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Dualizing complex of the face ring of a simplicial poset

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ABSTRACT

A finite poset P is called *simplicial* if it has the smallest element $\hat{0}$, and every interval $[\hat{0}, x]$ is a Boolean algebra. The face poset of a simplicial complex is a typical example. Generalizing the Stanley–Reisner ring of a simplicial complex, Stanley assigned the graded ring A_P to P. This ring has been studied from both combinatorial and topological perspectives. In this paper, we will give a concise description of a dualizing complex of A_P, which has many applications.

1. Introduction

All posets (partially ordered sets) in this paper will be assumed to be finite. By the order given by inclusions, the power set of a finite set becomes a poset called a *Boolean algebra*. We say that a poset P is *simplicial* if it admits the smallest element $\hat{0}$, and the interval $[\hat{0}, x] := \{ y \in P \mid y \leq x \}$ is isomorphic to a Boolean algebra for all $x \in P$. For simplicity, we denote rank(x) of $x \in P$ just by $\rho(x)$. If P is simplicial and $\rho(x) = m$, then $[\hat{0}, x]$ is isomorphic to the Boolean algebra $2^{\{1,\dots,m\}}$.

Let Δ be a finite simplicial complex (with $\emptyset \in \Delta$). Its face poset (i.e., the set of the faces of Δ with the order given by the inclusion) is a simplicial poset. Any simplicial poset P is the face (cell) poset of a regular cell complex, which we denote by $\Gamma(P)$. For $\hat{0} \neq x \in P$, $c(x) \in \Gamma(P)$ denotes that the open cell corresponds to x. Clearly, $\dim c(x) = \rho(x) - 1$. While the closure $\overline{c(x)}$ of $c(x)$ is always a simplex, the intersection $\overline{c(x)} \cap \overline{c(y)}$ for $x, y \in P$ is not necessarily a simplex. For example, if two d-simplices are glued along their boundaries, then it is not a simplicial complex, but it gives a simplicial poset.

Let P be a simplicial poset. For $x, y \in P$, set

$$[x \vee y] := \text{the set of minimal elements of } \{ z \in P \mid z \geq x, y \}.$$

More generally, for $x_1, \dots, x_m \in P$, $[x_1 \vee \cdots \vee x_m]$ denotes the set of minimal elements of the common upper bounds of $x_1, \dots, x_m$.

Set $\{ y \in P \mid \rho(y) = 1 \} = \{y_1, \dots, y_n\}$. For $U \subset [n] := \{1, \dots, n\}$, we simply denote $[\bigvee_{i \in U} y_i]$ by $[U]$. Here, $[\emptyset] = \{\hat{0}\}$. If $x \in [U]$, then $\rho(x) = \#U$. For each $x \in P$, there exists a unique U such that $x \in [U]$. Let $x, x' \in P$ with $x \geq x'$ and $\rho(x) = \rho(x') + 1$, and take $U, U' \subset [n]$ such that $x \in [U]$ and $x' \in [U']$. Since $U = U' \coprod \{i\}$ for some i in this case, we can set

$$\alpha(i, U) := \#\{ j \in U \mid j < i \} \quad \text{and} \quad \epsilon(x, x') := (-1)^{\alpha(i,U)}.$$

Then ϵ gives an incidence function of the cell complex $\Gamma(P)$; that is, for all $x, y \in P$ with $x > y$ and $\rho(x) = \rho(y) + 2$, we have

$$\epsilon(x, z) \cdot \epsilon(z, y) + \epsilon(x, z') \cdot \epsilon(z', y) = 0,$$

where $\{z, z'\} = \{ w \in P \mid x > w > y \}$.

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Stanley [9] assigned the commutative ring A_P to a simplicial poset P. We remark that, if $[x\vee y] \neq \emptyset$, then $\{ z \in P \mid z \leq x, y \}$ has the largest element $x \wedge y$. Let $\Bbbk$ be a field, and let $S := \Bbbk[\, t_x \mid x \in P \,]$ be the polynomial ring in the variables t_x. Consider the ideal

$$I_P := \left(t_x t_y - t_{x\wedge y} \sum_{z\in[x\vee y]} t_z \mid x, y \in P \right) + (\, t_{\hat{0}} - 1 \,)$$

of S (if $[x \vee y] = \emptyset$, we interpret that $t_x t_y - t_{x\wedge y} \sum_{z\in[x\vee y]} t_z = t_x t_y$), and set

$$A_P := S/I_P.$$

We denote A_P just by A, if there is no danger of confusion. Clearly, $\dim A_P = \operatorname{rank} P = \dim \Gamma(P) + 1$. For a rank 1 element $y_i \in P$, set $t_i := t_{y_i}$. If $\{x\} = [U]$ for some $U \subset [n]$ with $\#U \geq 2$, then $t_x = \prod_{i\in U} t_i$ in A, and t_x is a "dummy variable". Since I_P is a homogeneous ideal under the grading given by $\deg(t_x) = \rho(x)$, A is a graded ring. If $\Gamma(P)$ is a simplicial complex, then A_P is generated by degree 1 elements, and coincides with the Stanley–Reisner ring of $\Gamma(P)$.

Note that A also has a $\mathbb{Z}^n$-grading such that $\deg t_i \in \mathbb{N}^n$ is the ith unit vector. For each $x \in P$, the ideal

$$\mathfrak{p}_x := (t_z \mid z \not\leq x)$$

of A is a ($\mathbb{Z}^n$-graded) prime ideal with $\dim A/\mathfrak{p}_x = \rho(x)$, since $A/\mathfrak{p}_x \cong \Bbbk[t_i \mid y_i \leq x]$.

In [1], Duval adapted classical arguments on Stanley–Reisner rings for A_P, and got basic results. Recently, Masuda and his co-workers studied A_P from the viewpoint of *toric topology*, since the *equivariant cohomology* ring of a torus manifold is of the form A_P (see [4,5]). In this paper, we will introduce another approach.

Let R be a Noetherian commutative ring, $\operatorname{Mod} R$ the category of R-modules, and $\operatorname{mod} R$ its full subcategory consisting of finitely generated modules. The *dualizing complex* $D_R^\bullet$ of R gives the important duality $\mathrm{R}\operatorname{Hom}_R(-, D_R^\bullet)$ on the bounded derived category $\mathsf{D}^b(\operatorname{mod} R)$ (see [2]). If R is a (graded) local ring with the maximal ideal $\mathfrak{m}$, then the (graded) Matlis dual of $H^{-i}(D_R^\bullet)$ is the local cohomology $H^i_{\mathfrak{m}}(R)$.

We have a concise description of a dualizing complex A_P as follows. This result refines Duval's computation of $H^i_{\mathfrak{m}}(A)$ ([1, Theorem 5.9]).

Theorem 1.1. *Let P be a simplicial poset with $d = \operatorname{rank} P$, and set $A := A_P$. The complex*

$$I_A^\bullet : 0 \to I_A^{-d} \to I_A^{-d+1} \to \cdots \to I_A^0 \to 0,$$

given by

$$I_A^{-i} := \bigoplus_{\substack{x\in P,\\ \rho(x)=i}} A/\mathfrak{p}_x,$$

and

$$\partial^{-i}_{I_A^\bullet} : I_A^{-i} \supset A/\mathfrak{p}_x \ni 1_{A/\mathfrak{p}_x} \longmapsto \sum_{\substack{\rho(y)=i-1,\\ y\leq x}} \epsilon(x, y) \cdot 1_{A/\mathfrak{p}_y} \in \bigoplus_{\substack{\rho(y)=i-1,\\ y\leq x}} A/\mathfrak{p}_y \subset I_A^{-i+1}$$

is isomorphic to a dualizing complex $D_A^\bullet$ of A in $\mathsf{D}^b(\operatorname{Mod} A)$.

