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Zeroes of Primary Summand Functions on Compact Solvmanifolds

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The fact that continuous functions in primary summands of the Heisenberg manifold must vanish somewhere was proven by L. Auslander and R. Tolimieri, who deduced from this theorem the classical results on the vanishing of theta functions, as well as important applications to wavelets and radar ambiguity functions. The Heisenberg theorem seemed to depend on the presence of a central character, but the result is here extended to include primary summand functions on all compact nilmanifolds and to three-dimensional compact solvmanifolds which are not n-tori.

Let G be a solvable, connected and simply connected Lie group, with Lie algebra $\mathbf{g}$ and with cocompact discrete subgroup Γ. By a representation π of G we shall mean a strongly continuous, unitary representation of G in some separable Hilbert space H_π; π will be called irreducible if the space H_π contains no proper closed nontrivial subspace invariant under π.

Let M be the space of right cosets Γg of Γ in G, endowed with the quotient topology. Then G acts on $L^2(M)$ by right translation; i.e., $g \mapsto R(g)$, where $[R(g)f](\Gamma x) = f(\Gamma xg)$ for $f \in L^2(M)$ (here M has the G-invariant probability measure inherited from Haar measure on G). R is called the quasiregular representation of G on $L^2(M)$.

It is well known that $L^2(M)$ decomposes into the direct sum $\oplus H_\pi$, where the spaces H_π are mutually orthogonal $R(G)$-invariant subspaces, and R on the space H_π is a finite multiple of the irreducible representation π [GGP, Sect. I.2]. We let $(\Gamma\backslash G)^\wedge$ denote the set of irreducible representations appearing in the quasiregular representation R of G on $L^2(M)$. $(\Gamma\backslash S)^\wedge_\infty$ will denote the set of those representations $\pi \in (\Gamma\backslash S)^\wedge$ which are infinite dimensional. Then the orthogonal projection P_π of $L^2(M)$ onto H_π is L^2-continuous and preserves $C^\infty(M)$ [Aus-Bre, Theorem 5], and is given by convolution with a bounded Borel measure σ_π.

Now let N be a nilpotent Lie group, connected and simply connected, with Lie algebra $\mathbf{n}$ and cocompact discrete subgroup Γ.

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If the coadjoint orbits of the action of N on the dual $\mathfrak{n}^*$ are linear varieties, then $\Gamma\backslash N$ possesses the property that the orthogonal projections P_π of $L^2(\Gamma\backslash N)$ onto H_π preserve continuity [Ri1, Bre1]. These flat-orbit nilmanifolds share this property with compact quotients of the 3-dimensional solvable group S_R by discrete subgroups. Here S_R denotes the semidirect product $\mathbf{R} \propto \mathbf{R}^2$, where $\mathbf{R}$ acts on $\mathbf{R}^2$ via a one-parameter subgroup of rotations.

This paper was motivated by a theorem of L. Auslander and R. Tolimieri. Let H_3 be the 3-dimensional Heisenberg group, $\mathbf{R}^3$ endowed with the multiplication $(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy')$, and let Γ be the discrete group of integer points in H_3. Let f be a continuous function in $H_\pi \subset L^2(\Gamma\backslash H_3)$, where π is an irreducible, unitary, infinite-dimensional representation in $(\Gamma\backslash H_3)^\wedge$. Then f must have at least one zero on $\Gamma\backslash H_3$ [Aus-Tol, Theorem II.2]. It is shown in Chapter II of [Aus-Tol] that the vanishing of theta functions follows as a direct corollary of this theorem. In work by L. Auslander, R. Tolimieri, I. Daubechies, A. Janssen, D. Gabor, and others, this result has been shown to have important consequences for wavelet theory, and applications to problems involving the radar ambiguity function (see, for example, [Dau]).

The phenomenon of vanishing arises from a rather surprising interaction between the representation theory of H_3 (which determines the primary summand H_π) and the topology of the manifold $\Gamma\backslash H_3$. In this paper, we generalize this theorem to all 3-dimensional compact solvmanifolds, using techniques of harmonic analysis on solvmanifolds developed by L. Auslander, J. Brezin, L. Richardson, and others.

The proof that all continuous primary summand functions on compact nilmanifolds have zeros is an adaptation of Auslander and Tolimieri's original proof, using induction and relying heavily upon the central covariance which all such functions possess; this covariance appears to be at the heart of the result in the nilpotent case. However, since 3-dimensional non-nilpotent solvable Lie groups with cocompact discrete subgroups have trivial centers [AGH, Chap. 3], completely new techniques are needed to show that most 3-dimensional compact solvmanifolds do possess the property that their continuous π-primary functions (hereafter referred to as primary functions) must vanish, for infinite-dimensional π. A noteworthy exception is one compact quotient of S_R which is actually homeomorphic to the 3-torus T^3; here one finds plenty of continuous primary functions which do not vanish, as one would expect. However, for three of four remaining compact quotients of S_R, it is shown that continuous primary functions must have zeros. For the fourth compact quotient of S_R, we have shown that continuous functions in certain subspaces of a primary summand H_π must have zeros. As of this writing, however, it is conjectured but not known that all continuous primary summand functions on this manifold must have zeros.

Let S_H be the semidirect product $\mathbf{R} \propto \mathbf{R}^2$, where $\mathbf{R}$ acts upon $\mathbf{R}^2$ via the one-parameter subgroup $t \mapsto \left[\begin{smallmatrix} \lambda^t & \\ & \lambda^{-t}\end{smallmatrix}\right]$ in $SL_2(\mathbf{R})$, where $\lambda + \lambda^{-1} = k+1$ for any integer $k \geqslant 2$. It is shown in this paper that for all compact quotients of S_H, continuous primary functions must have zeros. This exhausts the compact solvmanifolds of dimension three.

Thus the interplay of topology and representation theory which produces zeros of continuous primary functions is seen to be more than a nilpotent phenomenon, but the extent of this interaction remains obscure. There is the possibility of a generalization of this theorem to a larger class of compact solvmanifolds.

I express my thanks to Leonard Richardson; the contents of this paper are my doctoral dissertation, done under his direction at Louisiana State University.

1. Preliminaries

Let G be a connected, simply connected Lie group with Lie algebra $\mathbf{g}$, and let $\mathbf{g}^*$ be the vector space of linear functionals on $\mathbf{g}$. We define a sequence of ideals of the Lie algebra $\mathbf{g}$ by $\mathbf{g}^{(0)} = \mathbf{g}$, $\mathbf{g}^{(k)} = [\mathbf{g}^{(k-1)}, \mathbf{g}^{(k-1)}]$; this is called the derived series of $\mathbf{g}$, and $\mathbf{g}$ is said to be solvable if $\mathbf{g}^{(n)} = 0$ for some $n \in \mathbf{N}$. We define another sequence of ideals of the Lie algebra $\mathbf{g}$ by $\mathbf{g}_{(0)} = \mathbf{g}$, $\mathbf{g}_{(k)} = [\mathbf{g}_{(k-1)}, \mathbf{g}]$; this is called the lower central series of $\mathbf{g}$, and $\mathbf{g}$ is said to be nilpotent if $\mathbf{g}_{(n)} = 0$ for some $n \in \mathbf{N}$ (see [Hum, Sect. 3]). The term "nilmanifold" ("solvmanifold") will refer to compact spaces $\Gamma \backslash G$, where G is nilpotent (solvable) and Γ is discrete and cocompact.

The adjoint representation of the group G in the vector space $\mathbf{g}$, written Ad, is defined as follows; for each element $x \in G$, $\mathrm{Ad}(x)\colon \mathbf{g} \to \mathbf{g}$ is the differential at the identity of G of the group automorphism $I(x)$, inner conjugation by $x \in G$. $\mathrm{Ad}(x)$ satisfies

$$x(\exp X)x^{-1} = \exp[\mathrm{Ad}(x)X] \tag{1}$$

for each $x \in G$, $X \in \mathbf{g}$.

The coadjoint representation of G is of central importance in the representation theory of nilpotent and solvable Lie groups. The set of equivalence classes of irreducible representations of a nilpotent Lie group G is naturally parametrized by the orbit space $\mathbf{g}^*/\mathrm{Ad}^*\, G$; this is also true for the (completely) solvable Lie groups examined in this work. This parametrization, due to A. A. Kirillov, is freely drawn upon here; for details, see [CG, Chap. II].

As described in the introduction, there are two 3-dimensional, solvable, non-nilpotent Lie groups with cocompact discrete subgroups, the groups S_H and S_R. Their Lie algebras are three-dimensional vector spaces spanned by the vectors T, X, and Y, where $\exp sT = (s, 0, 0)$, $\exp sX = (0, s, 0)$ and $\exp sY = (0, 0, s)$.

We have the following five compact quotients of S_R, with convenient coordinatization (see [AGH, Sect. 2.2]).

1. $\Gamma_{R,1} \backslash S_{R,1} = M_{R,1}$, where $S_{R,1} = \mathbf{R} \propto \mathbf{R}^2$, $\mathbf{R}$ acts on $\mathbf{R}^2$ via the one-parameter subgroup $\sigma_1(t) = \left[\begin{smallmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{smallmatrix}\right]$, and $\Gamma_{R,1} = \{(p, m, n) \in S_{R,1};\ p, m, n \in \mathbf{Z}\}$. Here $\Gamma_{R,1}$ is isomorphic to the abelian group $\mathbf{Z}^3$, and so $M_{R,1} \cong T^3$ [Mos, Theorem A].

2. $\Gamma_{R,2} \backslash S_{R,2} = M_{R,2}$, where $S_{R,2} = \mathbf{R} \propto \mathbf{R}^2$, $\mathbf{R}$ acts on $\mathbf{R}^2$ via the one-parameter subgroup $\sigma_2(t) = \left[\begin{smallmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{smallmatrix}\right]$, and $\Gamma_{R,2} = \{(p, m, n) \in S_{R,2};\ p, m, n \in \mathbf{Z}\}$.

3. $\Gamma_{R,3} \backslash S_{R,3} = M_{R,3}$, where $S_{R,3} = \mathbf{R} \propto \mathbf{R}^2$, $\mathbf{R}$ acts on $\mathbf{R}^2$ via the one-parameter subgroup $\sigma_3(t)$ with $\sigma_3(1) = \left[\begin{smallmatrix} & 1 \\ -1 & -1 \end{smallmatrix}\right]$, $\sigma_3(t)$ is isomorphic to the subgroup $Rot(2\pi t/3) = \left[\begin{smallmatrix} \cos 2\pi t/3 & \sin 2\pi t/3 \\ -\sin 2\pi t/3 & \cos 2\pi t/3 \end{smallmatrix}\right]$, and $\Gamma_{R,3} = \{(p, m, n) \in S_{R,3};\ p, m, n \in \mathbf{Z}\}$.

4. $\Gamma_{R,4} \backslash S_{R,4} = M_{R,4}$, where $S_{R,4} = \mathbf{R} \propto \mathbf{R}^2$, $\mathbf{R}$ acts on $\mathbf{R}^2$ via the one-parameter subgroup $\sigma_4(t) = \left[\begin{smallmatrix} \cos \pi t/2 & \sin \pi t/2 \\ -\sin \pi t/2 & \cos \pi t/2 \end{smallmatrix}\right]$, and $\Gamma_{R,4} = \{(p, m, n) \in S_{R,4};\ p, m, n \in \mathbf{Z}\}$.

5. $\Gamma_{R,6} \backslash S_{R,6} = M_{R,6}$, where $S_{R,6} = \mathbf{R} \propto \mathbf{R}^2$, $\mathbf{R}$ acts on $\mathbf{R}^2$ via the one-parameter subgroup $\sigma_6(t)$ in $SL_2(\mathbf{R})$ with $\sigma_6(1) = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right]$, and $\Gamma_{R,6} = \{(p, m, n) \in S_{R,6};\ p, m, n \in \mathbf{Z}\}$.

