Zeroes of Primary Summand Functions on Compact Solvmanifolds

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The fact that continuous functions in primary summands of the Heisenberg manifold must vanish somewhere was proven by L. Auslander and R. Tolimieri, who deduced from this theorem the classical results on the vanishing of theta functions, as well as important applications to wavelets and radar ambiguity functions. The Heisenberg theorem seemed to depend on the presence of a central character, but the result is here extended to include primary summand functions on all compact nilmanifolds and to three-dimensional compact solvmanifolds which are not n-tori.

Let $G$ be a solvable, connected and simply connected Lie group, with Lie algebra $\mathfrak{g}$ and with cocompact discrete subgroup $\Gamma$. By a representation $\pi$ of $G$ we shall mean a strongly continuous, unitary representation of $G$ in some separable Hilbert space $H_\pi$; $\pi$ will be called irreducible if the space $H_\pi$ contains no proper closed nontrivial subspace invariant under $\pi$.

Let $M$ be the space of right cosets $\Gamma g$ of $\Gamma$ in $G$, endowed with the quotient topology. Then $G$ acts on $L^2(M)$ by right translation; i.e., $g \mapsto R(g)$, where $[R(g)f](\Gamma x) = f(\Gamma xg)$ for $f \in L^2(M)$ (here $M$ has the $G$-invariant probability measure inherited from Haar measure on $G$). $R$ is called the quasiregular representation of $G$ on $L^2(M)$.

It is well known that $L^2(M)$ decomposes into the direct sum $\bigoplus H_\pi$, where the spaces $H_\pi$ are mutually orthogonal $R(G)$-invariant subspaces, and $R$ on the space $H_\pi$ is a finite multiple of the irreducible representation $\pi$ [GGP, Sect. 1.2]. We let $(\Gamma \backslash G)\wedge$ denote the set of irreducible representations appearing in the quasiregular representation $R$ of $G$ on $L^2(M)$. $(\Gamma \backslash S)\wedge_\infty$ will denote the set of those representations $\pi \in (\Gamma \backslash S)\wedge$ which are infinite dimensional. Then the orthogonal projection $P_\pi$ of $L^2(M)$ onto $H_\pi$ is $L^2$-continuous and preserves $C^\infty(M)$ [Aus-Bre, Theorem 5], and is given by convolution with a bounded Borel measure $\sigma_\pi$.

Now let $N$ be a nilpotent Lie group, connected and simply connected, with Lie algebra $\mathfrak{n}$ and cocompact discrete subgroup $\Gamma$. 161
If the coadjoint orbits of the action of $N$ on the dual $n^*$ are linear varieties, then $\Gamma \backslash N$ possesses the property that the orthogonal projections $P_x$ of $L^2(\Gamma \backslash N)$ onto $H_x$ preserve continuity [Ril, Bre1]. These flat-orbit nilmanifolds share this property with compact quotients of the 3-dimensional solvable group $S_R$ by discrete subgroups. Here $S_R$ denotes the semidirect product $\mathbb{R} \ltimes \mathbb{R}^2$, where $\mathbb{R}$ acts on $\mathbb{R}^2$ via a one-parameter subgroup of rotations.

This paper was motivated by a theorem of L. Auslander and R. Tolimieri. Let $H_3$ be the 3-dimensional Heisenberg group, $\mathbb{R}^3$ endowed with the multiplication $(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy')$, and let $\Gamma$ be the discrete group of integer points in $H_3$. Let $f$ be a continuous function in $H_\pi \subset L^2(\Gamma \backslash H_3)$, where $\pi$ is an irreducible, unitary, infinite-dimensional representation in $(\Gamma \backslash H_3)$. Then $f$ must have at least one zero on $\Gamma \backslash H_3$ [Aus-Tol, Theorem II.2]. It is shown in Chapter II of [Aus-Tol] that the vanishing of theta functions follows as a direct corollary of this theorem. In work by L. Auslander, R. Tolimieri, I. Daubechies, A. Janssen, D. Gabor, and others, this result has been shown to have important consequences for wavelet theory, and applications to problems involving the radar ambiguity function (see, for example, [Dau]).

The phenomenon of vanishing arises from a rather surprising interaction between the representation theory of $H_4$ (which determines the primary summand $H_\pi$) and the topology of the manifold $\Gamma \backslash H_3$. In this paper, we generalize this theorem to all 3-dimensional compact solvmanifolds, using techniques of harmonic analysis on solvmanifolds developed by L. Auslander, J. Brezin, L. Richardson, and others.

The proof that all continuous primary summand functions on compact nilmanifolds have zeros is an adaptation of Auslander and Tolimieri's original proof, using induction and relying heavily upon the central covariance which all such functions possess; this covariance appears to be at the heart of the result in the nilpotent case. However, since 3-dimensional non-nilpotent solvable Lie groups with cocompact discrete subgroups have trivial centers [AGH, Chap. 3], completely new techniques are needed to show that most 3-dimensional compact solvmanifolds do possess the property that their continuous $\pi$-primary functions (hereafter referred to as primary functions) must vanish, for infinite-dimensional $\pi$. A noteworthy exception is one compact quotient of $S_R$ which is actually homeomorphic to the 3-torus $T^3$; here one finds plenty of continuous primary functions which do not vanish, as one would expect. However, for three of four remaining compact quotients of $S_R$, it is shown that continuous primary functions must have zeros. For the fourth compact quotient of $S_R$, we have shown that continuous functions in certain subspaces of a primary summand $H_\pi$ must have zeros. As of this writing,
however, it is conjectured but not known that all continuous primary summand functions on this manifold must have zeros.

Let $S_H$ be the semidirect product $\mathbb{R} \rtimes \mathbb{R}^2$, where $\mathbb{R}$ acts upon $\mathbb{R}^2$ via the one-parameter subgroup $t \mapsto \left[ \begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array} \right]$ in $SL_2(\mathbb{R})$, where $\lambda + \lambda^{-1} = k + 1$ for any integer $k \geq 2$. It is shown in this paper that for all compact quotients of $S_H$, continuous primary functions must have zeros. This exhausts the compact solvmanifolds of dimension three.

Thus the interplay of topology and representation theory which produces zeros of continuous primary functions is seen to be more than a nilpotent phenomenon, but the extent of this interaction remains obscure. There is the possibility of a generalization of this theorem to a larger class of compact solvmanifolds.

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1. Preliminaries

Let $G$ be a connected, simply connected Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{g}^*$ be the vector space of linear functionals on $\mathfrak{g}$. We define a sequence of ideals of the Lie algebra $\mathfrak{g}$ by $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}]$; this is called the derived series of $\mathfrak{g}$, and $\mathfrak{g}$ is said to be solvable if $\mathfrak{g}^{(n)} = 0$ for some $n \in \mathbb{N}$. We define another sequence of ideals of the Lie algebra $\mathfrak{g}$ by $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}]$; this is called the lower central series of $\mathfrak{g}$, and $\mathfrak{g}$ is said to be nilpotent if $\mathfrak{g}^{(n)} = 0$ for some $n \in \mathbb{N}$ (see [Hum, Sect. 3]). The term “nilmanifold” (“solvmanifold”) will refer to compact spaces $\Gamma \backslash G$, where $G$ is nilpotent (solvable) and $\Gamma$ is discrete and cocompact.

The adjoint representation of the group $G$ in the vector space $\mathfrak{g}$, written $\text{Ad}$, is defined as follows; for each element $x \in G$, $\text{Ad}(x): \mathfrak{g} \to \mathfrak{g}$ is the differential at the identity of $G$ of the group automorphism $Z(x)$, inner conjugation by $x \in G$. $\text{Ad}(x)$ satisfies

$$x(\exp X)x^{-1} = \exp[\text{Ad}(x)X]$$

for each $x \in G$, $X \in \mathfrak{g}$.

The coadjoint representation of $G$ is of central importance in the representation theory of nilpotent and solvable Lie groups. The set of equivalence classes of irreducible representations of a nilpotent Lie group $G$ is naturally parametrized by the orbit space $\mathfrak{g}^*/\text{Ad}^* G$; this is also true for the (completely) solvable Lie groups examined in this work. This parametrization, due to A. A. Kirillov, is freely drawn upon here; for details, see [CG, Chap. II].
As described in the introduction, there are two \( j \)-dimensional, solvable, non-nilpotent Lie groups with cocompact discrete subgroups, the groups \( S_H \) and \( S_R \). Their Lie algebras are three-dimensional vector spaces spanned by the vectors \( T, X, \) and \( Y \), where \( \exp sT = (s, 0, 0) \), \( \exp sX = (0, s, 0) \) and \( \exp sY = (0, 0, s) \).

We have the following five compact quotients of \( S_R \), with convenient coordinatization (see [AGH, Sect. 2.2]).

1. \( \Gamma_{R,1} \setminus S_{R,1} = M_{R,1} \), where \( S_{R,1} = \mathbb{R} \ltimes \mathbb{R}^2 \), \( \mathbb{R} \) acts on \( \mathbb{R}^2 \) via the one-parameter subgroup \( \sigma_1(t) = \left[ \begin{array}{cc} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{array} \right] \), and \( \Gamma_{R,1} = \{(p, m, n) \in S_{R,1}; p, m, n \in \mathbb{Z}\} \). Here \( \Gamma_{R,1} \) is isomorphic to the abelian group \( \mathbb{Z}^3 \), and so \( M_{R,1} \cong T^3 \) [Mos, Theorem A].

2. \( \Gamma_{R,2} \setminus S_{R,2} = M_{R,2} \), where \( S_{R,2} = \mathbb{R} \ltimes \mathbb{R}^2 \), \( \mathbb{R} \) acts on \( \mathbb{R}^2 \) via the one-parameter subgroup \( \sigma_2(t) = \left[ \begin{array}{cc} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{array} \right] \), and \( \Gamma_{R,2} = \{(p, m, n) \in S_{R,2}; p, m, n \in \mathbb{Z}\} \).

3. \( \Gamma_{R,3} \setminus S_{R,3} = M_{R,3} \), where \( S_{R,3} = \mathbb{R} \ltimes \mathbb{R}^2 \), \( \mathbb{R} \) acts on \( \mathbb{R}^2 \) via the one-parameter subgroup \( \sigma_3(t) \) with \( \sigma_3(1) = \left[ \begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array} \right] \), \( \sigma_3(t) \) is isomorphic to the subgroup \( \text{Rot}(2\pi t/3) = \left[ \begin{array}{cc} \cos 2\pi t/3 & \sin 2\pi t/3 \\ -\sin 2\pi t/3 & \cos 2\pi t/3 \end{array} \right] \), and \( \Gamma_{R,3} = \{(p, m, n) \in S_{R,3}; p, m, n \in \mathbb{Z}\} \).

4. \( \Gamma_{R,4} \setminus S_{R,4} = M_{R,4} \), where \( S_{R,4} = \mathbb{R} \ltimes \mathbb{R}^2 \), \( \mathbb{R} \) acts on \( \mathbb{R}^2 \) via the one-parameter subgroup \( \sigma_4(t) = \left[ \begin{array}{cc} \cos \pi t/2 & \sin \pi t/2 \\ -\sin \pi t/2 & \cos \pi t/2 \end{array} \right] \), and \( \Gamma_{R,4} = \{(p, m, n) \in S_{R,4}; p, m, n \in \mathbb{Z}\} \).

5. \( \Gamma_{R,6} \setminus S_{R,6} = M_{R,6} \), where \( S_{R,6} = \mathbb{R} \ltimes \mathbb{R}^2 \), \( \mathbb{R} \) acts on \( \mathbb{R}^2 \) via the one-parameter subgroup \( \sigma_6(t) \) in \( SL_2(\mathbb{R}) \) with \( \sigma_6(1) = \left[ \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right] \), and \( \Gamma_{R,6} = \{(p, m, n) \in S_{R,6}; p, m, n \in \mathbb{Z}\} \).

We also have the following compact quotients of \( S_H \), with convenient coordinatizations.