In [11], the author defined a *squarefree module* over a polynomial ring, and many applications have been found. This idea is also useful for our study. In fact, regarding A as a squarefree module over the polynomial ring $\operatorname{Sym} A_1$, Duval's formula of $H^i_{\mathfrak{m}}(A)$ can be proved quickly (Remark 2.6). Moreover, we can show that a theorem of Murai and Terai [7] on the h-vectors of simplicial complexes also holds for simplicial posets (Theorem 5.6). In the present paper, we will define a squarefree module over A to study the interaction between the topological properties of $\Gamma(P)$ and the homological properties of A.

The category $\operatorname{Sq} A$ of squarefree A-modules is an Abelian category with enough injectives, and $A/\mathfrak{p}_x$ is an injective object. Hence, $I_A^\bullet \in \mathsf{D}^b(\operatorname{Sq} A)$, and $\mathbb{D}(-) := \underline{\operatorname{Hom}}_A^\bullet(-, I_A^\bullet)$ gives a duality on $\mathsf{K}^b(\text{Inj-Sq})$ ($\cong \mathsf{D}^b(\operatorname{Sq} A)$), where Inj-Sq denotes the full subcategory of $\operatorname{Sq} A$ consisting of all injective objects (i.e., finite direct sums of copies of $A/\mathfrak{p}_x$ for various $x \in P$). Via the forgetful functor $\operatorname{Sq} A \to \operatorname{mod} A$, $\mathbb{D}$ coincides with the usual duality $\mathrm{R}\operatorname{Hom}_A(-, D_A^\bullet)$ on $\mathsf{D}^b(\operatorname{mod} A)$.

By [13], to a squarefree A-module M we can assign the constructible sheaf M^+ on (the underlying space X of) $\Gamma(P)$. In this context, the duality $\mathbb{D}$ corresponds to the Poincaré–Verdier duality on the derived category of the constructible sheaves on X up to translation, as in [8,13]. In particular, the sheafification of the complex $I_A^\bullet[-1]$ coincides with the Verdier dualizing complex of X with coefficients in $\Bbbk$, where $[-1]$ represents translation by -1. Using this argument, we can show the following. At least for the Cohen–Macaulay property, the next result has been shown in Duval [1]. However, our proof gives a new perspective.

Corollary 1.2 (*See Theorem 4.4*)**.** *The Cohen–Macaulay, Gorenstein* and Buchsbaum properties, and Serre's condition (S_i) of A_P, depend only on the topology of the underlying space of $\Gamma(P)$ and* $\operatorname{char}(\Bbbk)$. *Here, we say that A_P is Gorenstein* if A_P is Gorenstein and the graded canonical module ω_{A_P} is generated by its degree* 0 *part.*

While Theorem 1.1 and the results in Section 4 are similar to the corresponding ones for *toric face rings* [8], the construction of a toric face ring and that of A_P are not so similar. Both of them are generalizations of the notion of Stanley–Reisner rings, but the directions of the generalizations are almost opposite (for example, Proposition 5.1 does not hold for toric face rings). The prototype of the results in [8] and the present paper is found in [13]. However, the subject there is "sheaves on a poset", and the connection to our rings is not so straightforward.

2. The proof of Theorem 1.1

In the rest of the paper, P is a simplicial poset with rank $P = d$. We use the same conventions as in the preceding section; in particular, $A = A_P$, $\{ y \in P \mid \rho(y) = 1 \} = \{y_1, \dots, y_n\}$, and $t_i := t_{y_i} \in A$.

For a subset $U \subset [n] = \{ 1, \dots, n \}$, A_U denotes the localization of A by the multiplicatively closed set $\{ \prod_{i \in U} t_i^{a_i} \mid a_i \geq 0 \}$.

Lemma 2.1. *For $x \in [U]$,*

$$u_x := \frac{t_x}{\prod_{i \in U} t_i} \in A_U$$

is an idempotent. Moreover, $u_x \cdot u_{x'} = 0$ for $x, x' \in [U]$ with $x \neq x'$, and

$$1_{A_U} = \sum_{x \in [U]} u_x. \tag{2.1}$$

Hence, we have a $\mathbb{Z}^n$-graded direct sum decomposition

$$A_U = \bigoplus_{x \in [U]} A_U \cdot u_x$$

(if $[U] = \emptyset$, then $A_U = 0$).

Proof. Since $\prod_{i \in U} t_i = \sum_{x \in [U]} t_x$ in A, the Eq. (2.1) is clear. For $x, x' \in [U]$ with $x \neq x'$, we have $[x \vee x'] = \emptyset$ and $t_x \cdot t_{x'} = 0$. Hence, $u_x \cdot u_{x'} = 0$ and

$$u_x = u_x \cdot 1_{A_U} = u_x \cdot \sum_{x'' \in [U]} u_{x''} = u_x \cdot u_x.$$

Now, the last assertion is clear. □

Let $\mathrm{Gr}\,A$ be the category of $\mathbb{Z}^n$-graded A-modules, and $\mathrm{gr}A$ its full subcategory consisting of finitely generated modules. Here, a morphism $f : M \to N$ in $\mathrm{Gr}\,A$ is an A-homomorphism with $f(M_{\mathbf{a}}) \subset N_{\mathbf{a}}$ for all $\mathbf{a} \in \mathbb{Z}^n$. As usual, for M and $\mathbf{a} \in \mathbb{Z}^n$, $M(\mathbf{a})$ denotes the shifted module of M with $M(\mathbf{a})_{\mathbf{b}} = M_{\mathbf{a}+\mathbf{b}}$. For $M, N \in \mathrm{Gr}\,A$,

$$\underline{\mathrm{Hom}}_A(M, N) := \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} \mathrm{Hom}_{\mathrm{Gr}\,A}(M, N(\mathbf{a}))$$

has a $\mathbb{Z}^n$-graded A-module structure. Similarly, $\underline{\mathrm{Ext}}^i_A(M, N) \in \mathrm{Gr}\,A$ can be defined. If $M \in \mathrm{gr}A$, the underlying module of $\underline{\mathrm{Hom}}_A(M, N)$ is isomorphic to $\mathrm{Hom}_A(M, N)$, and the same is true for $\underline{\mathrm{Ext}}^i_A(M, N)$.

If $M \in \mathrm{Gr}\,A$, then $M^\vee := \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} \mathrm{Hom}_{\Bbbk}(M_{-\mathbf{a}}, \Bbbk)$ can be regarded as a $\mathbb{Z}^n$-graded A-module, and $(-)^\vee$ gives an exact contravariant functor from $\mathrm{Gr}\,A$ to itself, which is called the *graded Matlis duality functor*. For $M \in \mathrm{Gr}\,A$, it is *Matlis reflexive* (i.e., $M^{\vee\vee} \cong M$) if and only if $\dim_{\Bbbk} M_{\mathbf{a}} < \infty$ for all $\mathbf{a} \in \mathbb{Z}^n$.

