frac{(\alpha_n + \alpha_{n+1})}{p'\alpha_n \alpha_{n+1}} b \right\|_{\bar{F}_{\bar{\alpha},p;J}} \leqslant \left(\frac{\alpha_n + \alpha_{n+1}}{p'\alpha_n \alpha_{n+1}}\right)^{1/p} \|b\|_{\bar{E}_{\beta,p;J}}.$$

This shows that (2.14) is in fact an equality and establishes (2.12), completing the proof of the lemma for $1 < p < \infty$.

For $p = \infty$ we simply use an alternative version of (2.17) where we (necessarily!) do not raise to the power p. The case $p = 1$ calls, however, for a different definition of the function $v(t_2,..., t_{n+1})$. Instead of (2.15) we take

$$v(t_2,..., t_{n+1}) = \frac{\chi_{[1/r,r]}(t_n/t_{n+1})}{2 \log r} u(t_2,..., t_{n-1}, \max(t_n, t_{n+1})),$$

where $r>1$ is a fixed number. Since $\int_0^\infty \cdots \int_0^\infty v(t_2,\ldots,t_{n+1})(dt_2/t_2)\cdots(dt_{n+1}/t_{n+1})=b$ we deduce that $b\in \bar{F}_{\bar{\alpha},1;J}$ and $\|b\|_{\bar{F}_{\bar{\alpha},1;J}}$ is bounded by a number arbitrarily close to

$$([(r^{\alpha_n}-1)/\alpha_n+(r^{\alpha_{n+1}}-1)/\alpha_{n+1}]/2\log r)\|b\|_{E_{\beta,1;J}}.$$

We obtain (2.12) by letting r tend to 1 (cf. also (2.14)). ∎

Lemma 2.10 can now be used in an exact analogue of the simple argument given above for Favini–Lions spaces to obtain the inclusions (2.6) and (2.7) for $\mathrm{M}=K, p$ and $\mathrm{M}=J, p$ where $1\leqslant p\leqslant\infty$.

D. *Definitions and Elementary Properties of Interpolation Spaces Obtained by "Integration" of the Infinite Family* $\{A(\gamma)\mid\gamma\in\Gamma\}$

We shall now define the "lower" spaces $\mathrm{L_M}(A,Z)$ and "upper" spaces $\mathrm{U_M}(A,Z)$ for $\mathrm{M}=\mathrm{FL}$, St.L, J,p, or K,p. These correspond to the lower and upper exponentiated integrals $\mathrm{L}(A,Z)$, $\mathrm{U}(A,Z)$ of our scalar model. We set $\mathrm{L_M}(A,Z)=\sup_{\bar\Gamma\in\mathscr{P}}\mathrm{L_M}(A,\bar\Gamma,Z)$ and $\mathrm{U_M}(A,Z)=\inf_{\bar\Gamma\in\mathscr{P}}\mathrm{U_M}(A,\bar\Gamma,Z)$. Analogously to the definitions above of $\inf_{\gamma\in\Gamma_j}A(\gamma)$ and $\sup_{\gamma\in\Gamma_j}A(\gamma)$ this means that $\mathrm{U_M}(A,Z)$ is the Banach space of all elements $a\in\bigcap_{\bar\Gamma\in\mathscr{P}}\mathrm{U_M}(A,\bar\Gamma,Z)$ for which $\|a\|_{\mathrm{U_M}(A,Z)}=\sup_{\bar\Gamma\in\mathscr{P}}\|a\|_{\mathrm{U_M}(A,\bar\Gamma,Z)}<\infty$, and $\mathrm{L_M}(A,Z)$ is the Banach space of all elements $a\in\mathscr{U}$ of the form $a=\Sigma_{\bar\Gamma\in\mathscr{P}}u(\bar\Gamma)$ (convergence in $\mathscr{U}$) where $u(\bar\Gamma)\in\mathrm{L_M}(A,\bar\Gamma,Z)$ for each $\bar\Gamma\in\mathscr{P}$ and $\Sigma_{\bar\Gamma\in\mathscr{P}}\|u(\bar\Gamma)\|_{\mathrm{L_M}(A,\bar\Gamma,Z)}<\infty$. $\mathrm{L_M}(A,Z)$ is normed by $\|a\|_{\mathrm{L_M}(A,Z)}=\inf\Sigma_{\bar\Gamma\in\mathscr{P}}\|u(\bar\Gamma)\|_{\mathrm{L_M}(A,\bar\Gamma,Z)}$, where the infimum is taken over all representations $\Sigma_{\bar\Gamma\in\mathscr{P}}u(\bar\Gamma)$ of the above sort for a.

Analogously to the scalar case where the relations (2.4) and (2.5) follow from (2.2) and (2.3), we can now use (2.7) to obtain that $\|a\|_{\mathrm{U_M}(A,Z)}=\lim_{\bar\Gamma\in\mathscr{P}}\|a\|_{\mathrm{U_M}(A,\bar\Gamma,Z)}$ for all $a\in\mathrm{U_M}(A,Z)$. This could be expressed symbolically by writing $\mathrm{U_M}(A,Z)=\lim_{\bar\Gamma\in\mathscr{P}}\mathrm{U_M}(A,\bar\Gamma,Z)$ (cf. (2.5)). The analogous result for lower spaces, which corresponds in some sense to the formula $\mathrm{L_M}(A,Z)=\lim_{\bar\Gamma\in\mathscr{P}}\mathrm{L_M}(A,\bar\Gamma,Z)$ (cf. (2.4)), is a little more complicated and may be stated as follows:

> For each $a\in\mathrm{L_M}(A,Z)$ there exists a sequence $(a_n)_{n=1}^\infty$, $a_n\in\bigcup_{\bar\Gamma\in\mathscr{P}}\mathrm{L_M}(A,\bar\Gamma,Z)$, such that $\|a-a_n\|_{\mathrm{L_M}(A,Z)}\to 0$ and $\lim_{n\to\infty}(\lim_{\bar\Gamma\in\mathscr{P}}\|a_n\|_{\mathrm{L_M}(A,\bar\Gamma,Z)})=\|a\|_{\mathrm{L_M}(A,Z)}$. (2.18)

To establish (2.18) let us first introduce the notation $\Lambda_\mathrm{M}(A,Z)=\bigcup_{\bar\Gamma\in\mathscr{P}}\mathrm{L_M}(A,\bar\Gamma,Z)$ and $\|b\|_{\Lambda_\mathrm{M}}=\inf_{\bar\Gamma\in\mathscr{P}}\|b\|_{\mathrm{L_M}(A,\bar\Gamma,Z)}$ for each $b\in\Lambda_\mathrm{M}(A,Z)$. By (2.6) $\|b\|_{\Lambda_\mathrm{M}}=\lim_{\bar\Gamma\in\mathscr{P}}\|b\|_{\mathrm{L_M}(A,\bar\Gamma,Z)}$ and Λ_M is clearly a normed space satisfying $\Lambda_\mathrm{M}\leqslant L_\mathrm{M}$.

Given $a\in\mathrm{L_M}(A,Z)$, for each positive integer n there is a decomposition $a=\sum_{\bar\Gamma\in\mathscr{P}}u_n(\bar\Gamma)$ for which $\sum_{\bar\Gamma\in\mathscr{P}}\|u_n(\bar\Gamma)\|_{\mathrm{L_M}(A,\bar\Gamma,Z)}\leqslant\|a\|_{\mathrm{L_M}(A,Z)}+1/n$. We define $a_n=\sum_{\bar\Gamma\in\mathscr{P}_n}u_n(\bar\Gamma)$ where $\mathscr{P}_n$ is a finite subset of $\mathscr{P}$ such that

$\sum_{\bar{\Gamma} \in \mathscr{P} \setminus \mathscr{P}_n} \|u_n(\bar{\Gamma})\|_{\mathrm{L_M}(A, \bar{\Gamma}, Z)} \leqslant 1/n$. Thus $\|a - a_n\|_{\mathrm{L_M}(A,Z)} \to 0$ and so $\lim_{n \to \infty} \|a_n\|_{\mathrm{L_M}(A,Z)} = \|a\|_{\mathrm{L_M}(A,Z)}$. But $\|a_n\|_{\mathrm{L_M}(A,Z)} \leqslant \|a_n\|_{\Lambda_{\mathrm{M}}} \leqslant \sum_{\bar{\Gamma} \in \mathscr{P}} \|u_n(\bar{\Gamma})\|_{\Lambda_{\mathrm{M}}} \leqslant \|a\|_{\mathrm{L_M}(A,Z)} + 1/n$, from which we see that the sequence $(a_n)_{n=1}^{\infty}$ has the properties required in (2.18).

2.19. *Remark.* The above argument also shows that the unit ball of Λ_{M} is a dense subset of the unit ball of $\mathrm{L_M}$. However, in general, Λ_{M} is not complete and Λ_{M} may be smaller than $\mathrm{L_M}$ (see Remark 2.39). We do not know whether the norms of Λ_{M} and $\mathrm{L_M}$ always coincide on Λ_{M} (i.e., whether $\mathrm{L_M}$ is the completion of Λ_{M}).

The preceding discussion leads us to define "measurability" of the "function" $A(\gamma)$ as follows:

2.20. Definition. Let M be one of the methods FL, St.L, K,p, or J,p. We shall say that a bounded family $\{A(\gamma) \mid \gamma \in \Gamma\}$ is *M, Z measurable* if $\mathrm{L_M}(A, Z) = \mathrm{U_M}(A, Z)$. In this case we can use the notation $I_{\mathrm{M}}(A, Z)$ for $\mathrm{L_M}(A, Z) = \mathrm{U_M}(A, Z)$.

We defer further discussion of M, Z measurability and the spaces $I_{\mathrm{M}}(A, Z)$ to Subsection F. We now consider interpolation properties of the various spaces we have defined.

2.21. Theorem. *Let $\{A(\gamma) \mid \gamma \in \Gamma\}$ and $\{B(\gamma) \mid \gamma \in \Gamma\}$ each be bounded families of Banach spaces on Γ. Let $\mathscr{U}$ and $\mathscr{V}$ denote the fixed Banach spaces such that $A(\gamma) \leqslant \mathscr{U}$ and $B(\gamma) \leqslant \mathscr{V}$ for all $\gamma \in \Gamma$. Let T be a bounded linear operator from $\mathscr{U}$ into $\mathscr{V}$ whose restriction to $A(\gamma)$ is a map into $B(\gamma)$ with $\|T\|_{A(\gamma), B(\gamma)} \leqslant N(\gamma)$ for all $\gamma \in \Gamma$. Suppose that $N(\gamma)$ is bounded above by a positive constant and is measurable with respect to Z on Γ. Then T maps $\mathrm{L_M}(A, Z)$ into $\mathrm{L_M}(B, Z)$, $\Lambda_{\mathrm{M}}(A, Z)$ into $\Lambda_{\mathrm{M}}(B, Z)$ and also $\mathrm{U_M}(A, Z)$ into $\mathrm{U_M}(B, Z)$, and in each case its norm does not exceed* $\exp \int_\Gamma \log N(\gamma)\, dZ(\gamma)$.

Proof. For each partition $\bar{\Gamma} = \{\Gamma_1, \Gamma_2, ..., \Gamma_n\} \in \mathscr{P}$ and each $j = 1, 2, ..., n$, T maps $\inf_{\gamma \in \Gamma_j} A(\gamma)$ into $\inf_{\gamma \in \Gamma_j} B(\gamma)$ and also $\sup_{\gamma \in \Gamma_j} A(\gamma)$ into $\sup_{\gamma \in \Gamma_j} B(\gamma)$, in each case with norm not exceeding $\sup_{\gamma \in \Gamma_j} N(\gamma) = N_j$. Thus, according to whether $M =$ FL, St.L, K,p, or J,p, we invoke the interpolation theorem of [Fa, p. 246; C3, p. 216, Theorem 4.1(2); Sp, p. 260; Sp, p. 262] and obtain that T maps $\mathrm{L_M}(A, \bar{\Gamma}, Z)$ into $\mathrm{L_M}(B, \bar{\Gamma}, Z)$ and also $\mathrm{U_M}(A, \bar{\Gamma}, Z)$ into $\mathrm{U_M}(B, \bar{\Gamma}, Z)$, in each case with norm not exceeding $N_1^{\theta_1} N_2^{\theta_2} \cdots N_n^{\theta_n} = \exp \int_\Gamma \log N^{\bar{\Gamma}}(\gamma)\, dZ(\gamma)$. (Here, as before, $\theta_j = Z(\Gamma_j)$, $j = 1, 2, ..., n$.)

Given any $\varepsilon > 0$, if $a \in \Lambda_{\mathrm{M}}(A, Z)$ choose a sufficiently fine partition $\bar{\Gamma}$ so that

$$\exp \int_\Gamma \log N^{\bar{\Gamma}}(\gamma)\, dZ(\gamma) \leqslant (1 + \varepsilon) \exp \int_\Gamma \log N(\gamma)\, dZ(\gamma) \tag{2.22}$$

and also so that $\|a\|_{L_M(A,\bar{\Gamma},Z)} \leqslant (1+\varepsilon)\|a\|_{\Lambda_M(A,Z)}$. Then $Ta \in L_M(B,\bar{\Gamma},Z)$ and therefore $Ta \in \Lambda_M(B,Z)$ with norm $\|Ta\|_{\Lambda_M(B,Z)}$ not exceeding

$$\|Ta\|_{L_M(B,\bar{\Gamma},Z)} \leqslant (1+\varepsilon)^2 \exp \int_\Gamma \log N(\gamma)\, dZ(\gamma) \|a\|_{\Lambda_M(A,Z)}.$$

Alternatively if $a \in L_M(A,Z)$ we may write

$$a = \sum_{\bar{\Omega} \in \mathscr{P}} u(\bar{\Omega}), \qquad \text{where} \quad \sum_{\bar{\Omega} \in \mathscr{P}} \|u(\bar{\Omega})\|_{L_M(A,\bar{\Omega},Z)} \leqslant (1+\varepsilon)\|a\|_{L_M(A,Z)}.$$

In view of (2.6) we may assume that each of the nonzero elements $u(\bar{\Omega})$ appearing in the above sum corresponds to a partition $\bar{\Omega}$ satisfying $\bar{\Omega} \succ \bar{\Gamma}$ where $\bar{\Gamma}$ is chosen to satisfy (2.22). (If not, we simply "permute" the terms of the sum $\Sigma u(\bar{\Omega})$ so that each nonzero $u(\bar{\Omega})$ is now associated with a possibly different partition which is a common refinement of $\bar{\Omega}$ and $\bar{\Gamma}$.) Since T is bounded from $\mathscr{U}$ into $\mathscr{V}$, $Ta = \Sigma_{\bar{\Omega} \in \mathscr{P}} Tu(\bar{\Omega})$ (convergence in $\mathscr{V}$) and

$$\sum_{\bar{\Omega} \in \mathscr{P}} \|Tu(\bar{\Omega})\|_{L_M(B,\bar{\Omega},Z)} \leqslant \sum_{\bar{\Omega} \in \mathscr{P}} \exp \int_\Gamma \log N^{\bar{\Omega}}(\gamma)\, dZ(\gamma)\, \|u(\bar{\Omega})\|_{L_M(A,\bar{\Omega},Z)}$$
$$\leqslant (1+\varepsilon)^2 \exp \int_\Gamma \log N(\gamma)\, dZ(\gamma) \|a\|_{L_M(A,Z)}.$$

It follows that $Ta \in L_M(B,Z)$ and satisfies the required norm estimate.