Suppose \( k \in \mathbb{Z}, \ k \geq 2 \). Define \( S_{H,k} = \mathbb{R} \ltimes \mathbb{R}^2 \), where \( \mathbb{R} \) acts on \( \mathbb{R}^2 \) via the one-parameter subgroup \( \sigma_k(t) \) in \( SL_2(\mathbb{R}) \) with \( \sigma_k(1) = \left[ \begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array} \right] \). Then \( S_{H,k} \cong S_H \) for each \( k \). Let \( \Gamma_{H,k} = \{(p, m, n) \in S_{H,k}; p, m, n \in \mathbb{Z}\} \); then each \( \Gamma_{H,k} \setminus S_{H,k} = M_{H,k} \) is a distinct compact quotient of \( S_H \).

Thus there are 5 distinct, non-homeomorphic compact quotients of \( S_R \), and infinitely many distinct compact quotients of \( S_H \).

It will be convenient to use several different coordinatizations of \( S_R \) and \( S_H \). The coordinatizations of \( S_R \) just described will be called integral coordinatizations of \( S_{R,p} \). Let \( A \in GL_2(\mathbb{R}) \) be such that

\[
A\sigma_p(t)A^{-1} = R(2\pi t/p) = \left[ \begin{array}{cc} \cos 2\pi t/p & \sin 2\pi t/p \\ -\sin 2\pi t/p & \cos 2\pi t/p \end{array} \right].
\]

If we recoordinatize \( N \) so that the action of \( \mathbb{R} \) on \( N \) is given by \( R(2\pi t/p) \), then \( \Gamma_{R,p} \cap N = A(\mathbb{Z}^2) \) (note that in the case of \( \Gamma_{R,1}, \Gamma_{R,2}, \) and \( \Gamma_{R,4}, \)
A = I). In this coordinatization of $S_{R,p}$, the nondegenerate coadjoint orbits of $S_{R,p}$ are circular cylinders, $x^2 + y^2 = \lambda^2$, for some $\lambda \in \mathbb{R}$. For the groups $S_{R,3}$ and $S_{R,6}$, the 2-torus $N \cap \Gamma_{R,p} \setminus N$ will be a non-standard torus in this coordinatization. We will call these coordinatizations the circular coordinatizations of $S_{R,p}$.

The coordinatizations of the solvmanifolds $S_{H,k}$ just described will be referred to as the integral coordinatizations of $S_{H,k}$. Let $A \in GL_2(\mathbb{R})$ be such that $A \sigma_k(t) A^{-1} = \left[ \begin{array}{cc} \lambda & 0 \\ 0 & \lambda^{-1} \end{array} \right]$ where $\lambda + \lambda^{-1} = k + 1$; if we recoordinatize $N$ so that the action of $\mathbb{R}$ on $N$ is given by this one parameter subgroup, then $\Gamma_{H,k} \cap N = A(\mathbb{Z}^2)$; the nondegenerate coadjoint orbits in this case are hyperbolic cylinders of the form $x y = \lambda, \lambda \in \mathbb{R}$. The 2-torus $N \cap \Gamma_{H,k} \setminus N$ in this coordinatization will be a non-standard torus, for all $k \geq 2$. This coordinatization of $S_{H,k}$ will be referred to as the hyperbolic coordinatization.

For each solvmanifold $M_{H,k}(M_{R,i})$, the group $S_{H,k}(S_{R,i})$ is a simply connected cover of $M_{H,k}(M_{R,i})$ and $\Pi_1(M_{H,k})=\Pi_1(M_{R,i})=\Gamma_{R,i}$. The $M_{H,k}$ and $M_{R,i}$ are bundles over the circle with 2-torus fiber; the projection maps are

$$
\mu_H : M_{H,k} \rightarrow \mathbb{Z} \setminus \mathbb{R}
$$

$$
\Gamma_{H,k}(t, u, v) \mapsto \mathbb{Z} + t
$$

and

$$
\mu_R : M_{R,p} \rightarrow \mathbb{Z} \setminus \mathbb{R}
$$

$$
\Gamma_{R,p}(t, u, v) \mapsto \mathbb{Z} + t
$$

A convenient decomposition of $H_\pi$ into irreducible subspaces will be used throughout this paper; however, no canonical decomposition of $H_\pi$ into irreducible subspaces exists. The irreducible subspaces of the chosen decomposition of $H_\pi$ will be referred to as the constructible irreducible subspaces of $H_\pi$.

We will now describe those functions on $M_{H,k}(M_{R,i})$ which are primary functions. We will use integral coordinatizations of $S_{R,p}$ and $S_{H,k}$.

In the integral coordinatization of $S_{H,k}$, the coadjoint orbits satisfy

$$(k - 1) x^2 + (k - 1) x y - y^2 = \lambda$$

so that the orbits are saturated in the $T^*$-direction. We will call $\lambda$ an integral functional if $\lambda|_n = \alpha x^* + \beta y^*$, $\alpha, \beta \in \mathbb{Z}$, and denote by $\mathcal{O}_\pi$ the orbit of $\lambda$ in $s_{H,k}^*$.  

Fix some nonzero integral functional $\lambda \in \mathcal{O}_\pi$. We define the character $\chi_\lambda$ on the (abelian) nilradical $n$ follows: if $\lambda|_n = \alpha x^* + \beta y^*$, then

$$
\chi_\lambda(0, r, s) = e^{2\pi i (\alpha r + \beta s)}.
$$
We seek a maximal subgroup $M$ of $S_{H,k}$ such that

(i) $M$ contains $N$;

(ii) $\chi_:\lambda$ may be extended to a character of $M$, i.e., $\chi_:\lambda[M, M] = 1$, where $[M, M]$ is the commutator subgroup of $M$. $M$ will be called a maximal subgroup subordinate to $\lambda$. We will call this extension $\tilde{\chi}$, the map extension of $\chi$ to $M$. To this end, it suffices to examine the values of $\chi_\lambda$ on terms in $[S_{H,k}, S_{H,k}]$ of the form

$$(t, 0, 0)(0, x, y)(-t, 0, 0)(0, -x, -y) = (0, x, y) - \sigma_\lambda(t)(x, y)).$$

Since $\sigma_\lambda(t)$ has eigenvalues $\lambda^+$ and $\lambda^-$, where $\lambda^+ = k + 1$, and since $\lambda$ is nonzero,

$$\chi_\lambda(0, (x, y) - \sigma_\lambda(t)(x, y)) = \exp 2\pi i \lambda((x, y) - \sigma_\lambda(t)(x, y))$$

is 1 for all $(x, y)$ if and only if $t = 0$. Thus $N$ itself is maximal subordinate to $\lambda$ for all nonzero $\lambda$.

We define the Mackey space $M(\lambda)$ for $\lambda$ as

$$M(\lambda) = \{f: S_{H,k} \to \mathbb{C} | \text{if is measurable}, |f| \in L^2(N \setminus S_{H,k}), \quad f(n g) = \chi_\lambda(n)f(g) \text{ } n \in N, g \in S_{H,k}\}.$$  

It is well known that the action of $S_{H,k}$ on $M(\lambda)$ by right translation is an irreducible representation $\pi$. We note that the functions $f$ in $M(\lambda)$ are left $\Gamma_{H,k} \cap N$-invariant, and define the homogenizing (lift) map $L: M(\lambda) \to L^2(\Gamma_{H,k} \setminus S_{H,k})$ as follows: for $f \in M(\lambda)$,

$$Lf(\Gamma_{H,k}(t, x, y)) = \sum_{\gamma \in \Gamma_{H,k} \cap N \setminus \Gamma_{H,k}} (f \cdot \gamma)(t, x, y),$$

where $(f \cdot \gamma)(g) = f(\gamma g)$, for $\gamma, g \in S_{H,k}$.

Note that the sum in (8) is well defined with respect to equivalence classes of $\Gamma_{H,k} \cap N \setminus H_k$. For all $f \in M(\lambda)$, $Lf$ is a well-defined element of $L^2(\Gamma_{H,k} \setminus S_{H,k})$, and the map $L$ is $S_{H,k}$-equivariant, so that the image in $L^2(\Gamma_{H,k} \setminus S_{H,k})$ of $M(\lambda)$ is an irreducible $\pi$-subspace of $L^2(\Gamma_{H,k} \setminus S_{H,k})$, and therefore a subspace of the $\pi$-primary summand $H_\pi$.

Those subspaces of $H_\pi$ which are the images of homogenizing maps $L$ from Mackey spaces $M(\lambda)$ will be referred to as the constructible irreducible subspaces of $H_\pi$; there are finitely many such subspaces in each $H_\pi$. The maximum number of such mutually orthogonal subspaces will be referred to as the multiplicity of $\pi$, $\text{mul } \pi$. There is one constructible irreducible subspace in $H_\pi$ for each distinct $\Gamma_{H,k}$-orbit in the integral functionals of $\mathcal{O}_\pi$, and so $\text{mul } \pi$ is given by the number of such orbits.
The constructible irreducible subspaces are not canonically determined irreducible subspaces of $H_\pi$; nevertheless, the $\pi$-primary summand is the orthogonal direct sum of these spaces, and so we may represent a typical member of the $\pi$-primary summand $H_\pi$ as follows. Let \( \{ \lambda_i \}_{i=1}^{\text{mul}_\pi} \) be a set of representatives of distinct $\Gamma_{H,k}$-orbits in the integral functionals of $C_\pi$, $\lambda_i$ an integral functional for each $i$. Let $f_i \in M(\lambda_i)$; $L_i$, the lift map from $M(\lambda_i)$ to $L^2$. Then a typical member of $H_\pi$ has the form

$$F = \sum_{i=1}^{\text{mul}_\pi} L_i f_i = \sum_{i=1}^{\text{mul}_\pi} \left\{ \sum_{y \in \Gamma_{H,k} \cap N \setminus \Gamma_{H,k}} (f_i \cdot \gamma) \right\}. \quad (9)$$

If $f_i \in M(\lambda_i)$, then $f_i$ may be written

$$f_i(t, x, y) = \chi_{\lambda_i}(0, x, y) \tilde{f}_i(t), \quad (10)$$

where $\tilde{f}_i \in L^2(\mathbb{R})$. Thus, if we choose the elements $(n, 0, 0) \in \Gamma_{H,k}, n \in \mathbb{Z}$, to represent the equivalence classes of $\Gamma_{H,k} \cap N \setminus \Gamma_{H,k}$, we have

$$L f_i(\Gamma_{H,k}(t, x, y)) = \sum_{n \in \mathbb{Z}} [f_i(n, 0, 0)](t, x, y)$$

$$= \sum_{n \in \mathbb{Z}} f_i((n, 0, 0) \cdot (t, x, y))$$

$$= \sum_{n \in \mathbb{Z}} f_i(n + t, \sigma_k(n)(x, y))$$

$$= \sum_{n \in \mathbb{Z}} \tilde{f}_i(n + t) \chi_{\lambda_i}(0, \sigma_k(n)(x, y))$$

$$= \sum_{n \in \mathbb{Z}} \tilde{f}_i(n + t) \chi_{\sigma_k(n)\lambda_i}(0, x, y). \quad (11)$$

Thus a typical member of $H_\pi$ is of the form

$$\sum_{i=1}^{\text{mul}_\pi} \sum_{n \in \mathbb{Z}} \tilde{f}_i(n + t) \chi_{\sigma_k(n)\lambda_i}(0, x, y), \quad (12)$$

where all elements $\sigma_k(n)\lambda_i$ satisfy Eq. (4).