Lemma 2.2. *$A_U \cdot u_x$ is Matlis reflexive, and $E_A(x) := (A_U \cdot u_x)^\vee$ is injective in $\mathrm{Gr}\,A$. Moreover, any indecomposable injective in $\mathrm{Gr}\,A$ is isomorphic to $E_A(x)(\mathbf{a})$ for some $x \in P$ and $\mathbf{a} \in \mathbb{Z}^n$.*

Proof. Clearly, $A_U \cdot u_x$ is a $\mathbb{Z}^n$-graded free $\Bbbk[\, t_i^{\pm 1} \mid i \in U \,]$-module. For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, let $\mathbf{a}' \in \mathbb{Z}^n$ be the vector whose ith coordinate is

$$a'_i = \begin{cases} a_i & \text{if } i \notin U, \\ 1 & \text{otherwise.} \end{cases}$$

Then, we have $\dim_{\Bbbk}(A_U \cdot u_x)_{\mathbf{a}} = \dim_{\Bbbk}(A \cdot t_x)_{\mathbf{a}'} \leq \dim_{\Bbbk} A_{\mathbf{a}'} < \infty$, and $A_U \cdot u_x$ is Matlis reflexive.

The injectivity of $E_A(x)$ follows from the same argument as [6, Lemma 11.23]. In fact, we have a natural isomorphism

$$\underline{\mathrm{Hom}}_A(M, E_A(x)) \cong (M \otimes_A E_A(x)^\vee)^\vee$$

for $M \in \mathrm{Gr}\,A$ by [6, Lemma 11.16]. Since $E_A(x)^\vee$ ($\cong A_U \cdot u_x$) is a flat A-module, $\underline{\mathrm{Hom}}_A(-, E_A(x))$ gives an exact functor.

Since $E_A(x)$ is the injective envelope of $A/\mathfrak{p}_x$ in $\mathrm{Gr}\,A$, and an associated prime of $M \in \mathrm{Gr}\,A$ is $\mathfrak{p}_x$ for some $x \in P$, the last assertion follows. □

If $(A_U \cdot u_x)_{-\mathbf{a}} \neq 0$ for $\mathbf{a} \in \mathbb{N}^n$, then it is obvious that $\mathbf{a} \in \mathbb{N}^U$ (i.e., $a_i = 0$ for $i \notin U$). As shown in the proof above, we have $\dim_\Bbbk (A_U \cdot u_x)_{-\mathbf{a}} = 1$ with $t^{-\mathbf{a}} \cdot u_x := u_x / \prod_{i \in U} t_i^{a_i} \in (A_U \cdot u_x)_{-\mathbf{a}}$ in this case.

For $M \in \operatorname{Gr} A$, its "$\mathbb{N}^n$-graded part" $M_{\geq \mathbf{0}} := \bigoplus_{\mathbf{a} \in \mathbb{N}^n} M_{\mathbf{a}}$ is a submodule of M. Then we have a canonical injection

$$\phi_x : A/\mathfrak{p}_x \longrightarrow E_A(x)$$

defined as follows. The set of the monomials $t^{\mathbf{a}} := \prod_{i \in U} t_i^{a_i} \in A/\mathfrak{p}_x \cong \Bbbk[\, t_i \mid i \in U \,]$ with $\mathbf{a} \in \mathbb{N}^U$ forms a $\Bbbk$-basis of $A/\mathfrak{p}_x$ ($\prod_{i \in U} t_i = t_x$ here), and $\phi_x(t^{\mathbf{a}}) \in (E_A(x))_{\mathbf{a}} = \operatorname{Hom}_\Bbbk(\, (A_U \cdot u_x)_{-\mathbf{a}},\, \Bbbk\,)$ for $\mathbf{a} \in \mathbb{N}^U$ is simply given by $t^{-\mathbf{a}} \cdot u_x \longmapsto 1$. Note that ϕ_x induces the isomorphism

$$A/\mathfrak{p}_x \cong E_A(x)_{\geq \mathbf{0}}. \tag{2.2}$$

The Cěch complex $C^\bullet$ of A with respect to $t_1, \ldots, t_n$ is of the form

$$0 \to C^0 \to C^1 \to \cdots \to C^d \to 0 \quad \text{with } C^i = \bigoplus_{\substack{U \subset [n] \\ \#U = i}} A_U$$

(note that if $\#U > d = \dim A$ then $A_U = 0$). The differential map is given by

$$C^i \supset A_U \ni a \longmapsto \sum_{\substack{U' \supset U \\ \#U' = i+1}} (-1)^{\alpha(U' \setminus U, U)} f_{U',U}(a) \in \bigoplus_{\substack{U' \supset U \\ \#U' = i+1}} A_{U'} \subset C^{i+1},$$

where $f_{U',U} : A_U \to A_{U'}$ is the natural map.

Since the radical of the ideal $(t_1, \ldots, t_n)$ is the graded maximal ideal $\mathfrak{m} := (t_x \mid \hat{0} \neq x \in P)$, the cohomology $H^i(C^\bullet)$ of $C^\bullet$ is isomorphic to the local cohomology $H^i_{\mathfrak{m}}(A)$. Moreover, $C^\bullet$ is isomorphic to $\mathrm{R}\Gamma_{\mathfrak{m}} A$ in the bounded derived category $\mathsf{D}^b(\operatorname{Mod} A)$. Here, $\mathrm{R}\Gamma_{\mathfrak{m}} : \mathsf{D}^b(\operatorname{Mod} A) \to \mathsf{D}^b(\operatorname{Mod} A)$ is the right derived functor of $\Gamma_{\mathfrak{m}} : \operatorname{Mod} A \to \operatorname{Mod} A$ given by $\Gamma_{\mathfrak{m}}(M) = \{ s \in M \mid \mathfrak{m}^i s = 0 \text{ for } i \gg 0 \}$.

The same is true in the $\mathbb{Z}^n$-graded context. We may regard $\Gamma_{\mathfrak{m}}$ as a functor from $\operatorname{Gr} A$ to itself, and let ${}^*\mathrm{R}\Gamma_{\mathfrak{m}} : \mathsf{D}^b(\operatorname{Gr} A) \to \mathsf{D}^b(\operatorname{Gr} A)$ be its right derived functor. Then $C^\bullet \cong {}^*\mathrm{R}\Gamma_{\mathfrak{m}}(A)$ in $\mathsf{D}^b(\operatorname{Gr} A)$.

Let ${}^*\mathcal{D}_A^\bullet$ be the $\mathbb{Z}^n$-graded normalized dualizing complex of A. By the $\mathbb{Z}^n$-graded version of the local duality theorem [2, Theorem V.6.2], $({}^*\mathcal{D}_A^\bullet)^\vee \cong {}^*\mathrm{R}\Gamma_{\mathfrak{m}}(A)$ in $\mathsf{D}^b(\operatorname{Gr} A)$. Since ${}^*\mathcal{D}_A^\bullet \in \mathsf{D}^b_{\operatorname{gr} A}(\operatorname{Gr} A)$, it is Matlis reflexive, and we have

$${}^*\mathcal{D}_A^\bullet \cong ({}^*\mathcal{D}_A^\bullet)^{\vee\vee} \cong {}^*\mathrm{R}\Gamma_{\mathfrak{m}}(A)^\vee \cong (C^\bullet)^\vee.$$

Since each $(C^i)^\vee$ is isomorphic to the injective object

$$\bigoplus_{\substack{x \in P \\ \rho(x) = i}} E_A(x)$$

in $\operatorname{Gr} A$, $(C^\bullet)^\vee$ actually coincides with ${}^*\mathcal{D}_A^\bullet$. Hence, ${}^*\mathcal{D}_A^\bullet$ is of the form

$$0 \to \bigoplus_{\substack{x \in P \\ \rho(x) = d}} E_A(x) \to \bigoplus_{\substack{x \in P \\ \rho(x) = d-1}} E_A(x) \to \cdots \to E_A(\hat{0}) \to 0,$$

where the cohomological degree is given by the same way as $I_A^\bullet$.