Finally, if $a \in U_M(A,Z)$ then, since $\exp \int_\Gamma N^{\bar{\Gamma}}(\gamma)\, dZ(\gamma) \leqslant \sup_{\gamma \in \Gamma} N(\gamma) < \infty$, we have $Ta \in U_M(B,Z)$. This time we shall choose $\bar{\Gamma}$ so that (2.22) holds and also $\|Ta\|_{U_M(B,Z)} \leqslant (1+\varepsilon)\|Ta\|_{U_M(B,\bar{\Gamma},Z]}$. The rest of the proof is obvious. ∎

2.23. *Remark.* One might expect that (cf. [C3]) it could perhaps be possible to extend the construction of the spaces $L_M(A,Z)$, $U_M(A,Z)$ to the case where $A(\gamma) \leqslant k(\gamma)\,\mathscr{U}$ (i.e., $k(\gamma)\|a\|_{\mathscr{U}} \leqslant \|a\|_{A(\gamma)}$ for all $a \in A(\gamma)$) where $\int_\Gamma |\log k(\gamma)|\, dZ(\gamma) < \infty$, and subsequently to also obtain a version of the preceding theorem which requires only that $\int_\Gamma \log^+ N(\gamma)\, dZ(\gamma) < \infty$ rather than the boundedness of $N(\gamma)$. (cf. [C3, Theorem 4.1], cf. also [J2]). The following simple example indicates that some problems can arise here:

Let $a(\gamma)$ be a real measurable function with $a(\gamma) \geqslant 1$. We shall take $A(\gamma) = \alpha(\gamma)\,\mathbb{C}$ (i.e., one-dimensional space with $\|x\|_{A(\gamma)} = a(\gamma)|x|$). Thus $\{A(\gamma) \mid \gamma \in \Gamma\}$ is a bounded family. If $a(\gamma)$ is bounded, then $L_M = U_M = e^{\int \log a(\gamma)\, dZ(\gamma)}\mathbb{C}$. (This can be shown by applying Theorem 2.21 with $B(\gamma) = \mathbb{C}$ to the cases where T is the identity operator from $A(\gamma)$ to $B(\gamma)$ or alternatively from $B(\gamma)$ to $A(\gamma)$.) However, suppose that $\operatorname{ess\,sup} a(\gamma) = \infty$ but

$\int_\Gamma \log a(\gamma)\, dZ(\gamma) < \infty$. Then, since $\inf_{\gamma \in \Gamma_j} A(\gamma) = (\sup_{\gamma \in \Gamma_j} A(\gamma))\, \mathbb{C} = \{0\}$ if $\sup_{\gamma \in \Gamma_j} a(\gamma) = \infty$, at least one of the spaces $\inf_{\gamma \in \Gamma_j} A(\gamma)$ degenerates to $\{0\}$ for every $\bar{\Gamma} = \{\Gamma_1, ..., \Gamma_n\} \in \mathscr{P}$. Consequently $\mathrm{L_M}(A, \bar{\Gamma}, Z) = \{0\}$. (On the other hand $\mathrm{U_M}(A, Z) = e^{\int \log a(\gamma)\, dZ(\gamma)}\mathbb{C}$ as before.) Considering the identity operator T: $B(\gamma) \to A(\gamma)$ which has norm $N(\gamma) = a(\gamma)$ we see that Theorem 2.21 does not hold for unbounded $N(\gamma)$. It seems then that a successful variant of the theory which applies to unbounded $N(\gamma)$ would require a different definition of $\mathrm{L_M}(A, Z)$ or perhaps other postulates relating to a nontrivial space $\mathscr{A}$ contained in all $A(\gamma)$ (cf. the "log-intersection" space of [C3]; cf. also Theorem 2.41).

The inclusions $(A_1, A_2)_{\theta,1} \subset [A_1, A_2]_\theta \subset (A_1, A_2)_{\theta,\infty}$, which generalize to the case of n-tuples as shown in Theorems 1.15, 1.17, and 1.23, can be further extended to the context of bounded families. Indeed from the above theorems for n-tuples it follows immediately that:

$$\begin{aligned}
\mathrm{L}_{J,1}(A, Z) &\leqslant \mathrm{L_{FL}}(A, Z), & U_{J,1}(A, Z) &\leqslant \mathrm{U_{FL}}(A, Z),\\
\mathrm{L_{FL}}(A, Z) &\leqslant \mathrm{L}_{K,\infty}(A, Z), & \mathrm{U_{FL}}(A, Z) &\leqslant \mathrm{U}_{K,\infty}(A, Z),\\
\mathrm{L_{FL}}(A, P_z) &\leqslant \mathrm{L_{St.L}}(A, P_z), & \mathrm{U_{FL}}(A, P_z) &\leqslant \mathrm{U_{St.L}}(A, P_z),\\
\mathrm{L_{St.L}}(A, P_z) &\leqslant \mathrm{L}_{K,\infty}(A, P_z), & \mathrm{U_{St.L}}(A, P_z) &\leqslant \mathrm{U}_{K,\infty}(A, P_z).
\end{aligned}$$

One may also seek generalizations of the inclusions $(A_1, A_2)_{\theta,p} \subset (A_1, A_2)_{\theta,q}$ which hold for $1 \leqslant p \leqslant q \leqslant \infty$. We present some partial results in this direction.

2.24. PROPOSITION. *For any bounded family $\{A(\gamma) \mid \gamma \in \Gamma\}$ and any probability measure Z on Γ, the inclusions*

$$\mathrm{L}_{K,p}(A, Z) \leqslant \mathrm{L}_{K,\infty}(A, Z), \qquad U_{K,p}(A, Z) \leqslant \mathrm{U}_{K,\infty}(A, Z), \tag{2.25}$$

and

$$\mathrm{L}_{J,1}(A, Z) \leqslant \mathrm{L}_{J,p}(A, Z), \qquad \mathrm{U}_{J,1}(A, Z) \leqslant \mathrm{U}_{J,p}(A, Z) \tag{2.26}$$

hold for all p, $1 \leqslant p < \infty$.

Proof. To obtain (2.25) it suffices to show that, for any Banach n-tuple $\bar{A}$ and for all $a \in \bar{A}_{\theta,p;K}$,

$$\|a\|_{\bar{A}_{\theta,\infty;K}} \leqslant \left(p^{n-1} \prod_{j=1}^{n} \theta_j \right)^{1/p} \|a\|_{\bar{A}_{\theta,p;K}}. \tag{2.27}$$

To establish (2.27) we begin with the inequality

$$\min(s_1/t_1, s_2/t_2, ..., s_n/t_n)\, K(t_1, t_2, ..., t_n, a; \bar{A}) \leqslant K(s_1, s_2, ..., s_n, a; \bar{A})$$

(cf. [Sp, p. 252]). Multiply both sides by $\prod_{j=1}^{n} s_j^{-\theta_j} = \prod_{j=1}^{n} t_j^{-\theta_j}(s_j/t_j)^{-\theta_j}$, raise them to the power p and integrate on the set $E = \{(s_1, s_2, ..., s_n) \mid s_1 = 1,\ 0 < s_j < \infty,\ j = 2, ..., n\}$ with respect to $d\mu(\bar{s}) = (ds_2/s_2)(ds_3/s_3) \cdots (ds_n/s_n)$. We encounter the integral $I = \int_E [\prod_{j=1}^{n} s_j^{-\theta_j} \min(1, s_2, ..., s_n)]^p (ds_2/s_2) \cdots (ds_n/s_n)$ whose straightforward (if slightly tedious) calculation is already implicit in Lemma 2.10 (see (2.11)). In fact if $A(\gamma) = \mathbb{C}$ for all $\gamma \in \Gamma$ then $\mathrm{L_M}(A, \bar{\Gamma}, Z) = \mathrm{U_M}(A, \bar{\Gamma}, Z)$ for all $\bar{\Gamma} \in \mathscr{P}$ and it follows from (2.6) and (2.7) that $\mathrm{L_M}(A, \bar{\Gamma}, Z) = \mathrm{L_M}(A, \bar{\Omega}, Z)$ for all $\bar{\Omega} \succ \bar{\Gamma}$. Thus, for $M = K, p$, $\bar{\Gamma} = \{\Gamma\}$ and $\bar{\Omega} = \{\Omega_1, \Omega_2, ..., \Omega_n\}$ with $Z(\Omega_j) = \theta_j$, we obtain that $\|1\|_{\mathrm{L_M}(A, \bar{\Omega}, Z)} = \|1\|_{\mathrm{L_M}(A, \bar{\Gamma}, Z)}$. Since $K(t_1, t_2, ..., t_n, 1) = \min(t_1, t_2, ..., t_n)$ this is equivalent to $(p^{n-1} \prod_{j=1}^{n} \theta_j)^{1/p} I^{1/p} = 1$. Thus $I = 1/(p^{n-1} \prod_{j=1}^{n} \theta_j)$ and so (2.27) and then also (2.25) follow immediately.

Rather than obtaining (2.26) by dualizing, we shall deduce if from the inequality

$$\left(p'^{n-1} \prod_{j=1}^{n} \theta_j\right)^{-1/p'} \|a\|_{\bar{A}_{\theta,p;J}} \leqslant \|a\|_{\bar{A}_{\theta,1;J}} \tag{2.28}$$

which will now be established for all $a \in \bar{A}_{\theta,1;J}$. We represent each such a in the form $a = \int_E u(\bar{t})\, d\mu(\bar{t})$ where $\int_E \bar{t}^{-\theta} J(\bar{t}, u(\bar{t}); \bar{A})\, d\mu(\bar{t})$ is arbitrarily close to $\|a\|_{\bar{A}_{\theta,1;J}}$. (Here E and μ are as above and the notation is thus a trivial modification of that in Definition 1.6, (cf. Remark 1.9(ii).) It will be convenient to use the notation (cf. [Sp]) $\bar{s}/\bar{t} = (s_1/t_1,\ s_2/t_2, ..., s_n/t_n)$ and to denote $\min(\bar{s}) = \min(s_1, s_2, ..., s_n)$, for each $\bar{s} = (s_1, s_2, ..., s_n)$, $\bar{t} = (t_1, ..., t_n)$ in $\mathbb{R}_+^n$.

We define the $\Delta(\bar{A})$ valued function.

$$v(\bar{t}) = \int_E [(\bar{s}/\bar{t})^{-\theta} \min(\bar{s}/\bar{t})]^{p'} u(\bar{s})\, d\mu(\bar{s}).$$

From the calculation of I above we have that $\int_E v(\bar{t})\, d\mu(\bar{t}) = (p'^{n-1} \prod_{j=1}^{n} \theta_j)^{-1} a$. Furthermore,

$$\begin{aligned}
&\bar{t}^{-\theta} J(\bar{t}, v(\bar{t}); \bar{A}) \\
&\quad \leqslant \int_E \bar{t}^{-\theta} J(\bar{t}, u(\bar{s}); \bar{A}) [(\bar{s}/\bar{t})^{-\theta} \min(\bar{s}/\bar{t})]^{p'}\, d\mu(\bar{s}) \\
&\quad \leqslant \int_E \bar{s}^{-\theta} J(\bar{s}, u(\bar{s}); \bar{A}) [(\bar{s}/\bar{t})^{-\theta} \min(\bar{s}/\bar{t})]^{p'-1}\, d\mu(\bar{s})
\end{aligned}$$

(since $\min(\bar{s}/\bar{t})\, J(\bar{t}, u(\bar{s}); \bar{A}) \leqslant J(\bar{s}, u(\bar{s}); \bar{A})$.)

By Young's inequality we deduce that

$$\left(\int_E (\bar{t}^{-\theta} J(\bar{t}, v(\bar{t}); \bar{A})^p \, d\mu(\bar{t})\right)^{1/p}$$

$$\leqslant \int_E (\bar{s}^{-\theta} J(\bar{s}, u(\bar{s}); \bar{A}) \, d\mu(s) \left(\int_E (\bar{s}^{-\theta} \min(\bar{s}))^{(p'-1)p} \, d\mu(s)\right)^{1/p}.$$

This in turn implies that $(p'^{n-1} \prod_{j=1}^n \theta_j)^{-1} \|a\|_{\bar{A}_{\theta,p;J}} \leqslant \|a\|_{\bar{A}_{\theta,1;J}} ([(p'-1)p]^{n-1} \prod_{j=1}^n \theta_j)^{-1/p}$ which immediately yields (2.28). ▮

2.29. *Remark.* By considering the special case $A_1 = A_2 = \cdots = A_n$ for which $K(t_1, t_2, ..., t_n, a; \bar{A}) = \min(t_1, t_2, ..., t_n)\|a\|_{A_1}$ and for which also $\|a\|_{\bar{A}_{\theta,p;J}} = (p'^{n-1} \prod_{j=1}^n \theta_j)^{1/p'} \|a\|_{A_1}$ we see that the constants in the inequalities (2.27) and (2.28) are best possible. It is natural to conjecture that, analogously to (2.25) and (2.26), similar inclusions may hold between the spaces $\mathrm{L}_{K,p}(A, Z)$ and $\mathrm{L}_{K,q}(A, Z)$, etc. and also between the spaces $\mathrm{L}_{J,p}(A, Z)$ and $L_{J,q}(A, Z)$, etc. for all $1 \leqslant p \leqslant q \leqslant \infty$. Here again the above special case shows that the norm of the inclusion map cannot be less than 1. For an analogue of (2.27) for $q < \infty$ in the case of couples see [BL, p. 84, note 3.14.4].

We can also consider the possibility of generalizing the inclusions $\bar{A}_{\theta,p;J} \subset \bar{A}_{\theta,p;K}$ of [Sp, Proposition 5.1, p. 265]. An examination of the constant appearing in the proof of that proposition shows that in fact it implies $\mathrm{L}_{J,p}(A, Z) \leqslant \mathrm{L}_{K,p}(A, Z)$ and $\mathrm{U}_{J,p}(A, Z) \leqslant \mathrm{U}_{K,p}(A, Z)$ for $p = 1$ and $p = \infty$. We do not know whether this result is true for other values of p.

2.30. *Remark.* For some purposes in real interpolation of n-tuples $\bar{A}$ it is convenient to replace the J- and K-functionals by their "l^q" counterparts:

$$J_q(\bar{t}, a; \bar{A}) = \left(\sum_{j=1}^n (t_j \|a\|_{A_j})^q\right)^{1/q},$$

$$K_q(\bar{t}, a; \bar{A}) = \inf\left\{\left(\sum_{j=1}^n (t_j \|a_j\|_{A_j})^q\right)^{1/q} \,\middle|\, a = \sum_{j=1}^n a_j\right\}$$

for some $q \in [1, \infty]$. (For example, in Sect. 3 when dealing with a triple of Hilbert spaces we take $q = 2$.)

For fixed n the spaces $\bar{A}_{\theta,p;J_q}$ and $\bar{A}_{\theta,p;K_q}$ obtained using these modified functionals coincide with $\bar{A}_{\theta,p;J}$ and $\bar{A}_{\theta,p;K}$, respectively, to within equivalence of norms. However, if we wish to define infinite family versions of these spaces $\mathrm{L}_\mathrm{M}(A, Z)$, $\mathrm{U}_\mathrm{M}(A, Z)$, $I_\mathrm{M}(A, Z)$ where $\mathrm{M} = J_q, p$ or K_q, p it is necessary to change the constants in the definitions of $\mathrm{L}_\mathrm{M}(A, \bar{\Gamma}, Z)$ and $\mathrm{U}_\mathrm{M}(A, \bar{\Gamma}, Z)$.

As in the cases $\mathrm{M} = K,p$, $\mathrm{M} = J,p$ we are guided by the need for an analogue of Lemma 2.10 to hold so that $\mathrm{L_M}(A, \bar{\Gamma}, Z) = \mathrm{U_M}(A, \bar{\Gamma}, Z) = A$ isometrically if $A(\gamma) = A$ is constant. It turns out that, for example, in the case $\mathrm{M} = K_q, p$ we must take

$$\|a\|_{\mathrm{L}_{K_q,p}(A,\bar{\Gamma},Z)} = \left(q'^{n-1} \Gamma(p/q') \Big/ \prod_{j=1}^{n} \Gamma(p\theta_j/q') \right)^{1/p} \|a\|_{(A_{\Gamma}(\bar{\Gamma}))_{\theta,p;K_q}}$$

and similarly for $U_{K_q,p}(A, \bar{\Gamma}, Z)$. (Note that apart from its other preceding roles Γ here also stands for the Euler gamma function.)

E. *An Example*

Let us consider a simple example with weighted l^1 spaces. Specifically let $A(\gamma) = l^1_{w(\gamma)}$, with $\|(x_m)_{m=1}^{\infty}\|_{A(\gamma)} = \sum_{m=1}^{\infty} |x_m|\, w_m(\gamma)$, where $(w_m(\gamma))_{m=1}^{\infty}$ is a sequence of positive measurable functions on Γ.