Suppose we have $\lambda \in \mathfrak{n}^*$, $\lambda = \alpha X^* + \beta Y^*$ for some $\alpha, \beta \in \mathbb{Z}$. Then

$$\chi_\lambda(0, x, y) = \exp 2\pi i(\alpha x + \beta y). \quad (13)$$

If we set $z_1 = e^{2\pi i x}$, $z_2 = e^{2\pi i y}$, we may write $\chi_\lambda(0, x, y) = e^{2\pi i (\alpha x + \beta y)} = \text{exp} (2\pi i (\alpha x + \beta y))$. \]
so that \( F \in H_\pi \) may be thought of as a function of \( t, z_1, z_2 \), i.e., as \( \tilde{F} \), where

\[
\tilde{F}(t, z_1, z_2) = \sum_{i=1}^{\mu \pi} \sum_{n \in \mathbb{Z}} f'_i(n + t) z_1^{\alpha_{n,i}} z_2^{\beta_{n,i}}
\]

(14)

for \( \sigma_k^*(n) \lambda_i = \alpha_{n,i} X^* + \beta_{n,i} Y^* \). Fixing \( t_0 \in \mathbb{R} \), we may define a cross section

\[
\tilde{F}_{t_0}(z_1, z_2) \equiv \tilde{F}(t_0, z_1, z_2)
\]

(15)

so that \( \tilde{F} \) is a function from \( T^2 \) to \( \mathbb{C} \), for \( z_1, z_2 \) of modulus 1. Let the integral functionals \( \{ \lambda_i \}_{i=1}^{\mu \pi} \) be a set of \( \Gamma_{H,k} \)-orbit representatives in \( \pi^* \); we define \( H_\pi \) to be the image of the lift map

\[
L: M(\lambda_i) \to L^2(\Gamma_{H,k} \backslash S_{H,k})
\]

(\( H_\pi \) is the \( i \)-th constructible irreducible subspace of \( H_\pi \), the \( \pi \)-primary summand).

In the integral coordinatization of \( S_{R,p} \), the coadjoint orbits satisfy

\[
(i) \quad x^2 + \frac{1}{4} y^2 = k^2 \quad \text{for some } k \in \mathbb{R} \text{ if } p = 1, 2, 4
\]

\[
(ii) \quad x^2 + xy + \frac{1}{4} y^2 = k^2 \quad \text{for some } k \in \mathbb{R} \text{ if } p = 3, 6,
\]

(16) (17)

so that the orbits are saturated in the \( T^* \)-direction; fix some coadjoint orbit \( \mathcal{O}_\pi \subset S_{R,p}^* \), and some nonzero integral functional \( \lambda \in \mathcal{O}_\pi \).

We define the character \( \chi_\lambda \) on the nilradical \( N \) as

\[
\chi_\lambda(0, r, s) = \exp 2\pi i \lambda(r, s) = \exp 2\pi i (\alpha r + \beta s),
\]

where \( \lambda = \alpha X^* + \beta Y^* \). We seek a maximal extension of the character \( \chi_\lambda \) on \( M \); we examine the values of \( \chi_\lambda \) on terms of the form

\[
(t, 0, 0)(0, x, y)(-t, 0, 0)(0, -x, -y) = (0, x, y) - \sigma_p(t)(x, y),
\]

in the commutator \([S_{R,p}, S_{R,p}]\). Since \( \sigma_p(t) \) has eigenvalues \( \exp \pm 2\pi it/p \), and since \( \sigma_p(p \mathbb{Z}) \equiv I \), we may extend the character \( \chi_\lambda \) to a character on the subgroup

\[
M_p = \{ (n, x, y): n = pk \text{ for some } k \in \mathbb{Z}, x, y \in \mathbb{R} \}.
\]

Then \( M_p \) is called maximal subordinate to the functional \( \lambda \).

We define the Mackey space \( M(\lambda) \) for \( \lambda \) as

\[
M(\lambda) = \{ f: S_{R,p} \to \mathbb{C} | f \text{ is measurable, } |f| \in L^2(\Gamma_{R,p} \backslash S_{Z,p}),
\]

\[
f(mg) = \chi_\lambda(m) f(g) \quad m \in M_p, \quad g \in S_{R,p}\}.
\]

(20)
The action of $S_{\mathbb{R}, p}$ on $M(\lambda)$ by right translation is an irreducible representation $\pi$, independent (up to equivalence) of the choice of $\lambda \in \mathcal{O}_\pi$.

The functions $f \in M(\lambda)$ are left $\Gamma_{\mathbb{R}, p} \cap M_{\mathbb{R}, p}$-invariant. We define the lift map $L: M(\lambda) \to L^2(\Gamma_{\mathbb{R}, p} \setminus S_{\mathbb{R}, p})$ as follows: for $f \in M(\lambda)$,

$$Lf(\Gamma_{\mathbb{R}, p}(t, x, y)) = \sum_{\gamma \in \Gamma_{\mathbb{R}, p} \cap M_{\mathbb{R}, p}} (f \cdot \gamma)(t, x, y). \quad (21)$$

Note that the sume (21) is a sum of $p$ terms, and is a left $\Gamma_{\mathbb{R}, p}$-invariant function in $L^2(\Gamma_{\mathbb{R}, p} \setminus S_{\mathbb{R}, p})$. $L$ is an $S_{\mathbb{R}, p}$-equivariant map, so that the image in $L^2(\Gamma_{\mathbb{R}, p} \setminus S_{\mathbb{R}, p})$ of $M(\lambda)$ is an irreducible $\pi$-subspace of $L^2(\Gamma_{\mathbb{R}, p} \setminus S_{\mathbb{R}, p})$, and therefore a subspace of the $\pi$-primary summand $H_{\pi} \subset L^2(\Gamma_{\mathbb{R}, p} \setminus S_{\mathbb{R}, p})$; this subspace will also be referred to as a constructible irreducible subspace of $H_{\pi}$. The number of such mutually orthogonal subspaces of $H_{\pi}$ is equal to the number of disjoint $\Gamma_{\mathbb{R}, p}$-orbits in the set of integral functions in $\mathcal{O}_\pi$. $H_{\pi}$ is the orthogonal direct sum of these subspaces.

We may represent a typical element of the $\pi$-primary summand $H_{\pi}$ as follows. Let $\{\lambda_i\}_{i=1}^{\text{mul } \pi}$ be a set of representatives of distinct $\Gamma_{\mathbb{R}, p}$-orbits of integral functionals in $\mathfrak{n}^* \cap \mathcal{O}_\pi$. If $f_i \in M(\lambda_i)$, a typical member of $H_{\pi}$ has the form

$$F = \sum_{i=1}^{\text{mul } \pi} Lf_i = \sum_{i=1}^{\text{mul } \pi} \left\{ \sum_{\gamma \in M_{\mathbb{R}, p} \cap \Gamma_{\mathbb{R}, p}} f_i \cdot \gamma \right\}. \quad (22)$$

If $f_i \in M(\lambda_i)$, then $f_i$ may be written

$$f_i(t, x, y) = \chi_{\lambda_i}(0, x, y) \tilde{f}_i(t), \quad (23)$$

where $\tilde{f}_i \in L^2(p\mathbb{Z} \setminus \mathbb{R})$ ($\tilde{f}_i$ is to be thought of as a function on $\mathbb{R}$ with period $p$). Thus, if we choose the elements $(0, 0, 0) \in \Gamma_{\mathbb{R}, p}$, $n = 0, 1, 2, \ldots, p - 1$ to represent the equivalence classes of $M_{\mathbb{R}, p} \cap \Gamma_{\mathbb{R}, p}$, we have

$$Lf_i(\Gamma_{\mathbb{R}, p}(t, x, y)) = \sum_{n=0}^{p-1} (f_i \cdot (n, 0, 0))(t, x, y)$$

$$= \sum_{n=0}^{p-1} f_i(n + t, \sigma_k(n))(x, y)$$

$$- \sum_{n=0}^{p-1} (\tilde{f}_i(n + t) \chi_{\lambda_i}(\sigma_k(n))(x, y))$$

$$= \sum_{n=0}^{p-1} (\tilde{f}_i(n + t) \chi_{\lambda_i \sigma_k(n)}(x, y)). \quad (24)$$
Thus a typical member of $H_\lambda$ is of the form

$$
\sum_{i=1}^{\text{mul}(\sigma)} \sum_{n \in \mathbb{Z}} \mathcal{f}_i(n + t) \chi_{\lambda_i}(n, 0, x, y).
$$

(25)

Recall that all elements $\sigma_i^*(n)\lambda_i$ satisfy Eq. (16) or (17).

Suppose we have $\lambda \in \mathfrak{n}^*$, $\lambda = \alpha X^* + \beta Y^*$ for some $\alpha, \beta \in \mathbb{Z}$. Then

$$
\chi_{\lambda}(0, x, y) = \exp 2\pi i(\alpha x + \beta y).
$$

(26)

If we set $z_1 = e^{2\pi i x}$, $z_2 = e^{2\pi i y}$, we may define $\mathcal{F}$ and $\mathcal{F}_i$ as in Eqs. (14) and (15) for $F \in H_\lambda$. If the integral functionals $\{\lambda_i\}_{i=1}^{\text{mul} \sigma}$ are a set of $\Gamma_{\mathfrak{r}, p}$-orbit representatives in $\mathfrak{n}^*$, we define $H_{\pi i}$ to be the image of the lift map $L(M(\lambda_i)) \to L^2(\Gamma_{\mathfrak{r}, p} \setminus \mathcal{S}_{\mathfrak{r}, p})$. $H_{\pi i}$ is the the $i$th constructible irreducible subspace of $H_\lambda$.

We end this section with a fact, and a lemma.

1. $S_{H, k}(\mathcal{S}_{R, p})$ in its integral coordinatization has the fundamental domain $[0, 1]^3$; since $T^3$ has the same fundamental domain and since the invariant measure of the boundary is zero, the identification of fundamental domain produces a Borel isomorphism of the measure spaces and an isometry between $L^2(T^3)$ and $L^2(\Gamma_{\mathfrak{r}, k} \setminus \mathcal{S}_{\mathfrak{r}, k})$. Since each character $f_{\alpha, \beta, \gamma}(t, x, y) = \exp 2\pi i(\alpha x + \beta y + \gamma t)$, $\alpha, \beta, \gamma \in \mathbb{Z}$, appears in the summand $H_\pi$ for which $\alpha X^* + \beta Y^* \in \mathfrak{c}_\pi$, we have that the $\pi$-primary summands $H_\pi$, together with the constant functions, form a complete set of orthonormal subspaces in $L^2(\Gamma_{\mathfrak{r}, k} \setminus \mathcal{S}_{\mathfrak{r}, k})[L^2(\Gamma_{\mathfrak{r}, k} \setminus \mathcal{S}_{\mathfrak{r}, k})]$.

2. We define

$$
P_{\pi i}: L^2(\Gamma_{\mathfrak{r}, k} \setminus \mathcal{S}_{\mathfrak{r}, p}) \to H_{\pi i}
$$

to be the orthogonal projection of $L^2(\Gamma_{\mathfrak{r}, p} \setminus \mathcal{S}_{\mathfrak{r}, p})$ onto $H_{\pi i}$. We have for $f \in L^2(\Gamma_{\mathfrak{r}, p} \setminus \mathcal{S}_{\mathfrak{r}, p})$,

$$
P_{\pi i}(f)(\Gamma_{\mathfrak{r}, p}(t, x, y)) = \sum_{N=0}^{p-1} f(t, \cdot, \cdot)^* \sigma_i^*(N) \chi_{\lambda_i}(0, x, y),
$$

(27)

where $f(t, \cdot, \cdot)^*$ is the standard Fourier transform in the variables $x$ and $y$ for fixed $t$ (note that for fixed $t$, $P_{\pi i}f(t, x, y)$ is a function on $(\mathcal{N} \cap \Gamma_{\mathfrak{r}, p} \setminus N) \cong T^2$).

**Lemma 1.** Suppose $f$ is continuous on $M_{\mathfrak{r}, p}$. Then $P_{\pi i}f = L\mathcal{f}$ for some continuous $\mathcal{f}$ in $M(\lambda_i)$.

**Proof.** We need to produce a continuous function $\mathcal{f} \in M(\lambda_i)$ such that $L\mathcal{f} = P_{\pi i}f$. 