For each $i \in \mathbb{Z}$, we have an injection $\phi^i : I_A^i \to {}^*\mathcal{D}_A^i$ given by

$$I_A^i = \bigoplus_{\rho(x) = -i} A/\mathfrak{p}_x \supset A/\mathfrak{p}_x \xrightarrow{\phi_x} E_A(x) \subset \bigoplus_{\rho(x) = -i} E_A(x) = {}^*\mathcal{D}_A^i.$$

By the definition of $\phi_x : A/\mathfrak{p}_x \longrightarrow E_A(x) = (A_U \cdot u_x)^\vee$, we have a cochain map

$$\phi^\bullet : I_A^\bullet \to {}^*\mathcal{D}_A^\bullet.$$

Lemma 2.3. *For all i, the cohomology $H^i({}^*\mathcal{D}_A^\bullet)$ of ${}^*\mathcal{D}_A^\bullet$ is $\mathbb{N}^n$-graded.*

This lemma immediately follows from Duval's description of $H^i_{\mathfrak{m}}(A)$ [1, Theorem 5.9], but we give another proof using the notion of *squarefree modules*. This approach makes our proof more self-contained, and we will extend this idea in the following sections.

Let $S = \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring, and regard it as a $\mathbb{Z}^n$-graded ring. For $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, let $x^{\mathbf{a}}$ denote the monomial $\prod x_i^{a_i} \in S$.

Definition 2.4 ([11]). With the above notation, a $\mathbb{Z}^n$-graded S-module M is called *squarefree* if it is finitely generated, $\mathbb{N}^n$-graded (i.e., $M = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} M_{\mathbf{a}}$), and the multiplication map $M_{\mathbf{a}} \ni s \longmapsto x_i s \in M_{\mathbf{a}+\mathbf{e}_i}$ is bijective for all $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and all i with $a_i > 0$. Here, $\mathbf{e}_i \in \mathbb{N}^n$ is the ith unit vector.

The following lemma is easy, and we omit the proof.

Lemma 2.5. *Consider the polynomial ring $T := \mathrm{Sym}\, A_1 \cong \Bbbk[t_1, \dots, t_n]$ (note that T is not a subring of A). Then A is a squarefree T-module.*

Remark 2.6. Since A is a squarefree T-module, Duval's formula on $H^i_{\mathfrak{m}}(A)$ immediately follows from [11, Lemma 2.9]. However, since $H^i_{\mathfrak{m}}(A)$ has a finer "grading" (see [1] or Corollary 3.6 below), the formula will be mentioned in Corollary 4.3.

The proof of Lemma 2.3. Let T be as in Lemma 2.5. For $\mathbf{1} := (1, 1, \dots, 1) \in \mathbb{N}^n$, $T(-\mathbf{1})$ is the ($\mathbb{Z}^n$-graded) canonical module of T. By the local duality theorem, we have

$$H^i({}^*\mathcal{D}^\bullet_A) \cong \underline{\mathrm{Ext}}^i_A(A, {}^*\mathcal{D}^\bullet_A) \cong \underline{\mathrm{Ext}}^{n+i}_T(A, T(-\mathbf{1})).$$

By [11, Theorem 2.6], $\underline{\mathrm{Ext}}^{n+i}_T(A, T(-\mathbf{1}))$ is a squarefree module; in particular, $\mathbb{N}^n$-graded. □

The proof of Theorem 1.1. Recall the cochain map $\phi^\bullet : I^\bullet_A \to {}^*\mathcal{D}^\bullet_A$ constructed before Lemma 2.3. By (2.2), $\phi^\bullet(I^\bullet_A)$ coincides with $({}^*\mathcal{D}^\bullet_A)_{\geq \mathbf{0}}$. Hence, $\phi^\bullet$ is a quasi-isomorphism by Lemma 2.3. Since ${}^*\mathcal{D}^\bullet_A \cong \mathcal{D}^\bullet_A$ in $\mathsf{D}^b(\mathrm{Mod}\, A)$, we are done. □

3. Squarefree modules over A_P

In this section, we will define a squarefree module over the face ring $A = A_P$ of a simplicial poset P. For this purpose, we equip A with a finer "grading", where the index set is no longer a monoid (a similar idea has appeared in [1,8]).

Recall the convention that $\{ y \in P \mid \rho(y) = 1 \} = \{y_1, \dots, y_n\}$ and $t_i = t_{y_i} \in A$. For each $x \in P$, set

$$\mathbb{M}(x) := \bigoplus_{y_i \leq x} \mathbb{N}\, \mathbf{e}^x_i,$$

where $\mathbf{e}^x_i$ is a basis element. So $\mathbb{M}(x) \cong \mathbb{N}^{\rho(x)}$ as additive monoids. For x, z with $x \leq z$, we have an injection $\iota_{z,x} : \mathbb{M}(x) \ni \mathbf{e}^x_i \mapsto \mathbf{e}^z_i \in \mathbb{M}(z)$ of monoids. Set

$$\mathbb{M} := \varinjlim_{x \in P} \mathbb{M}(x),$$

where the direct limit is taken in the category of sets with respect to $\iota_{z,x} : \mathbb{M}(x) \to \mathbb{M}(z)$ for $x, z \in P$ with $x \leq z$. Note that $\mathbb{M}$ is no longer a monoid. Since all $\iota_{z,x}$ is injective, we can regard $\mathbb{M}(x)$ as a subset of $\mathbb{M}$. For each $\underline{\mathbf{a}} \in \mathbb{M}$, $\{x \in P \mid \underline{\mathbf{a}} \in \mathbb{M}(x)\}$ has the smallest element, which is denoted by $\sigma(\underline{\mathbf{a}})$.

We say that a monomial

$$\mathsf{m} = \prod_{x \in P} t_x^{n_x} \in A \quad (n_x \in \mathbb{N})$$

is *standard* if $\{ x \in P \mid n_x \neq 0 \}$ is a chain. In this case, set $\sigma(\mathsf{m}) := \max\{ x \in P \mid n_x \neq 0\}$. If $n_x = 0$ for all $x \neq \hat{0}$, then $\mathsf{m} = 1$. Hence, 1 is a standard monomial with $\sigma(1) = \hat{0}$. As shown in [9], the set of standard monomials forms a $\Bbbk$-basis of A.

There is a one-to-one correspondence between the elements of $\mathbb{M}$ and the standard monomials of A. For a standard monomial m, set $U := \{ i \in [n] \mid y_i \leq \sigma(\mathsf{m}) \}$. Then we have $\sigma(\mathsf{m}) \in [U]$. There is $\mathbf{a} \in \mathbb{N}^U$ such that the image of m in $A/\mathfrak{p}_{\sigma(\mathsf{m})} \cong \Bbbk[\, t_i \mid i \in U\,]$ is a monomial of the form $\prod_{i \in U} t_i^{a_i}$. So m corresponds to $\underline{\mathbf{a}} \in \mathbb{M}(\sigma(\mathsf{m}))\ (= \bigoplus_{i \in U} \mathbb{N}\, \mathbf{e}_i^{\sigma(\mathsf{m})}) \subset \mathbb{M}$ whose $\mathbf{e}_i^{\sigma(\mathsf{m})}$-coordinate is a_i. We denote this m by $t^{\underline{\mathbf{a}}}$.