To have a bounded family we shall require that $\inf_{\gamma \in \Gamma} w_m(\gamma) = u_m > 0$ for each m. Thus we can take $\mathscr{U} = l^1_u$ where $u = (u_m)_{m=1}^{\infty}$. In order to avoid the sort of problems encountered in Remark 2.21 we shall also require that $\sup_{\gamma \in \Gamma} w_m(\gamma) = v_m < \infty$ for each m, and so $l^1_v \leqslant A(\gamma)$ for all γ where $v = (v_m)_{m=1}^{\infty}$.

We shall show that $A(\gamma)$ is M, Z measurable for each of the methods $\mathrm{M} = \mathrm{FL}$, St.L, J,1, and K,1 and that in each of these cases $I_{\mathrm{M}}(A, Z) = l^1_w$ where the weight sequence $w = (w_m)_{m=1}^{\infty}$ is given by $w_m = \exp \int_{\Gamma} \log w_m(\gamma)\, dZ(\gamma)$.

For each $\bar{\Gamma} = \{\Gamma_1, \Gamma_2, ..., \Gamma_n\} \in \mathscr{P}$ let $u_m(j) = \inf_{\gamma \in \Gamma_j} w_m(\gamma)$ and $v_m(j) = \sup_{\gamma \in \Gamma_j} w_m(\gamma)$ for $j = 1, 2, ..., n$, and denote $u(j) = (u_m(j))_{m=1}^{\infty}$, $v(j) = (v_m(j))_{m=1}^{\infty}$. Let $\bar{B}_{\Gamma}$ and $\bar{B}^{\Gamma}$ be the n-tuples $(l^1_{v(1)}, l^1_{v(2)}, ..., l^1_{v(n)})$ and $(l^1_{u(1)}, l^1_{u(2)}, ..., l^1_{u(n)})$, respectively. Let $\bar{\theta} = (\theta_j)_{j=1}^{n}$ where $\theta_j = Z(\Gamma_j)$. Then

$$\begin{aligned} l^1_v &\leqslant (\bar{B}_{\Gamma})_{\theta,1;J} \leqslant \mathrm{L_M}(A, \bar{\Gamma}, Z) \leqslant \mathrm{L_M}(A, Z) \\ &\leqslant \mathrm{U_M}(A, Z) \leqslant \mathrm{U_M}(A, \bar{\Gamma}, Z) \leqslant \prod_{j=1}^{n} \theta_j (\bar{B}^{\Gamma})_{\theta,1;K} + (\bar{B}^{\Gamma})_{[z],\Gamma} \\ &\leqslant l^1_u \end{aligned} \tag{2.31}$$

for each of the methods $\mathrm{M} = \mathrm{FL}$, St.L, J,1, and K,1. (This follows from Theorems 1.15 and 1.17 and the proof of Sparr discussed in Remark 2.29. If $\mathrm{M} \neq$ St. L and (Γ, Z) is not a contour equipped with harmonic measure then $(\bar{B}^{\Gamma})_{[z],\Gamma}$ will be interpreted as $(\bar{B}^{\Gamma})_{[0],\bar{T}}$ where $\bar{T} = \{T_1, T_2, ..., T_n\}$ is a decomposition of the unit circle into disjoint arcs T_j of lengths $2\pi\theta_j$, $j = 1, 2, ..., n$.)

Our next step is to show that

$$l^1_{v(\Gamma)} \leqslant (\bar{B}_{\Gamma})_{\theta,1;J}, \tag{2.32}$$

$$(\bar{B}^{\Gamma})_{[z],\Gamma} \leqslant l^1_{u(\Gamma)}, \tag{2.33}$$

and

$$\prod_{j=1}^{n} \theta_j(\bar{B}^{\Gamma})_{\theta,1;K} \leqslant l^1_{u(\Gamma)}, \tag{2.34}$$

where $v(\Gamma) = (v_m(\Gamma))_{m=1}^{\infty}$ and $u(\Gamma) = u_m(\Gamma))_{m=1}^{\infty}$ are defined by $v_m(\Gamma) = \prod_{j=1}^{n} (v_m(j))^{\theta_j}$, $u_m(\Gamma) = \prod_{j=1}^{n}(u_m(j))^{\theta_j}$. (It will be clear from subsequent calculations or the case where $u(j) = v(j)$ that these inclusions are in fact equalities.)

We first establish (2.32). Let the set E and the measure μ be as in the proof of Proposition 2.24. Let $\alpha = (\alpha_2, \alpha_3, ..., \alpha_n)$ denote an $(n-1)$-dimensional multi-index where α_j may assume both positive and negative integer values. Suppose that $x = (x_m)_{m=1}^{\infty} \in l^1_{v(\Gamma)}$ and let $\lambda > 1$ be a fixed number. For each multi-index α define the set of integers

$$M_\alpha = \{m \mid \lambda^{\alpha_j} < v_m(j)/v_m(1) \leqslant \lambda^{\alpha_j + 1}, j = 2, 3, ..., n\}$$

and the subset of E

$$E_\alpha = \{\bar{t} = (1, t_2, t_3, ..., t_n) \mid \lambda^{-\alpha_j - 1} \leqslant t_j < \lambda^{-\alpha_j}, j = 2, 3, ..., n\}.$$

We define a sequence valued function $y(\bar{t}) = (y_m(\bar{t}))_{m=1}^{\infty}$ on E by taking $y_m(\bar{t}) = x_m \chi_{E_\alpha}(\bar{t})/(\log \lambda)^{n-1}$ for each $m \in M_\alpha$. Clearly $\int_E y_m(\bar{t})\, d\mu(\bar{t}) = x_m$ for each m, meaning that $\int_E y(\bar{t})\, d\mu(\bar{t}) = x$. (The required absolute integrability of $y(\bar{t})$ on compact subsets of E (Definition 1.6) is assured since each such subset is contained in the union of finitely many sets E_α and on each of these the constant value assumed by $y(\bar{t})$ will be shown to be in $\Delta(\bar{B}_\Gamma)$.)

For each $\bar{t} \in E_\alpha$ (so that $t_1 = 1$) we have

$$\begin{aligned}
&\bar{t}^{-\theta} J(\bar{t}, y(\bar{t}); \bar{B}_\Gamma) \\
&\quad = (\log \lambda)^{1-n} \prod_{j=1}^{n} t_j^{-\theta_j} \max_{j=1}^{n} \left(t_j \sum_{m \in M_\alpha} |x_m|\, v_m(j) \right) \\
&\quad \leqslant (\log \lambda)^{1-n} \prod_{j=2}^{n} \lambda^{(\alpha_j+1)\theta_j} \max_{j=1}^{n} \sum_{m \in M_\alpha} |x_m|\, \lambda^{-\alpha_j} \lambda^{\alpha_j+1} v_m(1) \\
&\quad = (\log \lambda)^{1-n} \lambda^{(\sum_{j=2}^{n} \theta_j)+1} \sum_{m \in M_\alpha} |x_m| \prod_{j=2}^{n} \lambda^{\alpha_j \theta_j} v_m(1) \\
&\quad \leqslant (\log \lambda)^{1-n} \lambda^2 \sum_{m \in M_\alpha} |x_m|\, v_m(\Gamma).
\end{aligned}$$

Thus $\int_{E_\alpha} \bar{t}^{-\theta} J(\bar{t}, y(\bar{t}); \bar{B}_\Gamma)\, d\mu(\bar{t}) \leqslant \lambda^2 \sum_{m \in M_\alpha} |x_m|\, v_m(\Gamma)$. Summing over all the possible values of α and bearing in mind that λ can be chosen arbitrarily close to 1, we obtain that $x \in (\bar{B}_\Gamma)_{\theta,1;J}$ and $\|x\|_{(\bar{B}_\Gamma)_{\theta,1;J}} \leqslant \|x\|_{l^1_{v(\Gamma)}}$, proving (2.32).

The inclusion (2.33) is a special case of a result of Hernandez (see Sect. 6.1 of [H1]). It can also be easily established by using the maximum principle to estimate the l^1 norm of the analytic l^1-valued function $\Phi(\zeta) = (\Phi_m(\zeta))_{m=1}^{\infty} = (f_m(\zeta)\ U_m(\zeta))_{m=1}^{\infty}$. $\Phi(\zeta)$ is defined on the domain D bounded by Γ by taking $f(\zeta) = (f_m(\zeta))_{m=1}^{\infty}$ to be a "good" representative in $\mathscr{F}(\bar{B}^{\Gamma}, \bar{\Gamma})$ of a (finitely supported) sequence $x = (x_m)$ in $(\bar{B}^{\Gamma})_{[z],\Gamma}$, and letting $U_m(\zeta) = \prod_{j=1}^{n} (u_m(j))^{z_j(\zeta)}$ where $z_j(\zeta) = P_\zeta(\Gamma_j) + iP_\zeta(\Gamma_j)^{\sim}$ with $\operatorname{Im} z_j(z) = 0$ almost exactly as in the proof of Theorem 1.17 in Subsection 4C.

For the inclusion (2.34) let $x = (x_m) \in (\bar{B}^{\Gamma})_{\theta,1;K}$. Then

$$K(\bar{t}, x; \bar{B}^{\Gamma}) = \sum_{m=1}^{\infty} |x_m| \min_{j=1}^{n} t_j u_m(j)$$

and so

$$\begin{aligned}
\|x\|_{(\bar{B}^{\Gamma})_{\theta,1;K}} &= \int_E \sum_{m=1}^{\infty} \prod_{j=1}^{n} (t_j u_m(j))^{-\theta_j} \min_{j=1}^{n} [t_j u_m(j)]\ u_m(\bar{\Gamma}) |x_m|\ d\mu(\bar{t}) \\
&= \int_E \sum_{m=1}^{\infty} \prod_{j=1}^{n} (t_j)^{-\theta_j} \min_{j=1}^{n} (t_j)\ u_m(\bar{\Gamma})\ |x_m|\ d\mu(\bar{t}) \\
&= \|x\|_{l^1_{u(\Gamma)}} \Big/ \prod_{j=1}^{n} \theta_j
\end{aligned}$$

(cf. the calculation of I in the proof of Proposition 2.24.) This establishes (2.34).

Inclusions (2.31), (2.32), (2.33), and (2.34) now imply that, for $\mathrm{M} = \mathrm{FL}$, St.L, $J, 1$, or $K, 1$,

$$\begin{aligned}
l^1_v \leqslant l^1_{v(\Gamma)} &\leqslant \mathrm{L}_{\mathrm{M}}(A, \bar{\Gamma}, Z) \leqslant \mathrm{L}_{\mathrm{M}}(A, Z) \leqslant \mathrm{U}_{\mathrm{M}}(A, Z) \\
&\leqslant \mathrm{U}_{\mathrm{M}}(A, \bar{\Gamma}, Z) \leqslant l^1_{u(\Gamma)} \leqslant l^1_u. \qquad (2.35)
\end{aligned}$$

It is also clear from the definitions of $u(\bar{\Gamma})$, $v(\bar{\Gamma})$, and w that

$$l^1_{v(\Gamma)} \leqslant l^1_w \leqslant l^1_{u(\Gamma)} \qquad \text{for all} \quad \bar{\Gamma} \in \mathscr{P}. \qquad (2.36)$$

Thus our final step, which will establish M, Z measurability and show that $I_{\mathrm{M}}(A, Z) = l^1_w$, will be to show that

$$\inf_{\Gamma \in \mathscr{P}} l^1_{u(\Gamma)} \leqslant \sup_{\Gamma \in \mathscr{P}} l^1_{v(\Gamma)}, \qquad (2.37)$$

where here as before inf and sup denote the uniformly bounded intersection and hull, respectively, of the given collection of spaces.

For each m let $(v_m^k(\gamma))_{k=1}^{\infty}$ be a decreasing sequence of simple functions and $(u_m^k(\gamma))_{k=1}^{\infty}$ be an increasing sequence of simple functions both of which converge to $w_m(\gamma)$ for a.e. $\gamma \in \Gamma$. Thus $\lim_{k \to \infty} \exp \int_\Gamma \log v_m^k(\gamma)\, dZ(\gamma) = \lim_{k \to \infty} \exp \int_\Gamma \log u_m^k(\gamma)\, dZ(\gamma) = w_m$. The sets of constancy of u_m^k which have positive measure constitute a partition in $\mathscr{P}$, as do the sets of constancy of v_m^k with positive measure. Let $\bar{\Gamma}(k, m) \in \mathscr{P}$ be a common refinement of these two partitions and, for each integer $v \geqslant 1$, let $\bar{\Gamma}(v)$ be a common refinement of all the partitions $\bar{\Gamma}(k, m)$ for $k \leqslant v$, $m \leqslant v$.

For each m and each $v \geqslant m$

$$\exp \int_\Gamma \log u_m^v(\gamma)\, dZ(\gamma) \leqslant u_m(\bar{\Gamma}(v)) \leqslant w_m \leqslant v_m(\bar{\Gamma}(v))$$

$$\leqslant \exp \int_\Gamma \log v_m^v(\gamma)\, dZ(\gamma).$$

Therefore, for each m,

$$\lim_{v \to \infty} u_m(\bar{\Gamma}(v)) = \lim_{v \to \infty} v_m(\bar{\Gamma}(v)) = w_m. \tag{2.38}$$

Now let $x = (x_m)_{m=1}^{\infty} \in \inf_{\bar{\Gamma} \in \mathscr{P}} l^1_{u(\bar{\Gamma})}$ with norm 1. Then it is clear from the preceding that for each integer N $\sum_{m=1}^{N} |x_m|\, w_m = \lim_{v \to \infty} \sum_{m=1}^{N} |x_m|\, u_m(\bar{\Gamma}(v)) \leqslant 1$ and so $\sum_{m=1}^{\infty} |x_m|\, w_m \leqslant 1$. Fix $\varepsilon > 0$ and let $\delta^m = (\delta_n^m)_{n=1}^{\infty}$ be the sequence whose mth term is 1 and all others zero. Then, for suitable integers v_m, $\|\delta^m\|_{l^1_{v(\bar{\Gamma}(v_m))}} \leqslant (1+\varepsilon)\, w_m$. Consequently, writing $x = \sum_{m=1}^{\infty} x_m \delta^m$, we have

$$\|x\|_{\sup_{\bar{\Gamma} \in \mathscr{P}} l^1_{v(\bar{\Gamma})}} < \sum_{m=1}^{\infty} \|x_m \delta^m\|_{l^1_{v(\bar{\Gamma}(v_m))}} \leqslant 1 + \varepsilon.$$

This proves (2.37) and completes our discussion.

2.39. *Remark.* By considering a special case of the above example we can see that in general the space $A_{\mathrm{M}}(A, Z)$ is not complete. Let Z be standard Lebesgue measure on $\Gamma = (0, 1)$ and

$$w_m(\gamma) = \begin{cases} 1, & \gamma \geqslant 1/m \\ m^{-m}, & \gamma < 1/m. \end{cases}$$

Then $w = (w_m)_{m=1}^{\infty} = (1/m)_{m=1}^{\infty}$. Note that every space $A(\gamma) = l^1_{w(\gamma)}$ equals l^1 up to equivalence of norms. It is also easy to see that for any $\bar{\Gamma} = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\} \in \mathscr{P}$ each of the spaces $\inf_{\gamma \in \Gamma_j} A(\gamma)$ coincides with l^1 up to equivalence of norms and hence so does $\mathrm{L_M}(A, \bar{\Gamma}, Z)$. It follows that $A_{\mathrm{M}}(A, Z)$ as a set equals $l^1 \neq l^1_w = \mathrm{L_M}(A, Z)$.

Furthermore, as we shall see,

$$\|x\|_{\Lambda_{\mathrm{M}}(A,Z)} = \|x\|_{\mathrm{L_M}(A,Z)} \tag{2.40}$$

for all $x \in \Lambda_{\mathrm{M}}(A, Z)$. Hence $\Lambda_{\mathrm{M}}(A, Z)$ is not complete. (To prove (2.40) note first that it holds for all sequences x having finitely many nonzero elements (e.g., use (2.35), (2.36), and (2.38)). Then for arbitrary $x \in \Lambda_{\mathrm{M}} = l^1$ let x^N be the Nth truncation of x ($x_n^N = x_n$ if $n \leqslant N$, $x_n = 0$ otherwise). Since $\|x - x^N\|_{A(\gamma)} \leqslant \|x - x^N\|_{l^1}$ for all $\gamma \in \Gamma$ it follows that $\|x - x^N\|_{\mathrm{L_M}(A,A)}$ and $\|x - x^N\|_{\Lambda_{\mathrm{M}}(A,Z)}$ both tend to zero as N tends to ∞ (e.g., by Theorem 2.21) and this immediately yields (2.40).)