Let
\[ \tilde{f}(t, x, y) = f(t, \cdot, \cdot) \wedge (\lambda_i) \chi_{\lambda_i}. \]  
(28)

To see that \( L\tilde{f} = P_{n,f} \), we must demonstrate
1. that \( \tilde{f} \in M(\lambda_i) \) and \( \tilde{f} \) is continuous in \((t, x, y)\);
2. that \( \tilde{f}(t + k, \sigma_p(k)(x, y)) = f(t + k, \cdot, \cdot) \wedge (\sigma_p^*(k)\lambda_i) \chi_{\sigma_p^*(k)\lambda_i}(x, y) \); i.e.,
the \( k \)th terms in each sum are identical. By definition,
\[ \tilde{f}(t + k, \sigma_p(k)(x, y)) = f(t + k, \cdot, \cdot) \wedge (\lambda_i) \chi_{\sigma_p(k)(x, y)}. \]  
(29)

Since \( \chi_{\sigma_p^*(k)\lambda_i}(x, y) = \chi_{\lambda_i}(\sigma_p(k)(x, y)) \), to demonstrate part 2 we need only show that
\[ f((t + k), \cdot, \cdot) \wedge (\cdot) = f(t, \cdot, \cdot) \wedge (\sigma_p^*(k)\lambda_i). \]  
(30)

By definition,
\[ f((t + k), \cdot, \cdot) \wedge (\lambda_i) = f(t + k, \cdot, \cdot) \wedge (\lambda_i) \]
\[ = \int_{N \cap \Gamma_{R,p}\setminus N} f(t + k, x, y) \chi_{\lambda_i}(x, y) \, dx \, dy. \]  
(31)

Since \( f \) is continuous on \( M_{R,p} \) and is therefore left \( \Gamma_{R,p} \)-invariant, we have
\[ \int_{N \cap \Gamma_{R,p}\setminus N} f(t + k, x, y) \chi_{\lambda_i} \, dx \, dy \]
\[ = \int_{N \cap \Gamma_{R,p}\setminus N} f(t, \sigma_p(-k)(x, y)) \chi_{\lambda_i}(x, y) \, dx \, dy \]
\[ = \int_{N \cap \Gamma_{R,p}\setminus N} f(t, x, y) \chi_{\lambda_i}(\sigma_p(k)(x, y)) \, |\det \sigma_p(k)| \, dx \, dy \]
\[ = f(t, x, y) \chi_{\sigma_p^*(k)\lambda_i}(x, y) \, dx \, dy \]
\[ = f(t, \cdot, \cdot) \wedge (\sigma_p^*(k)\lambda_i). \]

\( \tilde{f} \) has the desired left-\( M_p \)-invariance, and is therefore in \( M(\lambda_i) \); since \( f \) itself is continuous in \( t \), \( f(t, \cdot, \cdot) \wedge (\lambda_i) \) is continuous in \( t \) and therefore \( \tilde{f} \) is a continuous function in \( M(\lambda_i) \). This completes the proof of Lemma 1.
2. ZEROS OF CONTINUOUS FUNCTIONS IN $H_n$ OF A COMPACT NILMANIFOLD

We have the following generalization of a theorem of L. Auslander and R. Tolimieri [Aus-Tol, Theorem II.2].

**THEOREM 2.** Let $N$ be a nilpotent Lie group with cocompact discrete subgroup $\Gamma$, $\Gamma \backslash N$ not isomorphic to $T^n$ for any $n \in \mathbb{N}$. If $f$ is a continuous function in $H_n \subseteq L^2(\Gamma \backslash N)$ for $\pi \in (\Gamma \backslash N)_{\infty}$, then $f$ has at least one zero on $\Gamma \backslash N$.

**Proof.** We proceed by induction on $\dim N$.

We begin with $\dim N = 3$, where the 3-dimensional Heisenberg group $H_3$ is the only example of a nilpotent group with quotient manifolds that are not isomorphic to $T^3$. Theorem 2 for this case was proved by L. Auslander and R. Tolimieri in [Aus-Tol, Theorem II.2].

**LEMMA 3.** Let $\Gamma' = \{ (p, m, n) \in H_3 : p, m, n \in \mathbb{Z} \}$, $\Gamma'$ contains a subgroup isomorphic to $\Gamma$, and its index in $\Gamma'$ is finite.

This lemma follows immediately from the results of A. I. Malcev in [Mal].

We are given $\Gamma \backslash N$ compact, and the map

$$\Phi : L^2(\Gamma' \backslash N) \rightarrow L^2(\Gamma \backslash N)$$

defined by $\Phi f(\Gamma x) = f(\Gamma' x)$ is a well-defined, $N$-equivariant isometry of $L^2(\Gamma' \backslash N)$ with its image in $L^2(\Gamma \backslash N)$.

Suppose $\pi \in (\Gamma \backslash N)_{\infty}$ and that $f$ is a continuous function in $H_n \subseteq L^2(\Gamma \backslash N)$. Then $\Phi f$ is continuous in $L^2(\Gamma \backslash N)$. Since $\Phi$ is an $N$-equivariant isometry, $\Phi(H_n)$ is contained in the $\pi$-primary summand of $L^2(\Gamma \backslash N)$. By Theorem II.2 in [Aus-Tol], then, $\Phi f$ must have a zero. This completes the first step of the induction.

Suppose that the Lie algebra center $z(n)$ has a nontrivial subspace on which $\chi_{\pi}$ is zero, where the character $\chi_{\lambda_n}$ induces to $\pi$; then $k = z(n) \cap \ker \lambda_n$ is a nonzero, rational subspace of $n$, and if $K = \exp k$, then functions in $H_n$ are $K$-invariant. Therefore $\pi$ is actually a representation of a lower dimensional group $\bar{N} = N/K$, $H_n$ may be imbedded in $L^2(\bar{\Gamma} \backslash \bar{N})$ where $\bar{\Gamma}$ is the image in $\bar{N}$ of $\Gamma$, and thus continuous functions in $H_n$ must have zeros by the induction hypothesis.

Therefore we suppose that $z(n)$ is 1-dimensional, and that $\chi_{\pi}$ inducing $\pi$ is nontrivial on $z(n)$. 


Suppose \( \{X_1, \ldots, X_n\} \) is a strong Malcev basis through \( z(n) \), such that \( z(n) = R X_n \), and such that

\[
\Gamma = \exp ZX_n \cdot \exp ZX_{n-1} \cdots \exp ZX_1
\]  
(32)

(see [CG, Theorem V.1.6]).

Suppose \( F \) is a continuous, nonvanishing function in \( H_p \). Then

\[
F(x_1, \ldots, x_n) = \exp 2\pi i p x_n F(x_1, \ldots, x_{n-1}, 0), \quad \text{since } \pi \in (\Gamma \backslash N)^\sim, \quad p \in \mathbb{Z}. 
\]

Consider the function

\[
G(x_1, \ldots, x_n) = \frac{F(x_1, \ldots, x_n)}{|F(x_1, \ldots, x_n)|}.
\]

This function is continuous and nonvanishing on \( \Gamma \backslash N \), and possesses the same \( \mathbb{Z}(N) \)-covariance as \( F \). Let \( \Gamma_p \) be defined as

\[
\Gamma_p = \exp \frac{Z}{p} X_n \cdot \exp ZX_{n-1} \cdot \exp ZX_{n-2} \cdots \exp ZX_1.
\]  
(33)

Since \( F \) is left \( \Gamma_p \)-invariant, so is \( G \), and both are defined on \( \Gamma_p \backslash N \); note \( \Gamma_p \) is uniform in \( N \), since \( \Gamma \subseteq \Gamma_p \). Let

\[
\mu: N \to \mathbb{Z}(N) \backslash N
\]

be the natural map, and let \( \bar{N}, \bar{\Gamma}_p \) be the images of \( N \) and \( \Gamma_p \) under \( \mu \).

Define

\[
\Omega: \Gamma_p \backslash N \to \bar{\Gamma}_p \backslash \bar{N} \times T
\]

by

\[
\Gamma_p(x_1, \ldots, x_n) \mapsto (\bar{\Gamma}_p(x_1, \ldots, x_{n-1}), G(x_1, \ldots, x_n)).
\]  
(34)

Then \( \Omega \) is continuous on \( \Gamma_p \backslash N \) since \( G \) is; it is 1–1 since \( G \) takes on the value 1 exactly once on every fiber over \( \bar{\Gamma}_p \backslash \bar{N} \). \( \Omega \) is clearly onto, and since \( \bar{\Gamma}_p \backslash \bar{N} \) is compact, \( \Omega \) is a homeomorphism of \( \Gamma_p \backslash N \) and \( \bar{\Gamma}_p \backslash \bar{N} \times T \) (note: \( \bar{\Gamma}_p \backslash \bar{N} \) is compact since \( \mathbb{Z}(N) \) is a rational subgroup).

However, if \( \Gamma_p \) is a \( k \)-step nilpotent group, then \( \bar{\Gamma}_p \times \mathbb{Z} \) is a \( k - 1 \)-step nilpotent group (recall that \( \Gamma_p \) is not actually abelian). Therefore, since these groups are respectively the fundamental groups of \( \Gamma_p \backslash N \) and \( \bar{\Gamma}_p \backslash \bar{N} \times T \), we arrive at a contradiction.

### 3. Homotopy Classes of Functions on Solvmanifolds

In Section 4 we use homotopy classes of functions from solvmanifolds to the circle to show that \( \pi \)-primary summand functions which are continuous must have zeros.
In this section, we demonstrate that functions nonvanishing on the solvmanifolds $M_{R,p}$, $p = 2, 3$, and $M_{H,k}$ for all $k \geq 2$, must be null-homotopic on 2-torus fibers of the bundles $M_{R,p} \to T$ and $M_{H,k} \to T$, where $T$ is the circle group. (Note: this is also true of the bundles $M_{R,p}$, $p = 4$ and 6, but this fact is not used in Section 4.)

We consider first the solvmanifolds $M_{R,p}$, $p = 2, 3$.

**Theorem 4.** For the manifolds $M_{R,p}$, $p = 2, 3$, the functions $f_p : M_{R,p} \to T$ defined by $f_p(\Gamma_p(t, x, y)) = e^{2 \pi it}$ are continuous and generate the groups of homotopy classes of functions from $M_{R,p}$ to $T$.

**Proof.** We first state a few relevant facts [G-H].

Denote by $[M, T]$ the set of homotopy equivalence classes of continuous functions from $M$ to $T$.

1. For all solvmanifolds under consideration, we have that $H^1(M) = [M, T]$ via the map

$$\ast : [M, T] \to H^1(M);$$

$$f \mapsto f_\#(\omega),$$

where $\omega$ is a generating cocycle in $H^1(T)$, and $f_\#(\omega)$ is the class in $H^1(M)$ of the cocycle $\sigma$ which satisfies

$$\sigma(\gamma) = \omega(f \circ \gamma)$$

for all 1-simplices $\gamma$.

2. For all solvmanifolds under consideration, we have $H^1(M) \cong \text{Hom}(H_1(M), \mathbb{Z})$ via the isomorphism

$$\alpha : H^1(M) \to \text{Hom}(H_1(M), \mathbb{Z}); \alpha(\sigma) = \bar{\sigma},$$

where for a cycle $\gamma \in H_1(M)$, $\bar{\sigma}(\gamma) = [\sigma, \gamma]$. This follows from the existence of the exact sequence

$$0 \to \text{Ext}_\mathbb{Z}(H_{n-1}(M), \mathbb{Z}) \to H^n(M) \overset{\alpha_n}{\longrightarrow} \text{Hom}(H_n(M), \mathbb{Z}) \to 0$$

for all $n \in \mathbb{Z}^+$ (Universal Coefficient Theorem). $H_0(M)$ is always a projective $\mathbb{Z}$-module, and so $\text{Ext}_\mathbb{Z}(H_0(M), \mathbb{Z})$ is zero. Therefore $\alpha$ is an isomorphism.