Let $\underline{\mathbf{a}}, \underline{\mathbf{b}} \in \mathbb{M}$. If $[\sigma(\underline{\mathbf{a}}) \vee \sigma(\underline{\mathbf{b}})] \neq \emptyset$, then we can take the sum $\underline{\mathbf{a}} + \underline{\mathbf{b}} \in \mathbb{M}(x)$ for each $x \in [\sigma(\underline{\mathbf{a}}) \vee \sigma(\underline{\mathbf{b}})]$. Unless $[\sigma(\underline{\mathbf{a}}) \vee \sigma(\underline{\mathbf{b}})]$ consists of a single element, we cannot define $\underline{\mathbf{a}} + \underline{\mathbf{b}} \in \mathbb{M}$. Hence, we denote each $\underline{\mathbf{a}} + \underline{\mathbf{b}} \in \mathbb{M}(x)$ by $(\underline{\mathbf{a}} + \underline{\mathbf{b}})|x$.

Definition 3.1. $M \in \mathrm{Mod}\, A$ is said to be $\mathbb{M}$-*graded* if the following are satisfied.

(1) $M = \bigoplus_{\underline{\mathbf{a}} \in \mathbb{M}} M_{\underline{\mathbf{a}}}$ as $\Bbbk$-vector spaces.
(2) For $\underline{\mathbf{a}}, \underline{\mathbf{b}} \in \mathbb{M}$, we have

$$t^{\underline{\mathbf{a}}} M_{\underline{\mathbf{b}}} \subset \bigoplus_{x \in [\sigma(\underline{\mathbf{a}}) \vee \sigma(\underline{\mathbf{b}})]} M_{(\underline{\mathbf{a}} + \underline{\mathbf{b}})|x}.$$

Hence, if $[\sigma(\underline{\mathbf{a}}) \vee \sigma(\underline{\mathbf{b}})] = \emptyset$, then $t^{\underline{\mathbf{a}}} M_{\underline{\mathbf{b}}} = 0$.

Clearly, A itself is an $\mathbb{M}$-graded module with $A_{\underline{\mathbf{a}}} = \Bbbk\, t^{\underline{\mathbf{a}}}$. Since there is a natural map $\mathbb{M} \to \mathbb{N}^n$, an $\mathbb{M}$-graded module can be seen as an $\mathbb{N}^n$-graded module.

If M is an $\mathbb{M}$-graded A-module, then

$$M_{\not\geq x} := \bigoplus_{\underline{\mathbf{a}} \notin \mathbb{M}(x)} M_{\underline{\mathbf{a}}}$$

is an $\mathbb{M}$-graded submodule for all $x \in P$, and

$$M_{\leq x} := M/M_{\not\leq x}$$

is a $\mathbb{Z}^{\rho(x)}$-graded module over $A/\mathfrak{p}_x \cong \Bbbk[\, t_i \mid y_i \leq x\,]$.

Definition 3.2. We say that an $\mathbb{M}$-graded A-module M is *squarefree* if $M_{\leq x}$ is a squarefree module over the polynomial ring $A/\mathfrak{p}_x \cong \Bbbk[\, t_i \mid y_i \leq x\,]$ for all $x \in P$.

Note that squarefree A-modules are automatically finitely generated, and can be seen as squarefree modules over $T = \operatorname{Sym} A_1$.

Clearly, A itself, and $\mathfrak{p}_x$ and $A/\mathfrak{p}_x$ for $x \in P$, are squarefree. Let $\operatorname{Sq} A$ be the category of squarefree A-modules and their A-homomorphisms $f : M \to M'$ with $f(M_{\underline{\mathbf{a}}}) \subset M'_{\underline{\mathbf{a}}}$ for all $\underline{\mathbf{a}} \in \mathbb{M}$. For example, $I_A^\bullet$ is a complex in $\operatorname{Sq} A$. To see the basic properties of $\operatorname{Sq} A$, we introduce the *incidence algebra* of the poset P as in [13] (see [13] for further information).

The incidence algebra Λ of P over $\Bbbk$ is the finite-dimensional associative $\Bbbk$-algebra with basis $\{\, e_{x,y} \mid x, y \in P,\ x \geq y\,\}$ whose multiplication is defined by

$$e_{x,y} \cdot e_{z,w} = \delta_{y,z}\, e_{x,w},$$

where $\delta_{y,z}$ denotes Kronecker's delta.

Set $e_x := e_{x,x}$ for $x \in P$. Each e_x is an idempotent, and Λe_x is indecomposable as a left Λ-module. Clearly, $e_x \cdot e_y = 0$ for $x \neq y$, and $1_\Lambda = \sum_{x \in P} e_x$. Let $\operatorname{mod} \Lambda$ be the category of finitely generated left Λ-modules. As a $\Bbbk$-vector space, $N \in \operatorname{mod} \Lambda$ has the decomposition $N = \bigoplus_{x\in P} e_x N$. Henceforth, we set $N_x := e_x N$. Clearly, $e_{x,y} N_y \subset N_x$, and $e_{x,y} N_z = 0$ if $y \neq z$.

For each $x \in P$, we can construct a left Λ-module as follows. Set

$$E_\Lambda(x) := \bigoplus_{y\in P,\ y\leq x} \Bbbk\, \bar{e}_y,$$

where the $\bar{e}_y$ are basis elements. The module structure of $E_\Lambda(x)$ is defined by

$$e_{z,\,w} \cdot \bar{e}_y = \begin{cases} \bar{e}_z & \text{if } w = y \text{ and } z \leq x; \\ 0 & \text{otherwise.} \end{cases}$$

Then $E_\Lambda(x)$ is indecomposable and injective in $\operatorname{mod} \Lambda$. Conversely, any indecomposable injective is of this form. Moreover, $\operatorname{mod} \Lambda$ is an Abelian category with enough injectives, and the injective dimension of each object is at most d.

Proposition 3.3. *There is an equivalence between* $\operatorname{Sq} A$ *and* $\operatorname{mod} \Lambda$*. Hence,* $\operatorname{Sq} A$ *is an Abelian category with enough injectives, and the injective dimension of each object is at most* d*. An object* $M \in \operatorname{Sq} A$ *is an indecomposable injective if and only if* $M \cong A/\mathfrak{p}_x$ *for some* $x \in P$.

Proof. Let $N \in \operatorname{mod} \Lambda$. To each $\underline{\mathbf{a}} \in \mathbb{M}$, we assign a $\Bbbk$-vector space $M_{\underline{\mathbf{a}}}$ with a bijection $\mu_{\underline{\mathbf{a}}} : N_{\sigma(\underline{\mathbf{a}})} \to M_{\underline{\mathbf{a}}}$. We put an $\mathbb{M}$-graded A-module structure on $M := \bigoplus_{\underline{\mathbf{a}}\in\mathbb{M}} M_{\underline{\mathbf{a}}}$ by

$$t^{\underline{\mathbf{a}}} s = \sum_{x\in[\sigma(\underline{\mathbf{a}})\vee\sigma(\underline{\mathbf{b}})]} \mu_{(\underline{\mathbf{a}}+\underline{\mathbf{b}})|x}(e_{x,\sigma(\underline{\mathbf{b}})} \cdot \mu_{\underline{\mathbf{b}}}^{-1}(s)) \quad \text{for } s \in M_{\underline{\mathbf{b}}}.$$

To see that M is actually an A-module, note that both $(t^{\underline{\mathbf{a}}}\, t^{\underline{\mathbf{b}}})\, s$ and $t^{\underline{\mathbf{a}}}\, (t^{\underline{\mathbf{b}}}\, s)$ equal

$$\sum_{x\in[\sigma(\underline{\mathbf{a}})\vee\sigma(\underline{\mathbf{b}})\vee\sigma(\underline{\mathbf{c}})]} \mu_{(\underline{\mathbf{a}}+\underline{\mathbf{b}}+\underline{\mathbf{c}})|x}(e_{x,\sigma(\underline{\mathbf{c}})} \cdot \mu_{\underline{\mathbf{c}}}^{-1}(s)) \quad \text{for } s \in M_{\underline{\mathbf{c}}}.$$

We can also show that M is squarefree.