F. *Some Remarks Concerning M, Z Measurability*

We do not know of more concrete conditions on an arbitrary bounded family $\{A(\gamma) \mid \gamma \in \Gamma\}$ which guarantee M, Z measurability of the family (Definition 2.20) in general. Such conditions could also conceivably be different for different methods M. However, for the case where all of the spaces $A(\gamma)$ are are all the same finite dimensional space with possibly different norms (i.e., the context of [C1, C2]) we can give such conditions:

2.41. THEOREM. *Let $\{A(\gamma) \mid \gamma \in \Gamma\}$ be a bounded family such that each $A(\gamma)$ and the containing space $\mathscr{U}$ are all $\mathbb{C}^d$ equipped with possibly different norms. Suppose that there exists a non trivial normed space $\mathscr{A}$, which is also $\mathbb{C}^d$ renormed appropriately, such that $\mathscr{A} \leqslant A(\gamma)$ for all $\gamma \in \Gamma$. If for each $a \in \mathbb{C}^d$, $\|a\|_{A(\gamma)}$ is a Z-measurable function of γ on Γ, then $\{A(\gamma) \mid \gamma \in \Gamma\}$ is M, Z measurable for each of the methods* $\mathrm{M} = K,p$, J,p, FL, *and* St.L.

Proof. Consider the family $\{f_\gamma\}_{\gamma \in \Gamma}$ of functions on the compact set $K = \{a \in \mathbb{C}^d \mid \|a\|_{\mathscr{A}} = 1\}$ defined by $f_\gamma(a) = \log\|a\|_{A(\gamma)}$. This is a bounded equicontinuous family since $\|a\|_{\mathscr{A}} \leqslant c\|a\|_{\mathscr{U}}$ for some fixed $c > 0$ and so $-\log c \leqslant f_\gamma(a) \leqslant 0$, and furthermore $|f_\gamma(a) - f_\gamma(b)| \leqslant \log(1 + c\|b - a\|_{\mathscr{A}})$ for all $\gamma \in \Gamma$ and all $a, b \in K$. Therefore, by the Arzelà–Ascoli theorem, for each $\varepsilon > 0$ there exists a finite sequence $\gamma_1, \gamma_2, \ldots, \gamma_n$ of points in Γ such that for each $\gamma \in \Gamma$ there exists an integer j, $1 \leqslant j \leqslant n$, for which $|f_\gamma(a) - f_{\gamma_j}(a)| \leqslant \varepsilon$ for all $a \in K$, or equivalently,

$$e^{-\varepsilon} \leqslant \|a\|_{A(\gamma)} / \|a\|_{A(\gamma_j)} < e^{\varepsilon}. \tag{2.42}$$

But K contains a countable dense subset K_1 and the set E_j of points γ such that (2.42) holds for all $a \in K_1$ coincides with the set for which (2.42) holds for all $a \in K$. Thus E_j is Z-measurable and $\bigcup_{j=1}^{n} E_j = \Gamma$. Define $\bar{\Gamma} =$

$\{\Gamma_1,\ldots,\Gamma_n\}\in\mathscr{P}$ by $\Gamma_1=E_1$ and $\Gamma_j=E_j\backslash\bigcup_{k<j}\Gamma_k$, $j=2,\ldots,n$. It is easy to see that for all $\bar{\Omega}\in\mathscr{P}$ with $\bar{\Omega}\succ\bar{\Gamma}$ and for each of the four methods M,

$$\|a\|_{\mathrm{L}_{\mathrm{M}}(A,\bar{\Omega},Z)}\leqslant e^{2\varepsilon}\|a\|_{\mathrm{U}_{\mathrm{M}}(A,\bar{\Omega},Z)}$$

for all $a\in\mathbb{C}^d$. Since ε is arbitrary it follows that $\{A(\gamma)\mid\gamma\in\Gamma\}$ is M, Z measurable. ∎

2.43. *Remarks.* In particular it is obvious that, in the case $d=1$, M, Z measurability is equivalent to the Z-measurability of the real valued function $\varphi(\gamma)=\|1\|_{A(\gamma)}$. In the context of infinite dimensional spaces another trivial instance of M, Z measurability is of course when $A(\gamma)$ assumes only finitely many different "values," each of them on a Z-measurable set. Note also that if $\{A(\gamma)\mid\gamma\in\Gamma\}$ is St. L, P_z measurable then $\mathrm{U}_{\mathrm{St.L}}(A, P_z)$ and $\mathrm{L}_{\mathrm{St.L}}(A, P_z)$ coincide with $A[z]$, assuming that $\{A(\gamma)\mid\gamma\in\Gamma\}$ is also an interpolation family as defined in [C3].

3. The Inclusion $\bar{A}_{\theta,2;J}\subset\bar{A}_{\theta,2;K}$ is Strict

In this section we give the details of the construction of the triples $\bar{A}^r$ of two dimensional Hilbert spaces (Example 1.10) which enable us to deduce (Corollary 1.11) that the spaces $\bar{A}_{\theta,p;J}$ and $\bar{A}_{\theta,p;K}$ do not coincide in general, even if $\Delta(\bar{A})$ is dense in A_j for each j. We choose to work with spaces over the complex field in order to facilitate comparison of the spaces $\bar{A}_{\theta,p;J}$ and $\bar{A}_{\theta,p;K}$ with certain complex interpolation spaces. (See Remark 3.15 at the end of this section.) Not surprisingly, via trivial modifications, we can obtain triples of two-dimensional real Hilbert spaces with analogous properties.

We begin with some observations of a more general nature concerning n-tuples of finite dimensional Hilbert spaces. Given any Hermitian positive definite $d\times d$ matrix M we can define a Hilbert norm on $\mathbb{C}^d$ by $\|a\|=\sqrt{\langle a, Ma\rangle}=\sqrt{\langle\sqrt{M}a,\sqrt{M}a\rangle}=\sqrt{(\sqrt{M}a,\sqrt{M}a)}$. Here we use the notation $\langle a, b\rangle=\sum_{k=1}^d a_k\bar{b}_k$ and $(a, b)=\sum_{k=1}^d a_k b_k$ for $a=(a_1, a_2,\ldots, a_d)$ and $b=(b_1, b_2,\ldots, b_d)$ in $\mathbb{C}^d$. For our purposes it will be convenient to always take M to be a matrix over the reals so that the two natural definitions of dual norm coincide. More specificially, $\|a\|'=\sup_{b\neq0}|\langle a, b\rangle|/\|b\|=\sup_{b\neq0}|(a, b)|/\|b\|=\sqrt{\langle a, M^{-1}a\rangle}$. We consider an n-tuple $\bar{A}=(A_1, A_2,\ldots, A_n)$ where each A_j is $\mathbb{C}^d$ and $\|a\|_{A_j}=\sqrt{\langle a, M_ja\rangle}$ for some real symmetric positive definite matrix M_j, $j=1, 2,\ldots, n$. Of course in this context the spaces $\bar{A}_{\theta,p;J}$ and $\bar{A}_{\theta,p;K}$ all coincide with $\mathbb{C}^d$ and all norms are equivalent. We shall be concerned with inequalities between these various norms. It will be convenient to modify the definitions of $\bar{A}_{\theta,p;J}$ and

$\bar{A}_{\theta,p;K}$ and their norms by replacing the J- and K-functionals in the formulae (1.8) and (1.4) by their "l^2" counterparts

$$J_2(\bar{t}, a; \bar{A}) = \left(\sum_{j=1}^{n} (t_j \|a\|_{A_j})^2 \right)^{1/2}$$

and

$$K_2(\bar{t}, a; \bar{A}) = \inf\left\{ \left(\sum_{j=1}^{n} (t_j \|a_j\|_{A_j})^2 \right)^{1/2} \middle| a = \sum_{j=1}^{n} a_j \right\}.$$

The ratio between the new and original $\bar{A}_{\theta,p;K}$ norms is clearly bounded above and below by constants depending only on n. The same is true for $\bar{A}_{\theta,p;J}$. Thus for the remainder of this section we shall use (i.e., abuse) the notation $\| \ \|_{\bar{A}_{\theta,p;J}}$ and $\| \ \|_{\bar{A}_{\theta,p;K}}$ to denote the new norms defined via $J_2(\bar{t}, a; \bar{A})$ and $K_2(\bar{t}, a; \bar{A})$. (In fact we shall be concerned almost exclusively with the case $n = 3$, $d = 2$, $p = 2$.)

In the present context (in contrast to that discussed in Theorem 1.32 and Sect. 6; see Remark 6.1(iv)) the proof that $(A_1 + A_2)' = A_1' \cap A_2'$ (see, e.g., [BL, p. 32]) can be easily adapted to show that the norms $J_2(\bar{t}, \cdot; \bar{A})$ and $K_2(\bar{t}^{-1}, \cdot; \bar{A}')$ are dual to each other, where $\bar{t}^{-1} = (1/t_1, 1/t_2, ..., 1/t_n)$ and $\bar{A}' = (A_1', A_2', ..., A_n')$, i.e., the norm of A_j' is generated by M_j^{-1}. Since of course $J_2(\bar{t}, \cdot; \bar{A})$ is a Hilbert norm with $J_2(\bar{t}, a; \bar{A})^2 = \langle a, \sum_{j=1}^n t_j^2 M_j a \rangle$ it follows that $K_2(\bar{t}, a; \bar{A})^2 = \langle a, (\sum_{j=1}^n t_j^{-2} M_j^{-1})^{-1} a \rangle$ and so also $\|a\|^2_{\bar{A}_{\theta,2;K}} = \langle a, M_{\theta,2;K} a \rangle$, i.e., the (new Hilbert) norm of $\bar{A}_{\theta,2;K}$ is generated by the matrix

$$M_{\theta,2;K} = \int_E (\bar{t}^{-\theta})^2 \left(\sum_{j=1}^{n} t_j^{-2} M_j^{-1} \right)^{-1} d\mu. \tag{3.1}$$

Here $E \subset \mathbb{R}^n$ and the measure μ on E are as in Definition 1.3 or its variants (Remark 1.5).

From the above duality of norms we can deduce that the norms $\| \ \|_{\bar{A}_{\theta,2;J}}$ and $\| \ \|_{\bar{A}'_{\theta,2;K}}$ are also dual to each other. (Once again this is a straightforward adaptation of analogous arguments for couples.) Consequently $\|a\|^2_{\bar{A}_{\theta,2;J}} = \langle a, M_{\theta,2;J} a \rangle$ where

$$M_{\theta,2;J} = \left(\int_E (\bar{t}^{-\theta})^2 \left(\sum_{j=1}^{n} t_j^{-2} M_j \right)^{-1} d\mu \right)^{-1}. \tag{3.2}$$

We can now turn to an explicit description of our counterexample. Let $n = 3$ and $d = 2$. Thus we consider a triple $\bar{A} = (A_1, A_2, A_3)$ of two-dimensional Hilbert spaces defined by the real symmetric positive definite 2×2 matrices M_1, M_2, M_3. We shall compare the "volumes" V_J and V_K of the unit balls of the two spaces $\bar{A}_{\theta,2:J}$ and $\bar{A}_{\theta,2:K}$ and show that the ratio V_K/V_J can be arbitrarily large. To simplify the calculations we shall only consider the case $\bar{\theta}=(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ although the arguments work for all $\bar{\theta}\in H^3_+$.

We shall use the notation

$$M_{\theta,2:J}=M_J=M_J(M_1,M_2,M_3) \tag{3.3}$$

and

$$M_{\theta,2:K}=M_K=M_K(M_1,M_2,M_3) \tag{3.4}$$

for the matrices defining the norms of $\bar{A}_{\theta,2:J}$ and $\bar{A}_{\theta,2:K}$. More specifically, for all values of the parameters $a\in(0,\infty)$ and $b\in(-1,1)$ we let

$$M_J(a,b)=M_J\left(\begin{pmatrix}a&0\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&a\end{pmatrix},\begin{pmatrix}1&b\\b&1\end{pmatrix}\right) \tag{3.5}$$

and

$$M_K(a,b)=M_K\left(\begin{pmatrix}a&0\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&a\end{pmatrix},\begin{pmatrix}1&b\\b&1\end{pmatrix}\right). \tag{3.6}$$

Now we choose $E=\{(t_1,t_2,1)\mid t_1,t_2>0\}$ and $d\mu=dt_1\,dt_2/t_1t_2$. Setting $x_1=t_1^{-2}$, $x_2=t_2^{-2}$ and using (3.2) we obtain that

$$M_J(M_1,M_2,M_3)=\left(\frac{1}{4}\int_0^\infty\int_0^\infty(x_1x_2)^{1/3}(x_1M_1+x_2M_2+M_3)^{-1}\frac{dx_1\,dx_2}{x_1x_2}\right)^{-1} \tag{3.7}$$

and thus, in particular,

$$\begin{aligned}
&4M_J^{-1}(a,b)\\
&=\int_0^\infty\int_0^\infty(x_1x_2)^{1/3}\left(x_1\begin{pmatrix}a&0\\0&1\end{pmatrix}+x_2\begin{pmatrix}1&0\\0&a\end{pmatrix}+\begin{pmatrix}1&b\\b&1\end{pmatrix}\right)^{-1}\frac{dx_1dx_2}{x_1x_2}\\
&=\int_0^\infty\int_0^\infty\begin{pmatrix}ax_1+x_2+1&b\\b&x_1+ax_2+1\end{pmatrix}^{-1}x_1^{-2/3}x_2^{-2/3}\,dx_1\,dx_2\\
&=\int_0^\infty\int_0^\infty[(ax_1+x_2+1)(x_1+ax_2+1)-b^2]^{-1}\\
&\quad\times\begin{pmatrix}x_1+ax_2+1&-b\\-b&ax_1+x_2+1\end{pmatrix}x_1^{-2/3}x_2^{-2/3}\,dx_1\,dx_2\\
&=\begin{pmatrix}I_1+aI_2+I_3&-bI_3\\-bI_3&aI_1+I_2+I_3\end{pmatrix},
\end{aligned}$$

where

$$I_j = I_j(a, b) = \int_0^\infty \int_0^\infty x_j \frac{x_1^{-2/3} x_2^{-2/3}\, dx_1\, dx_2}{(ax_1 + x_2 + 1)(x_1 + ax_2 + 1) - b^2}$$

for $j = 1, 2, 3$, with $x_3 \equiv 1$.

(In fact by symmetry $I_1 = I_2$.) Since $V_J(a, b) = v_2 \det M_J^{-1}(a, b)$, where $v_2 = \pi^2/4$ is the volume (four-dimensional Lebesgue measure) of the euclidean unit ball of $\mathbb{C}^2 = \mathbb{R}^4$, we may write

$$V_J(a, b) \sim (I_1 + aI_2 + I_3)(aI_1 + I_2 + I_3) - b^2 I_3^2. \tag{3.8}$$

(Here, and in the rest of this section, the notation $f \sim g$ shall mean that the quotient f/g is bounded above and below by positive numbers which do not depend on a or b.)