We begin by demonstrating that for $M_{R,p}$, $p = 2, 3$, we have
$H^1(M_{R,p}) \cong \mathbb{Z}$, generated by the cocycle $\lambda_1$ for which $\tilde{\lambda}_1(\gamma_1) = 1$ (here $\gamma_1$ is the 1-simplex $t \in [0, 1] \rightarrow \Gamma_p(t, 0, 0)$). We also define the simplices

$$\gamma_2: t \in [0, 1] \rightarrow \Gamma_p(0, t, 0)$$

and

$$\gamma_3: t \in [0, 1] \rightarrow \Gamma_p(0, 0, t)$$

and note that $\gamma_1$, $\gamma_2$, and $\gamma_3$ generate the group $\pi_1(M_{R,p})$. Furthermore, $\pi_1(M_{R,p})$ is isomorphic to $\Gamma_p$.

**Case 1.** $H^1(M_{R,2}) = \mathbb{Z} \cdot \gamma_1 \oplus \mathbb{Z} \cdot \gamma_2 \oplus \mathbb{Z} \cdot \gamma_3$.

This follows from the fact that $[\pi_1(M_{R,2}), \pi_1(M_{R,2})]$ is generated by the elements $\gamma_2^2$ and $\gamma_3^3$ in $\pi_1(M_{R,2})$.

**Case 2.** $H^1(M_{R,3}) = \mathbb{Z} \cdot \gamma_1 \oplus \mathbb{Z} \cdot \gamma_2$.

Here we use the fact that $[\pi_1(M_{R,3}), \pi_1(M_{R,3})]$ is generated in $\pi_1(M_{R,3})$ by the elements $\gamma_2 \gamma_3$ and $\gamma_3^3$.

We now compute $H^1(M_{R,p})$, $p = 2, 3$, using the fact that $H^1(M_{R,p}) \cong \text{Hom}(H_1(M_{R,p}), \mathbb{Z})$.

**Case 1.** $H^1(M_{R,2}) \cong \text{Hom}(\mathbb{Z} \cdot \gamma_1 \oplus \mathbb{Z} \cdot \gamma_2 \oplus \mathbb{Z} \cdot \gamma_3, \mathbb{Z}) \cong \mathbb{Z} \cdot \tilde{\lambda}_1$, where $\tilde{\lambda}_1$ is the cocycle in $H^1(M_{R,2})$ satisfying $\tilde{\lambda}_1(\gamma_1) = 1$, $\tilde{\lambda}_1(\gamma_2) = \tilde{\lambda}_1(\gamma_3) = 0$.

**Case 2.** $H^1(M_{R,3}) \cong \text{Hom}(\mathbb{Z} \cdot \gamma_1 \oplus \mathbb{Z} \cdot \gamma_2 \oplus \mathbb{Z} \cdot \gamma_3, \mathbb{Z}) \cong \mathbb{Z} \cdot \tilde{\lambda}_1$, where $\tilde{\lambda}_1$ is the cocycle in $M_{R,3}$ satisfying $\tilde{\lambda}_1(\gamma_1) = 1$, $\tilde{\lambda}_1(\gamma_2) = \tilde{\lambda}_1(\gamma_3) = 0$.

Finally, if we suppose that $\omega$ is any cocycle generating $H^1(T)$, then $[M_{R,p}, T]$ is generated by any continuous function $f$ on $M_{R,p}$ satisfying $f_*(\omega) = \tilde{\lambda}_1$. Therefore we must have $f_*(\omega)(\gamma_1) = 1$, $f_*(\omega)(\gamma_2) = f_*(\omega)(\gamma_3) = 0$.

Since $f_*$ in the statement of Theorem 4 satisfies these conditions and is continuous on $M_{R,p}$, $f_*$ generates $[M_{R,p}, T]$ for $p = 2, 3$. This completes the proof of Theorem 4.

**THEOREM 5.** For the manifolds $M_{H,k}$, $k = 3, 4, 5, \ldots$, the functions $f_k: M_{H,k} \rightarrow T$ defined by

$$f_k(\Gamma_k(t, x, y)) = e^{2\pi it}$$

are continuous on $M_{H,k}$ and generate the groups of homotopy classes of functions from $M_{H,k}$ to $T$.

**Proof.** Facts 1 and 2 following the statement of Theorem 4 also apply here. We begin by demonstrating that for $M_{H,k}$, $k \geq 3$, we have $H^1(M_{H,k}) \cong \mathbb{Z}$, generated by the cocycle $\lambda_1$ for which $\tilde{\lambda}_1(\gamma_1) = 1$ (here $\gamma_1$ is the 1-simplex $t \in [0, 1] \rightarrow \Gamma_k(t, 0, 0)$).
Again we define the simplices
\[ \gamma_2 : t \in [0, 1] \mapsto \Gamma_k(0, t, 0) \]
\[ \gamma_3 : t \in [0, 1] \mapsto \Gamma_k(0, 0, t) \]
and note that \( \gamma_1, \gamma_2, \) and \( \gamma_3 \) generate the group \( \pi_1(M_{H,k}) \). We also have \( \pi_1(M_{H,k}) \cong \Gamma_k \).

**Case 1.** \( H_1(M_{H,2}) = \mathbb{Z} \cdot \gamma_1 \).

This follows from the fact that \([\pi_1(M_{H,2}), \pi_1(M_{H,2})]\) is generated in \( \pi_1(M_{H,2}) \) by the elements \( \gamma_2 \) and \( \gamma_2 \gamma_3 \), which together generate all terms of the form \( \gamma_2 \gamma_3 \).

**Case 2.** \( H_1(M_{H,k}) = \mathbb{Z} \cdot \gamma_1 \oplus \mathbb{Z}_{k-1} \cdot \gamma_2 \) for \( k \geq 3 \).

In the case \( k \geq 3 \), we have \([\pi_1(M_{H,k}), \pi_2(M_{H,k})]\) generated by \( \gamma_2^{k-1} \) and \( \gamma_3 \) in \( \pi_1(M_{H,k}) \).

We now compute \( H^1(M_{H,k}) \) for \( k \geq 2 \).

**Case 1.** \( H^1(M_{H,2}) \cong \text{Hom}(H_1(M_{H,2}), \mathbb{Z}) = \text{Hom}(\mathbb{Z} \cdot \gamma_1, \mathbb{Z} \cdot \lambda_1, \mathbb{Z}) \cong \mathbb{Z} \cdot \lambda_1 \), where \( \lambda_1 \) is the cocycle satisfying \( \tilde{\lambda}_1(\gamma_1) = 1, \tilde{\lambda}_1(\gamma_2) = \lambda_1(\gamma_3) = 0 \).

**Case 2.** \( H^1(M_{H,k}) \cong \text{Hom}(H_1(M_{H,k}), \mathbb{Z}) = \text{Hom}(\mathbb{Z} \cdot \gamma_1 \oplus \mathbb{Z}_{k-1} \cdot \gamma_2, \mathbb{Z}) \) for \( k \geq 3 \). However, this is \( \text{Hom}(\mathbb{Z} \cdot \gamma_1, \mathbb{Z}) \cong \mathbb{Z} \cdot \lambda_1 \), where \( \lambda_1 \) is the cocycle satisfying \( \tilde{\lambda}_1(\gamma_1) = 1, \tilde{\lambda}_1(\gamma_2) = \tilde{\lambda}_1(\gamma_3) = 0 \).

If we suppose again that \( \omega \) is a cocycle generating \( H^1(T) \), then \([M_{H,k}, T]\) is generated by any continuous \( f \) on \( M_{H,k} \), \( k \geq 2 \), satisfying \( f_*(\omega) = \lambda_1 \). The rest of the argument proceeds as for Theorem 4.

4. **Zeros of Continuous Functions in \( H_\pi \subseteq L^2(M_{R,p}) \)**

We begin this section by demonstrating that \( M_{R,1} \cong T^3 \) possesses \( \pi \)-primary functions which are nonvanishing, as one would expect.

Suppose \( \lambda_\pi \in \mathcal{P}_\pi \cap \mathcal{O}_\pi \); then the character \( \chi_{\lambda_\pi} \) defined on a maximal subgroup \( M \) of \( S_{R,1} \) gives rise to the Mackey space \( M(\lambda_\pi) \).

Functions in the image of the lift map \( L : M(\lambda_\pi) \rightarrow L^2(M_{R,1}) \) are of the form \( f : M_{R,1} \rightarrow \mathbb{C}, f(\Gamma_1(t, x, y)) = \bar{f}(t) \chi_{\lambda_\pi}(0, x, y) \) for some \( L^2 \) function \( \bar{f} \) with period 1. Clearly \( f \) is continuous if \( \bar{f} \) is.

Thus we see that the \( \pi \)-primary summand contains the character
\[ \psi(\Gamma_1(t, x, y)) = e^{2\pi i \lambda_\pi(x, y)} \]
(40)
which is continuous and nonvanishing on \( M_{R,1} \).

We wish to emphasize here that the manifold \( M_{R,1} \cong T^3 \) is the only compact 3-dimensional solvmanifold arising from a non-nilpotent Lie group
known to possess continuous, nonvanishing $\pi$-primary summand functions. We demonstrate in what follows that for $M_{r,n}$, $p = 2, 4, 6$, continuous functions in the infinite-dimensional $\pi$-primary summands must have zeros. In the case of $M_{r,3}$, continuous functions in certain constructible subspaces which span the $\pi$-primary summands are known to have zeros, but the complete answer for $M_{r,3}$ is not known.

We begin by examining the situation for $M_{r,2}$.

**Theorem 6.** Suppose $f$ is a continuous function in $H_\pi \subset L^2(M_{r,2})$, for $\pi \in (\Gamma_{r,2} \setminus S_{r,2})_\infty$. Then $f$ has at least one zero on $M_{r,2}$.

**Proof.** Recall that $S_{r,2} = \mathbb{R} \ltimes \mathbb{R}^2$, with $\mathbb{R}$ acting on $\mathbb{R}^2$ via the 1-parameter subgroup $\sigma(t) = \text{Rot}(\pi t)$, and that $\Gamma_{r,2}$ is the subgroup of integer points in $S_{r,2}$. The coadjoint orbits are therefore circular cylinders in $s_{r,2}^*$, saturated in the $T^*$-direction. The coadjoint orbit associated with $\pi \in (M_{r,2})_\infty$ is that containing an integral functional $\lambda_\pi = \alpha X^* + \beta Y^*$ for which $\chi_{\lambda_\pi}$ induces $\pi$.

Let $P_\pi$ be orthogonal projection of $L^2(M_{r,2})$ onto the irreducible subspace $H_{\pi,i}$ which is the image of the lift map $L_i$ from $M(\lambda_\pi)$ to $L^2(M_{r,2})$. We note that if $f$ is continuous on $M_{r,2}$, then $P_\pi f$ is continuous on $M_{r,2}$ [Ril], and

$$P_\pi f = \sum_{j=0}^{1} f(t, \cdot , \cdot )^\wedge (\sigma(j)(\alpha, \beta)) \chi_{\sigma(j)(\alpha, \beta)}$$

is equal to $L_1 f'$ for some continuous $f'$ in $M(\lambda_\pi)$ (Lemma 1).

Thus, in order to prove Theorem 6, it suffices to look at sums of functions of the form $\sum_{i=1}^{\mu(\pi)} L_i f_i$, for $f_i$ continuous in $M(\lambda_{\pi})$, where $M(\lambda_{\pi})$ lifts to $H_{\pi,i}$.

Let $S = \{ \lambda : \alpha^2 + \beta^2 = \lambda^2, (\alpha, \beta) \in \mathbb{Z}^2, (\alpha, \beta) \neq (0, 0) \}$. Order the elements of $S$ so that $\lambda_k > \lambda_{k-1}$. The proof of Theorem 6 is by induction on the elements of $S$.

Case 1. $\lambda_1 = 1$, $\mu(\pi_{\lambda_1}) = 2$.

This case falls into the category of odd $\lambda_{\pi}$, which is treated in the induction step.