We also have the following compact quotients of S_H, with convenient coordinatizations.

Suppose $k \in \mathbf{Z}$, $k \geqslant 2$. Define $S_{H,k} = \mathbf{R} \propto \mathbf{R}^2$, where $\mathbf{R}$ acts on $\mathbf{R}^2$ via the one-parameter subgroup $\sigma_k(t)$ in $SL_2(\mathbf{R})$ with $\sigma_k(1) = \left[\begin{smallmatrix} 1 & 1 \\ k-1 & k \end{smallmatrix}\right]$. Then $S_{H,k} \cong S_H$ for each k. Let $\Gamma_{H,k} = \{(p, m, n) \in S_{H,k};\ p, m, n \in \mathbf{Z}\}$; then each $\Gamma_{H,k} \backslash S_{H,k} = M_{H,k}$ is a distinct compact quotient of S_H.

Thus there are 5 distinct, non-homeomorphic compact quotients of S_R, and infinitely many distinct compact quotients of S_H.

It will be convenient to use several different coordinatizations of S_R and S_H. The coordinatizations of S_R just described will be called *integral coordinatizations of* $S_{R,p}$. Let $A \in GL_2(\mathbf{R})$ be such that

$$A\sigma_p(t)A^{-1} = R(2\pi t/p) = \begin{bmatrix} \cos 2\pi t/p & \sin 2\pi t/p \\ -\sin 2\pi t/p & \cos 2\pi t/p \end{bmatrix}.$$

If we recoordinatize N so that the action of $\mathbf{R}$ on N is given by $R(2\pi t/p)$, then $\Gamma_{R,p} \cap N = A(\mathbf{Z}^2)$ (note that in the case of $\Gamma_{R,1}$, $\Gamma_{R,2}$, and $\Gamma_{R,4}$,

$A=I$). In this coordinatization of $S_{R,p}$, the nondegenerate coadjoint orbits of $S_{R,p}$ are circular cylinders, $x^2+y^2=\lambda^2$, for some $\lambda \in \mathbf{R}$. For the groups $S_{R,3}$ and $S_{R,6}$, the 2-torus $N \cap \Gamma_{R,p} \backslash N$ will be a non-standard torus in this coordinatization. We will call these coordinatizations the *circular coordinatizations* of $S_{R,p}$.

The coordinatizations of the solvmanifolds $S_{H,k}$ just described will be referred to as the integral coordinatizations of $S_{H,k}$. Let $A \in GL_2(\mathbf{R})$ be such that $A\sigma_k(t)A^{-1}=\left[\begin{smallmatrix}\lambda^t & 0\\ 0 & \lambda^{-t}\end{smallmatrix}\right]$ where $\lambda+\lambda^{-1}=k+1$; if we recoordinatize N so that the action of $\mathbf{R}$ on N is given by this one parameter subgroup, then $\Gamma_{H,k} \cap N = A(\mathbf{Z}^2)$; the nondegenerate coadjoint orbits in this case are hyperbolic cylinders of the form $xy=\lambda$, $\lambda \in \mathbf{R}$. The 2-torus $N \cap \Gamma_{H,k} \backslash N$ in this coordinatization will be a non-standard torus, for all $k \geqslant 2$. This coordinatization of $S_{H,k}$ will be referred to as the *hyperbolic coordinatization.*

For each solvmanifold $M_{H,k}(M_{R,i})$, the group $S_{H,k}(S_{R,i})$ is a simply connected cover of $M_{H,k}(M_{R,i})$ and $\Gamma_{H,k}(\Gamma_{R,i})$ is the group of covering transformations of $S_{H,k}(S_{R,i})$. Thus we have $\Pi_1(M_{H,k})=\Gamma_{H,k}$, $\Pi_1(M_{R,i})=\Gamma_{R,i}$. The $M_{H,k}$ and $M_{R,i}$ are bundles over the circle with 2-torus fiber; the projection maps are

$$\begin{aligned}\mu_H : M_{H,k} &\to \mathbf{Z}\backslash\mathbf{R}\\ \Gamma_{H,k}(t,u,v) &\mapsto \mathbf{Z}+t\end{aligned} \tag{2}$$

and

$$\begin{aligned}\mu_R : M_{R,p} &\to \mathbf{Z}\backslash\mathbf{R}\\ \Gamma_{R,p}(t,u,v) &\mapsto \mathbf{Z}+t.\end{aligned} \tag{3}$$

A convenient decomposition of H_π into irreducible subspaces will be used throughout this paper; however, no canonical decomposition of H_π into irreducible subspaces exists. The irreducible subspaces of the chosen decomposition of H_π will be referred to as the constructible irreducible subspaces of H_π.

We will now describe those functions on $M_{H,k}(M_{R,p})$ which are primary functions. We will use integral coordinatizations of $S_{R,p}$ and $S_{H,k}$.

In the integral coordinatization of $S_{H,k}$, the coadjoint orbits satisfy

$$(k-1)x^2+(k-1)\,xy-y^2=\lambda \tag{4}$$

so that the orbits are saturated in the T^*-direction. We will call λ an integral functional if $\lambda|_{\mathfrak{n}}=\alpha X^*+\beta Y^*$, α, $\beta \in \mathbf{Z}$, and denote by $\mathcal{O}_\pi$ the orbit of λ in $s^*_{H,k}$.

Fix some nonzero integral functional $\lambda \in \mathcal{O}_\pi$. We define the character χ_λ on the (abelian) nilradical $\mathfrak{n}$ follows: if $\lambda|_{\mathfrak{n}}=\alpha X^*+\beta Y^*$, then

$$\chi_\lambda(0,r,s)=e^{2\pi i(\alpha r+\beta s)}. \tag{5}$$

We seek a maximal subgroup M of $S_{H,k}$ such that

(i) M contains N;

(ii) χ_λ may be extended to a character of M, i.e., $\chi_\lambda[M, M] = 1$, where $[M, M]$ is the commutator subgroup of M. M will be called a maximal subgroup subordinate to λ. We will call this extension $\tilde{\chi}$, the map extension of χ to M. To this end, it suffices to examine the values of χ_λ on terms in $[S_{H,k}, S_{H,k}]$ of the form

$$(t, 0, 0)(0, x, y)(-t, 0, 0)(0, -x, -y) = (0, (x, y) - \sigma_k(t)(x, y)). \quad (6)$$

Since $\sigma_k(t)$ has eigenvalues λ^t and λ^{-t}, where $\lambda + \lambda^{-1} = k+1$, and since λ is nonzero,

$$\chi_\lambda(0, (x, y) - \sigma_k(t)(x, y)) = \exp 2\pi i\lambda((x, y) - \sigma_k(t)(x, y))$$

is 1 for all (x, y) if and only if $t = 0$. Thus N itself is maximal subordinate to λ for all nonzero λ.

We define the Mackey space $M(\lambda)$ for λ as

$$M(\lambda) = \{f\colon S_{H,k} \to \mathbf{C} \mid f \text{ is measurable, } |f| \in L^2(N\backslash S_{H,k}),$$
$$f(ng) = \chi_\lambda(n) f(g)\ n \in N,\ g \in S_{H,k}\}. \quad (7)$$

It is well known that the action of $S_{H,k}$ on $M(\lambda)$ by right translation is an irreducible representation π. We note that the functions f in $M(\lambda)$ are left $\Gamma_{H,k} \cap N$-invariant, and define the homogenizing (lift) map L: $M(\lambda) \to L^2(\Gamma_{H,k}\backslash S_{H,k})$ as follows: for $f \in M(\lambda)$,

$$Lf(\Gamma_{H,k}(t, x, y)) = \sum_{\gamma \in \Gamma_{H,k} \cap N\backslash\Gamma_{H,k}} (f\cdot\gamma)(t, x, y), \quad (8)$$

where $(f\cdot\gamma)(g) = f(\gamma g)$, for γ, $g \in S_{H,k}$.

Note that the sum in (8) is well defined with respect to equivalence classes of $\Gamma_{H,k} \cap N\backslash_{H,k}$. For all $f \in M(\lambda)$, Lf is a well-defined element of $L^2(\Gamma_{H,k}\backslash S_{H,k})$, and the map L is $S_{H,k}$-equivariant, so that the image in $L^2(\Gamma_{H,k}\backslash S_{H,k})$ of $M(\lambda)$ is an irreducible π-subspace of $L^2(\Gamma_{H,k}\backslash S_{H,k})$, and therefore a subspace of the π-primary summand H_π.

Those subspaces of H_π which are the images of homogenizing maps L from Mackey spaces $M(\lambda)$ will be referred to as the constructible irreducible subspaces of H_π; there are finitely many such subspaces in each H_π. The maximum number of such mutually orthogonal subspaces will be referred to as the multiplicity of π, $\operatorname{mul}\pi$. There is one constructible irreducible subspace in H_π for each distinct $\Gamma_{H,k}$-orbit in the integral functionals of $\mathcal{O}_\pi$, and so $\operatorname{mul}\pi$ is given by the number of such orbits.

The constructible irreducible subspaces are not canonically determined irreducible subspaces of H_π; nevertheless, the π-primary summand is the orthogonal direct sum of these spaces, and so we may represent a typical member of the π-primary summand H_π as follows. Let $\{\lambda_i\}_{i=1}^{\operatorname{mul}\pi}$ be a set of representatives of distinct $\Gamma_{H,k}$-orbits in the integral functionals of $\mathcal{O}_\pi$, λ_i an integral functional for each i. Let $f_i \in M(\lambda_i)$; L_i, the lift map from $M(\lambda_i)$ to L^2. Then a typical member of H_π has the form

$$F = \sum_{i=1}^{\operatorname{mul}\pi} L_i f_i = \sum_{i=1}^{\operatorname{mul}\pi} \left\{ \sum_{\gamma \in \Gamma_{H,k} \cap N \backslash \Gamma_{H,k}} (f_i \cdot \gamma) \right\}. \tag{9}$$

If $f_i \in M(\lambda_i)$, then f_i may be written

$$f_i(t, x, y) = \chi_{\lambda_i}(0, x, y)\, \tilde{f}_i(t), \tag{10}$$

where $\tilde{f}_i \in L^2(\mathbf{R})$. Thus, if we choose the elements $(n, 0, 0) \in \Gamma_{H,k}$, $n \in \mathbf{Z}$, to represent the equivalence classes of $\Gamma_{H,k} \cap N \backslash_{H,k}$, we have

$$\begin{aligned} Lf_i(\Gamma_{H,k}(t, x, y)) &= \sum_{n \in \mathbf{Z}} [f_i(n, 0, 0)](t, x, y) \\ &= \sum_{n \in \mathbf{Z}} f_i((n, 0, 0) \cdot (t, x, y)) \\ &= \sum_{n \in \mathbf{Z}} f_i(n + t, \sigma_k(n)(x, y)) \\ &= \sum_{n \in \mathbf{Z}} \tilde{f}_i(n + t)\, \chi_{\lambda_i}(0, \sigma_k(n)(x, y)) \\ &= \sum_{n \in \mathbf{Z}} \tilde{f}_i(n + t)\, \chi_{\sigma_k^*(n)\lambda_i}(0, x, y). \end{aligned} \tag{11}$$

Thus a typical member of H_π is of the form

$$\sum_{i=1}^{\operatorname{mul}\pi} \sum_{n \in \mathbf{Z}} \tilde{f}_i(n + t)\, \chi_{\sigma_k^*(n)\lambda_i}(0, x, y), \tag{12}$$

where all elements $\sigma_k^*(n)\lambda_i$ satisfy Eq. (4).