Via the change of variables $x_j = (1 - b^2)\, y_j/(1 + a)$, $j = 1, 2$, we obtain that

$$I_3(a, b) = \left[\frac{(1-b)^2}{(1+a)}\right]^{2/3} \times \int_0^\infty \int_0^\infty \frac{y_1^{-2/3} y_2^{-2/3}\, dy_1\, dy_2}{\left(\begin{array}{c}((1-b^2)^2/(1+a)^2)(ay_1 + y_2)(y_1 + ay_2) \\ + (1-b)^2(y_1 + y_2) + 1 - b^2\end{array}\right)}.$$

Hence, by monotone convergence, the expression

$$(1 - b^2)^{1/3} I_3(a, b) = (1+a)^{-2/3} \int_0^\infty \int_0^\infty \frac{y_1^{-2/3} y_2^{-2/3}\, dy_1\, dy_2}{((1-b^2)/(1+a)^2)(ay_1 + y_2)(y_1 + ay_2) + y_1 + y_2 + 1}$$

converges as b tends to 1, for each fixed positive a, to the integral

$$(1+a)^{-2/3} \int_0^\infty \int_0^\infty \frac{y_1^{-2/3} y_2^{-2/3}\, dy_1\, dy_2}{y_1 + y_2 + 1}. \tag{3.9}$$

As can be readily verified (e.g., with the help of polar coordinates) this last integral is finite. From this point onwards we shall restrict a to the range $0 < a \leqslant 1$ and so the above calculation shows in fact that

$$\lim_{b \to 1} (1 - b^2)^{1/3} I_3(a, b) \sim 1. \tag{3.10}$$

We next estimate the expression

$$\begin{aligned}
&\lim_{b\to 1}\,(I_1+I_2)\\
&\quad=\lim_{b\to 1}\int_0^\infty\int_0^\infty \frac{(x_1+x_2)(x_1x_2)^{-2/3}\,dx_1\,dx_2}{(ax_1+x_2+1)(x_1+ax_2+1)-b^2}\\
&\quad=\int_0^\infty\int_0^\infty \frac{(x_1+x_2)(x_1x_2)^{-2/3}\,dx_1\,dx_2}{a(x_1^2+x_2^2)+(1+a^2)\,x_1x_2+(1+a)(x_1+x_2)}\\
&\qquad\text{by monotone convergence}\\
&\quad\sim\int_0^\infty\int_0^\infty \frac{(x_1+x_2)(x_1x_2)^{-2/3}\,dx_1\,dx_2}{a(x_1^2+x_2^2)+x_1x_2+(x_1+x_2)}=I_4.
\end{aligned}$$

The integrand of I_4 is dominated by

$$\frac{x_1(x_1x_2)^{-2/3}}{ax_1^2+x_1x_2+x_1}+\frac{x_2(x_1x_2)^{-2/3}}{ax_2^2+x_1x_2+x_2}=\frac{(x_1x_2)^{-2/3}}{ax_1+x_2+1}+\frac{(x_1x_2)^{-2/3}}{ax_2+x_1+1}.$$

Thus

$$\begin{aligned}
I_4&\leqslant 2\int_0^\infty\int_0^\infty \frac{(x_1x_2)^{-2/3}\,dx_1\,dx_2}{ax_1+x_2+1}\\
&=2a^{-1/3}\int_0^\infty\int_0^\infty \frac{(y_1y_2)^{-2/3}\,dy_1\,dy_2}{y_1+y_2+1},
\end{aligned}$$

where we have used the transformation $y_1=ax_1$, $y_2=x_2$. We recall (cf. (3.9)) that this last integral is finite.

But also

$$\begin{aligned}
I_4&\geqslant\int_0^\infty\left(\int_0^{ax_1}\frac{x_1(x_1x_2)^{-2/3}\,dx_2}{a(x_1^2+x_2^2)+x_1x_2+(x_1+x_2)}\right)dx_1\\
&\sim\int_0^\infty\left(\int_0^{ax_1}\frac{(x_1x_2)^{-2/3}\,dx_2}{ax_1+x_2+1}\right)dx_1\\
&=a^{-1/3}\int_0^\infty\left(\int_0^{y_1}\frac{(y_1y_2)^{-2/3}\,dy_2}{y_1+y_2+1}\right)dy_1,
\end{aligned}$$

using the same transformation as previously. We have thus shown that

$$\lim_{b\to 1}\,(I_1+I_2)\sim a^{-1/3} \tag{3.11}$$

which of course also implies the finiteness of the limits $\lim_{b\to 1} I_1$ and

$\lim_{b\to 1} I_2$ for each a, $0<a\leqslant 1$. Combining this with (3.8) and (3.10) we can deduce that

$$\lim_{b\to 1}(1-b^2)^{1/3}V_J(a,b)\sim\lim_{b\to 1}\,[(I_1+aI_2+aI_1+I_2)(1-b^2)^{1/3}I_3+(1-b^2)^{4/3}I_3^2]$$
$$\sim\lim_{b\to 1}\,(1+a)(I_1+I_2)\sim a^{-1/3}. \tag{3.12}$$

We now turn to determining the corresponding asymptotic behavior of $V_K(a,b)$. It turns out to be possible to deduce this from (3.12) with the help of the following matrix identity.

$$M_K^{-1}(a,b)=a^{-2/3}(1-b^2)^{-1/3}M_J(a,-b). \tag{3.13}$$

Indeed, from our earlier discussion of duality, in terms of the notation (3.3) to (3.6),

$$M_K^{-1}(M_1,M_2,M_3)=M_J(M_1^{-1},M_2^{-1},M_3^{-1}).$$

More specifically,

$$\begin{aligned}M_K^{-1}(a,b)&=M_J\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}^{-1},\begin{pmatrix}1&0\\0&a\end{pmatrix}^{-1},\begin{pmatrix}1&b\\b&1\end{pmatrix}^{-1}\right)\\&=M_J\left(\frac{1}{a}\begin{pmatrix}1&0\\0&a\end{pmatrix},\frac{1}{a}\begin{pmatrix}a&0\\0&1\end{pmatrix},\frac{1}{1-b^2}\begin{pmatrix}1&-b\\-b&1\end{pmatrix}\right)\\&=\frac{1}{a}M_J\left(\begin{pmatrix}1&0\\0&a\end{pmatrix},\begin{pmatrix}a&0\\0&1\end{pmatrix},\frac{a}{1-b^2}\begin{pmatrix}1&-b\\-b&1\end{pmatrix}\right)\end{aligned}$$

by (3.7). Further, via the change of variables, $y_1=(1-b^2)\,x_1/a$, $y_2=(1-b^2)\,x_2/a$ in the integral (3.7), it follows that the above matrix equals

$$\begin{aligned}&a^{-2/3}(1-b^2)^{-1/3}M_J\left(\begin{pmatrix}1&0\\0&a\end{pmatrix},\begin{pmatrix}a&0\\0&1\end{pmatrix},\begin{pmatrix}1&-b\\-b&1\end{pmatrix}\right)\\&\quad=a^{-2/3}(1-b^2)^{-1/3}M_J\left(\begin{pmatrix}a&0\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&a\end{pmatrix},\begin{pmatrix}1&-b\\-b&1\end{pmatrix}\right)\end{aligned}$$

again by (3.7). (Indeed for $\bar{\theta}=(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ the space $\bar{A}_{\theta,p;J}$ is unchanged by any permutation of the order in which the spaces A_1, A_2, A_3 appear in $\bar{A}$.) Thus (3.13) is proved. From the formulae used above to obtain (3.8) it is clear that $V_J(a,-b)=V_J(a,b)$ and so

$$\begin{aligned}V_K(a,b)&\sim\det M_K^{-1}(a,b)=(a^{-2/3}(1-b^2)^{-1/3})^2\det M_J(a,-b)\\&\sim a^{-4/3}(1-b^2)^{-2/3}V_J(a,b)^{-1}.\end{aligned}$$

Consequently $\lim_{b\to 1}(1-b^2)^{1/3}V_K(a,b)\sim a^{-4/3}a^{1/3}=a^{-1}$. Indeed $\lim_{b\to 1} V_K(a,b)/V_J(a,b)\sim a^{-2/3}$ for all $a\in(0,1)$ so that clearly, for suitable choices of a and b, V_K may be arbitrary larger than V_J.

3.14. *Remark.* We briefly indicate some details of the proof of Corollary 1.11. For $1\leqslant p<\infty$ and any sequence of Banach spaces $\{B_m\}_{m=1}^{\infty}$ let $l^p\{B_m\}$ denote the space of sequences $\{b_m\}_{m=1}^{\infty}$ such that $b_m\in B_m$ and the norm $\|\{b_m\}\|_{l^p\{B_m\}}=(\sum_{m=1}^{\infty}\|b_m\|_{B_m}^p)^{1/p}$ is finite. Then for any sequence of Banach triples $\{\bar{A}^m\}=\{A_1^m, A_2^m, A_3^m)\}$ it is easy to see that $\bar{A}=(l^p\{A_1^m\}, l^p\{A_2^m\}, l^p\{A_3^m\})$ is also a Banach triple with containing space $l^p\{A_1^m+A_2^m+A_3^m\}$. Taking A_1^m, A_2^m, and A_3^m to all be $\mathbb{C}^2$ as above with the property that $V_K\geqslant mV_J$, it is easy to see that $\bar{A}_{\theta,2;J}$ is strictly contained in $\bar{A}_{\theta,2;K}$. Furthermore by taking $p=2$ we may obtain $\bar{A}$ as a triple of Hilbert spaces. (For examples of similar applications of direct sums cf., e.g., [J1, Lemma 1, p. 52] or [Cw, Lemma 2, pp. 221, 222].)

3.15. *Remark.* It is of interest to compare the estimates obtained above for $\lim_{b\to 1}(1-b^2)^{1/3}V_J$ and $\lim_{b\to 1}(1-b^2)^{1/3}V_K$ with a corresponding estimate which we shall obtain now for $V_{\mathrm{St.L}}$, the volume of the unit ball of the analogous St. Louis complex interpolation space $\bar{A}_{[\zeta],\Gamma}$ where $\bar{A}=(A_1, A_2, A_3)$ is the same triple of two-dimensional Hilbert spaces as above and the spaces $\bar{A}_{[\zeta],\Gamma}$ are defined as in Section 1 using a domain D with boundary Γ, and a decomposition of Γ into three disjoint subsets $\Gamma_1, \Gamma_2, \Gamma_3$, each having harmonic measure $\frac{1}{3}$ at $\zeta\in D$. By an argument given in [C1, p. 279] (essentially an alternative proof of the Masani–Wiener theorem) we have that for all $a\in\mathbb{C}^2$

$$\|a\|^2_{\bar{A}_{[\zeta],\Gamma}}=\langle\beta(\zeta)\,a, \beta(\zeta)\,a\rangle,$$

where $\beta(z)$ is a nonsingular matrix valued analytic function of z on D with nontangential limits $\beta(\gamma)$ for a.e. $\gamma\in\Gamma$ satisfying $\langle\beta(\gamma)\,a, \beta(\gamma)\,a\rangle=\langle a, M_j a\rangle$ for a.e. $\gamma\in\Gamma_j$, $j=1,2,3$, and all $a\in\mathbb{C}^2$. Consequently $\det\beta(z)$ is a bounded nonvanishing analytic function on D. We claim that in fact $\det\beta(z)$ is an outer function, since $1/\det\beta(z)$ is also bounded (cf. [G, Theorem 5.5, Corollary 5.6, p. 74]). This can be seen, e.g., by applying the Masani–Wiener theorem as above also to the dual couple (A_1', A_2', A_3') generated by the matrices $M_1^{-1}, M_2^{-1}, M_3^{-1}$, and using the duality theorem ([C3, Remark 3.2, p. 214] also [C2, p. 135]).

We can now assert, using notation as in Section 1, that

$$|\det\beta(\zeta)|=\exp\int_\Gamma\log|\det\beta(\gamma)|\,dP_\zeta(\gamma)=|\det M_1\det M_2\det M_3|^{1/6}.$$

Thus $V_{\mathrm{St.L}}\sim|\det\beta(\zeta)|^{-2}=a^{-2/3}(1-b^2)^{-1/3}$ and $\lim_{b\to 1}(1-b^2)^{1/3} V_{\mathrm{St.L}}\sim a^{-2/3}$.

By comparison with earlier estimates we see that for a sufficiently close to 0 and b sufficiently close to 1

$$V_J \ll V_{\mathrm{St.L}} \ll V_K.$$

4. The Favini–Lions and the St. Louis Complex Interpolation Spaces

4A. *The Inclusion* $A_{\theta,1;J} \subset [\bar{A}]_\theta$: *Theorem* 1.15

For each $a \in \Delta(\bar{A})$ let $f(z_1, z_2,..., z_{n-1}) = g(z_1,..., z_n) = (\prod_{j=1}^n \|a\|_{A_j}^{\theta_j - z_j})\, a$ where, as above, $z_n = 1 - \sum_{j=1}^{n-1} z_j$. Then clearly $f \in \mathscr{H}_1(\bar{A})$ (see Remark 1.14) and, since $\sum_{j=1}^n \theta_j = 1$, $f(\theta_1,..., \theta_{n-1}) = a$. Furthermore, for any $k = 1, 2,..., n$, if $z = (z_1, z_2,..., z_{n-1}) \in \partial\Omega_k$ then $\|f(z)\|_{A_k} = \prod_{j=1}^n \|a\|_{A_j}^{\theta_j}$. Consequently $\|a\|_{[\bar{A}]_\theta} \leqslant \prod_{j=1}^n \|a\|_{A_j}^{\theta_j}$. Now, given any $b \in \bar{A}_{\theta,1;J}$, we have for suitable functions $u(\bar{t})$ (see (1.7) and (1.8)) that

$$\begin{aligned}
&\|b\|_{[\bar{A}]_\theta} \\
&\quad = \left\| \int_E u(\bar{t})\, d\mu(\bar{t}) \right\|_{[\bar{A}]_\theta} \leqslant \int_E \|u(\bar{t})\|_{[\bar{A}]_\theta}\, d\mu(\bar{t}) \\
&\quad \leqslant \int_E \prod_{j=1}^n \|u(\bar{t})\|_{A_j}^{\theta_j}\, d\mu(t) = \int_E \bar{t}^{-\theta} \prod_{j=1}^n (t_j \|u(\bar{\theta})\|_{A_j})^{\theta_j}\, d\mu(\bar{t}) \\
&\quad \leqslant \int_E \bar{t}^{-\theta} J(\bar{t}, u(\bar{t}); \bar{A})\, d\mu(\bar{t}).
\end{aligned}$$

Taking the infimum over all such functions $u(\bar{t})$, we deduce that $\|b\|_{[\bar{A}]_\theta} \leqslant \|b\|_{\bar{A}_{\theta,1;J}}$ which completes the proof.

4B. *Proof of Lemma* 1.16

Given $f \in \mathscr{H}(\bar{A})$ and arbitrary $\varepsilon > 0$ let $f_1 = e_\delta f$ where $e_\delta(z) = e^{\delta \sum_{j=1}^{n-1} z_j^2}$ and $\delta > 0$ is chosen sufficiently small to ensure that $\|f_1 - f\|_{\mathscr{H}(\bar{A})} \leqslant \varepsilon/4$. Let φ be a complex valued C^∞ function on $\mathbb{R}^{n-1}$ whose Fourier transform $\hat{\varphi}$, $\hat{\varphi}(s) = \int_{\mathbb{R}^{n-1}} e^{i(s,t)} \varphi(t)\, dt$, is a C^∞ function of compact support such that $\hat{\varphi}(s) = 1$ for all s, $|s| \leqslant 1$. (Here of course $s = (s_1, s_2,..., s_{n-1})$ and $t = (t_1, t_2,..., t_{n-1})$ are in $\mathbb{R}^{n-1}$, $|s| = (\sum_{j=1}^{n-1} s_j^2)^{1/2}$ and $(s, t) = \sum_{j=1}^{n-1} s_j t_j$.)