For the induction step, we suppose that if $f$ is continuous on $M_{r,\gamma}$, $f \in H_{\pi_k}$, for $\lambda_k < \lambda_{\pi}$, then $f$ must have a zero.

Case 1. $\lambda_{\pi}^2$ is odd.

Let $\{ (\alpha_i, \beta_i) \}_{i=1}^{\mu(\pi_{\lambda_{\pi}})}$ be a complete set of $\Gamma_2$-orbit representatives in $O_{\lambda_{\pi}} \cap \mathbb{Z}^*$; for convenience we may take them to be in the set $\{ (\alpha, \beta) : \alpha, \beta \in \mathbb{Z}, \alpha + \beta > 0 \}$. Note that $\alpha + \beta$ is odd whenever $\alpha^2 + \beta^2 = \lambda_{\pi}^2$ is odd.

Let $P = \{ p = \alpha_i + \beta_i \}$. Order $P$ so that $p_k > p_{k-1}$. Note that the $p$ are positive.
Let $Q_k = \{i: \alpha_i + \beta_i = p_k\}$, $k = 1, \ldots, m$. Note that each $Q_k$ has cardinality 2; this follows from the fact that if $\alpha_i + \beta_i = \alpha_j + \beta_j$ and $\alpha_i^2 + \beta_i^2 = \alpha_j^2 + \beta_j^2$ (orbit condition), then if $\alpha_i \neq \alpha_j$, we must have $\alpha_i = \beta_j$. Thus $Q_k$ has cardinality 2, one for each of $(\alpha, \beta)$ and $(\beta, \alpha)$. Let $\{f_i: \mathbb{R} \to \mathbb{C}\}_{\text{mult}(\pi)}$ be a set of continuous functions with period 2.

Then as demonstrated in Section 1, a typical continuous $\phi$ in $H_{\pi}$ has the form (setting $z_1 = e^{2\pi i x}$, $z_2 = e^{2\pi i y}$)

$$
\phi(t, z_1, z_2) = \left\{ \sum_{Q_k} \left[ \sum_{i \in Q_k} f_i(t) z_1^{\alpha_i} z_2^{\beta_i} + f_i(t + 1) z_1^{-\alpha_i} z_2^{-\beta_i} \right] \right\}. \quad (42)
$$

Fix $t$ and define $\phi(z_1, z_2) = \phi(t, z_1, z_2)$, so that $\phi: \mathbb{N} \cap \Gamma_2 \setminus \mathbb{N} \cong T^2 \to \mathbb{C}$.

Suppose $\phi$ is nonvanishing on $M_{R, 2}$. Then $\phi: T^2 \to \mathbb{C}$ is nullhomotopic. Consider $\Phi_t$, the restriction of $\phi$ to the closed curve $z_1 = z_2$ in $T^2$. Then $\Phi_t$, defined by

$$
\Phi_t(z) = \phi_t(z, z) = \sum_{Q_k} \left[ \sum_{i \in Q_k} f_i(t) \right] z^{pk} + \left[ \sum_{i \in Q_k} f_i(t + 1) \right] z^{-pk} \quad (43)
$$

is a curve in $\mathbb{C} \setminus \{0\}$ which has winding number zero.

Clearly $\Phi_t$ may be viewed as the restriction to $T$ of a meromorphic function $\Phi_t: \mathbb{C} \to \mathbb{C}$,

$$
\Phi_t(\omega) = \sum_{Q_k} \left[ \sum_{i \in Q_k} f_i(t) \right] \omega^{pk} + \left[ \sum_{i \in Q_k} f_i(t + 1) \right] \omega^{-pk}. \quad (44)
$$

We claim that $\Phi_t$ has a pole of odd order at $\omega \equiv 0$. If not, then the coefficients in $\Phi_t$ of negative exponents are zero; but since $\Phi_t$ contains only terms with odd exponents, $\Phi_t$ has no constant term, and thus we would have a polynomial $\Phi_t$ with $\Phi_t(0) = 0$. Thus $\Phi_t$ would wind at least once on the circle $T$, so $\Phi_t$ would wind, a contradiction. Let $\gamma \geq 1$ be the order of the pole at zero. Then we may write

$$
\Phi_t(z) = z^{-\gamma} \left\{ \sum_{Q_k} \sum_{i \in Q_k} f_i(t) \right\} z^{pk + \gamma} + \left[ \sum_{i \in Q_k} f_i(t + 1) \right] z^{-pk - \gamma} \quad (45)
$$

where $p$ is a polynomial in $z$ with even exponents.

But then the zeros of $p(z)$ (and so the zeros of $\Phi_t$) occur in pairs of equal modulus, so that the number of zeros inside the unit circle is even. Define $\Gamma: [0, 1] \to T$ by $\Gamma(t) = e^{2\pi it}$. Then we have

$$
\text{Ind}_\Gamma \Phi_t = N_{\Phi_t} - P_{\Phi_t} \neq 0, \quad (46)
$$
where $N_{\Phi_i}$ and $P_{\Phi_i}$ are, respectively, the number of zeros and poles of $\Phi_i$ inside $\Gamma$ [Ru, Chap. 10]. Thus the winding number of $\Phi_i$ cannot be zero, and we arrive at a contradiction.

**Case 2.** $\lambda_n^2 \equiv 0 \mod 4$.

Note that if $\alpha^2 + \beta^2 = \lambda_n^2 \equiv 0 \mod 4$, then both $\alpha$ and $\beta$ must be even.

Let $\left\{ (\alpha_i, \beta_i) \right\}_{i=1}^{\text{mul}(m)}$ be a set of $\Gamma_{R,2}$-orbit representatives, and let $\left\{ f_i : \mathbb{R} \to \mathbb{C} \right\}_{i=1}^{\text{mul}(m)}$ be continuous with period 2. Then a typical $\phi \in H_{\pi_{n,2}}$ may be written

$$\phi(t, z_1, z_2) = \sum_{i=1}^{\text{mul}(m)} f_i(t) z_1^{\alpha_i} z_2^{\beta_i} + f_i(t+1) z_1^{-\alpha_i} z_2^{-\beta_i},$$

for $z_1 = e^{2\pi i x}$, $z_2 = e^{2\pi i y}$.

We define

$$\psi(t, z_1, z_2) = \sum_{i=1}^{\text{mul}(m)} f_i(t) z_1^{\alpha_i/2} z_2^{\beta_i/2} + f_i(t+1) z_1^{-\alpha_i/2} z_2^{-\beta_i/2}.$$

Note that since the $\alpha_i$ and $\beta_i$ are all divisible by 2, $\psi$ has integer exponents; therefore, $\psi$ is continuous, $\Gamma_{R,2}$-invariant, and lives in $H_{n,2}$ where $\lambda_n^2 = \lambda_n^2/4$. By the induction hypothesis, $\psi$ has a zero. Since $\phi(t, z_1, z_2) = \psi(t, z_1^2, z_2^2)$, $\phi$ must also have a zero.

**Case 3.** $\lambda_n^2 \equiv 2 \mod 4$.

Let $\left\{ (\alpha_i, \beta_i) \right\}_{i=1}^{\text{mul}(m)}$ be a complete set of $\Gamma_{R,2}$-orbit representatives from $\mathcal{O}_{\pi_{n,2}} \cap \mathcal{O}^*$, satisfying $\alpha_i > 0$ for each $i$. Note that $\alpha_i^2 + \beta_i^2 = \lambda_n^2 - 2 \mod 4$ implies that $\alpha_i$ and $\beta_i$ are both odd for each $i$.

Let $P = \left\{ p = \alpha_i \text{ for some } i \right\}$. Order $P$ so that $p_k > p_{k-1}$. Let $Q_k = \left\{ i : \alpha_i = p_k \right\}$, and note that each $Q_k$ has cardinality 2. Let $\left\{ f_i : \mathbb{R} \to \mathbb{C} \right\}_{i=1}^{\text{mul}(m)}$ be a set of continuous functions of period 2. Then a typical continuous $\phi$ in $H_{\pi_{n,2}}$ has the form of Eq. (42).

Fix $t$ and define $\phi_i(z_1, z_2) = \phi(t, z_1, z_2)$, $\phi_j : N \cap \Gamma_2 \backslash N \cong T^2 \to \mathbb{C}$. Suppose that $\phi$ is nonvanishing, so that $\phi_i$ must be nullhomotopic on $N \cap \Gamma_2 \backslash N$.

Define $\widetilde{\phi}_i(z) = \phi_i(z, 1)$, the restriction of $\phi_i$ to the curve $z_2 \equiv 1$ in $N \cap \Gamma_2 \backslash N$. Then we have

$$\widetilde{\phi}_i(z) = \sum_{Q_k} \left\{ \sum_{i \in Q_k} f_i(t) \right\} z^{p_k} + \left[ \sum_{i \in Q_k} f_i(t+1) \right] z^{-p_k}.$$ 

Since each $\alpha_i$ is odd, each $p_k$ is odd; by choice of $\alpha_i$, we have $p_k > 0$ for all $k$.

From here we proceed, as in Case 1, to demonstrate that $\widetilde{\phi}_i$ must wind on the circle $T$, a contradiction. This completes Case 3 and finishes the proof of Theorem 6.
In Theorem 7, we show that continuous functions in the \( \pi \)-primary summands of \( L^2(M_{R,4}) \) and \( L^2(M_{R,6}) \) must have zeros.

**THEOREM 7.** Let \( f \) be a continuous function in \( H_\pi \subseteq L^2(M_{R,i}) \), for \( i = 4 \) or 6, \( \pi \in (\Gamma_{R,i} \setminus S_{R,i})_\infty \). Then \( f \) has at least one zero on \( M_{R,i} \).

**Proof.** Define the groups

\[
\Gamma'_4 = \{(m, n, p) \in \Gamma_{R,4} \subseteq S_{R,4} : M = 2k, \text{ for some } k \in \mathbb{Z}\}
\]

\[
\Gamma'_6 = \{(m, n, p) \in \Gamma_{R,6} \subseteq S_{R,6} : M = 3k, \text{ for some } k \in \mathbb{Z}\}.
\]

Then \( \Gamma'_4 \) and \( \Gamma'_6 \) are subgroups of \( \Gamma_{R,4} \) and \( \Gamma_{R,6} \), respectively, of finite index; thus \( \Gamma'_i \) is cocompact in \( S_{R,i} \) for each \( i \), and it is straightforward to verify that \( \Gamma'_i \cong \Gamma_{R,2}, \ i = 4, 6 \).

We prove Theorem 7 for \( M_{R,4} \); the proof for \( M_{R,6} \) is analogous in every respect.

Since \( \Gamma'_4 \cong \Gamma_{R,2} \) and is cocompact in \( S_{R,4} \), we have \( \Pi_1(\Gamma'_4 \setminus S_{R,4}) \cong \Gamma_{R,2} \). Therefore we have \( \Gamma'_4 \setminus S_{R,4} \cong M_{R,2} \) [Mos, Theorem A].

Functions which are \( \Gamma_{R,4} \)-periodic are \( \Gamma'_4 \)-periodic, so \( L^2(M_{R,4}) \) embeds isometrically in \( L^2(M_{R,2}) \). Furthermore this embedding is \( S_{R,4} \)-equivariant with respect to the quasi-regular representation, and so takes \( \pi \)-spaces to \( \pi \)-spaces.

Let \( \Phi \) be the isometric embedding of \( L^2(M_{R,4}) \) in \( L^2(M_{R,2}) \). Then if \( f \) is a continuous function in \( H_\pi \subseteq L^2(M_{R,4}) \), \( \Phi f \) is continuous in \( H_\pi \subseteq L^2(M_{R,2}) \) and so must have a zero. However, if \( \Phi f(\Gamma_{R,2}(t, x, y)) = 0 \), then since \( f \) is \( \Gamma'_4 \)-invariant, \( f(\Gamma_{R,4}(t, x, y)) = 0 \); thus \( f \) must have a zero. This completes the proof of Theorem 7.

We finish this section with a theorem summarizing what is known for \( M_{R,3} \).

