Path numbers of balanced bipartite tournaments

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Abstract

A path decomposition of a digraph $D$ is a partition of its edge set into edge disjoint simple paths. The minimal number of paths necessary to form a path decomposition is called the path number of $D$ and denoted by $pn(D)$. A bipartite tournament $T(A,B)$ with partition sets $A$ and $B$ is balanced if $|A| = |B| = n \geq 1$. We prove the following: (a) if $n$ is odd and $k$ is any odd integer from the interval $[n; n^2)$ or (b) if $n$ is even and $k$ is any even integer from the interval $[n/2+1; n^2]$, then there exists a balanced bipartite tournament $T(A,B)$, $|A| = |B| = n$, with $pn(T(A,B)) = k$.

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In general, the terminology is that of [2]. A path decomposition of a digraph $D$ is a partition of its edge set $E(D)$ into edge disjoint simple paths. The minimal number of paths necessary to form a path decomposition is called the path number of $D$ and denoted by $pn(D)$. The deficiency $d(v)$ of a vertex $v$ of $D$ is defined as $d(v) = d^+(v) - d^-(v)$, where $d^+(v)$ (resp. $d^-(v)$) is its outdegree (resp. indegree). The quantity $x(v) = \max\{0, d(v)\}$ is called excess of $v$. The excess of $D$ is defined by $x(D) = \sum_{v \in V(D)} x(v)$.

It has been shown in [2] that for any digraph $D$

$$x(D) \leq pn(D). \quad (1)$$

$D$ is consistent if equality holds in (1).

A bipartite tournament is an orientation of a complete bipartite graph. $T(A,B)$ will denote a bipartite tournament with partite sets $A$ and $B$. When no confusion can arise the shorten form $T$ will be used. A bipartite tournament $T(A,B)$ is balanced if $|A| = |B|$. An antidirected path (cycle) is an orientation of a path (cycle) such that each two adjacent edges are oriented oppositely.

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Theorem 3. Let \( T(A, B) \) be a balanced bipartite tournament with \( |A| = |B| = n \). If \( n \) is odd, then \( x(T) \geq n \).

**Proof.** Suppose that there is a balanced bipartite tournament \( T \) such that \( x(T) < n \). Since \( n \) is odd, \( d(v) \geq 1 \) for each \( v \in V(T) \). Let \( \{v_1, \ldots, v_k\} \cup \{v_{k+1}, \ldots, v_{2n}\} \), \( 1 \leq k \leq 2n - 1 \), be the partition of \( V(T) \) such that \( d(v_i) > 0 \) for \( i = 1, 2, \ldots, k \) and \( d(v_i) < 0 \) for \( i = k + 1, k + 2, \ldots, 2n \). Let \( d_i = d(v_i) \). From the identity \( \sum_{v \in V(T)} d^+(v) = \sum_{v \in V(T)} d^-(v) \) and the definition of vertex deficiency it follows \( d_1 + \cdots + d_k + d_{k+1} + \cdots + d_{2n} = 0 \). So,

\[
x(T) = d_1 + \cdots + d_k = |d_{k+1}| + \cdots + |d_{2n}|
\]

Since \( x(T) < n \) and \( d_i \geq 1 \) for \( i = 1, 2, \ldots, k, k < n \). It implies \( 2n - k > n \) and since \( |d_i| \geq 1 \), \( x(T) = |d_{k+1}| + \cdots + |d_{2n}| > n \) contradicting the assumption. \( \Box \)

As \( \text{pn}(T) \leq n^2 \) obviously holds we have:

**Corollary 1.** Let \( T(A, B) \) be a balanced bipartite tournament with \( |A| = |B| = n \). If \( n \) is odd, then \( n \leq \text{pn}(T) \leq n^2 \).

**Theorem 4.** Let \( T(A, B) \) be a balanced bipartite tournament with \( |A| = |B| = n \). If \( n \) is even, then \( n/2 + 1 \leq \text{pn}(T) \leq n^2 \).

**Proof.** The length of the longest path in \( T \) is at most \( 2n - 1 \). Hence \( \text{pn}(T) \geq [n^2 / 2n - 1] = n/2 + 1 \). The upper bound is the same as for \( n \) odd. \( \Box \)

The main results are contained in next two theorems.

**Theorem 5.** For each odd positive integer \( n \) and each odd \( k \), \( n \leq k \leq n^2 \), there exists a balanced bipartite tournament \( T(A, B) \), \( |A| = |B| = n \), such that \( \text{pn}(T) = k \).
Proof. Let $n = 2s + 1$, $s \geq 0$. Denoted by $T$ the bipartite tournament with partition sets $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$. The arc set of $T$ is given by

$$a_i \to \{b_i, b_{i+2}, \ldots, b_{i+2s}\}, \quad i = 1, 2, \ldots, n,$$

(2)

where all indices are taken modulo $n$ and all unspecified arcs are understood to be oriented from $B$ to $A$. We claim that $T$ is consistent and $pn(T) = n$.