For each positive integer m let $g_m(z) = \int_{\mathbb{R}^{n-1}} m^{n-1} \varphi(ms) f_1(z - is)\, ds$. Since e_δ is arbitrarily small on the complements of suitably large compact subsets of $\bar{\Omega}$ it follows that f_1 is a uniformly continuous function from $\bar{\Omega}$ into $\Sigma(\bar{A})$. Furthermore its restriction to $\partial\Omega_j$ is a uniformly continuous map into A_j for each $j = 1, 2,..., n$. Thus $g_m \in \mathscr{H}(\bar{A})$ and $\lim_{m \to \infty} \|g_m - f_1\|_{\mathscr{H}(\bar{A})} = 0$. We let $f_2 = g_m$ where m is chosen such that $\|g_m - f_1\|_{\mathscr{H}(\bar{A})} = \|f_2 - f_1\|_{\mathscr{H}(\bar{A})} \leqslant \varepsilon/4$. Now let $u_x(s) = \int_{\mathbb{R}^{n-1}} e^{(x+it,s)}$

$f_2(x+it)\,dt$ where $x, s, t \in \mathbb{R}^{n-1}$ and $x+it = (x_1+it_1, x_2+it_2, ..., x_{n-1}+it_{n-1}) \in \bar{\Omega}$. For each fixed s the integrand, considered as a function of $x+it$, maps $\bar{\Omega}$ continuously into $\Sigma(\bar{A})$ and is analytic in Ω. Also $\lim_{\alpha\to\infty} \sup\{\|e^{(x+it,s)}f_2(x+it)\|_{\Sigma(\bar{A})} \mid |t| > \alpha\} = 0$. It follows by Cauchy's theorem that $u_x(s) = u_{x'}(s)$ for all $x, x' \in \bar{\Omega}$ and henceforth we shall use the notation $u(s)$ for this function. Furthermore since we can take x or x' to be in $\partial\Omega_j$ for any choice of $j = 1, 2, ..., n$, we see that u maps $\mathbb{R}^{n-1}$ continuously into $\Delta(\bar{A})$. (The continuity follows from the fact that for $x \in \partial\Omega_j$ the integral

$$\int_{\mathbb{R}^{n-1}} \|f_2(x+it)\|_{A_j}\,dt \leqslant \int_{\mathbb{R}^{n-1}} \|f_1(x+it)\|_{A_j}\,dt$$

is finite.) Since $\hat{\varphi}$ has compact support so does u. Applying the inverse Fourier transform to $e^{-(x,s)}u(s) = \int_{\mathbb{R}^{n-1}} e^{(it,s)} f_2(x+it)\,dt$ yields that, for all $z = x+it \in \bar{\Omega}$,

$$\begin{aligned} f_2(z) = f_2(x+it) &= (2\pi)^{-(n-1)} \int_{\mathbb{R}^{n-1}} e^{-(it,s)} e^{-(x,s)} u(s)\,ds \\ &= (2\pi)^{-(n-1)} \int_{\mathbb{R}^{n-1}} e^{-(x+it,s)} u(s)\,ds \\ &= (2\pi)^{-(n-1)} \int_{\mathbb{R}^{n-1}} e^{-(z,s)} u(s)\,ds. \end{aligned}$$

Let $f_3 = e_{\delta'} f_2$ where $\delta' > 0$ is chosen sufficiently small to ensure that $\|f_3 - f_2\|_{\mathscr{H}(\bar{A})} \leqslant \varepsilon/4$. Let Q be a fixed cube in $\mathbb{R}^{n-1}$ which contains the support of u. Let $\{Q_1, Q_2, ..., Q_N\}$ be an arbitrary finite collection of non overlapping cubes whose union is Q. For $k = 1, 2, ..., N$ we shall denote the centre of Q_k by $\lambda_k = (\lambda_{k,1}, \lambda_{k,2}, ..., \lambda_{k,n-1})$. Let

$$f_4(z) = (2\pi)^{-(n-1)} \sum_{k=1}^{N} e^{-(z,\lambda_k)} \int_{Q_k} u(s)\,ds.$$

For all $z = x+it \in \bar{\Omega}$ and $s \in \mathbb{R}^{n-1}$, $|e^{-(z,s)}| = |e^{-(x,s)}| \leqslant e^{|s_1|+|s_2|+\cdots+|s_{n-1}|}$ and, since $\|u(s)\|_{\Delta(\bar{A})}$ is a bounded function supported in Q, we deduce that the numbers $\sup_{z\in\bar{\Omega}} \|f_2(z)\|_{\Delta(\bar{A})}$ and $\sup_{z\in\bar{\Omega}} \|f_4(z)\|_{\Delta(\bar{A})}$ are both bounded by a fixed constant M which depends only on the quantities $\sup_{s\in Q}|s|$ and $\sup_{s\in Q}\|u(s)\|_{\Delta(\bar{A})}$. In particular M is independent of the choice of decomposition of Q, $\{Q_1, ..., Q_N\}$. Let $g = e_{\delta'} f_4$. Then

$$\begin{aligned} &\|g(z) - f_3(z)\|_{\Delta(\bar{A})} \\ &\quad = |e_{\delta'}(z)|\ \|f_4(z) - f_2(z)\|_{\Delta(\bar{A})} \leqslant 2M \exp\Big(-\delta' \sum_{j=1}^{n-1} (\operatorname{Im} z_j)^2\Big). \end{aligned}$$

Thus

$$\|g(z)-f_3(z)\|_{\Delta(\bar{A})}\leqslant \varepsilon/4 \qquad \text{for all} \quad z\in\bar{\Omega}\backslash B_\rho,$$

where

$$B_\rho=\{z=(z_1,..., z_{n-1})\in\bar{\Omega}\mid |\operatorname{Im} z_j|\leqslant \rho, j=1, 2,..., n-1\}$$

and ρ is chosen sufficiently large. On the other hand, for $z\in B_\rho$,

$$\begin{aligned}\|g(z)-f_3(z)\|_{\Delta(\bar{A})}&\\ &=|e_{\delta'}(z)|\ \|f_2(z)-f_4(z)\|_{\Delta(\bar{A})}\\ &\leqslant e^{\delta'(n-1)}(2\pi)^{-(n-1)}\sum_{k=1}^{N}\int_{Q_k}|e^{-(z,s)}-e^{-(z,\lambda_k)}|\ \|u(s)\|_{\Delta(\bar{A})}\,ds.\end{aligned}$$

Now $e^{-(z,s)}$ is a uniformly continuous function (of $2n-2$ variables $(z_1, z_2,..., z_{n-1}, s_1, s_2,..., s_{n-1})$) on the compact set $B_\rho\times Q$. Thus if we choose each of the cubes Q_k with sufficiently small side length we will obtain that $\|g(z)-f_3(z)\|_{\Delta(\bar{A})}\leqslant\varepsilon/4$ for $z\in B_\rho$ and so for all $z\in\bar{\Omega}$. This implies that

$$\|g-f_3\|_{\mathscr{H}(\bar{A})}\leqslant\varepsilon/4$$

and

$$\begin{aligned}\|g-f\|_{\mathscr{H}(\bar{A})}\leqslant \|g-f_3\|_{\mathscr{H}(\bar{A})}+\|f_3-f_2\|_{\mathscr{H}(\bar{A})}&\\ +\|f_2-f_1\|_{\mathscr{H}(\bar{A})}+\|f_1-f\|_{\mathscr{H}(\bar{A})}\leqslant\varepsilon.&\end{aligned}$$

Since g is a function of the required form, the proof is complete.

4C. *Proof of Theorem* 1.17, $[\bar{A}]_\theta\subset\bar{A}_{[\zeta],\Gamma}$

For each $j=1, 2,..., n$, let u_j be the harmonic function on D $u_j(w)=P_w(\Gamma_j)$ and let v_j be the harmonic conjugate of u_j chosen so that $v_j(\zeta)=0$ at the constant point $\zeta\in D$ which is used to define $A_{[\zeta],\Gamma}$ and $\bar{\theta}$. Thus $z_j(w)=u_j(w)+iv_j(w)$ is analytic in D and has a nontangential limit $z_j(\gamma)=\lim_{w\rhd\gamma} z_j(w)$ for a.e. $\gamma\in\Gamma$. In particular $\operatorname{Re} z_j(\gamma)=1$ for a.e. $\gamma\in\Gamma_j$ and $\operatorname{Re} z_j(\gamma)=0$ for a.e. $\gamma\in\Gamma\backslash\Gamma_j$. Note also that $z_n(w)=1-\sum_{j=1}^{n-1} z_j(w)$.

Let $a\in[\bar{A}]_\theta$. By Lemma 1.16 there exists a Cauchy sequence $\{g_m\}$ in $\mathscr{H}(\bar{A})$ such that $\lim_{m\to\infty}\|g_m\|_{\mathscr{H}(\bar{A})}=\|a\|_{[\bar{A}]_\theta}$, $\lim_{m\to\infty}\|a_m-a\|_{[\bar{A}]_\theta}=0$, where $a_m=g_m(\theta_1, \theta_2,..., \theta_{n-1})$ and each g_m is of the form $g_m(z)=\sum_{k=1}^{N_m}\psi_{k,m}(z)\,b_{k,m}$ where $\psi_{k,m}(z)$ is a scalar valued analytic function for all

$z=(z_1, z_2,..., z_{n-1}) \in \mathbb{C}^{n-1}$ and is bounded on $\bar{\Omega}$, and where $b_{k,m} \in \Delta(\bar{A})$. Define the function f_m on D by

$$f_m(w) = g_m(z_1(w), z_2(w),..., z_{n-1}(w)) = \sum_{k=1}^{N_m} \varphi_{k,m}(w)\, b_{k,m},$$

where $\varphi_{k,m}(w) = \psi_{k,m}(z_1(w), z_2(w),..., z_{n-1}(w))$. Then $f_m \in \mathscr{G}(\bar{A}, \bar{\Gamma})$ and for a.e. $\gamma \in \Gamma_j$

$(z_1(\gamma), z_2(\gamma),..., z_{n-1}(\gamma)) \in \partial\Omega_j$ and $f_m(\gamma) = \lim_{w \rhd \gamma} f_m(w) = g_m(z_1(\gamma),..., z_{n-1}(\gamma))$. Thus $\|f_m\|_{\mathscr{F}(\bar{A},\bar{\Gamma})} \leqslant \|g_m\|_{\mathscr{H}(\bar{A})}$ and $\|f_m - f_{m'}\|_{\mathscr{F}(\bar{A},\bar{\Gamma})} \leqslant \|g_m - g_{m'}\|_{\mathscr{H}(\bar{A})}$ for all positive integers m, m', and so the elements $a_m = g_m(\theta_1, \theta_2,..., \theta_{n-1}) = f_m(\zeta) \in \Delta(\bar{A})$ form a Cauchy sequence in $\bar{A}_{[\zeta],\Gamma}$ as well as in $\bar{A}_{[\theta]}$ and $\Sigma(\bar{A})$. Since $\bar{A}_{[\zeta],\Gamma}$ is complete and continuously embedded in $\Sigma(\bar{A})$ we deduce that $a \in \bar{A}_{[\zeta],\Gamma}$ and $\|a\|_{\bar{A}_{[\zeta],\Gamma}} \leqslant \|a\|_{[\bar{A}]_\theta}$ which completes the proof.

4D. *St. Louis Spaces Are Not "Rearrangement Invariant"*

We begin by indicating the general strategy behind the construction of our example and by fixing some notation. Throughout this subsection Γ will denote the unit circle. We shall start by specifying two infinite interpolation families $\{A(\gamma)\}_{\gamma \in \Gamma}$ and $\{B(\gamma)\}_{\gamma \in \Gamma}$ (for the definition of interpolation families, see [C3, p. 206]) such that, for each $\gamma \in \Gamma$, $A(\gamma)$ is the same space $\mathscr{U}$ equipped, however, with a norm which varies continuously with γ. $\{B(\gamma)\}_{\gamma \in \Gamma}$ will be obtained by taking $B(\gamma) = A(1/\gamma)$ for all $\gamma \in \Gamma$. (In this subsection '$=$' signifies that the two norms are equal.) $\{B(\gamma)\}_{\gamma \in \Gamma}$ is thus a "measure preserving rearrangement" of $\{A(\gamma)\}_{\gamma \in \Gamma}$ with respect to harmonic measure on Γ at 0. Of course the spaces $A[0]$ and $B[0]$ (as defined in [C3, p. 209]) both coincide algebraically with $\mathscr{U}$. However, we will be able to arrange for the ratio $\|x\|_{A[0]}/\|x\|_{B[0]}$ to be arbitrarily large for suitable elements x and for suitable choices of a certain parameter in the definition of $\{A(\gamma)\}_{\gamma \in \Gamma}$.

To obtain counterexamples in the setting of n-tuples we must "discretize" the above situation. Thus we shall divide Γ into n arcs of equal length $\Gamma_1, \Gamma_2,..., \Gamma_n$, where $\Gamma_j = \{e^{it} \mid 2\pi(j-1)/n \leqslant t < 2\pi j/n\}$, and let $\bar{A} = (A_1, A_2,..., A_n)$ be defined by $A_j = A(\gamma_j)$ where $\gamma_j = e^{i2\pi(j-1/2)/n}$. Clearly $\bar{A}_{[0],\Gamma} = E[0]$ where $\{E(\gamma)\}_{\gamma \in \Gamma}$ is the interpolation family defined by $E(\gamma) = A(\gamma_j)$ for $\gamma \in \Gamma_j$ and $\bar{\Gamma} = \{\Gamma_1, \Gamma_2,..., \Gamma_n\}$. Similarly, if $\bar{\Gamma}^* = \{\Gamma_1^*, \Gamma_1^*,..., \Gamma_n^*\}$ where for each j the set $\Gamma_j^* = \{\gamma \in \Gamma \mid 1/\gamma \in \Gamma_j\}$ coincides, except for its endpoints which may be neglected, with Γ_{n+1-j}, then $\bar{A}_{[0],\Gamma^*} = F[0]$ where $F(\gamma) = E(1/\gamma)$. An appropriate adaptation of the preceding estimates for norms in $A[0]$ and $B[0]$ will show, as required, that the ratio $\|x\|_{F[0]}/\|x\|_{E[0]} = \|x\|_{\bar{A}_{[0],\Gamma^*}}/\|x\|_{\bar{A}_{[0],\Gamma}}$ can also be made arbitrarily large for suitable elements x.

We now turn to the specific details of the construction. Let $\mathscr{U}$ be the space of complex valued functions on the unit disc D obtained as the closure of the analytic polynomials with respect to the norm $\|\varphi\|_{\mathscr{U}} = ((1/2\pi)\int_0^{2\pi} |\varphi(e^{it})|^p\, dt)^{1/p}$. Thus $\mathscr{U}$ is the Hardy space $H^p(D)$ if $1 \leqslant p < \infty$ (see, e.g., [G, p. 59]) and $\mathscr{U}$ is the disc algebra if $p = \infty$. For each $\gamma \in \Gamma$ we define $A(\gamma)$ to be $\mathscr{U}$ renormed equivalently by

$$\|\varphi\|_{A(\gamma)} = \left(\frac{1}{2\pi}\int_0^{2\pi} |\varphi(e^{it})\, e^{\lambda\gamma e^{it}}|^p\, dt\right)^{1/p},$$

where λ is an arbitrary positive constant. Clearly

$$e^{-\lambda}\|\varphi\|_{A(\gamma)} \leqslant \|\varphi\|_{\mathscr{U}} \leqslant e^{\lambda}\|\varphi\|_{A(\gamma)}$$

and

$$e^{-\lambda|\gamma-\gamma'|}\|\varphi\|_{A(\gamma')} \leqslant \|\varphi\|_{A(\gamma)} \leqslant e^{\lambda|\gamma-\gamma'|}\|\varphi\|_{A(\gamma')} \tag{4.1}$$

for all $\gamma, \gamma' \in \Gamma$, and $\varphi \in \mathscr{U}$. It follows immediately that $\{A(\gamma)\}_{\gamma\in\Gamma}$ and $\{B(\gamma)\}_{\gamma\in\Gamma}$ are interpolation families. We shall now estimate the norms $\|1\|_{B[0]}$, $\|1\|_{F[0]}$, and subsequently $\|1\|_{A[0]}$ and $\|1\|_{E[0]}$, of the function which assumes the constant value 1 on $D \cup \Gamma$. We shall first show that

$$\|1\|_{B[0]} \geqslant e^{\lambda} \tag{4.2}$$

if $p = \infty$ and

$$\|1\|_{B[0]} \geqslant (\varepsilon/2\pi)^{1/p} e^{\lambda\sqrt{1-\varepsilon^2}} \tag{4.3}$$

for every ε, $0 < \varepsilon < 1$, if $1 \leqslant p < \infty$.

The calculations here are a "model" for our main step which will be to obtain the estimates

$$\|1\|_{F[0]} \geqslant e^{(\lambda n/\pi)\sin(\pi/n)} \tag{4.4}$$

for $p = \infty$ and

$$\|1\|_{F[0]} \geqslant (\varepsilon/2\pi)^{1/p} e^{\lambda\sqrt{1-\varepsilon^2}(n/\pi)\sin(\pi/n)} \tag{4.5}$$

for $1 \leqslant p < \infty$ and every ε, $0 < \varepsilon < 1$.