**THEOREM 8.** Let \( f \) be a continuous element of a constructible, irreducible subspace of a \( \pi \)-primary summand \( H_\pi \subseteq L^2(M_{R,3}) \). Then \( f \) has at least one zero on \( M_{R,3} \).

**Proof.** Suppose \( \lambda_\pi \in L^* \cap \mathcal{O}_\pi \), an integral functional in \( s_{R,3}^* \); then the character \( \chi_{\lambda_\pi} \) defined on a maximal subgroup \( M \) of \( S_{R,3} \) gives rise to the Mackey space, \( M(\lambda_\pi) \). The constructible irreducible subspace corresponding to \( \lambda_\pi = (\alpha, \beta) \) is the image of the lift map \( L: M(\lambda_\pi) \to L^2(M_{R,3}) \), an \( S_{R,3} \)-invariant isometry.
A typical continuous element of this constructible irreducible subspace of $H_x$ has the form

$$\mathcal{J}(\Gamma_3(t, x, y)) = f(t, z_1, z_2)$$

$$= f(t) z_1^\alpha z_2^\beta + f(t + 1) z_1^\alpha z_2^\beta(z^\alpha + \beta + f(t + 2) z_1^\alpha z_2^\beta$$ (49)

for $z_1 = e^{2\pi i x}$, $z_2 = e^{2\pi i y}$, and $f: \mathbb{R} \to \mathbb{C}$ a continuous function of period 3.

Suppose $\mathcal{J}$ is nonvanishing on $M_{R,3}$. Then $\mathcal{J}$ must be nullhomotopic when restricted to $T^2$-fibers of the bundle $M_{R,3} \to T$.

We examine the functions $\mathcal{J}$ on a case-by-case basis.

Case 1. $\alpha \equiv \beta \equiv 1 \text{ mod 3}$, or $\alpha \equiv \beta \equiv 2 \text{ mod 3}$.

We define $\phi_r(z) = f(t, z, 1)$ for fixed $t \in \mathbb{R}$; $\phi_r$ must have winding number zero on $T$, since $\mathcal{J}(t, z_1, z_2)$ is nullhomotopic on $T^2$ for fixed $t$. We have

$$\phi_r(z) = f(t) z^\alpha + f(t + 1) z^\beta + f(t + 2) z^{-(\alpha + \beta)}.$$ (50)

Clearly one of $\alpha$, $\beta$, and $-(\alpha + \beta)$ must be negative. If we view $\phi_r$ as the restriction to the set $T = \{ |z| = 1, z \in \mathbb{C} \}$ of the meromorphic function

$$\Phi_r(\omega) = f(t) \omega^\alpha + f(t + 1) \omega^\beta + f(t + 2) \omega^{-(\alpha + \beta)}$$ (51)

we see that $\Phi_r$ has a pole at $\omega = 0$.

Let $\gamma$ be the order of the pole at zero. Then we may write

$$\Phi_r(\omega) = \omega^{-\gamma} \{ f(t) \omega^{\alpha + \gamma} \cdot f(t + 1)^{\beta + \gamma} + f(t + 2)^{-(\alpha + \beta) + \gamma} \}$$

$$= \omega^{-\gamma} p(\omega),$$ (52)

where $p(\omega)$ is a polynomial. Note that the exponents of $p(\omega)$ must all be divisible by 3. Since $\alpha + \beta \equiv 1$ or $2 \text{ mod 3}$, we must have $\gamma \equiv 1$ or $2 \text{ mod 3}$, so that $\alpha + \gamma \equiv \beta + \gamma \equiv - (\alpha + \beta) + \gamma \equiv 0 \text{ mod 3}$. Thus the zeros of $p(\omega)$ are grouped as triples of equal modulus; in particular, the number of zeros of $p(\omega)$ (and hence of $\Phi_r(\omega)$) inside $T$ is a multiple of 3. However, the pole of $\Phi_r$ at $\omega = 0$ is not a multiple of 3, and therefore, referring to (46), we see that $\Phi_r$ must wind on the curve $T$, and therefore that $\Phi_r$ cannot be nullhomotopic, a contradiction.

Case 2. $\alpha \equiv 1 \text{ mod 3}$, $\beta \equiv 2 \text{ mod 3}$.

We define $\phi_r(z) = f(t, z, z^{-1}) = f(t) z^\alpha - \beta + f(t + 1) z^{2\beta + \alpha} + f(t + 2) z^{-(2\alpha + \beta)}$ as in Case 1 and note that $\phi_r$ is $\mathcal{J}$ restricted to the curve $z_2 = z_1^{-1}$ in the $T^2$-fiber over $\Gamma_3(t, 0, 0)$.

Clearly one of $\alpha - \beta$, $2\beta + \alpha$, and $-(2\alpha + \beta)$ must be negative, since $(\alpha - \beta) + (2\beta + \alpha) = 2\alpha + \beta$. All are congruent to 2 mod 3, so none can be zero. If we view $\phi_r$ as the restriction to $T$ of the meromorphic function

$$\Phi_r(\omega) = f(t) \omega^{\alpha - \beta} + f(t + 1) \omega^{2\beta + \alpha} + f(t + 2) \omega^{-(2\alpha + \beta)},$$

we see that $\Phi_r$ has a
pole at zero. Let \( \gamma \) be the order of the pole at zero; then \( -\gamma \equiv 2 \mod 3 \), and we may write

\[
\Phi_i(\omega) = \omega^{-\gamma} \left\{ \frac{f(t)}{\omega^{\alpha - \beta + \gamma}} + \frac{f(t + 1)}{\omega^{2\beta + \alpha + \gamma}} + \frac{f(t + 2)}{\omega^{-(2\alpha + \beta) + \gamma}} \right\}
\]

\[= \omega^{-\gamma} p(\omega), \tag{53}\]

where \( p(\omega) \) is a polynomial. Note that since \( -\gamma \equiv 2 \mod 3 \), and since all of \( \alpha - \beta, 2\beta + \alpha, \) and \( -(2\alpha + \beta) \) are congruent to 2 as well, the function \( p(\omega) \) contains only terms with exponents divisible by 3, and so the number of zeros inside \( T \) is a multiple of 3. Since \( \gamma \) is not a multiple of 3, we have \( \text{Ind}_r \Phi_i \neq 0 \), and again we arrive at a contradiction.

**Case 3.** \( \alpha \equiv 2 \mod 3, \beta \equiv 1 \mod 3 \).

Proceeding as before, we define \( \Phi_i(z) = f(t, z, z^{-1}) = f(t)z^{\alpha - \beta} + f(t + 1)z^{2\beta + \alpha} + f(t + 2)(z^{-(2\alpha + \beta)}), \) and note that \( \Phi_i \) is \( f \) restricted to the curve \( z^2 = z^{-1} \) in the \( T^2 \)-fiber over \( \Gamma_\alpha(t, 0, 0) \).

Again, one of \( \alpha - \beta, 2\beta + \alpha, \) and \( -(2\alpha + \beta) \) must be negative, and all are congruent to 1 mod 3, so that none is zero. If we view \( \Phi_i \) as the restriction to \( T \) of the meromorphic function

\[
\Phi_i(\omega) = f(t)\omega^{\alpha - \beta} + f(t + 1)\omega^{2\beta + \alpha} + f(t + 2)\omega^{-(2\alpha + \beta)}
\]

we see that \( \Phi_i \) has a pole at zero. Let \( \gamma \) be the order of the pole at zero; then \( -\gamma \equiv 1 \mod 3 \), and we may write

\[
\Phi_i(\omega) = \omega^{-\gamma} \left\{ \frac{f(t)}{\omega^{\alpha - \beta + \gamma}} + \frac{f(t + 1)}{\omega^{2\beta + \alpha + \gamma}} + \frac{f(t + 2)}{\omega^{-(2\alpha + \beta) + \gamma}} \right\}
\]

\[= \omega^{-\gamma} p(\omega), \]

where \( p(\omega) \) is a polynomial. Note that since \( -\gamma \equiv 1 \mod 3 \), and since all of \( \alpha - \beta, 2\beta + \alpha, \) and \( -(2\alpha + \beta) \) are as well, the function \( p(\omega) \) contains only terms with exponents divisible by 3. The number of zeros inside \( T \) is therefore a multiple of 3. Since \( \gamma \) is not a multiple of 3, we have \( \text{Ind}_r \Phi_i \neq 0 \), and we arrive at a contradiction.

**Case 4.** \( \alpha \equiv 0 \mod 3, \beta \equiv 1 \mod 3 \) or \( \alpha \equiv 2 \mod 3, \beta \equiv 0 \mod 3 \).

If \( (\alpha, \beta) \) is such that \( \alpha \equiv 0 \mod 3 \) and \( \beta \equiv 1 \mod 3 \), then the \( \Gamma_\alpha \)-orbit containing \( (\alpha, \beta) \) also contains the point \( (\beta, -(\alpha + \beta)) \), which is of the type dealt with in Case 2. Since the function \( f \) is independent of the base point chosen from the \( \Gamma_\alpha \)-orbit, \( f \) must have a zero. Similarly, if \( (\alpha, \beta) \) is of the type \( \alpha \equiv 2 \mod 3, \beta \equiv 0 \mod 3 \), then the point \( (-(\alpha + \beta), \alpha) \) is of the type dealt with in Case 2.

**Case 5.** \( \alpha \equiv 0 \mod 3, \beta \equiv 2 \mod 3 \) or \( \alpha \equiv 1 \mod 3, \beta \equiv 0 \mod 3 \).

If \( (\alpha, \beta) \) is such that \( \alpha \equiv 0 \mod 3, \beta \equiv 2 \mod 3 \), then \( (\beta, -(\alpha, \beta)) \) is in the same \( \Gamma_\alpha \)-orbit as \( (\alpha, \beta) \) and is of the type dealt with in Case 3. If \( (\alpha, \beta) \) is
of the type $\alpha \equiv 1 \mod 3$, $\beta \equiv 0 \mod 3$, then the point $(-\langle \alpha, \beta \rangle, \alpha)$ is of the type dealt with in Case 3.

Case 6. $\alpha \equiv \beta \equiv 0 \mod 3$.

We have $\alpha = 3^k \alpha'$ and $\beta = 3^k \beta'$ for some $k \neq 0$ and some pair $(\alpha', \beta')$, not both congruent to $0 \mod 3$. We may therefore define

$$\tilde{g}(t, z_1, z_2) = f(t) z_1^{\alpha'} z_2^{\beta'} + f(t + 1) z_1^{\beta'} z_2^{-(\alpha' + \beta')} + f(t + 2) z_1^{-(\alpha' + \beta')} z_2^{\alpha'},$$

which is clearly the lift of a function in the Mackey space $M(\lambda'_n)$ for $\lambda'_n = (\alpha', \beta')$; since not both $\alpha'$ and $\beta'$ are congruent to $0 \mod 3$, $\tilde{g}$ must have a zero, since one of Cases 1–5 applies. Since $\tilde{g}$ has a zero, and since we have $\tilde{g}(t, z_1^3, z_2^3) = \tilde{f}(t, z_1, z_2)$, $\tilde{f}$ must have a zero on $M_{R,3}$.

This completes the proof of Theorem 8.

5. ZEROS OF CONTINUOUS FUNCTIONS IN $H_\pi \subseteq L^2(M_{H,k})$

In this section, we demonstrate that functions in a uniformly dense subspace $K_\pi$ of continuous functions in $H^0_\pi = C(M_{H,k}) \cap H_\pi$ must have zeros on $M_{H,k}$, for all hyperbolic solvmanifolds $M_{H,k}$. It then follows easily that all continuous functions on $H_\pi$ must have zeros.