To construct the inverse correspondence, for $x \in P$ with $r = \rho(x)$, set $\underline{\mathbf{a}}(x) := (r, r, \ldots, r) \in \mathbb{N}^r \cong \mathbb{M}(x) \subset \mathbb{M}$. If $x \geq y$, then there is $\underline{\mathbf{a}}(x) - \underline{\mathbf{a}}(y) \in \mathbb{M}(x) \subset \mathbb{M}$ such that $t^{\underline{\mathbf{a}}(x)-\underline{\mathbf{a}}(y)} \cdot t^{\underline{\mathbf{a}}(y)} = t^{\underline{\mathbf{a}}(x)}$. (One might think a simpler definition $\underline{\mathbf{a}}(x) := (1, 1, \ldots, 1) \in \mathbb{N}^r$ works. However, this is not true. In this case, the candidate of $\underline{\mathbf{a}}(x)-\underline{\mathbf{a}}(y)$ belongs to $\mathbb{M}(z)$ for some $z \in P$ with $z < x$. So $(\underline{\mathbf{a}}(x)-\underline{\mathbf{a}}(y))+\underline{\mathbf{a}}(y)$ does not exist, unless $\#[y\vee z] = 1$.) Now we can construct $N \in \operatorname{mod} \Lambda$ from $M \in \operatorname{Sq} A$ as follows. Set $N_x := M_{\underline{\mathbf{a}}(x)}$, and define the multiplication map $N_y \ni s \mapsto e_{x,y} \cdot s \in N_x$ by $M_{\underline{\mathbf{a}}(y)} \ni s \mapsto t^{\underline{\mathbf{a}}(x)-\underline{\mathbf{a}}(y)} s \in M_{\underline{\mathbf{a}}(x)}$ for $x, y \in P$ with $x \geq y$.

By this correspondence, we have $\operatorname{Sq} A \cong \operatorname{mod} \Lambda$. For the last statement, note that $E_\Lambda(x) \in \operatorname{mod} \Lambda$ corresponds to $A/\mathfrak{p}_x \in \operatorname{Sq} A$. □

Let Inj-Sq be the full subcategory of $\operatorname{Sq} A$ consisting of all injective objects, that is, finite direct sums of copies of $A/\mathfrak{p}_x$ for various $x \in P$. As is well known, the bounded homotopy category $\mathsf{K}^b(\text{Inj-Sq})$ is equivalent to $\mathsf{D}^b(\operatorname{Sq} A)$. Since

$$\underline{\operatorname{Hom}}_A(A/\mathfrak{p}_x, A/\mathfrak{p}_y) = \begin{cases} A/\mathfrak{p}_y & \text{if } x \geq y, \\ 0 & \text{otherwise,} \end{cases}$$

we have $\underline{\operatorname{Hom}}_A^\bullet(J^\bullet, I_A^\bullet) \in \mathsf{K}^b(\text{Inj-Sq})$ for all $J^\bullet \in \mathsf{K}^b(\text{Inj-Sq})$. Moreover, $\underline{\operatorname{Hom}}_A^\bullet(-, I_A^\bullet)$ gives a functor

$$\mathbb{D} : \mathsf{K}^b(\text{Inj-Sq}) \to \mathsf{K}^b(\text{Inj-Sq})^{\mathsf{op}}.$$

Proposition 3.4. *Via the forgetful functor* $\mathbb{U} : \text{Inj-Sq} \to \operatorname{gr} A$, $\mathbb{D}$ *coincides with* $\mathrm{R}\underline{\mathrm{Hom}}_A(-, {}^*\mathcal{D}_A^\bullet)$. *More precisely, we have a natural isomorphism*

$$\Phi : \mathbb{U} \circ \mathbb{D} \xrightarrow{\cong} \mathrm{R}\underline{\mathrm{Hom}}_A(-, {}^*\mathcal{D}_A^\bullet) \circ \mathbb{U}.$$

Here, both $\mathbb{U} \circ \mathbb{D}$ *and* $\mathrm{R}\underline{\mathrm{Hom}}_A(-, {}^*\mathcal{D}_A^\bullet) \circ \mathbb{U}$ *are functors from* $\mathsf{K}^b(\text{Inj-Sq})$ *to* $\mathsf{D}^b(\operatorname{gr} A)$.

Proof. The cochain map $\phi^\bullet : I_A^\bullet \to {}^*\mathcal{D}_A^\bullet$ induces the natural transformation Φ. It remains to prove that $\Phi(J^\bullet) : \mathbb{D}(J^\bullet) \to \mathrm{R}\underline{\mathrm{Hom}}_A(J^\bullet, {}^*\mathcal{D}_A^\bullet)$ is a quasi-isomorphism for all $J^\bullet \in \mathsf{K}^b(\text{Inj-Sq})$. For this fact, we use a similar argument to the final steps of the previous section (while the same argument as the proof of [8, Proposition 5.4] also works here). Note that $J^\bullet$ is a complex of squarefree modules over the polynomial ring $T := \operatorname{Sym} A_1$. Since $\mathrm{R}\underline{\mathrm{Hom}}_A(J^\bullet, {}^*\mathcal{D}_A^\bullet) \cong \mathrm{R}\underline{\mathrm{Hom}}_T(J^\bullet, {}^*\mathcal{D}_T^\bullet)$ by the local duality theorem, the cohomologies of $\mathrm{R}\underline{\mathrm{Hom}}_A(J^\bullet, {}^*\mathcal{D}_A^\bullet)$ are squarefree T-modules, in particular, $\mathbb{N}^n$-graded. On the other hand, through Φ, $\mathbb{D}(J^\bullet)$ is isomorphic to the $\mathbb{N}^n$-graded part of $\mathrm{R}\underline{\mathrm{Hom}}_A(J^\bullet, {}^*\mathcal{D}_A^\bullet)$. □

Remark 3.5. By the equivalence $\mathsf{K}^b(\text{Inj-Sq}) \cong \mathsf{D}^b(\operatorname{Sq} A)$, $\mathbb{D}$ can be regarded as a contravariant functor from $\mathsf{D}^b(\operatorname{Sq} A)$ to itself. Then, through the equivalence $\operatorname{Sq} R \cong \operatorname{mod} \Lambda$, $\mathbb{D}$ coincides with the functor $\mathbf{D} : \mathsf{D}^b(\operatorname{mod} \Lambda) \to \mathsf{D}^b(\operatorname{mod} \Lambda)^{\mathrm{op}}$ defined in [13] up to translation. Hence, for $M^\bullet \in \mathsf{D}^b(\operatorname{Sq} A)$, the complex $\mathbb{D}(M^\bullet)$ has the following description. The term of cohomological degree p is

$$\mathbb{D}(M^\bullet)^p := \bigoplus_{i+\rho(x)=-p} (M^i_{\underline{\mathbf{a}}(x)})^* \otimes_{\Bbbk} A/\mathfrak{p}_x,$$

where $(-)^*$ denotes the $\Bbbk$-dual, and $\underline{\mathbf{a}}(x) \in \mathbb{M}(x) \subset \mathbb{M}$ is the one defined in the proof of Proposition 3.3. The differential is given by

$$(M^i_{\underline{\mathbf{a}}(x)})^* \otimes_{\Bbbk} A/\mathfrak{p}_x \ni f \otimes 1_{A/\mathfrak{p}_x} \longmapsto \sum_{\substack{y \le x,\\ \rho(y)=\rho(x)-1}} \epsilon(x,y) \cdot f_y \otimes 1_{A/\mathfrak{p}_y} + (-1)^p \cdot f \circ \partial^{i-1}_{M^\bullet} \otimes 1_{A/\mathfrak{p}_x},$$

where $f_y \in (M_{\underline{\mathbf{a}}(y)})^*$ denotes $M_{\underline{\mathbf{a}}(y)} \ni s \mapsto f(t^{\underline{\mathbf{a}}(x)-\underline{\mathbf{a}}(y)} \cdot s) \in \Bbbk$, and $\epsilon(x,y)$ is the incidence function. We also have $\mathbb{D} \circ \mathbb{D} \cong \mathrm{id}_{\mathsf{D}^b(\operatorname{Sq} A)}$.