Let $g(w) = \sum_{k=1}^N \varphi_k(w)\, a_k$ be an element of $\mathscr{G}(B(\cdot), \Gamma)$ [C3, p. 207] with $a_k \in \mathscr{U}$ and φ_k a bounded scalar valued analytic function on D. (We recall that, although $\mathscr{G}(B(\cdot), \Gamma)$ may contain such functions g for which the φ_k may be unbounded, by Proposition 2.5 of [C3, p. 210] it suffices to consider only bounded functions φ_k in the process for estimating the norm of $B[0]$.) We may consider g as a scalar valued function of two variables on $D \times D$, $g(w, z) = \sum_{k=1}^N \varphi_k(w)\, a_k(z)$. There are now two cases to be dealt with:

Case i. $p=\infty$. Here we define a bounded analytic function h on D by $h(w)=g(w,w)$. The nontangential limit of h, $h(\gamma)=\sum_{k=1}^{N}\lim_{w \rhd \gamma}\varphi_k(w)\, a_k(\gamma)=\sum_{k=1}^{N}\varphi_k(\gamma)\, a_k(\gamma)$ exists for a.e. $\gamma\in\Gamma$ and in fact

$$|h(\gamma)\, e^{\lambda}| \leqslant \sup_{z\in D}\left|\sum_{k=1}^{N}\varphi_k(\gamma)\, a_k(z)\, e^{\lambda z/\gamma}\right| = \|g(\gamma)\|_{B(\gamma)}. \tag{4.6}$$

Consequently $|g(0,0)|=|h(0)|\leqslant \operatorname{ess\,sup}_{\varphi\in\Gamma}|h(\gamma)|\leqslant e^{-\lambda}\|g\|_{\mathscr{G}(B(\cdot),\Gamma)}$. Let $f\in\mathscr{F}(B(\cdot),\Gamma)$ be the limit with respect to the norm $\|\ \|_{\mathscr{G}(B(\cdot),\Gamma)}$ of a sequence of functions in $\mathscr{G}(B(\cdot),\Gamma)$ of the above form such that $f(0)=1$ (constant function of z). The preceding inequality implies that $1\leqslant e^{-\lambda}\|f\|_{\mathscr{G}(B(\cdot),\Gamma)}$ for all such f. This proves (4.2). A rather similar argument will now give (4.4). Indeed for a.e. $\gamma\in\Gamma_j$, much as in (4.6),

$$\begin{aligned}|h(\gamma)\, e^{\lambda\gamma/\gamma_j}| &\leqslant \sup_{z\in D}\left|\sum_{k=1}^{N}\varphi_k(\gamma)\, a_k(z)\, e^{\lambda z/\gamma_j}\right| = \|g(\gamma)\|_{B(\gamma_j)}\\ &= \|g(\gamma)\|_{F(\gamma)}\leqslant \|g\|_{\mathscr{G}(F(\cdot),\Gamma)}.\end{aligned}$$

Consequently,

$$\begin{aligned}|h(0)| &\leqslant \exp\int_{\Gamma}\log|h(\gamma)|\, dP_0(\gamma)\\
&\leqslant \|g\|_{\mathscr{G}(F(\cdot),F)}\exp\left(\sum_{j=1}^{n}\int_{2\pi(j-1)/n}^{2\pi j/n} -\lambda\cos(t-2\pi(j-1/2)/n)\, dt/2\pi\right)\\
&= \|g\|_{\mathscr{G}(F(\cdot),\Gamma)}\exp\left(n\int_{-\pi/n}^{\pi/n} -\lambda\cos t\, dt/2\pi\right)\\
&= \|g\|_{\mathscr{G}(F(\cdot),\Gamma)}\exp\left(-\frac{n}{\pi}\lambda\sin\frac{\pi}{n}\right).\end{aligned}$$

From this inequality we deduce, analogously to before, that (4.4) holds.

Case ii. $1\leqslant p<\infty$. Here the function $g=g(w,z)\in\mathscr{G}(B(\cdot),\Gamma)$ is introduced as before, but instead of the function $h(w)=g(w,w)$ we use $k(w)=k_u(w)=g(w,e^{iu}w)$ where u is a (temporarily) constant real number. Clearly $k\in H^p(D)$ so that

$$|g(0,0)\, e^{\lambda e^{iu}}|^p = |k(0)\, e^{\lambda e^{iu}}|^p \leqslant \int_0^{2\pi}|k(e^{it})\, e^{\lambda e^{iu}}|^p\, dt/2\pi.$$

Integrating this inequality with respect to u now yields

$$\begin{aligned}
&|g(0,0)|^p \int_0^{2\pi} e^{\lambda p \cos u}\, du \\
&\leqslant \int_0^{2\pi} \int_0^{2\pi} |g(e^{it}, e^{i(t+u)})\, e^{\lambda e^{iu}}|^p\, dt\, du/2\pi \\
&= \int_0^{2\pi} \int_0^{2\pi} |g(e^{it}, e^{is})\, e^{\lambda e^{is}/e^{it}}|^p\, ds\, dt/2\pi \\
&= \int_\Gamma \int_0^{2\pi} |g(\gamma, e^{is})\, e^{\lambda e^{is}/\gamma}|^p\, ds\, dP_0(\gamma) \\
&= \int_\Gamma 2\pi \|g(\gamma)\|^p_{B(\gamma)}\, dP_0(\gamma) \leqslant 2\pi \|g\|^p_{\mathscr{G}(B(\cdot),\Gamma)}.
\end{aligned}$$

Now

$$\int_0^{2\pi} e^{\lambda p \cos u}\, du \geqslant \int_0^{\varepsilon} e^{\lambda p \cos \varepsilon}\, du \geqslant \varepsilon e^{\lambda p \sqrt{1-\varepsilon^2}} \qquad \text{for all} \quad 0 \leqslant \varepsilon \leqslant 1.$$

Thus $|g(0,0)| \leqslant (2\pi/\varepsilon)^{1/p} e^{-\lambda\sqrt{1-\varepsilon^2}} \|g\|_{\mathscr{G}(B(\cdot),\Gamma)}$ and a repetition of by now familiar arguments proves (4.3). The proof of (4.5) will proceed similarly as follows: Let $\alpha(\gamma) = \gamma_j$ for all $\gamma \in \Gamma_j$ and let $\varphi_u(w)$ be a (bounded) outer function on D such that $|\varphi_u(\gamma)| = |e^{\lambda e^{iu}\gamma/\alpha(\gamma)}|$ for a.e. $\gamma \in \Gamma$. Here again u is a (constant) real number.

$$|g(0,0)\, \varphi_u(0)|^p = |k(0)\, \varphi_u(0)| \leqslant \int_0^{2\pi} |k(e^{it})\, \varphi_u(e^{it})|^p\, dt/2\pi.$$

Similarly to before

$$\begin{aligned}
&|g(0,0)|^p \int_0^{2\pi} |\varphi_u(0)|^p\, du \\
&\leqslant \int_0^{2\pi} \int_0^{2\pi} |g(e^{it}, e^{i(t+u)})\, e^{\lambda e^{i(u+t)}/\alpha(e^{it})}|^p\, du\, dt/2\pi \\
&= \int_\Gamma \int_0^{2\pi} |g(\gamma, e^{is})\, e^{\lambda e^{is}/\alpha(\gamma)}|^p\, ds\, dP_0(\gamma) \\
&= \int_\Gamma 2\pi \|g(\gamma)\|^p_{F(\gamma)}\, dP_0(\gamma) \leqslant 2\pi \|g\|_{\mathscr{G}(B(\cdot),\Gamma)}.
\end{aligned}$$

Now

$$\begin{aligned}\varphi_u(0) &= \exp\left(\int_0^{2\pi} \log|\varphi_u(e^{it})|\,dt/2\pi\right)\\ &= \exp\left(\sum_{j=1}^{n}\int_{2\pi(j-1)/n}^{2\pi j/n} \log|e^{\lambda e^{i[u+t-2\pi(j-1/2)]/n}}|\,dt/2\pi\right)\\ &= \exp\left(n\int_{-\pi/n}^{\pi/n} \lambda\cos(u+t)\,dt/2\pi\right)\\ &= \exp\left(\frac{n}{\pi}\lambda\cos u\sin\frac{\pi}{n}\right).\end{aligned}$$

Thus, for $0<\varepsilon\leqslant 1$,

$$\begin{aligned}\int_0^{2\pi}|\varphi_u(0)|^p\,du &\geqslant \int_0^{\varepsilon}\exp\left(\frac{n\lambda p}{\pi}\sqrt{1-\varepsilon^2}\sin\frac{\pi}{n}\right)du\\ &= \varepsilon\exp\left(\lambda p(\sqrt{1-\varepsilon^2})\frac{n}{\pi}\sin\frac{\pi}{n}\right).\end{aligned}$$

Consequently $|g(0,0)|^p\leqslant(2\pi/\varepsilon)\exp(-\lambda p\sqrt{1-\varepsilon^2}\,(n/\pi)\sin(\pi/n))\,\|g\|_{\mathscr{G}(F(\cdot),\Gamma)}$ and (4.5) follows.

The estimates for $\|1\|_{A[0]}$ and $\|1\|_{E[0]}$ are rather more straightforward. We use the function $f(w,z)=e^{-\lambda wz}=\sum_{m=0}^{\infty}(-\lambda wz)^m/m!$ The partial sums of this series are of course in $\mathscr{G}(A(\cdot),\Gamma)$ and converge uniformly on $\bar{D}\times\bar{D}$ and therefore also in the norm of $\mathscr{G}(A(\cdot),\Gamma)$ for all $p\in[1,\infty]$. Thus $f\in\mathscr{F}(A(\cdot),\Gamma)$ and $f(0,z)=1$ and $\|1\|_{A[0]}\leqslant\|f\|_{\mathscr{G}(A(\cdot),\Gamma)}=\operatorname{ess\,sup}_{\gamma\in\Gamma}\|f(\gamma,z)\,e^{\lambda\gamma z}\|_{\mathscr{U}}=1$. Also the identity operator is bounded from $A(\gamma)$ to $E(\gamma)$ with norm $e^{\lambda|\gamma-\gamma_j|}=e^{\lambda|1-\gamma/\gamma_j|}$ (see (4.1)) for all $\gamma\in\Gamma_j$. So we can apply the interpolation theorem [C3, Theorem 4.1, p. 216] to obtain that

$$\begin{aligned}\|1\|_{E[0]} &\leqslant \exp\left(\sum_{j=1}^{n}\int_{\Gamma_j}\log e^{\lambda|1-\gamma/\gamma_j|}\,dP_0(\gamma)\right)\|1\|_{A[0]}\\ &\leqslant \exp\left(\frac{\lambda n}{2\pi}\int_{-\pi/n}^{\pi/n}\left|2\sin\frac{t}{2}\right|dt\right)=\exp\left[\frac{4\lambda n}{\pi}\left(1-\cos\frac{\pi}{2n}\right)\right].\end{aligned}$$

Finally we can combine all the above estimates to show that $\|1\|_{B[0]}/\|1\|_{A[0]}$ can be made arbitrarily large by suitable choices of λ (and of ε if $p<\infty$), and also that

$$\|1\|_{F[0]}/\|1\|_{E[0]}\geqslant\exp\left[\frac{\lambda n}{\pi}\left(\sin\frac{\pi}{n}-4\left(1-\cos\frac{\pi}{2n}\right)\right)\right]\qquad\text{if}\quad p=\infty$$

and otherwise

$$\|1\|_{F[0]}/\|1\|_{E[0]} \geqslant (\varepsilon/2\pi)^{1/p} \exp\left[\frac{\lambda n}{\pi}\left((\sqrt{1-\varepsilon^2}) \sin\frac{\pi}{n} - 4\left(1-\cos\frac{\pi}{2n}\right)\right)\right].$$

Therefore this ratio can be arbitrarily large, for suitable choices of λ and ε, provided that $\sin(\pi/n) - 4(1-\cos(\pi/2n)) > 0$. This holds for all $n \geqslant 3$ and indeed for $n \geqslant 2.1834$.

We shall next describe a rather simpler two-dimensional version of the interpolation family $\{A(\gamma)\}_{\gamma \in \Gamma}$. Although as before we obtain that the ratio $\|x\|_{B[0]}/\|x\|_{A[0]}$ can be arbitrarily large, we are not able to deduce the same here concerning $\|x\|_{F[0]}/\|x\|_{E[0]}$ for any choice of n. This naturally raises the question of whether for each m we can find a triple of two dimensional spaces $\bar{A}^m$ and elements x_m such that $\|x_m\|_{\bar{A}^m_{[0],\Gamma^*}}/\|x_m\|_{\bar{A}^m_{[0],\Gamma}} \geqslant m$. (One possible approach to such a construction might be to use a finite dimensional "approximation" to a triple of H^2 spaces suitably weighted as above. (Cf. also Remark 4.8.) In view of the Masani–Wiener theorem (see [C1, p. 279] and Remark 3.15) the complex interpolation spaces for such a triple of finite dimensional Hilbert spaces are determined via an analytic matrix valued function which can perhaps be composed with projections onto a suitable two-dimensional subspace to yield the required example.

4.7. Example. For each $\gamma \in \Gamma$ let $A(\gamma)$ be $\mathbb{C}^2$ renormed by $\|(z_1, z_2)\|_{A(\gamma)} = |z_1| + |z_2| + \lambda|z_1\gamma - z_2|$ where λ is an arbitrary positive constant. (We also take $\mathscr{U}$ to be $\mathbb{C}^2$ with any norm we please.) Then $g(w) = (1, w)$ is a $\mathbb{C}^2$ valued analytic function in the class $\mathscr{G}(A(\cdot), \Gamma)$ and $\|g(\gamma)\|_{A(\gamma)} = 1 + 1 + 0$. Consequently $\|(1,0)\|_{A[0]} = \|g(0)\|_{A[0]} \leqslant 2$. To estimate $\|(1,0)\|_{B[0]}$, where, as above, $B(\gamma) = A(1/\gamma)$, consider any function $f \in \mathscr{F}(B(\cdot), \Gamma)$ such that $f(0) = (1, 0)$. Then $f(w) = (\varphi_1(w), \varphi_2(w))$ where φ_1 and φ_2 are bounded scalar valued analytic functions on the unit disc D, also defined (via non tangential limits) for a.e. $\gamma \in \Gamma$, and $\varphi_1(0) = 1$, $\varphi_2(0) = 0$. Then, for a.e. $\gamma \in \Gamma$, $\|f(\gamma)\|_{B(\gamma)} = \|(\varphi_1(\gamma), \varphi_2(\gamma))\|_{B(\gamma)} \geqslant \lambda|\varphi_1(\gamma)/\gamma - \varphi_2(\gamma)| = \lambda|\varphi_1(\gamma) - \gamma\varphi_2(\gamma)|$. Applying the maximum modulus principle to the function $\varphi_1(w) - w\varphi_2(w)$ yields that

$$\lambda = \lambda|\varphi_1(0)| \leqslant \lambda \operatorname*{ess\,sup}_{\gamma \in \Gamma} |\varphi_1(\gamma) - \gamma\varphi_2(\gamma)| \leqslant \operatorname*{ess\,sup}_{\gamma \in \Gamma} \|f(\gamma)\|_{B(\gamma)}.$$

Consequently $\|(1,0)\|_{B[0]} \geqslant \lambda$ and the ratio $\|(1,0)\|_{B[0]}/\|(1,0)\|_{A[0]} \geqslant \lambda/2$ can be arbitrarily large.

4.8. *Remark.* As promised we shall construct, given any number $m > 0$, a triple of finite dimensional spaces $\bar{Y} = (Y_1, Y_2, Y_3)$ for which, for some nonzero $y \in Y_1 \cap Y_2 \cap Y_3$,

$$\|y\|_{[Y'_1, Y'_2, Y'_3]_\theta} / \|y\|_{[Y_1, Y_2, Y_3]'_\theta} > m. \tag{4.9}$$

To begin with, for each $j = 1, 2, 3$, we choose A_j to be the disc algebra with weight $|e^{\lambda z \gamma_j}|$ as in the example above, and let Y_j be the subspace of polynomials of degree at most N. By choosing first λ and then N large enough we obtain as above that

$$\|1\|_{\bar{Y}_{[0],\Gamma^*}} \geqslant \|1\|_{\bar{A}_{[0],\Gamma^*}} > m\|1\|_{\bar{Y}_{[0],\Gamma}}. \tag{4.10}$$

Y_1, Y_2 and Y_3 coincide algebraically, and so do Y'_1, Y'_2, and Y'_3. Suppose now that (4.9) is false for all y. Then the norm of the identity mapping $[\bar{Y}]'_\theta \to [\bar{Y}']_\theta$ does not exceed m. Furthermore, by Theorem 1.17, the norm of the identity mapping $[\bar{Y}']_\theta \to \bar{Y}'_{[0],\Gamma}$ is at most 1 and, by the duality theorem for St. Louis spaces [C3, pp. 214, 216, 228], $\bar{Y}'_{[0],\Gamma} = (\bar{Y}_{[0],\Gamma})'$ isometrically. Hence the identity mapping $[\bar{Y}]'_\theta \to (\bar{Y}_{[0],\Gamma})'$ has norm less than m. By duality and Theorem 1.17, the identity mapping $\bar{Y}_{[0],\Gamma} \to [\bar{Y}]_\theta \to \bar{Y}_{[0]\Gamma^*}$ has norm less than m. This contradicts (4.10), and shows that $\bar{Y}$ has the required property (4.9).