If $\{\lambda_{n_i}\}_{i=1}^{\text{mul}(\pi)}$ is a complete set of $\Gamma_{H,k}$-orbit representatives from $\mathcal{C}_n \cap \mathcal{L}^*$, then the set $\{L_i: M(\lambda_{n_i}) \to L^2(M_{H,k})\}_{i=1}^{\text{mul}(\pi)}$ is a complete set of lift maps into the constructible irreducible subspaces of $H_\pi$. Let $T_i: L^2(\mathbb{R}) \to M(\lambda_{n_i})$ be the isometry intertwining the Schrödinger model of $\pi$ on $L^2(\mathbb{R})$ and the induced model on $M(\lambda_{n_i})$; i.e., $T_i f = \tilde{f}$, where $\tilde{f}(t, x, y) = \chi_{\lambda_{n_i}}(x, y) f(t)$.

Then $L_i = L_i \circ T_i$ lifts $L^2(\mathbb{R})$ into the $i$th constructible irreducible subspace of $H_\pi$. We define

$$K_\pi = L'_1(C_0^\infty(\mathbb{R})) \oplus \cdots \oplus L'_{\text{mul}(\pi)}(C_0^\infty(\mathbb{R})) \subseteq H^\infty_\pi.$$

**Lemma 9.** $K_\pi$ is uniformly dense in $H^0_\pi$.

**Proof.** We first demonstrate that $K_\pi$ is uniformly dense in $H^\infty_\pi = C^\infty(M_{H,k}) \cap H_\pi$.

If $S_*(\mathbb{R})$ are the smooth vectors for the Schrödinger model of $\pi$ on $L^2(\mathbb{R})$, then we have

$$\overline{L'_i(C_0^\infty(\mathbb{R}))} \supseteq L'_i(S_*(\mathbb{R}))$$

in the sup-norm on $M_{H,k}$ [Bre1, Lemma 5].
We have also that \( \bigoplus_i H_{\pi_i}^\infty = H_{\pi}^\infty \), since \( L_i(S_{\pi}(R)) = H_{\pi_i}^\infty \) by preservation of smooth vectors under intertwining maps.

We have also that \( \bigoplus_i H_{\pi_i}^\infty = H_{\pi}^\infty \), since orthogonal projection onto \( S_{H,k} \)-invariant subspaces preserves infinite differentiability [Aus-Bre, Sect. 2]. Therefore, if \( \phi \in H_{\pi}^\infty \), we may uniformly approximate \( P_{\pi_i}(\phi) \) in each subspace by elements of \( L_i(C_0^\infty(R)) \). Thus, \( K_{\pi} \) is uniformly dense in \( H_{\pi}^\infty \).

We finish the proof of Lemma 9 by demonstrating that \( H_{\pi}^\infty \) is uniformly dense in \( H_{\pi}^0 \).

Let \( F \) be a fundamental domain for \( \Gamma_k \setminus S_{H,k} \) containing the identity; define a \( C^\infty \) approximate identity \( \{ \epsilon_n \}_{n=1}^\infty \) so that

1. \( 0 \leq \epsilon_n < \infty \) for each \( n \in \mathbb{Z}^+ \).
2. Support \( \epsilon_n \) is contained in the interior of \( F \) for each \( n \in \mathbb{Z}^+ \).

We define, for \( \phi \in H_{\pi}^0 \),

\[
\phi \ast \epsilon_n(t, x, y) = \int_{F \subseteq S_{H,k}} \phi( (t, x, y)(t', x', y')^{-1}) \epsilon_n(t', x', y') \, dt' \, dx' \, dy'.
\]

Then \( \phi \ast \epsilon_n \) is \( C^\infty \) for each \( n \in \mathbb{N} \), since \( \epsilon_n \) is \( C^\infty \) and \( S_{H,k} \) is unimodular, and in fact \( \phi \ast \epsilon_n \in H_{\pi}^\infty \) since \( \phi \ast \epsilon_n \) is the uniform limit of linear combinations of right translates of \( \phi \).

We now claim that \( \phi \ast \epsilon_n \) converges uniformly to \( \phi \) on \( M_{H,k} \). We have

\[
\| \phi \ast \epsilon_n - \phi \|_{\infty} \leq \sup_{(t, x, y) \in F} \int_F |\phi( (t, x, y)(t', x', y')^{-1}) - \phi(t, x, y)| \times \epsilon_n(t', x', y') \, dt' \, dx' \, dy'.
\]

Choose \( N \) so that for \( n \geq N \), we have \( |\phi( (t, x, y)(t', x', y')^{-1}) - \phi(t, x, y)| \leq \epsilon \) for all \( (t, x, y) \in F \) and all \( (t', x', y') \in \text{support}(\epsilon_n) \). Then \( \| \phi \ast \epsilon_n - \phi \|_{\infty} \leq \epsilon \), which completes the proof of uniform convergence.

Thus we have \( H_{\pi}^\infty \subseteq H_{\pi}^0 \) uniformly dense in \( H_{\pi}^0 \), completing the proof of Lemma 9.

Recall that \( S_{H,k} = R \ltimes R^2 \), with \( R \) acting on \( R^2 \) via the 1-parameter subgroup \( \sigma_k : R \to SL_2(R) \) satisfying \( \sigma_k(1) = [k^{-1} \, 1 \, 1 \, k] \). The one-parameter subgroup \( \sigma_k \) is conjugate to \( \sigma(\lambda) = [\lambda^{-1} \, 0 \, 0 \, \lambda] \), where \( \lambda + \lambda^{-1} = k + 1 \), \( \lambda \in R \). The coadjoint orbits in \( s_{H,k}^* \) are therefore hyperbolic cylinders, saturated in the \( T^* \)-direction, and satisfying the equation

\[
(k - 1)x^2 + (k - 1)xy - y^2 = \omega \tag{57}
\]

for some \( \omega \in R \). Note that two coadjoint orbits satisfy (57) for each value
of $\omega$, each being a connected component of the set in $s^*_H$ satisfying (57). The coadjoint orbit associated with $\pi \in (\Gamma_{H,k} \setminus S_{H,k})_\infty$ is that containing an integral functional $\lambda_\pi = aX^* + \beta Y^*$ for which $\chi_{\lambda_{\pi}}$ induces $\pi$.

**Theorem 10.** Suppose $f$ is a continuous function in $H_\pi \subseteq L^2(M_{H,k})$, for $\pi \in (\Gamma_{H,k} \setminus S_{H,k})_\infty$. Then $f$ has at least one zero on $M_{H,k}$.

**Proof:** We begin by proving Theorem 10 for functions $f \in K_\pi \subseteq H_\pi$. We have $K_\pi = L_1^1(C_0^\infty(R)) \oplus \cdots \oplus L_{\text{mul}(\pi)}^1(C_0^\infty(R))$, so that a typical element of $K_\pi$ has the form

$$\phi(\Gamma_k(t, x, y)) = \sum_{i=1}^{\text{mul}\pi} \sum_{n \in \mathbb{Z}} f_i(t + n) \exp 2\pi i (\alpha_{n,i} x + \beta_{n,i} y),$$

where $\{(x_{0,i}, \beta_{0,i})\}_{i=1}^{\text{mul}(\pi)}$ is a set of distinct $\Gamma_k$-orbit representatives in $C_\pi \cap C^*, (\alpha_{n,i}, \beta_{n,i}) = \phi_k(n)(x_{0,i}, \beta_{0,i})$, and where for each $i = 1, \ldots, \text{mul}(\pi)$, and for fixed $t$, the sum over $n$ in (58) is finite. Suppose $\phi$ is nonvanishing on $M_{H,k}$. By setting $z_1 = e^{2\pi i x}, z_2 = e^{2\pi i y}$, we may define

$$\bar{\phi}(t, z_1, z_2) = \phi(\Gamma_k(t, x, y)) = \sum_{i=1}^{\text{mul}\pi} \sum_{n \in \mathbb{Z}} f_i(t + n) z_1^{\alpha_{n,i}} z_2^{\beta_{n,i}}.$$  

For any fixed $t$, we must have $\bar{\phi}$ nullhomotopic on $N \cap \Gamma_k \cong T^2 \setminus \mathbb{N}$ by Theorem 5.

We note at this point that if $(\alpha, \beta)$ satisfies $(k - 1)\alpha^2 + (k - 1)\alpha \beta - \beta^2 = \omega$ for $\omega > 0$, then either all points in $O_{(\alpha, \beta)}$ satisfy $\alpha > 0$, or they all satisfy $\alpha < 0$. If not, then since $O_{(\alpha, \beta)}$ is connected, $O_{(\alpha, \beta)}$ must intersect the $y$-axis, so that $\alpha = 0$ and $-\beta^2 = \omega$, a contradiction. Similarly, if $O_{(\alpha, \beta)}$ satisfies $(k - 1)\alpha^2 + (k - 1)\alpha \beta - \beta^2 = \omega$ for $\omega < 0$, then either all points in $O_{(\alpha, \beta)}$ satisfy $\beta > 0$, or they all satisfy $\beta < 0$.

Suppose $O_{(\alpha, \beta)}$ satisfies $(k - 1)\alpha^2 + (k - 1)\alpha \beta - \beta^2 = \omega$ for $\omega > 0$.

**Case 1.** Suppose all $(\alpha, \beta) \in O_{(\alpha, \beta)}$ satisfy $\alpha > 0$. Then we have

$$\phi(t, z_1, z_2) = \sum_{i=1}^{\text{mul}\pi} \sum_{n \in \mathbb{Z}} f_i(t + n) z_1^{\alpha_{n,i}} z_2^{\beta_{n,i}},$$

where $\alpha_{n,i} > 0$ for all $n, i$. Fixing $t$, we have

$$\phi_\pi(z) = \phi(t, z, 1) = \sum_{i=1}^{\text{mul}\pi} \sum_{n \in \mathbb{Z}} f_i(t + n) z^{\omega_{n,i}},$$

a curve of winding number zero on the circle $T$. The sum in (60) is always finite.

We may consider $\Phi_\pi$ to be the restriction of a polynomial $\Phi_\pi$ on $C$ to $T$, i.e., $\Phi_\pi(n) = \sum_{i=1}^{\text{mul}\pi} \sum_{n \in \mathbb{Z}} f_i(t + n) \omega_{n,i}$. Since $\alpha_{n,i}$ is never zero, $\Phi_\pi$ has no constant term and so has a zero at $\omega = 0$. Since $\Phi_\pi$ has no pole inside $T$, the coadjoint orbit associated with $R \in (\Gamma_{H,k} \setminus S_{H,k})_\infty$ is that containing an integral functional $\lambda = aX^* + \beta Y^*$ for which $\chi_{\lambda}$ induces $\pi$.
we see, referring to (2.20) that \( \Phi_t \), and hence \( \phi_t \), cannot have winding number zero on \( T \). Therefore \( \phi_t \) cannot be nullhomotopic on \( N \cap \Gamma \setminus \{ \} \) for fixed \( t \), and so \( \phi \) cannot be nonvanishing.

**Case 2.** Suppose all \( (\alpha, \beta) \in \mathcal{C}_{(\alpha, \beta)} \) satisfy \( \alpha < 0 \). Then set \( \phi_t(z) = \phi(t, z^{-1}, 1) \) for fixed \( t \) and proceed as in Case 1.

**Case 3.** Let \( \mathcal{C}_{(\alpha, \beta)} \) satisfy \( (k-1)\alpha^2 + (k-1)\alpha \beta - \beta^2 = \omega \) for \( \omega < 0 \). Suppose all \( (\alpha, \beta) \in \mathcal{C}_{(\alpha, \beta)} \) satisfy \( \beta > 0 \). Then set \( \phi_t(z) = \phi(t, 1, z) \) for fixed \( t \) and proceed as in Case 1.

**Case 4.** Suppose all \( (\alpha, \beta) \in \mathcal{C}_{(\alpha, \beta)} \) satisfy \( \beta > 0 \). Then set \( \phi_t(z) = \phi(t, 1, z^{-1}) \) and proceed as in Case 1.

Thus we have proof that all \( f \in K_n \) have zeros, and so we have shown that all functions in a uniformly dense subspace of \( H^0_n \) have zeros. Therefore all functions in \( H^0_n \) must have zeros.

**References**