Since $H^{-i}(\mathbb{D}(M)) \cong \underline{\mathrm{Ext}}_A^{-i}(M, {}^*\mathcal{D}_A^\bullet) \cong H^i_{\mathfrak{m}}(M)^\vee$ in $\operatorname{Gr} A$, we have the following.

Corollary 3.6. *If* $M \in \operatorname{Sq} A$, *then the local cohomology* $H^i_{\mathfrak{m}}(M)^\vee$ *can be seen as a squarefree A-module.*

4. Sheaves and Poincaré–Verdier duality

The results in this section are parallel to those in [8, Section 6] (or the earlier work [12]). Although the relation between the rings treated there and our A_P is not so direct, the argument is very similar. So we omit the detail of some proofs here.

Recall that a simplicial poset P gives a regular cell complex $\Gamma(P)$. Let X be the underlying space of $\Gamma(P)$, and $c(x)$ the open cell corresponding to $\hat{0} \neq x \in P$. Hence, for each $x \in P$ with $\rho(x) \geq 2$, $c(x)$ is a subset of X homeomorphic to $\mathbb{R}^{\rho(x)-1}$ (if $\rho(x) = 1$, then $c(x)$ is a single point), and X is the disjoint union of the cells $c(x)$. Moreover, $x \geq y$ if and only if $\overline{c(x)} \supset c(y)$.

As in the preceding section, let Λ be the incidence algebra of P, and $\operatorname{mod} \Lambda$ the category of finitely generated left Λ-modules. In [13], we assigned the constructible sheaf $N^\dagger$ on X to $N \in \operatorname{mod} \Lambda$. By the equivalence $\operatorname{Sq} A \cong \operatorname{mod} \Lambda$, we have the constructible sheaf M^+ on X corresponding to $M \in \operatorname{Sq} A$. Here, we give a precise construction for the reader's convenience. For the sheaf theory, consult [3].

For $M \in \operatorname{Sq} A$, set

$$\operatorname{Sp\acute{e}}(M) := \bigcup_{\hat{0} \neq x \in P} c(x) \times M_{\underline{\mathbf{a}}(x)},$$

where $\underline{\mathbf{a}}(x) \in \mathbb{M}(x) \subset \mathbb{M}$ is the one defined in the proof of Proposition 3.3. Let $\pi : \operatorname{Sp\acute{e}}(M) \to X$ be the projection map which sends $(p, m) \in c(x) \times M_{\underline{\mathbf{a}}(x)} \subset \operatorname{Sp\acute{e}}(M)$ to $p \in c(x) \subset X$. For an open subset $U \subset X$ and a map $s : U \to \operatorname{Sp\acute{e}}(M)$, we will consider the following conditions.

(∗) $\pi \circ s = \mathrm{id}_U$ and $s_p = t^{\underline{\mathbf{a}}(x)-\underline{\mathbf{a}}(y)} \cdot s_q$ for all $p \in c(x) \cap U$, $q \in c(y) \cap U$ with $x \geq y$. Here, $s_p \in M_{\underline{\mathbf{a}}(x)}$ (resp. $s_q \in M_{\underline{\mathbf{a}}(y)}$) is the element with $s(p) = (p, s_p)$ (resp. $s(q) = (q, s_q)$).

(∗∗) There is an open covering $U = \bigcup_{i \in I} U_i$ such that the restriction of s to U_i satisfies (∗) for all $i \in I$.

Now we define a sheaf M^+ on X as follows. For an open set $U \subset X$, set

$$M^+(U) := \{\, s \mid s : U \to \operatorname{Sp\acute{e}}(M) \text{ is a map satisfying } (**) \,\},$$

and let the restriction map $M^+(U) \to M^+(V)$ for $U \supset V$ be the natural one. It is easy to see that M^+ is a constructible sheaf with respect to the cell decomposition $\Gamma(P)$. For example, A^+ is the $\Bbbk$-constant sheaf $\underline{\Bbbk}_X$ on X, and $(A/\mathfrak{p}_x)^+$ is (the extension to X of) the $\Bbbk$-constant sheaf on the closed cell $\overline{c(x)}$.

Let $\mathrm{Sh}(X)$ be the category of sheaves of finite- dimensional $\Bbbk$-vector spaces on X. The functor $(-)^+ : \mathrm{Sq}\, A \to \mathrm{Sh}(X)$ is exact.

As mentioned in the previous section, $\mathbb{D} : \mathsf{D}^b(\mathrm{Sq}\, A) \to \mathsf{D}^b(\mathrm{Sq}\, A)^{\mathrm{op}}$ corresponds to $\mathbf{T} \circ \mathbf{D} : \mathsf{D}^b(\mathrm{mod}\, \Lambda) \to \mathsf{D}^b(\mathrm{mod}\, \Lambda)^{\mathrm{op}}$, where $\mathbf{D}$ is the one defined in [13], and $\mathbf{T}$ is the translation functor (i.e., $\mathbf{T}(M^\bullet)^i = M^{i+1}$). Through $(-)^\dagger : \mathrm{mod}\, \Lambda \to \mathrm{Sh}(X)$, $\mathbf{D}$ gives the Poincaré–Verdier duality on $\mathsf{D}^b(\mathrm{Sh}(X))$, so we have the following.

Theorem 4.1. *For $M^\bullet \in \mathsf{D}^b(\mathrm{Sq}\, A)$, we have*

$$\mathbf{T}^{-1} \circ \mathbb{D}(M^\bullet)^+ \cong \mathrm{R}\,\mathcal{H}\mathrm{om}((M^\bullet)^+, \mathcal{D}_X^\bullet)$$

in $\mathsf{D}^b(\mathrm{Sh}(X))$. In particular, $\mathbf{T}^{-1}((I_A^\bullet)^+) \cong \mathcal{D}_X^\bullet$, where $I_A^\bullet$ is the complex constructed in Theorem 1.1, and $\mathcal{D}_X^\bullet$ is the Verdier dualizing complex of X with coefficients in $\Bbbk$.

The next result follows from results in [13] (see also [8, Theorem 6.2]).