4E. *The Inclusions* $[\bar{A}]_\theta \subset \bar{A}_{\theta,\infty;K}$, $\bar{A}_{[\zeta],\Gamma} \subset \bar{A}_{\theta,\infty;K}$: *Theorem* 1.23

We shall first prove the second inclusion. Given any element $a \in \bar{A}_{[\zeta],\Gamma}$ with norm less than 1, choose $f \in \mathscr{F}(\bar{A}, \Gamma)$ with $f(\zeta) = a$ and $\|f\|_{\mathscr{F}(\bar{A},\Gamma)} \leqslant 1$. Then for any $\bar{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n_+$ let $\varphi(z)$ be a bounded outer function on $D \cup \Gamma$ such that $|\varphi(\gamma)| = 1/t_j$ for a.e. $\gamma \in \Gamma_j$, $j = 1, 2, \ldots, n$. $\varphi(\zeta)\, a = \int_\Gamma \varphi(\gamma) f(\gamma)\, dP_\zeta(\gamma)$ so

$$a = \sum_{j=1}^n a_j, \qquad \text{where} \quad a_j = (1/\varphi(\zeta)) \int_{\Gamma_j} \varphi(\gamma) f(\gamma)\, dP_\zeta(\gamma).$$

(The above integrals are defined in the sense of Bochner for $\Sigma(\bar{A})$ valued and for A_j valued functions, respectively.)

$$\begin{aligned} &K(\bar{t}, a; \bar{A}) \\ &\leqslant \sum_{j=1}^n t_j \|a_j\|_{A_j} \\ &\leqslant |1/\varphi(\zeta)| \sum_{j=1}^n t_j \int_{\Gamma_j} \|f(\gamma)\|_{A_j} / t_j\, dP_\zeta(\gamma) \\ &\leqslant \exp \int_\Gamma \log |1/\varphi(\gamma)|\, dP_\zeta(\gamma) \sum_{j=1}^n \theta_j = t_1^{\theta_1} t_2^{\theta_2} \cdots t_n^{\theta_n}. \end{aligned}$$

This shows that $a \in \bar{A}_{\theta,\infty;K}$ with $\|a\|_{\bar{A}_{\theta,\infty;K}} \leqslant 1$ and proves that $\bar{A}_{[\zeta],\Gamma} \subset \bar{A}_{\theta,\infty;K}$.

The inclusion $[\bar{A}]_\theta \subset \bar{A}_{\theta,\infty;K}$ now follows as an immediate corollary. We simply choose any $\bar{\Gamma} = \{\Gamma_1, ..., \Gamma_n\}$ and $\zeta \in D$ such that $P_\zeta(\Gamma_j) = \theta_j$ (e.g., Γ can be the unit circle divided into disjoint arcs Γ_j of length $2\pi\theta_j$, $j = 1, 2, ..., n$ and $\zeta = 0$). Then we apply Theorem 1.17.

4F. *The Inclusion for n-tuples of Hilbert Spaces*, $[\bar{A}]_\theta \subset \bar{A}_{\theta,2;J}$

Let $g(z) = e_\delta(z) \sum_{k=1}^N e^{(\lambda_k, z)} a_k$ be an element in $\mathscr{H}(\bar{A})$ of the form defined in Lemma 1.16, where, as in the proof of that lemma, we use the notation $e_\delta(z) = e^{\delta \sum_{j=1}^{n-1} z_j^2}$. Let $g_1(z) = e_1(z)\, g(z)$ and, much as in the proof of Lemma 1.16, we use the function u: $\mathbb{R}^{n-1} \to \Delta(\bar{A})$ defined by

$$u(s) = \int_{\mathbb{R}^{n-1}} e^{(x+it, s)} g_1(x+it)\, dt$$

which is independent of x. In particular if $x \in \partial\Omega_j$ we can apply Parseval's equation for Hilbert space valued functions to show that

$$\left(\int_{\mathbb{R}^{n-1}} \|e^{-(x,s)} u(s)\|_{A_j}^2\, ds\right)^{1/2} = (2\pi)^{(n-1)/2} \left(\int_{\mathbb{R}^{n-1}} \|g_1(x+it)\|_{A_j}^2\, dt\right)^{1/2}$$
$$\leqslant C \|g\|_{\mathscr{H}(\bar{A})} \tag{4.11}$$

where C is a constant depending only on n. Also, as in the proof of Lemma 1.16,

$$g_1(z) = (2\pi)^{-(n-1)} \int_{\mathbb{R}^{n-1}} e^{-(z,s)} u(s)\, ds \qquad \text{for all} \quad z \in \bar{\Omega}.$$

We now define a continuous $\Delta(\bar{A})$ valued function $v(\bar{t})$ on $\mathbb{R}_+^n$. Actually we only need to know its value on the set $E \subset \mathbb{R}_+^n$ (Definition 1.6). Thus we take $v(t_1, t_2, ..., t_{n-1}, 1) = C_1 \prod_{j=1}^{n-1} t_j^{\theta_j} u(-\log t_1, ..., -\log t_{n-1})$ where the constant $C_1 = 1/(2\pi)^{n-1} e_1(\theta_1, \theta_2, ..., \theta_{n-1})$.

$$\int_E v(\bar{t})\, d\mu(\bar{t}) = C_1 \int_{\mathbb{R}^{n-1}} \prod_{j=1}^{n-1} e^{-\theta_j s_j} u(s)\, ds = C_1 (2\pi)^{n-1} g_1(\theta_1, ..., \theta_{n-1})$$
$$= g(\theta_1, ..., \theta_{n-1})$$

Now we can estimate the norm

$$\|g(\theta_1, ..., \theta_{n-1})\|_{\bar{A}_{\theta,2;J}}$$
$$\leqslant \left(\int_E (\bar{t}^{-\theta} J(\bar{t}, v(\bar{t}); \bar{A}))^2\, d\mu(\bar{t})\right)^{1/2}$$

$$\leqslant C_1\left(\int_E \sum_{j=1}^{n} t_j^2 \,\|u(-\log t_1,..., -\log t_{n-1})\|_{A_j}^2 \,d\mu(\bar{t})\right)^{1/2}$$

$$= C_1\left(\sum_{j=1}^{n} \int_{\mathbb{R}^{n-1}} \|e^{-s_j}u(s_1,..., s_{n-1})\|_{A_j}^2 \,ds\right)^{1/2}$$

$$\leqslant C_1 C\sqrt{n}\,\|g\|_{\mathscr{H}(\bar{A})} \qquad \text{by} \quad (4.11).$$

Using standard density and completeness arguments we can deduce from the above inequality that, for every $a \in [\bar{A}]_\theta$, $a \in \bar{A}_{\theta,2;J}$ with $\|a\|_{\bar{A}_{\theta,2;J}} \leqslant C_1 C\sqrt{n}\,\|a\|_{[\bar{A}]_\theta}$.

5. Calculations of Spaces of Fernandez for the 4-tuple (B_1, B_2, B_2, B_1)

Proof of (1.26). If $f(z_1, z_2) \in H(B_1, B_2, B_2, B_1)$ then the function $g(z) = f(z, z)$ is analytic in $\{z \mid 0 < \operatorname{re} z < 1\}$ and continuous up to the boundary on which it has values in B_1. Thus it follows that $f(z, z) \in B_1$ (using, e.g., a simpler version of the argument in Sects. 9.1 and 29.1 of [Ca]). Similarly $f(z, 1-z) \in B_2$ and so $f(\frac{1}{2}, \frac{1}{2}) \in B_1 \cap B_2$.

Proof of (1.27). This is left as an exercise for the reader.

Proof of (1.28), (1.29), *and* (1.30). Observe that

$$K(t_1, t_2, a, \bar{A}) = \min(1, t_1 t_2)\, K\left(\frac{\min(t_1, t_2)}{\min(1, t_1 t_2)}, a; B_1, B_2\right).$$

Therefore, if $t_1 = e^{u+v}$, $t_2 = e^{u-v}$, then $(t_1 t_2)^{-1/2} K(t_1, t_2, a; \bar{A}) = e^{-|u|}K(e^{|u|-|v|}, a; B_1, B_2)$. This gives (1.29), and (1.28) also follows by first using the variables u, v and then $|u|$, $|u|-|v|$ to calculate the double integral. Inequality (1.30) follows from (1.28), except in trivial cases such as $B_1 = B_2$. (In fact it can be shown that (1.30) fails if and only if the norms $\|\ \|_{B_1}$ and $\|\ \|_{B_2}$ are equivalent on $B_1 \cap B_2$.)

6. Dependence on the Containing Space: Proof of Theorem 1.32

Let $\mathscr{V}$ be the quotient space of $\mathscr{U}$ modulo the one dimensional subspace generated by a_* and let T be the quotient map from $\mathscr{U}$ onto $\mathscr{V}$. Let $B_j = TA_j$ with $\|b\|_{B_j} = \|a\|_{A_j}$ for every $b = Ta \in B_j$. (Of course, since $a_* \notin A_j$, every coset $b \in B_j$ is the image of a unique $a \in A_j$.) Since T maps A_j into B_j boundedly (isometrically) and also maps $\mathscr{U}$ continuously into $\mathscr{V}$, an appropriate interpolation theorem for the method Φ shows that $\|Ta\|_{B_\Phi} \leqslant \|a\|_{\bar{A}_\Phi}$ for all $a \in \bar{A}_\Phi$. In particular, if $(a_m)_{m=1}^\infty$ is a sequence in $\Delta(\bar{A})$ such that $\|a_m - a_*\|_{\bar{A}_\Phi} \to 0$, then for $b_m = Ta_m$, $\|b_m\|_{B_\Phi} = \|Ta_m - Ta_*\|_{B_\Phi} \to 0$.

We now define the operator S_j: $B_j \to A_j$ to be the inverse of the restriction of T to A_j. S_j is of course an isometry. Furthermore if $b \in B_j \cap B_k$ then $S_j b = S_k b$ since the difference of these elements must be a scalar multiple of a_* and in $A_j + A_k$. Thus the operator S, which is the common restriction to $\Delta(\bar{B})$ of the operators S_1, S_2, S_3, is well defined and satisfies $\|Sb_m\|_{\bar{A}_\Phi} / \|b_m\|_{B_\Phi} \to \infty$ proving Theorem 1.32.

6.1. *Remarks.* (i) We cannot apply the interpolation theorem to S since it is not defined from $\mathscr{V}$ into $\mathscr{U}$. We now see that such a requirement, which is imposed in the formulation of interpolation theorems in [Sp, C3], etc., is not superfluous. Note that S is also not defined from $\Sigma(\bar{B})$ into $\Sigma(\bar{A})$. Thus, from the present point of view, the simple algebraic argument mentioned in Sect. 1, which enables an operator defined from A_j to B_j, $j = 1, 2$, and consistently from $A_1 \cap A_2$ to $B_1 \cap B_2$ to be extended uniquely to an operator from $A_1 + A_2$ to $B_1 + B_2$, is a sort of "accident" which happens to work for $n = 2$ but for no n greater than 2.

(ii) It is not difficult to produce examples of triples $\bar{A} = (A_1, A_2, A_3)$ satisfying the hypothesis of Theorem 1.32. For example, let (X_1, X_2) be a Banach couple with an element $x \in [X_1, X_2]_{1/3} \backslash X_1$.

Let $A_1 = X_2 \oplus X_1 \oplus X_1$, $A_2 = X_1 \oplus X_2 \oplus X_1$, $A_3 = X_1 \oplus X_1 \oplus X_2$, with containing space $\mathscr{U} = (X_1 + X_2) \oplus (X_1 + X_2) \oplus (X_1 + X_2)$. Then if $P_0(\Gamma_1) = P_0(\Gamma_2) = P_0(\Gamma_3) = \frac{1}{3}$ it is not hard to show (cf. [C3, Theorem 5.1, p. 218]) that

$$\bar{A}_{[0],\Gamma} = [X_1, X_2]_{1/3} \oplus [X_1, X_2]_{1/3} \oplus [X_1, X_2]_{1/3},$$

and to see that a suitable choice for a_* is $a_* = (x, x, x)$.

(iii) There seems to be some connection between the present theorem and the counterexample in [C3, Appendix 1] showing that the St. Louis space $A[z]$ does not coincide in general with a second space $A\{z\}$ obtained by a similar construction which we shall not bother to define precisely here. Since $A\{z\}$ is obtained by abstract completion of the space $\Delta(\bar{A})$ with respect to a suitable norm, its construction does not depend on the containing space and so Theorem 1.32 gives us a general and simpler way for constructing further examples where $A[z]$ and $A\{z\}$ do not coincide. This leads naturally to a "converse" question: Can we always make the spaces $A[z]$ and $A\{z\}$ coincide (as they do in many examples) by choosing a different "more natural" embedding for the n-tuple or infinite family which generates them? We rephrase the question more precisely: Given any n-tuple $\bar{A}$ (or more generally an infinite interpolation family

$\{A(\gamma)\}_{\gamma \in \Gamma}$) is it possible to construct a second n-tuple $\bar{B}$ (or interpolation family $\{B(\gamma)\}_{\gamma \in \Gamma}$) having the property that $B\{z\} = B[z]$ and which is "isomorphic" to $\bar{A}$ (or $\{A(\gamma)\}_{\gamma \in \Gamma}$) in the sense that there exists an isomorphism between $\Delta(\bar{A})$ and $\Delta(\bar{B})$ (or the log-intersection spaces of the two infinite families) which extends to an isomorphism of A_j onto B_j for each j ($A(\gamma)$ onto $B(\gamma)$ for each $\gamma \in \Gamma$) and which is "pairwise consistent" as in the above theorem?

One reasonable way to construct such an n-tuple $\bar{B}$ might be to first take as containing space $\mathscr{V}$ the abstract completion of $\Delta(\bar{A})$ with respect to the norm

$$\|a\|_{\Sigma_0(\bar{A})} = \inf_{\substack{a = \sum_{j=1}^n a_j \\ a_j \in \Delta(\bar{A})}} \sum_{j=1}^n \|a_j\|_{A_j}$$

(cf. [C3, Appendix 1, p. 226]) then, provided $\Delta(\bar{A})$ is dense in A_j for each j, we can take B_j to be the subspace of $\mathscr{V}$ consisting of equivalence classes of Cauchy sequences for which at least one representative also converges with respect to $\| \ \|_{A_j}$. B_j and A_j will be isometrically isomorphic since they each contain the same dense subset $\Delta(\bar{A})$. Furthermore it can be shown that $\|a\|_{\Sigma_0(\bar{A})} = \|a\|_{\Sigma(\bar{B})}$ for all $a \in \Delta(\bar{A})$. In other words, the norm $\| \ \|_{\Sigma_0(\bar{A})}$ on $\Delta(\bar{A})$, which in general is not equivalent to the usual sum norm $\| \ \|_{\Sigma(\bar{A})}$ [C3, p. 226], nevertheless is the usual sum norm for a suitable different choice of containing space.

(iv) All the above also shows that the duality formulae $\Delta(\bar{A})' = \Sigma(\bar{A}')$ and $\Sigma(\bar{A})' = \Delta(\bar{A}')$, which are readily established for couples [BL], are not automatically valid for n-tuples if the containing spaces for the n-tuples $\bar{A}$ or $\bar{A}'$ are chosen "badly." In fact Jaak Peetre drew our attention to this difficulty some years ago.

Similarly the duality theory for St. Louis, Sparr, or Favini–Lions spaces needs careful formulation.

Note added in proof. In fact a more detailed study of the above duality formulae and related matters in the context of n-tuples can be found in G. Dore, D. Guidetti, and A. Venni. Some properties of the sum and intersection of normed spaces. *Atti Sem. Mat. Fis. Univ. Modena* **31** (1982), 325–331.

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