Suppose we have $\lambda \in \mathbf{n}^*$, $\lambda = \alpha X^* + \beta Y^*$ for some $\alpha, \beta \in \mathbf{Z}$. Then

$$\chi_\lambda(0, x, y) = \exp 2\pi i(\alpha x + \beta y). \tag{13}$$

If we set $z_1 = e^{2\pi i x}$, $z_2 = e^{2\pi i y}$, we may write $\chi_\lambda(0, x, y) = e^{2\pi i(\alpha x + \beta y)} =$

$z_1^{\alpha} z_2^{\beta}$, so that $F \in H_\pi$ may be thought of as a function of t, z_1, z_2, i.e., as $\tilde{F}$, where

$$\tilde{F}(t, z_1, z_2) = \sum_{i=1}^{\operatorname{mul} \pi} \sum_{n \in \mathbf{Z}} f_i'(n+t)\, z_1^{\alpha n_i} z_2^{\beta n_i} \tag{14}$$

for $\sigma_k^*(n) \lambda_i = \alpha_{n,i} X^* + \beta_{n,i} Y^*$. Fixing $t_0 \in \mathbf{R}$, we may define a cross section

$$\tilde{F}_{t_0}(z_1, z_2) \cong \tilde{F}(t_0, z_1, z_2) \tag{15}$$

so that $\tilde{F}_t$ is a function from T^2 to $\mathbf{C}$, for z_1, z_2 of modulus 1. Let the integral functionals $\{\lambda_i\}_{i=1}^{\operatorname{mul} \pi}$ be a set of $\Gamma_{H,k}$-orbit representatives in $\mathbf{n}^*$; we define H_{π_i} to be the image of the lift map

$$L: M(\lambda_i) \to L^2(\Gamma_{H,k} \backslash S_{H,k})$$

(H_{π_i} is the ith constructible irreducible subspace of H_π, the π-primary summand).

In the integral coordinatization of $S_{R,p}$, the coadjoint orbits satisfy

$$\text{(i)} \quad x^2 + y^2 = k^2 \qquad \text{for some } k \in \mathbf{R} \text{ if } p = 1, 2, 4 \tag{16}$$

$$\text{(ii)} \quad x^2 + xy + y^2 = k^2 \qquad \text{for some } k \in \mathbf{R} \text{ if } p = 3, 6, \tag{17}$$

so that the orbits are saturated in the T^*-direction; fix some coadjoint orbit $\mathcal{O}_\pi \subset \mathbf{s}_{R,p}^*$, and some nonzero integral functional $\lambda \in \mathcal{O}_\pi$.

We define the character χ_λ on the nilradical N as

$$\chi_\lambda(0, r, s) = \exp 2\pi i \lambda(r, s) = \exp 2\pi i(\alpha r + \beta s), \tag{18}$$

where $\lambda = \alpha X^* + \beta Y^*$. We seek a maximal extension of the character χ_λ on M; we examine the values of χ_λ on terms of the form

$$(t, 0, 0)(0, x, y)(-t, 0, 0)(0, -x, -y) = (0, (x, y) - \sigma_p(t)(x, y)),$$

in the commutator $[S_{R,p}, S_{R,p}]$. Since $\sigma_p(t)$ has eigenvalues $\exp \pm 2\pi i t/p$, and since $\sigma_p(p\mathbf{Z}) \equiv I$, we may extend the character χ_λ to a character on the subgroup

$$M_p = \{(n, x, y): n = pk \text{ for some } k \in \mathbf{Z},\ x, y \in \mathbf{R}\}. \tag{19}$$

Then M_p is called maximal subordinate to the functional λ.

We define the Mackey space $M(\lambda)$ for λ as

$$M(\lambda) = \{f: S_{R,p} \to \mathbf{C} \mid f \text{ is measurable}, |f| \in L^2(\Gamma_{R,p} \backslash S_{Z,p}),$$
$$f(mg) = \chi_\lambda(m) f(g)\ m \in M_p,\ g \in S_{R,p}\}. \tag{20}$$

The action of $S_{R,p}$ on $M(\lambda)$ by right translation is an irreducible representation π, independent (up to equivalence) of the choice of $\lambda \in \mathcal{O}_\pi$.

The functions $f \in M(\lambda)$ are left $\Gamma_{R,p} \cap M_p$-invariant. We define the lift map L: $M(\lambda) \to L^2(\Gamma_{R,p} \backslash S_{R,p})$ as follows: for $f \in M(\lambda)$,

$$Lf(\Gamma_{R,p}(t, x, y)) = \sum_{\gamma \in \Gamma_{R,p} \cap M_p \backslash \Gamma_{R,p}} (f \cdot \gamma)(t, x, y). \tag{21}$$

Note that the sume (21) is a sum of p terms, and is a left $\Gamma_{R,p}$-invariant function in $L^2(\Gamma_{R,p} \backslash S_{R,p})$. L is an $S_{R,p}$-equivariant map, so that the image in $L^2(\Gamma_{R,p} \backslash S_{R,p})$ of $M(\lambda)$ is an irreducible π-subspace of $L^2(\Gamma_{R,p} \backslash S_{R,p})$, and therefore a subspace of the π-primary summand $H_\pi \subset L^2(\Gamma_{R,p} \backslash S_{R,p})$; this subspace will also be referred to as a constructible irreducible subspace of H_π. The number of such mutually orthogonal subspaces of H_π is equal to the number of disjoint $\Gamma_{R,p}$-orbits in the set of integral functions in $\mathcal{O}_\pi$. H_π is the orthogonal direct sum of these subspaces.

We may represent a typical element of the π-primary summand H_π as follows. Let $\{\lambda_i\}_{i=1}^{\operatorname{mul} \pi}$ be a set of representatives of distinct $\Gamma_{R,p}$-orbits of integral functionals in $\mathbf{n}^* \cap \mathcal{O}_\pi$. If $f_i \in M(\lambda_i)$, a typical member of H_π has the form

$$F = \sum_{i=1}^{\operatorname{mul} \pi} Lf_i = \sum_{i=1}^{\operatorname{mul} \pi} \left\{ \sum_{\gamma \in M_p \cap \Gamma_{R,p} \backslash \Gamma_{R,p}} f_i \cdot \gamma \right\}. \tag{22}$$

If $f_i \in M(\lambda_i)$, then f_i may be written

$$f_i(t, x, y) = \chi_{\lambda_i}(0, x, y)\, \tilde{f}_i(t), \tag{23}$$

where $\tilde{f}_i \in L^2(p\mathbf{Z} \backslash \mathbf{R})$ ($\tilde{f}_i$ is to be thought of as a function on $\mathbf{R}$ with period p). Thus, if we choose the elements $(n, 0, 0) \in \Gamma_{R,p}$, $n = 0, 1, 2, ..., p-1$ to represent the equivalence classes of $M_p \cap \Gamma_{R,p} \backslash \Gamma_{R,p}$, we have

$$\begin{aligned} Lf_i(\Gamma_{R,p}(t, x, y)) &= \sum_{n=0}^{p-1} (f_i \cdot (n, 0, 0))(t, x, y) \\ &= \sum_{n=0}^{p-1} f_i(n + t, \sigma_k(n)(x, y)) \\ &= \sum_{n=0}^{p-1} (\tilde{f}_i(n + t)\, \chi_{\lambda_i}(\sigma_k(n)(x, y))) \\ &= \sum_{n=0}^{p-1} (\tilde{f}_i(n + t)\, \chi_{\sigma_k^*(n)\lambda_i}(x, y)). \end{aligned} \tag{24}$$

Thus a typical member of H_π is of the form

$$\sum_{i=1}^{\mathrm{mul}(\pi)} \sum_{n\in 0}^{p-1} \tilde{f}_i(n+t)\, \chi_{\sigma_k^*(n)\lambda_i}(0, x, y). \tag{25}$$

Recall that all elements $\sigma_k^*(n)\lambda_i$ satisfy Eq. (16) or (17).

Suppose we have $\lambda \in \mathbf{n}^*$, $\lambda = \alpha X^* + \beta Y^*$ for some $\alpha, \beta \in \mathbf{Z}$. Then

$$\chi_\lambda(0, x, y) = \exp 2\pi i(\alpha x + \beta y). \tag{26}$$

If we set $z_1 = e^{2\pi i x}$, $z_2 = e^{2\pi i y}$, we may define $\tilde{F}$ and $\tilde{F}_t$ as in Eqs. (14) and (15) for $F \in H_\pi$. If the integral functionals $\{\lambda_i\}_{i=1}^{\mathrm{mul}\,\pi}$ are a set of $\Gamma_{R,p}$-orbit representatives in $\mathbf{n}^*$, we define H_{π_i} to be the image of the lift map $L(M(\lambda_i)) \to L^2(\Gamma_{R,p}\backslash S_{R,p})$. H_{π_i} is the the ith constructible irreducible subspace of H_π.

We end this section with a fact, and a lemma.

1. $S_{H,k}(S_{R,p})$ in its integral coordinatization has the fundamental domain $[0, 1]^3$; since T^3 has the same fundamental domain and since the invariant measure of the boundary is zero, the identification of fundamental domain produces a Borel isomorphism of the measure spaces and an isometry between $L^2(T^3)$ and $L^2(\Gamma_{H,k}\backslash S_{H,k})[L^2(\Gamma_{R,p}\backslash S_{R,p})]$. Since each character $f_{\alpha,\beta,\gamma}(t, x, y) = \exp 2\pi i(\alpha x + \beta y + \gamma t)$, $\alpha, \beta, \gamma \in \mathbf{Z}$, appears in the summand H_π for which $\alpha X^* + \beta Y^* \in \mathcal{O}_\pi$, we have that the π-primary summands H_π, together with the constant functions, form a complete set of orthonormal subspaces in $L^2(\Gamma_{H,k}\backslash S_{H,k})[L^2(\Gamma_{R,p}\backslash S_{R,p})]$.

2. We define

$$P_{\pi_i}\colon L^2(\Gamma_{R,p}\backslash S_{R,p}) \to H_{\pi_i}$$

to be the orthogonal projection of $L^2(\Gamma_{R,p}\backslash S_{R,p})$ onto H_{π_i}. We have for $f \in L^2(\Gamma_{R,p}\backslash S_{R,p})$,

$$P_{\pi_i}(f)(\Gamma_{R,p}(t, x, y)) = \sum_{N=0}^{p-1} f(t,\cdot,\cdot)^\wedge\, (\sigma_p^*(N)\lambda_i)\, \chi_{\sigma_p^*(N)\lambda_i}(0, x, y), \tag{27}$$

where $f(t,\cdot,\cdot)^\wedge$ is the standard Fourier transform in the variables x and y for fixed t (note that for fixed t, $P_{\pi_i} f(t, x, y)$ is a function on $(N \cap \Gamma_{R,p}\backslash N) \cong T^2$).

LEMMA 1. *Suppose f is continuous on $M_{R,p}$. Then $P_{\pi_i} f = L\tilde{f}$ for some continuous $\tilde{f}$ in $M(\lambda_i)$.*

Proof. We need to produce a continuous function $\tilde{f} \in M(\lambda_i)$ such that $L\tilde{f} = P_{\pi_i} f$.