Since by (2) $d(a_i) = 1$ and $d(b_i) = -1$, $x(T) = n$. To prove the claim, it is sufficient to decompose $E(T)$ into $n$ edge disjoint paths. Let $M_0, M_1, \ldots, M_s$ be 1-factors from $A$ to $B$ defined by

$$M_i = \{a_1b_{1+2i}, a_2b_{2+2i}, \ldots, a_{2s}b_{2s+2i+2}\}, \quad i = 0, 1, \ldots, s.$$ 

Similarly let $N_1, N_2, \ldots, N_s$ be 1-factors from $B$ to $A$ defined by

$$N_i = \{b_1a_{1+2i}, b_2a_{2+2i}, \ldots, b_{2s}a_{2s+1+2i}\}, \quad i = 1, 2, \ldots, s.$$ 

All indices are taken modulo $n$. It is easy to see that $E(T)$ is a disjoint union of $M_0, M_1, \ldots, M_s, N_1, N_2, \ldots, N_s$.

Consider $s$ subdigraphs $C_{1,s}, C_{2,s-1}, \ldots, C_{s,1}$ of $T$ induced by $M_1 \cup N_s, M_2 \cup N_{s-1}, \ldots, M_s \cup N_1$, respectively. Since $C_{i,s-\ell+1} = a_1 \rightarrow b_{1+2i} \rightarrow a_2 \rightarrow b_{2+2i} \rightarrow \cdots \rightarrow a_{2s} \rightarrow b_{2s+1} \rightarrow b_{2s+2} \rightarrow a_1$, each of them is a Hamiltonian cycle of $T$. Thus, $M_0 \cup E(C_{1,s}) \cup \cdots \cup E(C_{s,1})$ is a partition of $E(T)$ consisting of an 1-factor and $s$ Hamiltonian cycles.

Removing the arc $b_1a_{2s+1}$ from the cycle $C_{1,s}, b_2a_{2s+2}, \ldots, b_{s}a_{s+2}$ we obtain $s$ Hamiltonian paths $H_1 = C_{1,s} - b_1a_{2s+1}$, $H_2 = C_{2,s-1} - b_2a_{2s}, \ldots, H_s = C_{s,1} - b_{s}a_{s+2}$. The removed arcs and those of $M_0$ form $s$ edge disjoint 4-paths $P_1 = a_1 \rightarrow b_1 \rightarrow a_{2s+1} \rightarrow b_{2s+1}, P_2 = a_2 \rightarrow b_2 \rightarrow a_{2s+2} \rightarrow b_{2s} \rightarrow P_s = a_s \rightarrow b_s \rightarrow a_{s+2} \rightarrow b_{s+2}$ and trivial $P_{s+1} = a_{s+1} \rightarrow b_{s+1}$. Thus, $E(H_1) \cup \cdots \cup E(H_s) \cup E(P_1) \cup \cdots \cup E(P_{s+1})$ is a decomposition of $E(T)$ into $2s + 1 = n$ edge-disjoint paths implying $pn(T) = n$. Since $x(T) = n$, it implies by (1) $pn(T) = n$. The tournament $T$ is consistent.

We now start reversing, one by one, those arcs of $T$ which are oriented from $B$ to $A$. By Lemma 1 each reversal results in a consistent tournament with the path number increased by 2. Since the total number of arcs to be reversed is $ns$, it yields the sequence of consistent tournaments $T_1, T_2, \ldots, T_n$ with path numbers $n+2, n+4, \ldots, n+2ns = n^2$.

The proof is complete. \(\square\)

For a corresponding result for $n$ even the following two simple lemmas are needed. Being almost obvious their proofs are omitted.

**Lemma 2.** Let a digraph $D$ be the antidirected path $v_1 \rightarrow v_2 \leftarrow v_3 \rightarrow \cdots \rightarrow v_{2n-1} \rightarrow v_{2n}$. Then the following statements hold:

(a) $D$ is consistent and $pn(D) = 2n - 1$;

(b) successively reversing the arcs $v_2 \leftarrow v_3, v_4 \leftarrow v_5, \ldots, v_{2n-2} \leftarrow v_{2n-1}$ yields a sequence of consistent digraphs $D_1, D_2, \ldots, D_n$ with path numbers $2n-3, 2n-5, \ldots, 3, 1$. 

Lemma 3. Let a digraph \( D \) be the antidirected path \( v_1 \to v_2 \leftarrow v_3 \to \cdots \to v_{2n} \leftarrow v_{2n+1} \). Then the following statements hold:

(a) \( D \) is consistent and \( \text{pn}(D) = 2n \);

(b) successively reversing the arcs \( v_2 \leftarrow v_3, v_4 \leftarrow v_5, \ldots, v_{2n-2} \leftarrow v_{2n-1} \) yields a sequence of consistent digraphs \( D_1, D_2, \ldots, D_{n-1} \) with path numbers \( 2n - 2, 2n - 4, \ldots, 4, 2 \).