Let

$$\tilde{f}(t, x, y) = f(t, \cdot, \cdot)^{\wedge}\ (\lambda_i)\, \chi_{\lambda_i}. \tag{28}$$

To see that $L\tilde{f} = P_{\pi_i} f$, we must demonstrate

1. that $\tilde{f} \in M(\lambda_i)$ and $\tilde{f}$ is continuous in (t, x, y);

2. that $\tilde{f}((t+k),\ \sigma_p(k)(x, y)) = f(t, \cdot, \cdot)^{\wedge}\ (\sigma_p^*(k)\lambda_i)\, \chi_{\sigma_p^*(k)\lambda_i}(x, y)$; i.e., the kth terms in each sum are identical. By definition,

$$\tilde{f}(t+k, \sigma_p(k)(x, y)) = f(t+k, \cdot, \cdot)^{\wedge}\ (\lambda_i)\, \chi_{\lambda_i}(\sigma_p(k)(x, y)). \tag{29}$$

Since $\chi_{\sigma_p^*(k)\lambda_i}(x, y) = \chi_{\lambda_i}(\sigma_p(k)(x, y))$, to demonstrate part 2 we need only show that

$$f((t+k), \cdot, \cdot)^{\wedge}\ (\lambda_i) = f(t, \cdot, \cdot)^{\wedge}\ (\sigma_p^*(k)\lambda_i). \tag{30}$$

By definition,

$$\begin{aligned} f((t+k), \cdot, \cdot)^{\wedge}\ (\lambda_i) &= f(t+k, \cdot, \cdot)^{\wedge}\ (\lambda_i) \\ &= \int_{N \cap \Gamma_{R,p} \backslash N} f(t+k, x, y)\, \chi_{\lambda_i}(x, y)\, dx\, dy. \end{aligned} \tag{31}$$

Since f is continuous on $M_{R,p}$ and is therefore left $\Gamma_{R,p}$-invariant, we have

$$\begin{aligned} &\int_{N \cap \Gamma_{R,p} \backslash N} f(t+k, x, y)\, \chi_{\lambda_i}\, dx\, dy \\ &\quad = \int_{N \cap \Gamma_{R,p} \backslash N} f(t, \sigma_p(-k)(x, y))\, \chi_{\lambda_i}(x, y)\, dx\, dy \\ &\quad = \int_{N \cap \Gamma_{R,p} \backslash N} f(t, x, y)\, \chi_{\lambda_i}(\sigma_p(k)(x, y))\, |\det \sigma_p(k)|\, dx\, dy \\ &\quad = f(t, x, y)\, \chi_{\sigma_p^*(k)\lambda_i}(x, y)\, dx\, dy \\ &\quad = f(t, \cdot, \cdot)^{\wedge}\ (\sigma_p^*(k)\lambda_i). \end{aligned}$$

$\tilde{f}$ has the desired left-M_p-invariance, and is therefore in $M(\lambda_i)$; since f itself is continuous in t, $f(t, \cdot, \cdot)^{\wedge}\ (\lambda_i)$ is continuous in t and therefore $\tilde{f}$ is a continuous function in $M(\lambda_i)$. This completes the proof of Lemma 1.

2. Zeros of Continuous Functions in H_π of a Compact Nilmanifold

We have the following generalization of a theorem of L. Auslander and R. Tolimieri [Aus-Tol, Theorem II.2].

THEOREM 2. *Let N be a nilpotent Lie group with cocompact discrete subgroup Γ, $\Gamma\backslash N$ not isomorphic to T^n for any $n \in \mathbf{N}$. If f is a continuous function in $H_\pi \subseteq L^2(\Gamma\backslash N)$ for $\pi \in (\Gamma\backslash N)^\wedge_\infty$, then f has at least one zero on $\Gamma\backslash N$.*

Proof. We proceed by induction on dim N.

We begin with dim $N = 3$, where the 3-dimensional Heisenberg group H_3 is the only example of a nilpotent group with quotient manifolds that are not isomorphic to T^3. Theorem 2 for this case was proved by L. Auslander and R. Tolimieri in [Aus-Tol, Theorem II.2].

LEMMA 3. *Let Γ' be a uniform subgroup of H_3. Then if $\Gamma = \{(p, m, n) \in H_3 : p, m, n \in \mathbf{Z}\}$, Γ' contains a subgroup isomorphic to Γ, and its index in Γ' is finite.*

This lemma follows immediately from the results of A. I. Malcev in [Mal].

We are given $\Gamma\backslash N$ compact, and the map

$$\Phi \colon L^2(\Gamma'\backslash N) \to L^2(\Gamma\backslash N)$$

defined by $\Phi f(\Gamma x) = f(\Gamma' x)$ is a well-defined, N-equivariant isometry of $L^2(\Gamma'\backslash N)$ with its image in $L^2(\Gamma\backslash N)$.

Suppose $\pi \in (\Gamma'\backslash N)^\wedge_\infty$ and that f is a continuous function in $H_\pi \subseteq L^2(\Gamma'\backslash N)$. Then Φf is continuous in $L^2(\Gamma\backslash N)$. Since Φ is an N-equivariant isometry, $\Phi(H_\pi)$ is contained in the π-primary summand of $L^2(\Gamma\backslash N)$. By Theorem II.2 in [Aus-Tol], then, Φf must have a zero. This completes the first step of the induction.

Suppose that the Lie algebra center $z(\mathbf{n})$ has a nontrivial subspace on which λ_π is zero, where the character χ_{λ_π} induces to π; then $\mathbf{k} = z(\mathbf{n}) \cap \ker \lambda_\pi$ is a nonzero, rational subspace of $\mathbf{n}$, and if $K = \exp \mathbf{k}$, then functions in H_π are K-invariant. Therefore π is actually a representation of a lower dimensional group $\bar{N} = N/K$, H_π may be imbedded in $L^2(\bar{\Gamma}\backslash\bar{N})$ where $\bar{\Gamma}$ is the image in $\bar{N}$ of Γ, and thus continuous functions in H_π must have zeros by the induction hypothesis.

Therefore we suppose that $z(\mathbf{n})$ is 1-dimensional, and that χ_π inducing π is nontrivial on $z(\mathbf{n})$.

Suppose $\{X_1, ..., X_n\}$ is a strong Malcev basis through $z(\mathbf{n})$, such that $z(\mathbf{n}) = \mathbf{R} X_n$, and such that

$$\Gamma = \exp \mathbf{Z} X_n \cdot \exp \mathbf{Z} X_{n-1} \cdots \exp \mathbf{Z} X_1 \tag{32}$$

(see [CG, Theorem V.1.6]).

Suppose F is a continuous, nonvanishing function in H_π. Then $F(x_1, ..., x_n) = \exp 2\pi i p x_n F(x_1, ..., x_{n-1}, 0)$, since $\pi \in (\Gamma \backslash N)^\wedge$, $p \in \mathbf{Z}$. Consider the function

$$G(x_1, ..., x_n) = \frac{F(x_1, ..., x_n)}{|F(x_1, ..., x_n)|}.$$

This function is continuous and nonvanishing on $\Gamma \backslash N$, and possesses the same $Z(N)$-covariance as F. Let Γ_p be defined as

$$\Gamma_p = \exp \frac{\mathbf{Z}}{p} X_n \cdot \exp \mathbf{Z} X_{n-1} \cdot \exp \mathbf{Z} X_{n-2} \cdots \exp \mathbf{Z} X_1. \tag{33}$$

Since F is left Γ_p-invariant, so is G, and both are defined on $\Gamma_p \backslash N$; note Γ_p is uniform in N, since $\Gamma \subseteq \Gamma_p$. Let

$$\mu \colon N \to Z(N) \backslash N$$

be the natural map, and let $\tilde{N}$, $\tilde{\Gamma}_p$ be the images of N and Γ_p under μ.

Define

$$\Omega \colon \Gamma_p \backslash N \to \tilde{\Gamma}_p \backslash \tilde{N} \times T$$

by

$$\Gamma_p(x_1, ..., x_n) \mapsto (\tilde{\Gamma}_p(x_1, ..., x_{n-1}), G(x_1, ..., x_n)). \tag{34}$$

Then Ω is continuous on $\Gamma_p \backslash N$ since G is; it is 1–1 since G takes on the value 1 exactly once on every fiber over $\tilde{\Gamma}_p \backslash \tilde{N}$. Ω is clearly onto, and since $\Gamma_p \backslash N$ is compact, Ω is a homeomorphism of $\Gamma_p \backslash N$ and $\tilde{\Gamma}_p \backslash \tilde{N} \times T$ (note: $\tilde{\Gamma}_p \backslash \tilde{N}$ is compact since $Z(N)$ is a rational subgroup).

However, if Γ_p is a k-step nilpotent group, then $\tilde{\Gamma}_p \times \mathbf{Z}$ is a $k-1$-step nilpotent group (recall that Γ_p is not actually abelian). Therefore, since these groups are respectively the fundamental groups of $\Gamma_p \backslash N$ and $\tilde{\Gamma}_p \backslash \tilde{N} \times T$, we arrive at a contradiction.

3. Homotopy Classes of Functions on Solvmanifolds

In Section 4 we use homotopy classes of functions from solvmanifolds to the circle to show that π-primary summand functions which are continuous must have zeros.

In this section, we demonstrate that functions nonvanishing on the solvmanifolds $M_{R,p}$, $p=2, 3$, and $M_{H,k}$ for all $k \geqslant 2$, must be null-homotopic on 2-torus fibers of the bundles $M_{R,p} \to T$ and $M_{H,k} \to T$, where T is the circle group. (Note: this is also true of the bundles $M_{R,p}$, $p=4$ and 6, but this fact is not used in Section 4.)

We consider first the solvmanifolds $M_{R,p}$, $p=2, 3$.

THEOREM 4. *For the manifolds $M_{R,p}$, $p=2,3$, the functions f_p: $M_{R,p} \to T$ defined by $f_p(\Gamma_p(t, x, y)) = e^{2\pi i t}$ are continuous and generate the groups of homotopy classes of functions from $M_{R,p}$ to T.*

Proof. We first state a few relevant facts [G-H].

Denote by $[M, T]$ the set of homotopy equivalence classes of continuous functions from M to T.

1. For all solvmanifolds under consideration, we have that $H^1(M) = [M, T]$ via the map

$$\begin{aligned} *: [M, T] &\to H^1(M); \\ f &\mapsto f_*(\omega), \end{aligned} \tag{35}$$

where ω is a generating cocycle in $H^1(T)$, and $f_*(\omega)$ is the class in $H^1(M)$ of the cocycle σ which satisfies

$$\sigma(\gamma) = \omega(f \circ \gamma) \tag{36}$$

for all 1-simplices γ.

2. For all solvmanifolds under consideration, we have $H^1(M) \cong \mathrm{Hom}(H_1(M), \mathbf{Z})$ via the isomorphism

$$\alpha: H^1(M) \to \mathrm{Hom}(H_1(M), \mathbf{Z}); \alpha(\sigma) = \tilde{\sigma}, \tag{37}$$

where for a cycle $\gamma \in H_1(M)$, $\tilde{\sigma}(\gamma) = [\sigma, \gamma]$. This follows from the existence of the exact sequence

$$0 \to \mathrm{Ext}_{\mathbf{Z}}(H_{n-1}(M), \mathbf{Z}) \to H^n(M) \xrightarrow{\alpha_n} \mathrm{Hom}(H_n(M), \mathbf{Z}) \to 0$$

for all $n \in \mathbf{Z}^+$ (Universal Coefficient Theorem). $H_0(M)$ is always a projective $\mathbf{Z}$-module, and so $\mathrm{Ext}_{\mathbf{Z}}(H_0(M), \mathbf{Z})$ is zero. Therefore α is an isomorphism.

We begin by demonstrating that for $M_{R,p}$, $p=2, 3$, we have

$H^1(M_{R,p}) \cong \mathbf{Z}$, generated by the cocycle λ_1 for which $\bar{\lambda}_1(\gamma_1) = 1$ (here γ_1 is the 1-simplex $t \in [0, 1) \to \Gamma_p(t, 0, 0)$). We also define the simplices

$$\gamma_2 : t \in [0, 1] \to \Gamma_p(0, t, 0) \tag{38}$$

$$\gamma_3 : t \in [0, 1] \to \Gamma_p(0, 0, t) \tag{39}$$

and note that γ_1, γ_2, and γ_3 generate the group $\pi_1(M_{R,p})$. Furthermore, $\pi_1(M_{R,p})$ is isomorphic to Γ_p.

Case 1. $H_1(M_{R,2}) = \mathbf{Z} \cdot \gamma_1 \oplus \mathbf{Z}_2 \cdot \gamma_2 \oplus \mathbf{Z}_2 \cdot \gamma_3$.

This follows from the fact that $[\pi_1(M_{R,2}), \pi_1(M_{R,2})]$ is generated by the elements γ_2^2 and γ_3^2 in $\pi_1(M_{R,2})$.

Case 2. $H_1(M_{R,3}) = \mathbf{Z} \cdot \gamma_1 \oplus \mathbf{Z}_3 \cdot \gamma_2$.

Here we use the fact that $[\pi_1(M_{R,3}), \pi_1(M_{R,3})]$ is generated in $\pi_1(M_{R,3})$ by the elements $\gamma_2 \gamma_3$ and γ_3^3.

We now compute $H^1(M_{R,p})$, $p = 2, 3$, using the fact that $H^1(M_{R,p}) \cong \operatorname{Hom}(H_1(M_{R,p}), \mathbf{Z})$.

Case 1. $H^1(M_{R,2}) \cong \operatorname{Hom}(\mathbf{Z} \cdot \gamma_1 \oplus \mathbf{Z}_2 \cdot \gamma_2 \oplus \mathbf{Z} \cdot \gamma_3, \mathbf{Z}) = \operatorname{Hom}(\mathbf{Z} \cdot \gamma_1, \mathbf{Z}) \cong \mathbf{Z} \cdot \lambda_1$, where λ_1 is the cocycle in $H^1(M_{R,2})$ satisfying $\bar{\lambda}_1(\gamma_1) = 1$, $\bar{\lambda}_1(\gamma_2) = \bar{\lambda}_1(\gamma_3) = 0$.

Case 2. $H^1(M_{R,3}) \cong \operatorname{Hom}(\mathbf{Z} \cdot \gamma_1 \oplus \mathbf{Z}_3 \cdot \gamma_2, \mathbf{Z}) = \operatorname{Hom}(\mathbf{Z} \cdot \gamma_1, \mathbf{Z}) \cong \mathbf{Z} \cdot \lambda_1$, where λ_1 is the cocycle in $M_{R,3}$ satisfying $\bar{\lambda}_1(\gamma_{1)=1,}\ \bar{\lambda}^1(\gamma_2) = \bar{\lambda}_1(\gamma_3) = 0$.

Finally, if we suppose that ω is any cocycle generating $H^1(T)$, then $[M_{R,p}, T]$ is generated by any continuous function f on $M_{R,p}$ satisfying $f_*(\omega) = \lambda_1$. Therefore we must have $f_*(\omega)(\gamma_1) = 1$, $f_*(\omega)(\gamma_2) = f_*(\omega)(\gamma_3) = 0$.

Since f_p in the statement of Theorem 4 satisfies these conditions and is continuous on $M_{R,p}$, f_p generates $[M_{R,p}, T]$ for $p = 2, 3$. This completes the proof of Theorem 4.

THEOREM 5. *For the manifolds $M_{H,k}$, $k = 3, 4, 5, \ldots$, the functions $f_k : M_{H,k} \to T$ defined by*

$$f_k(\Gamma_k(t, x, y)) = e^{2\pi i t}$$

are continuous on $M_{H,k}$ and generate the groups of homotopy classes of functions from $M_{H,k}$ to T.

Proof. Facts 1 and 2 following the statement of Theorem 4 also apply here. We begin by demonstrating that for $M_{H,k}$, $k \geqslant 3$, we have $H^1(M_{H,k}) \cong \mathbf{Z}$, generated by the cocycle λ_1 for which $\bar{\lambda}_1(\gamma_1) = 1$ (here γ_1 is the 1-simplex $t \in [0, 1] \mapsto \Gamma_k(t, 0, 0)$).

Again we define the simplices

$$\gamma_2 : t \in [0, 1] \mapsto \Gamma_k(0, t, 0)$$

$$\gamma_3 : t \in [0, 1] \mapsto \Gamma_k(0, 0, t)$$

and note that γ_1, γ_2, and γ_3 generate the group $\pi_1(M_{H,k})$. We also have $\pi_1(M_{H,k}) \cong \Gamma_k$.

Case 1. $H_1(M_{H,2}) = \mathbf{Z} \cdot \gamma_1$.
This follows from the fact that $[\pi_1(M_{H,2}), \pi_1(M_{H,2})]$ is generated in $\pi_1(M_{H,2})$ by the elements γ_2 and $\gamma_2\gamma_3$, which together generate all terms of the form $\gamma_2^M \gamma_3^N$.

Case 2. $H_1(M_{H,k}) = \mathbf{Z} \cdot \gamma_1 \oplus \mathbf{Z}_{k-1} \cdot \gamma_2$ for $k \geqslant 3$.
In the case $k \geqslant 3$, we have $[\pi_1(M_{H,k}), \pi_2(M_{H,k})]$ generated by γ_2^{k-1} and γ_3 in $\pi_1(M_{H,k})$.
We now compute $H^1(M_{H,k})$ for $k \geqslant 2$.

Case 1. $H^1(M_{H,2}) \cong \mathrm{Hom}(H_1(M_{H,2}), \mathbf{Z}) = \mathrm{Hom}(\mathbf{Z} \cdot \gamma_1, \mathbf{Z} \cdot \lambda_1, \mathbf{Z}) \cong \mathbf{Z} \cdot \lambda_1$, where λ_1 is the cocycle satisfying $\tilde{\lambda}_1(\gamma_1) = 1$, $\tilde{\lambda}_1(\gamma_2) = \tilde{\lambda}_1(\gamma_3) = 0$.

Case 2. $H^1(M_{H,k}) \cong \mathrm{Hom}(H_1(M_{H,k}), \mathbf{Z}) = \mathrm{Hom}(\mathbf{Z} \cdot \gamma_1 \oplus \mathbf{Z}_{k-1} \cdot \gamma_2, \mathbf{Z})$ for $k \geqslant 3$. However, this is $\mathrm{Hom}(\mathbf{Z} \cdot \gamma_1, \mathbf{Z}) \cong \mathbf{Z} \cdot \lambda_1$, where λ_1 is the cocycle satisfying $\tilde{\lambda}_1(\gamma_1) = 1$, $\tilde{\lambda}_1(\gamma_2) = \tilde{\lambda}_1(\gamma_3) = 0$.

If we suppose again that ω is a cocycle generating $H^1(T)$, then $[M_{H,k}, T]$ is generated by any continuous f on $M_{H,k}$, $k \geqslant 2$, satisfying $f_*(\omega) = \lambda_1$. The rest of the argument proceeds as for Theorem 4.

4. Zeros of Continuous Functions in $H_\pi \subseteq L^2(M_{R,p})$

We begin this section by demonstrating that $M_{R,1} \cong T^3$ possesses π-primary functions which are nonvanishing, as one would expect.

Suppose $\lambda_\pi \in \mathscr{L}^* \cap \mathcal{O}_\pi$; then the character χ_{λ_π} defined on a maximal subgroup M of $S_{R,1}$ gives rise to the Mackey space $M(\lambda_\pi)$.

Functions in the image of the lift map L: $M(\lambda_\pi) \to L^2(M_{R,1})$ are of the form f: $M_{R,1} \to \mathbf{C}$, $f(\Gamma_1(t, x, y)) = \tilde{f}(t)\, \chi_{\lambda_\pi}(0, x, y)$ for some L^2 function $\tilde{f}$ with period 1. Clearly f is continuous if $\tilde{f}$ is.

Thus we see that the π-primary summand contains the character

$$\psi(\Gamma_1(t, x, y)) = e^{2\pi i \lambda_\pi(x, y)} \tag{40}$$

which is continuous and nonvanishing on $M_{R,1}$.

We wish to emphasize here that the manifold $M_{R,1} \cong T^3$ is the only compact 3-dimensional solvmanifold arising from a non-nilpotent Lie group known to possess continuous, nonvanishing π-primary summand functions. We demonstrate in what follows that for $M_{R,p}$, $p=2$, 4, 6, continuous functions in the infinite-dimensional π-primary summands must have zeros. In the case of $M_{R,3}$, continuous functions in certain constructible subspaces which span the π-primary summands are known to have zeros, but the complete answer for $M_{R,3}$ is not known.

We begin by examining the situation for $M_{R,2}$.

THEOREM 6. *Suppose f is a continuous function in $H_\pi \subseteq L^2(M_{R,2})$, for $\pi \in (\Gamma_{R,2}\backslash S_{R,2})^\wedge_\infty$. Then f has at least one zero on $M_{R,2}$.*

Proof. Recall that $S_{R,2} = \mathbf{R} \propto \mathbf{R}^2$, with $\mathbf{R}$ acting on $\mathbf{R}^2$ via the 1-parameter subgroup $\sigma(t) = Rot(\pi t)$, and that $\Gamma_{R,2}$ is the subgroup of integer points in $S_{R,2}$. The coadjoint orbits are therefore circular cylinders in $\mathbf{s}^*_{R,2}$, saturated in the T^*-direction. The coadjoint orbit associated with $\pi \in (M_{R,2})^\wedge_\infty$ is that containing an integral functional $\lambda_\pi = \alpha X^* + \beta Y^*$ for which χ_{λ_π} induces π.

Let P_{π_i} be orthogonal projection of $L^2(M_{R,2})$ onto the irreducible subspace $H_{\pi,i}$ which is the image of the lift map L_i from $M(\lambda_\pi)$ to $L^2(M_{R,2})$. We note that if f is continuous on $M_{R,2}$, then $P_{\pi_i}f$ is continuous on $M_{R,2}$ [Ri1], and

$$P_{\pi_i}f = \sum_{j=0}^{1} f(t,\cdot,\cdot)^\wedge\ (\sigma(j)(\alpha,\beta))\, \chi_{\sigma(j)(\alpha,\beta)} \tag{41}$$

is equal to $L_i f'$ for some continuous f' in $M(\lambda_\pi)$ (Lemma 1).

Thus, in order to prove Theorem 6, it suffices to look at sums of functions of the form $\sum_{i=1}^{\mathrm{mul}(\pi)} L_i f_i$, for f_i continuous in $M(\lambda_\pi)$, where $M(\lambda_\pi)$ lifts to $H_{\pi,i}$.

Let $S = \{\lambda\colon \alpha^2+\beta^2=\lambda^2,\ (\alpha,\beta)\in\mathbf{Z}^2,\ (\alpha,\beta)\neq(0,0)\}$. Order the elements of S so that $\lambda_k > \lambda_{k-1}$. The proof of Theorem 6 is by induction on the elements of S.

Case 1. $\lambda_1 = 1$, $\mathrm{mul}(\pi_{\lambda_1}) = 2$.

This case falls into the category of odd λ_n, which is treated in the induction step.

For the induction step, we suppose that if f is continuous on $M_{R,2}$, $f\in H_{\pi_k}$ for $\lambda_k<\lambda_n$, then f must have a zero.

Case 1. λ_n^2 is odd.

Let $\{(\alpha_i,\beta_i)\}_{i=1}^{\mathrm{mul}(\pi_{\lambda_n})}$ be a complete set of Γ_2-orbit representatives in $\mathcal{O}_{\lambda_N}\cap\mathcal{L}^*$; for convenience we may take them to be in the set $\{(\alpha,\beta)\colon \alpha,\beta\in\mathbf{Z},\ \alpha+\beta>0\}$. Note that $\alpha+\beta$ is odd whenever $\alpha^2+\beta^2=\lambda_n^2$ is odd.

Let $P=\{p=\alpha_i+\beta_i\}$. Order P so that $p_k>p_{k-1}$. Note that the p are positive.

Let $Q_k = \{i: \alpha_i + \beta_i = p_k\}$, $k = 1, ..., m$. Note that each Q_k has cardinality 2; this follows from the fact that if $\alpha_i + \beta_i = \alpha_j + \beta_j$ and $\alpha_i^2 + \beta_i^2 = \alpha_j^2 + \beta_j^2$ (orbit condition), then if $\alpha_i \neq \alpha_j$, we must have $\alpha_i = \beta_j$. Thus Q_k has cardinality 2, one for each of (α, β) and (β, α). Let $\{f_i: \mathbf{R} \to \mathbf{C}\}_i^{\mathrm{mul}(\pi_n)}$ be a set of continuous functions with period 2.