Theorem 6. For each even positive integer \( n \) and each even \( k \), \( n/2 + 1 \leq k \leq n^2 \), there exists a balanced bipartite tournament \( T(A, B) \), \( |A| = |B| = n \), such that \( \text{pn}(T) = k \).

Proof. Let \( n = 2s \) and let \( T = T(A, B) \) be the bipartite tournament with partition sets \( A = \{a_1, \ldots, a_n\}, B = \{b_1, \ldots, b_n\} \) whose all arcs are oriented from \( A \) to \( B \). Obviously, \( x(T) = \text{pn}(T) = n^2 \) and \( T \) is consistent.

Denote by \( M_0, M_1, \ldots, M_{n-1} \) 1-factors of \( G \) given by \( M_i = \{a_1 b_{1+i}, a_2 b_{2+i}, \ldots, a_n b_{n+i}\} \). (All indices are taken modulo \( n \).) It is easy to see that \( C_i = M_{2i} \cup M_{2i+1} = a_1 \to b_{1+2i+1} \leftarrow a_2 \to b_{2+2i+1} \leftarrow \cdots \to a_n \to b_{n+2i+1} \to a_1 \) (\( i = 0, 1, \ldots, s-1 \)) \( n \) yields a sequence of consistent digraphs \( D_1, D_2, \ldots, D_{n-1} \) with path numbers \( 2n - 2, 2n - 4, \ldots, 4, 2 \). On the other hand, \( E(T_i) \) can be partitioned into \( n^2 - 2 \) paths. These are the 3-path \( a_1 \to b_1 \to a_n b_n \) plus the rest of \( n^2 - 3 \) single arcs all going from \( A \) to \( B \). So, \( \text{pn}(T_i) \leq n^2 - 2 \). By inequality (1) \( T_1 \) is consistent and \( \text{pn}(T) = n^2 - 2 \).

Next, we reverse \( a_{n-1} \to b_n, a_{n-2} \to b_{n-1}, a_2 \to b_3 \), in that order, leaving \( a_1 \to b_2 \) intact. After the \( i \)-th (\( i = 1, 2, \ldots, n \)) reversal the tournament \( T_i \) with \( x(T_i) = n^2 - 2i \) arises. The arcs set of \( T_i \) consists of arcs of the \( (2i + 1) \)-path \( a_1 \to b_1 \to a_n \to b_n \to a_{n-1} \to b_{n-1} \to \cdots \to a_{n-i+1} \to b_{n-i+1} \) and \( n^2 - (2i + 1) \) arcs all going from \( A \) to \( B \). It implies \( \text{pn}(T_i) \leq n^2 - 2i \). Since \( x(T_i) = n^2 - 2i \) \( T_i \) is consistent and \( \text{pn}(T_i) = n^2 - 2i \). In particular, \( T_{n-1} \) is consistent with \( \text{pn}(T_{n-1}) = n^2 - 2(n - 1) \).

We next apply the similar reversals on \( C_1, C_2, \ldots, C_{s-1} \) leaving one suitable selected arc of each unchanged. It results in a sequence of \( s(2s - 1) + 1 \) consistent tournaments, \( T, T_1, \ldots, T_{s(2s - 1)} \) with path numbers \( n^2, n^2 - 2, \ldots, n^2 - 2s(2s - 1) = n \), respectively. Thus, all even integers from the interval \([n, n^2]\) are covered. For the rest of even integers, those of \([n/2 + 1, n-2]\), two cases need to be considered.

Notice that \( s \) unchanged arcs, one from each of \( C_0, C_1, \ldots, C_{s-1} \), can be selected so that they form an antidirected \((s + 1)\)-path \( P = a_1 \to b_2 \leftarrow \cdots \).

(a) \( s = 2t \). Then \( P = a_1 \to b_2 \leftarrow a_2 a_{2t-1} \to b_4 \leftarrow a_{2t-3} \to b_6 \leftarrow \cdots \leftarrow a_{s+2} \to b_s \leftarrow a_{s+1} \). Reversing \( t - 1 \) arcs of \( P \), as in Lemma 3, we obtain a sequence of consistent tournaments \( T_{s(2s-1)+1}, T_{s(2s-1)+2}, \ldots, T_{s(2s-1)+(t-1)} \) with path numbers \( n-2, n-4, \ldots, n-2(t-1) = n/2 + 2 \).

(b) \( s = 2t + 1 \). Then \( P = a_1 \to b_2 \leftarrow a_{2t-1} \to b_4 \leftarrow a_{2t-3} \to b_6 \leftarrow \cdots \leftarrow b_{s-1} \leftarrow a_{s+2} \to b_{s+1} \). Reversing now \( t \) arcs of \( P \), as in Lemma 2, we obtain a sequence...
of consistent tournaments $T_s(2s-1)+1, T_s(2s-1)+2, \ldots, T_s(2s-1)+t$ with path numbers $n-2, n-4, \ldots, n-2t=n/2+1$.

This completes the proof. \hfill \Box

References