Then as demonstrated in Section 1, a typical continuous ϕ in H_{π_n} has the form (setting $z_1 = e^{2\pi i x}$, $z_2 = e^{2\pi i y}$)

$$\phi(t, z_1, z_2) = \left\{ \sum_{\{Q_k\}} \left[\sum_{i \in Q_k} f_i(t)\, z_1^{\alpha_i} z_2^{\beta_i} + f_i(t+1)\, z_1^{-\alpha_i} z_2^{-\beta_i} \right] \right\}. \tag{42}$$

Fix t and define $\phi_t(z_1, z_2) \equiv \phi(t, z_1, z_2)$, so that ϕ_t: $N \cap \Gamma_2 \backslash N \cong T^2 \to \mathbf{C}$.

Suppose ϕ is nonvanishing on $M_{R,2}$. Then ϕ_t: $T^2 \to \mathbf{C}$ is nullhomotopic. Consider $\tilde{\phi}_t$, the restriction of ϕ_t to the closed curve $z_1 = z_2$ in T^2. Then $\tilde{\phi}_t$, defined by

$$\tilde{\phi}_t(z) = \phi_t(z, z) = \sum_{\{Q_k\}} \left[\sum_{i \in Q_k} f_i(t) \right] z^{p_k} + \left[\sum_{i \in Q_k} f_i(t+1) \right] z^{-p_k} \tag{43}$$

is a curve in $\mathbf{C} \backslash \{0\}$ which has winding number zero.

Clearly $\tilde{\phi}_t$ may be viewed as the restriction to T of a meromorphic function Φ_t: $\mathbf{C} \to \mathbf{C}$,

$$\Phi_t(\omega) = \sum_{\{Q_k\}} \left[\sum_{i \in Q_k} f_i(t) \right] \omega^{p_k} + \left[\sum_{i \in Q_k} f_i(t+1) \right] \omega^{-p_k}. \tag{44}$$

We claim that Φ_t has a pole of odd order at $\omega \equiv 0$. If not, then the coefficients in Φ_t of negative exponents are zero; but since Φ_t contains only terms with odd exponents, Φ_t has no constant term, and thus we would have a polynomial Φ_t with $\Phi_t(0) = 0$. Thus Φ_t would wind at least once on the circle T, so $\tilde{\phi}_t$ would wind, a contradiction. Let $\gamma \geqslant 1$ be the order of the pole at zero. Then we may write

$$\begin{aligned} \Phi_t(z) &= z^{-\gamma} \left\{ \sum_{\{Q_k\}} \sum_{i \in Q_k} f_i(t) \right] z^{p_k + \gamma} + \left[\sum_{i \in Q_k} f_i(t+1)\right] z^{-p_k + \gamma} \right\} \\ &= z^{-\gamma} p(z), \end{aligned} \tag{45}$$

where p is a polynomial in z with even exponents.

But then the zeros of $p(z)$ (and so the zeros of Φ_t) occur in pairs of equal modulus, so that the number of zeros inside the unit circle is even. Define Γ: $[0, 1] \to T$ by $\Gamma(t) = e^{2\pi i t}$. Then we have

$$\mathrm{Ind}_\Gamma \Phi_t = N_{\Phi_t} - P_{\Phi_t} \neq 0, \tag{46}$$

where N_{Φ_t} and P_{Φ_t} are, respectively, the number of zeros and poles of Φ_t inside Γ [Ru, Chap. 10]. Thus the winding number of $\tilde{\phi}_t$ cannot be zero, and we arrive at a contradiction.

Case 2. $\lambda_n^2 \equiv 0 \bmod 4$.

Note that if $\alpha^2+\beta^2=\lambda_N^2 \equiv 0 \bmod 4$, then both α and β must be even.

Let $\{(\alpha_i, \beta_i)\}_{i=1}^{\mathrm{mul}(\pi_n)}$ be a set of $\Gamma_{R,2}$-orbit representatives, and let $\{f_i: \mathbf{R} \to \mathbf{C}\}_{i=1}^{\mathrm{mul}(\pi_n)}$ be continuous with period 2. Then a typical $\phi \in H_{\pi_{\lambda_N}}$ may be written

$$\phi(t, z, z_2) = \sum_{i=1}^{\mathrm{mul}(\pi_n)} f_i(t)\, z_1^{\alpha_i} z_2^{\beta_i} + f_i(t+1)\, z_1^{-\alpha_i} z_2^{-\beta_i}, \tag{47}$$

for $z_1 = e^{2\pi i x}$, $z_2 = e^{2\pi i y}$.

We define

$$\psi(t, z_1, z_2) = \sum_{i=1}^{\mathrm{mul}(\pi_n)} f_i(t)\, z_1^{\alpha_i/2} z_2^{\beta_i/2} + f_i(t+1)\, z_1^{-\alpha_i/2} z_2^{-\beta_i/2}. \tag{48}$$

Note that since the α_i and β_i are all divisible by 2, ψ has integer exponents; therefore, ψ is continuous, $\Gamma_{R,2}$-invariant, and lives in $H_{\pi_{\lambda_k}}$ where $\lambda_k^2 = \lambda_n^2/4$. By the induction hypothesis, ψ has a zero. Since $\phi(t, z_1, z_2) = \psi(t, z_1^2, z_2^2)$, ϕ must also have a zero.

Case 3. $\lambda_n^2 \equiv 2 \bmod 4$.

Let $\{(\alpha_i, \beta_i)\}_{i=1}^{\mathrm{mul}(\pi_{\lambda_n})}$ be a complete set of $\Gamma_{R,2}$-orbit representatives from $\mathcal{O}_{\pi_{\lambda_n}} \cap \mathcal{L}^*$, satisfying $\alpha_i > 0$ for each i. Note that $\alpha_i^2 + \beta_i^2 = \lambda_n^2 = 2 \bmod 4$ implies that α_i and β_i are both odd for each i.

Let $P = \{p = \alpha_i \text{ for some } i\}$. Order P so that $p_k > p_{k-1}$. Let $Q_k = \{i\colon \alpha_i = p_k\}$, and note that each Q_k has cardinality 2. Let $\{f_i: \mathbf{R} \to \mathbf{C}\}_{i=1}^{\mathrm{mul}(\pi_n)}$ be a set of continuous funtions of period 2. Then a typical continuous ϕ in $H_{\pi_{\lambda_n}}$ has the form of Eq. (42).

Fix t and define $\phi_t(z_1, z_2) = \phi(t, z_1, z_2)$, $\phi_t\colon N \cap \Gamma_2 \backslash N \cong T^2 \to \mathbf{C}$. Suppose that ϕ is nonvanishing, so that ϕ_t must be nullhomotopic on $N \cap \Gamma_2 \backslash N$.

Define $\tilde{\phi}_t(z) = \phi_t(z, 1)$, the restriction of ϕ_t to the curve $z_2 \equiv 1$ in $N \cap \Gamma_2 \backslash N$. Then we have

$$\tilde{\phi}_t(z) = \sum_{\{Q_k\}} \left\{ \left[\sum_{i \in Q_k} f_i(t) \right] z^{p_k} + \left[\sum_{i \in Q_k} f_i(t+1) \right] z^{-p_k} \right\}.$$

Since each α_i is odd, each p_k is odd; by choice of α_i, we have $p_k > 0$ for all k.

From here we proceed, as in Case 1, to demonstrate that $\tilde{\phi}_t$ must wind on the circle T, a contradiction. This completes Case 3 and finishes the proof of Theorem 6.

In Theorem 7, we show that continuous functions in the π-primary summands of $L^2(M_{R,4})$ and $L^2(M_{R,6})$ must have zeros.

THEOREM 7. *Let f be a continuous function in* $H_\pi \subseteq L^2(M_{R,i})$, *for* $i=4$ *or* 6, $\pi \in (\Gamma_{R,i} \backslash S_{R,i})^\wedge_\infty$. *Then f has at least one zero on* $M_{R,i}$.

Proof. Define the groups

$$\Gamma'_4 = \{(m, n, p) \in \Gamma_{R,4} \subseteq S_{R,4} : M = 2k, \text{ for some } k \in \mathbf{Z}\}$$

$$\Gamma'_6 = \{(m, n, p) \in \Gamma_{R,6} \subseteq S_{R,6} : M = 3k, \text{ for some } k \in \mathbf{Z}\}.$$

Then Γ'_4 and Γ'_6 are subgroups of $\Gamma_{R,4}$ and $\Gamma_{R,6}$, respectively, of finite index; thus Γ'_i is cocompact in $S_{R,i}$ for each i, and it is straightforward to verify that $\Gamma'_i \cong \Gamma_{R,2}$, $i=4, 6$.

We prove Theorem 7 for $M_{R,4}$; the proof for $M_{R,6}$ is analogous in every respect.

Since $\Gamma'_4 \cong \Gamma_{R,2}$ and is cocompact in $S_{R,4}$, we have $\Pi_1(\Gamma'_4 \backslash S_{R,4}) \cong \Gamma_{R,2}$. Therefore we have $\Gamma'_4 \backslash S_{R,4} \cong M_{R,2}$ [Mos, Theorem A].

Functions which are $\Gamma_{R,4}$-periodic are Γ'_4-periodic, so $L^2(M_{R,4})$ embeds isometrically in $L^2(M_{R,2})$. Furthermore this embedding is $S_{R,4}$-equivariant with respect to the quasi-regular representation, and so takes π-spaces to π-spaces.

Let Φ be the isometric embedding of $L^2(M_{R,4})$ in $L^2(M_{R,2})$. Then if f is a continuous function in $H_\pi \subseteq L^2(M_{R,4})$, Φf is continuous in $H_\pi \subseteq L^2(M_{R,2})$ and so must have a zero. However, if $\Phi f(\Gamma_{R,2}(t, x, y)) = 0$, then since f is Γ_4-invariant, $f(\Gamma_{R,4}(t, x, y)) = 0$; thus f must have a zero. This completes the proof of Theorem 7.

We finish this section with a theorem summarizing what is known for $M_{R,3}$.

THEOREM 8. *Let f be a continuous element of a constructible, irreducible subspace of a* π*-primary summand* $H_\pi \subseteq L^2(M_{R,3})$. *Then f has at least one zero on* $M_{R,3}$.

Proof. Suppose $\lambda_\pi \in \mathscr{L}^* \cap \mathcal{O}_\pi$, an integral functional in $\mathfrak{s}^*_{R,3}$; then the character χ_{λ_π} defined on a maximal subgroup M of $S_{R,3}$ gives rise to the Mackey space, $M(\lambda_\pi)$. The constructible irreducible subspace corresponding to $\lambda_\pi = (\alpha, \beta)$ is the image of the lift map L: $M(\lambda_\pi) \to L^2(M_{R,3})$, an $S_{R,3}$-invariant isometry.

A typical continuous element of this constructible irreducible subspace of H_π has the form

$$\tilde{f}(\Gamma_3(t,x,y)) = \tilde{f}(t,z_1,z_2)$$
$$= f(t)\, z_1^\alpha z_2^\beta + f(t+1)\, z_1^\beta z_2^{-(\alpha+\beta)} + f(t+2)\, z_1^{-(\alpha+\beta)} z_2^\alpha \quad (49)$$

for $z_1 = e^{2\pi i x}$, $z_2 = e^{2\pi i y}$, and $f\colon \mathbf{R} \to \mathbf{C}$ a continuous function of period 3.

Suppose $\tilde{f}$ is nonvanishing on $M_{R,3}$. Then $\tilde{f}$ must be nullhomotopic when restricted to T^2-fibers of the bundle $M_{R,3} \to T$.

We examine the functions $\tilde{f}$ on a case-by-case basis.

Case 1. $\alpha \equiv \beta \equiv 1 \bmod 3$, or $\alpha \equiv \beta \equiv 2 \bmod 3$.

We define $\phi_t(z) = \tilde{f}(t, z, 1)$ for fixed $t \in \mathbf{R}$; ϕ_t must have winding number zero on T, since $\tilde{f}(t, z_1, z_2)$ is nullhomotopic on T^2 for fixed t. We have

$$\phi_t(z) = f(t) z^\alpha + f(t+1) z^\beta + f(t+2) z^{-(\alpha+\beta)}. \quad (50)$$

Clearly one of α, β, and $-(\alpha+\beta)$ must be negative. If we view ϕ_t as the restriction to the set $T = \{|z| = 1,\ z \in \mathbf{C}\}$ of the meromorphic function

$$\Phi_t(\omega) = f(t)\omega^\alpha + f(t+1)\omega^\beta + f(t+2)\omega^{-(\alpha+\beta)} \quad (51)$$

we see that Φ_t has a pole at $\omega = 0$.

Let γ be the order of the pole at zero. Then we may write

$$\Phi_t(\omega) = \omega^{-\gamma}\{f(t)\omega^{\alpha+\gamma} \cdot f(t+1)^{\beta+\gamma} + f(t+2)^{-(\alpha+\beta)+\gamma}\}$$
$$= \omega^{-\gamma} p(\omega), \quad (52)$$

where $p(\omega)$ is a polynomial. Note that the exponents of $p(\omega)$ must all be divisible by 3. Since $\alpha + \beta \equiv 1$ or $2 \bmod 3$, we must have $\gamma \equiv 1$ or $2 \bmod 3$, so that $\alpha + \gamma \equiv \beta + \gamma \equiv -(\alpha+\beta) + \gamma \equiv 0 \bmod 3$. Thus the zeros of $p(\omega)$ are grouped as triples of equal modulus; in particular, the number of zeros of $p(\omega)$ (and hence of $\Phi_t(\omega)$) inside T is a multiple of 3. However, the pole of Φ_t at $\omega = 0$ is not a multiple of 3, and therefore, referring to (46), we see that Φ_t must wind on the curve T, and therefore that ϕ_t cannot be nullhomotopic, a contradiction.

Case 2. $\alpha \equiv 1 \bmod 3$, $\beta \equiv 2 \bmod 3$.

We define $\phi_t(z) = \tilde{f}(t, z, z^{-1}) = f(t) z^{\alpha-\beta} + f(t+1) z^{2\beta+\alpha} + f(t+2) z^{-(2\alpha+\beta)}$ as in Case 1 and note that ϕ_t is $\tilde{f}$ restricted to the curve $z_2 = z_1^{-1}$ in the T^2-fiber over $\Gamma_3(t, 0, 0)$.

Clearly one of $\alpha - \beta$, $2\beta + \alpha$, and $-(2\alpha + \beta)$ must be negative, since $(\alpha - \beta) + (2\beta + \alpha) = 2\alpha + \beta$. All are congruent to $2 \bmod 3$, so none can be zero. If we view ϕ_t as the restriction to T of the meromorphic function $\Phi_t(\omega) = f(t)\omega^{\alpha-\beta} + f(t+1)\omega^{2\beta+\alpha} + f(t+2)\omega^{-(2\alpha+\beta)}$, we see that Φ_t has a pole at zero. Let γ be the order of the pole at zero; then $-\gamma \equiv 2 \bmod 3$, and we may write

$$\begin{aligned}\Phi_t(\omega) &= \omega^{-\gamma}\{f(t)\omega^{\alpha-\beta+\gamma} + f(t+1)\omega^{2\beta+\alpha+\gamma} + f(t+2)\omega^{-(2\alpha+\beta)+\gamma}\} \\ &= \omega^{-\gamma}p(\omega), \end{aligned} \tag{53}$$

where $p(\omega)$ is a polynomial. Note that since $-\gamma \equiv 2 \bmod 3$, and since all of $\alpha-\beta$, $2\beta+\alpha$, and $-(2\alpha+\beta)$ are congruent to 2 as well, the function $p(\omega)$ contains only terms with exponents divisible by 3, and so the number of zeros inside T is a multiple of 3. Since γ is not a multiple of 3, we have $\operatorname{Ind}_\Gamma \Phi_t \neq 0$, and again we arrive at a contradiction.

Case 3. $\alpha \equiv 2 \bmod 3$, $\beta \equiv 1 \bmod 3$.

Proceeding as before, we define $\phi_t(z) = \tilde{f}(t, z, z^{-1}) = f(t)z^{\alpha-\beta} + f(t+1)z^{2\beta+\alpha} + f(t+2)z^{-(2\alpha+\beta)}$, and note that ϕ_t is $\tilde{f}$ restricted to the curve $z_2 = z_1^{-1}$ in the T^2-fiber over $\Gamma_3(t, 0, 0)$.

Again, one of $\alpha-\beta$, $2\beta+\alpha$, and $-(2\alpha+\beta)$ must be negative, and all are congruent to 1 mod 3, so that none is zero. If we view ϕ_t as the restriction to T of the meromorphic function

$$\Phi_t(\omega) = f(t)\omega^{\alpha-\beta} + f(t+1)\omega^{2\beta+\alpha} + f(t+2)\omega^{-(2\alpha+\beta)}$$

we see that Φ_t has a pole at zero. Let γ be the order of the pole at zero; then $-\gamma \equiv 1 \bmod 3$, and we may write

$$\begin{aligned}\Phi_t(\omega) &= \omega^{-\gamma}\{f(t)\omega^{\alpha-\beta+\gamma} + f(t+1)\omega^{2\beta+\alpha+\gamma} + f(t+2)\omega^{-(2\alpha+\beta)+\gamma}\} \\ &= \omega^{-\gamma}p(\omega),\end{aligned}$$

where $p(\omega)$ is a polynomial. Note that since $-\gamma \equiv 1 \bmod 3$, and since all of $\alpha-\beta$, $2\beta+\alpha$, and $-(2\alpha+\beta)$ are as well, the function $p(\omega)$ contains only terms with exponents divisible by 3. The number of zeros inside T is therefore a multiple of 3. Since γ is not a multiple of 3, we have $\operatorname{Ind}_\Gamma \Phi_t \neq 0$, and we arrive at a contradiction.

Case 4. $\alpha \equiv 0 \bmod 3$, $\beta \equiv 1 \bmod 3$ or $\alpha \equiv 2 \bmod 3$, $\beta \equiv 0 \bmod 3$.

If (α, β) is such that $\alpha \equiv 0 \bmod 3$ and $\beta \equiv 1 \bmod 3$, then the Γ_3-orbit containing (α, β) also contains the point $(\beta, -(\alpha+\beta))$, which is of the type dealt with in Case 2. Since the function $\tilde{f}$ is independent of the base point chosen from the Γ_3-orbit, $\tilde{f}$ must have a zero. Similarly, if (α, β) is of the type $\alpha \equiv 2 \bmod 3$, $\beta \equiv 0 \bmod 3$, then the point $(-(\alpha+\beta), \alpha)$ is of the type dealt with in Case 2.

Case 5. $\alpha \equiv 0 \bmod 3$, $\beta \equiv 2 \bmod 3$; or $\alpha \equiv 1 \bmod 3$, $\beta \equiv 0 \bmod 3$.

If (α, β) is such that $\alpha \equiv 0 \bmod 3$, $\beta \equiv 2 \bmod 3$, then $(\beta, -(\alpha, \beta))$ is in the same Γ_3-orbit as (α, β) and is of the type dealt with in Case 3. If (α, β) is of the type $\alpha \equiv 1 \bmod 3$, $\beta \equiv 0 \bmod 3$, then the point $(-(\alpha, \beta), \alpha)$ is of the type dealt with in Case 3.

Case 6. $\alpha \equiv \beta \equiv 0 \bmod 3$.

We have $\alpha = 3^k\alpha'$ and $\beta = 3^k\beta'$ for some $k \neq 0$ and some pair (α', β'), not both congruent to 0 mod 3. We may therefore define

$$\bar{g}(t, z_1, z_2) = f(t)\, z_1^{\alpha'} z_2^{\beta'} + f(t+1)\, z_1^{\beta'} z_2^{-(\alpha'+\beta')} + f(t+z)\, z_1^{-(\alpha'+\beta')} z_2^{\alpha'},$$

which is clearly the lift of a function in the Mackey space $M(\lambda'_\pi)$ for $\lambda'_\pi = (\alpha', \beta')$; since not both α' and β' are congruent to 0 mod 3, $\bar{g}$ must have a zero, since one of Cases 1–5 applies. Since $\bar{g}$ has a zero, and since we have $\bar{g}(t, z_1^{3^k}, z_2^{3^k}) = \bar{f}(t, z_1, z_2)$, $\bar{f}$ must have a zero on $M_{R,3}$.

This completes the proof of Theorem 8.

5. Zeros of Continuous Functions in $H_\pi \subseteq L^2(M_{H,k})$

In this section, we demonstrate that functions in a uniformly dense subspace K_π of continuous functions in $H_\pi^0 = C(M_{H,k}) \cap H_\pi$ must have zeros on $M_{H,k}$, for all hyperbolic solvmanifolds $M_{H,k}$. It then follows easily that all continuous functions on H_π must have zeros.

If $\{\lambda_{\pi_i}\}_{i=1}^{\mathrm{mul}(\pi)}$ is a complete set of $\Gamma_{H,k}$-orbit representatives from $\mathcal{O}_\pi \cap \mathcal{L}^*$, then the set $\{L_i\colon M(\lambda_{\pi_i}) \to L^2(M_{H,k})\}_{i=1}^{\mathrm{mul}(o)}$ is a complete set of lift maps into the constructible irreducible subspaces of H_π. Let T_i: $L^2(\mathbf{R}) \to M(\lambda_{\pi_i})$ be the isometry intertwining the Schrödinger model of π on $L^2(\mathbf{R})$ and the induced model on $M(\lambda_{\pi_i})$; i.e., $T_i f = \bar{f}$, where $\bar{f}(t, x, y) = \chi_{\lambda_{\pi_i}}(x, y)\, f(t)$.

Then $L'_i = L_i \circ T_i$ lifts $L^2(\mathbf{R})$ into the ith constructible irreducible subspace of H_π. We define

$$K_\pi = L'_1(C_0^\infty(\mathbf{R})) \oplus \cdots \oplus L'_{\mathrm{mul}(\pi)}(C_0^\infty(\mathbf{R})) \subseteq H_\pi^\infty.$$

Lemma 9. *K_π is uniformly dense in H_π^0.*

Proof. We first demonstrate that K_π is uniformly dense in $H_\pi^\infty = C^\infty(M_{H,k}) \cap H_\pi$.

If $S_*(\mathbf{R})$ are the smooth vectors for the Schrödinger model of π on $L^2(\mathbf{R})$, then we have

$$\overline{L'_i(C_0^\infty(\mathbf{R}))} \supseteq L'_i(S_*(\mathbf{R})) \tag{54}$$

in the sup-norm on $M_{H,k}$ [Bre1, Lemma 5].

We have also that $\bigoplus_i H^{\infty}_{\pi,i} = H^{\infty}_{\pi}$, since $L'_i(S_*(\mathbf{R})) = H^{\infty}_{\pi,i}$ by preservation of smooth vectors under intertwining maps.

We have also that $\bigoplus_i H^{\infty}_{\pi,i} = H^{\infty}_{\pi}$, since orthogonal projection onto $S_{H,k}$-invariant subspaces preserves infinite differentiability [Aus-Bre, Sect. 2]. Therefore if $\phi \in H^{\infty}_{\pi}$, we may uniformly approximate $P_{\pi,i}(\phi)$ in each subspace by elements of $L'_i(C^{\infty}_0(\mathbf{R}))$. Thus, K_π is uniformly dense in H^{∞}_{π}.

We finish the proof of Lemma 9 by demonstrating that H^{∞}_{π} is uniformly dense in H^{0}_{π}.

Let F be a fundamental domain for $\Gamma_k \backslash S_{H,k}$ containing the identity; define a C^{∞} approximate identity $\{\varepsilon_n\}_{n=1}^{\infty}$ so that

1. $0 \leqslant \varepsilon_n < \infty$ for each $n \in \mathbf{Z}^+$.

2. Support ε_n is contained in the interior of F for each $n \in \mathbf{Z}^+$. We define, for $\phi \in H^{0}_{\pi}$,

$$\phi * \varepsilon_n(t, x, y) = \int_{F \subseteq S_{H,k}} \phi((t, x, y)(t', x', y')^{-1})\, \varepsilon_n(t', x', y')\, dt'\, dx'\, dy'. \tag{55}$$

Then $\phi * \varepsilon_n$ is C^{∞} for each $n \in \mathbf{N}$, since ε_n is C^{∞} and $S_{H,k}$ is unimodular, and in fact $\phi * \varepsilon_n \in H^{\infty}_{\pi}$ since $\phi * \varepsilon_n$ is the uniform limit of linear combinations of right translates of ϕ.

We now claim that $\phi * \varepsilon_n$ converges uniformly to ϕ on $M_{H,k}$. We have

$$\|\phi * \varepsilon_n - \phi\|_{\infty} \leqslant \sup_{(t,x,y)} \int_F |\phi((t, x, y)(t', x', y')^{-1}) - \phi(t, x, y)| \times \varepsilon_n(t', x', y')\, dt'\, dx'\, dy'. \tag{56}$$

Choose N so that for $n \geqslant N$, we have $|\phi((t, x, y)(t', x', y')^{-1}) - \phi(t, x, y)| < \varepsilon$ for all $(t, x, y) \in F$ and all $(t', x', y') \in \text{support}(\varepsilon_n)$. Then $\|\phi * \varepsilon_n - \phi\|_{\infty} \leqslant \int \varepsilon \cdot \varepsilon_n(t', x', y')\, dt'\, dx'\, dy' = \varepsilon$, which completes the proof of uniform convergence.

Thus we have $H^{\infty}_{\pi} \subseteq H^{0}_{\pi}$ uniformly dense in H^{0}_{π}, completing the proof of Lemma 9.

Recall that $S_{H,k} = \mathbf{R} \propto \mathbf{R}^2$, with $\mathbf{R}$ acting on $\mathbf{R}^2$ via the 1-parameter subgroup $\sigma_k\colon \mathbf{R} \to SL_2(\mathbf{R})$ satisfying $\sigma_k(1) = \left[\begin{smallmatrix} 1 & 1 \\ k-1 & k \end{smallmatrix}\right]$. The one-parameter subgroup σ_k is conjugate to $\sigma(t) = \left[\begin{smallmatrix} \lambda^t & 0 \\ 0 & \lambda^{-t} \end{smallmatrix}\right]$, where $\lambda + \lambda^{-1} = k + 1$, $\lambda \in \mathbf{R}$. The coadjoint orbits in $\mathfrak{s}^*_{H,k}$ are therefore hyperbolic cylinders, saturated in the T^*-direction, and satisfying the equation

$$(k-1)x^2 + (k-1)\,xy - y^2 = \omega \tag{57}$$

for some $\omega \in \mathbf{R}$. Note that two coadjoint orbits satisfy (57) for each value of ω, each being a connected component of the set in $\mathbf{s}^*_{H,k}$ satisfying (57). The coadjoint orbit associated with $\pi \in (\Gamma_{H,k}\backslash S_{H,k})^\wedge_\infty$ is that containing an integral functional $\lambda_\pi = \alpha X^* + \beta Y^*$ for which χ_{λ_π} induces π.

THEOREM 10. *Suppose f is a continuous function in $H_\pi \subseteq L^2(M_{H,k})$, for $\pi \in (\Gamma_{H,k}\backslash S_{H,k})^\wedge_\infty$. Then f has at least one zero on $M_{H,k}$.*

Proof. We begin by proving Theorem 10 for functions $f \in K_\pi \subseteq H_\pi$. We have $K_\pi = L'_1(C_0^\infty(\mathbf{R})) \oplus \cdots \oplus L'_{\mathrm{mul}(\pi)}(C_0^\infty(\mathbf{R}))$, so that a typical element of K_π has the form

$$\phi(\Gamma_k(t, x, y)) = \sum_{i=1}^{\mathrm{mul}\,\pi} \sum_{n \in \mathbf{Z}} f_i(t+n) \exp 2\pi i(\alpha_{n,i} x + \beta_{n,i} y), \tag{58}$$

where $\{(\alpha_{0,i}, \beta_{0,i})\}_{i=1}^{\mathrm{mul}(\pi)}$ is a set of distinct Γ_k-orbit representatives in $\mathcal{O}_\pi \cap \mathcal{L}^*$, $(\alpha_{n,i}, \beta_{n,i}) = \sigma_k(n)(\alpha_{0,i}, \beta_{0,i})$, and where for each $i = 1, ..., \mathrm{mul}(\pi)$, and for fixed t, the sum over n in (58) is finite. Suppose ϕ is nonvanishing on $M_{H,k}$. By setting $z_1 = e^{2\pi i x}$, $z_2 = e^{2\pi i y}$, we may define

$$\tilde{\phi}(t, z_1, z_2) = \phi(\Gamma_k(t, x, y)) = \sum_{i=1}^{\mathrm{mul}\,\pi} \sum_{n \in \mathbf{Z}} f_i(t+n)\, z_1^{\alpha_{n,i}} z_2^{\beta_{n,i}}. \tag{59}$$

For any fixed t, we must have $\tilde{\phi}$ nullthomotopic on $N \cap \Gamma_k \cong T^2\backslash N$ by Theorem 5.

We note at this point that if (α, β) satisfies $(k-1)\alpha^2 + (k-1)\,\alpha\beta - \beta^2 = \omega$ for $\omega > 0$, then either all points in $\mathcal{O}_{(\alpha,\beta)}$ satisfy $\alpha > 0$, or they all satisfy $\alpha < 0$. If not, then since $\mathcal{O}_{(\alpha,\beta)}$ is connected, $\mathcal{O}_{(\alpha,\beta)}$ must intersect the y-axis, so that $\alpha = 0$ and $-\beta^2 = \omega$, a contradiction. Similarly, if $\mathcal{O}_{(\alpha,\beta)}$ satisfies $(k-1)\alpha^2 + (k-1)\,\alpha\beta - \beta^2 = \omega$ for $\omega < 0$, then either all points in $\mathcal{O}_{(\alpha,\beta)}$ satisfy $\beta > 0$, or they all satisfy $\beta < 0$.

Suppose $\mathcal{O}_{(\alpha,\beta)}$ satisfies $(k-1)\alpha^2 + (k-1)\,\alpha\beta - \beta^2 = \omega$ for $\omega > 0$.

Case 1. Suppose all $(\alpha, \beta) \in \mathcal{O}_{(\alpha,\beta)}$ satisfy $\alpha > 0$. Then we have

$$\tilde{\phi}(t, z_1, z_2) = \sum_{i=1}^{\mathrm{mul}\,\pi} \sum_{n \in \mathbf{Z}} f_i(t+n)\, z_1^{\alpha_{n,i}} z_2^{\beta_{n,i}},$$

where $\alpha_{n,i} > 0$ for all n, i. Fixing t, we have

$$\tilde{\phi}_t(z) = \tilde{\phi}(t, z, 1) = \sum_{i=1}^{\mathrm{mul}\,\pi} \sum_{n \in \mathbf{Z}} f_i(t+n) z^{\alpha_{n,i}} \tag{60}$$

a curve of winding number zero on the circle T. The sum in (60) is always finite.

We may consider $\tilde{\phi}_t$ to be the restriction of a polynomial Φ_t on $\mathbf{C}$ to T, i.e., $\Phi_t(\omega) = \sum_{i=1}^{\mathrm{mul}\,\pi} \sum_{n \in \mathbf{Z}} f_i(t+n)\,\omega^{\alpha_{n,i}}$. Since $\alpha_{n,i}$ is never zero, Φ_t has no constant term and so has a zero at $\omega = 0$. Since Φ_t has no pole inside T, we see, referring to (2.20) that Φ_t and hence $\bar{\phi}_t$ cannot have winding number zero on T. Therefore $\bar{\phi}$ cannot be nullhomotopic on $N \cap \Gamma_k \backslash N$ for fixed t, and so $\bar{\phi}$ cannot be nonvanishing.

Case 2. Suppose all $(\alpha, \beta) \in \mathcal{O}_{(\alpha,\beta)}$ satisfy $\alpha < 0$. Then set $\bar{\phi}_t(z) = \bar{\phi}(t, z^{-1}, 1)$ for fixed t and proceed as in Case 1.

Case 3. Let $\mathcal{O}_{(\alpha,\beta)}$ satisfy $(k-1)\alpha^2 + (k-1)\alpha\beta - \beta^2 = \omega$ for $\omega < 0$. Suppose all $(\alpha, \beta) \in \mathcal{O}_{(\alpha,\beta)}$ satisfy $\beta > 0$. Then set $\bar{\phi}_t(z) = \bar{\phi}(t, 1, z)$ for fixed t and proceed as in Case 1.

Case 4. Suppose all $(\alpha, \beta) \in \mathcal{O}_{(\alpha,\beta)}$ satisfy $\beta > 0$. Then set $\bar{\phi}_t(z) = \bar{\phi}(t, 1, z^{-1})$ and proceed as in Case 1.

Thus we have proof that all $f \in K_\pi$ have zeros, and so we have shown that all functions in a uniformly dense subspace of H_π^0 have zeros. Therefore all functions in H_π^0 must have zeros.

REFERENCES

[AGH] L. AUSLANDER *et al.*, Flows on homogeneous spaces, *in* "Annals of Mathematics Studies," Vol. 53, Princeton Univ. Press, Princeton, NJ, 1963.

[Aus-Bre] L. AUSLANDER AND J. BREZIN, Uniform distribution in solvmanifolds, *Adv. in Math.* 7 (1971), 111–144.

[Aus-Tol] L. AUSLANDER AND R. TOLIMIERI, Abelian harmonic analysis, theta functions and function algebras on a nilmanifold, *in* "Lecture Notes in Mathematics," Vol. 436, Springer-Verlag, New York/Berlin, 1975.

[Bre] J. BREZIN, Harmonic analysis on compact solvmanifolds, *in* "Lecture Notes in Mathematics," Vol. 602, Springer-Verlag, New York/Berlin, 1970.

[Bre1] J. BREZIN, Geometry and the method of Kirillov, *in* "Lecture Notes in Mathematics," Vol. 466, pp. 13–25, Springer-Verlag, New York/Berlin, 1975.

[CG] L. CORWIN AND F. GREENLEAF, Representations of nilpotent Lie groups and their applications. Part 1. Basic theory and examples.

[Dau] I. DAUBECHIES *et al.*, Painless non-orthogonal expansions, *J. Math. Phys.* **27** (1986), 1271–1283.

[GGP] I. M. GELFAND *et al.*, "Representation Theory and Automorphic Functions," Saunders, Philadelphia, 1969.

[G-H] M. GREENBERG AND J. HARPER, Algebraic topology, a first course, *in* "Mathematics Lecture Note Series," Vol. 58, Addison–Wesley, Reading, MA, 1981.

[Hum] J. HUMPHREYS, Introduction to Lie algebras and representation theory, *in* "Graduate Texts in Mathematics," Vol. 9, Springer-Verlag, New York/Berlin, 1972.

[Mal] A. I. MALCEV, On a class of homogeneous spaces, *in* "Amer. Math. Soc. Transl. Ser. 1," Vol. 39, Amer. Math. Soc., Providence, RI, 1951.

[Mos] G. D. MOSTOW, Factor spaces of solvable groups, *Ann. of Math.* **60** (1954), 1–27.

[Ri1] L. RICHARDSON, A class of idempotent measures on compact nilmanifolds, *Acta Math.* **135** (1975).

[Ru] W. RUDIN, "Real and Complex Analysis," 2nd ed., McGraw–Hill, New York, 1974.