Theorem 4.2. *For $M \in \mathrm{Sq}\, A$, we have the decomposition $H_{\mathfrak{m}}^i(M) = \bigoplus_{\underline{\mathbf{a}} \in \mathbb{M}} H_{\mathfrak{m}}^i(M)_{-\underline{\mathbf{a}}}$ by Corollary 3.6. Note that $\mathbb{M}$ has the element $\mathbf{0}$. Then the following hold.*

(a) *There is an isomorphism*

$$H^i(X, M^+) \cong H_{\mathfrak{m}}^{i+1}(M)_{\mathbf{0}} \quad \text{for all } i \geq 1,$$

and an exact sequence

$$0 \to H_{\mathfrak{m}}^0(M)_{\mathbf{0}} \to M_{\mathbf{0}} \to H^0(X, M^+) \to H_{\mathfrak{m}}^1(M)_{\mathbf{0}} \to 0.$$

(b) *If $\mathbf{0} \neq \underline{\mathbf{a}} \in \mathbb{M}$ with $x = \sigma(\underline{\mathbf{a}})$, then*

$$H_{\mathfrak{m}}^i(M)_{-\underline{\mathbf{a}}} \cong H_c^{i-1}(U_x, M^+|_{U_x})$$

for all $i \geq 0$. Here, $U_x = \bigcup_{z \geq x} c(z)$ is an open set of X, and $H_c^\bullet(-)$ stands for the cohomology with compact support.

Let $\tilde{H}^i(X; \Bbbk)$ denote the ith *reduced cohomology* of X with coefficients in $\Bbbk$. That is, $\tilde{H}^i(X; \Bbbk) \cong H^i(X; \Bbbk)$ for all $i \geq 1$, and $\tilde{H}^0(X; \Bbbk) \oplus \Bbbk \cong H^0(X; \Bbbk)$, where $H^i(X; \Bbbk)$ is the usual cohomology of X.

Corollary 4.3 (*Duval [1, Theorem 5.9]*)**.** *We have*

$$[H_{\mathfrak{m}}^i(A)]_{\mathbf{0}} \cong \tilde{H}^{i-1}(X; \Bbbk) \quad \text{and} \quad [H_{\mathfrak{m}}^i(A)]_{-\underline{\mathbf{a}}} \cong H_c^{i-1}(U_x; \Bbbk)$$

for all $i \geq 0$ and all $\mathbf{0} \neq \underline{\mathbf{a}} \in \mathbb{M}$ with $x = \sigma(\underline{\mathbf{a}})$.

For this $\underline{\mathbf{a}} \in \mathbb{M}$ (but $\underline{\mathbf{a}}$ can be $\mathbf{0}$ here), $[H_{\mathfrak{m}}^i(A)]_{-\underline{\mathbf{a}}}$ is also isomorphic to the ith cohomology of the cochain complex

$$K_x^\bullet : 0 \to K_x^{\rho(x)} \to K_x^{\rho(x)+1} \to \cdots \to K_x^d \to 0 \quad \text{with } K_x^i = \bigoplus_{\substack{z \geq x \\ \rho(z)=i}} \Bbbk\, b_z$$

(b_z is a basis element) whose differential map is given by

$$b_z \longmapsto \sum_{\substack{w \geq z \\ \rho(w)=\rho(z)+1}} \epsilon(w, z)\, b_w.$$

Duval uses the latter description, and he denotes $H^i(K_x^\bullet)$ by $H^{i-\rho(x)-1}(\mathrm{lk}\, x)$.

Proof. The former half follows from Theorem 4.2 by the same argument as [8, Corollary 6.3]. The latter part follows because $H_{\mathfrak{m}}^i(A) \cong H^{-i}(\mathbb{D}(A))^\vee$ and $(\mathbb{D}(A)^\vee)_{-\underline{\mathbf{a}}} = K_x^\bullet$ as complexes of $\Bbbk$-vector spaces. □

Theorem 4.4 (*c.f. Duval [1]*)**.** *Set $d := \mathrm{rank}\, P = \dim X + 1$. Then we have the following.*

(a) *A is Cohen–Macaulay if and only if $\mathcal{H}^i(\mathcal{D}_X^\bullet) = 0$ for all $i \neq -d+1$, and $\tilde{H}^i(X; \Bbbk) = 0$ for all $i \neq d-1$.*

(b) *Assume that A is Cohen–Macaulay and $d \geq 2$. Then A is Gorenstein* if and only if $\mathcal{H}^{-d+1}(\mathcal{D}_X^\bullet) \cong \underline{\Bbbk}_X$. (When $d = 1$, A is Gorenstein* if and only if X consists of exactly two points.)*

(c) *A is Buchsbaum if and only if $\mathcal{H}^i(\mathcal{D}_X^\bullet) = 0$ for all $i \neq -d+1$.*

(d) *Set*

$$d_i := \begin{cases} \dim(\mathrm{supp}\, \mathcal{H}^{-i}(\mathcal{D}_X^\bullet)) & \text{if } \mathcal{H}^{-i}(\mathcal{D}_X^\bullet) \neq 0, \\ -1 & \text{if } \mathcal{H}^{-i}(\mathcal{D}_X^\bullet) = 0 \text{ and } \tilde{H}^i(X; \Bbbk) \neq 0, \\ -\infty & \text{if } \mathcal{H}^{-i}(\mathcal{D}_X^\bullet) = 0 \text{ and } \tilde{H}^i(X; \Bbbk) = 0. \end{cases}$$

Here, $\mathrm{supp}\, \mathcal{F} = \{ p \in X \mid \mathcal{F}_p \neq 0 \}$ for a sheaf $\mathcal{F}$ on X. Then, for $r \geq 2$, A satisfies Serre's condition (S_r) if and only if $d_i \leq i - r$ for all $i < d-1$.

Hence, the Cohen–Macaulay (resp. Gorenstein, Buchsbaum) property and Serre's condition (S_r) of A are topological properties of X, while they may depend on $\mathrm{char}(\Bbbk)$.*

As far as the author knows, even in the Stanley–Reisner ring case, (d) has not been mentioned in literature yet.

Recall that we say that A satisfies Serre's condition (S_r) if $\operatorname{depth} A_{\mathfrak{p}} \geq \min\{\, r,\ \operatorname{ht}\mathfrak{p}\,\}$ for all prime ideal $\mathfrak{p}$ of A. The next fact is well known to the specialist, but we will sketch the proof here for the reader's convenience.

Lemma 4.5. *For $r \geq 2$, A satisfies the condition (S_r) if and only if $\dim H^{-i}(I_A^\bullet) \leq i - r$ for all $i < d$. Here, the dimension of the* 0 *module is $-\infty$.*

Proof. For a prime ideal $\mathfrak{p}$, the normalized dualizing complex of $A_{\mathfrak{p}}$ is quasi-isomorphic to $\mathbf{T}^{-\dim A/\mathfrak{p}}(I_A^\bullet \otimes_A A_{\mathfrak{p}})$. Hence, we have

$$\operatorname{depth} A_{\mathfrak{p}} = \min\{\, i \mid (H^{-i}(I_A^\bullet) \otimes_A A_{\mathfrak{p}}) \neq 0\,\} - \dim A/\mathfrak{p}. \tag{4.1}$$

Recall that, if A satisfies Serre's condition (S_2), then P is *pure* (equivalently, $\dim A/\mathfrak{p} = d$ for all minimal prime ideal $\mathfrak{p}$). Similarly, if $\dim H^{-i}(I_A^\bullet) < i$ for all $i < d$, then P is pure. (In fact, if $\mathfrak{p}$ is a minimal prime ideal of A with $i := \dim A/\mathfrak{p} < d$, then $\operatorname{depth} A_{\mathfrak{p}} = 0$ implies that $\mathfrak{p}$ is a minimal prime of $H^{-i}(I_A^\bullet)$. It follows that $\dim H^{-i}(I_A^\bullet) = i$. This is a contradiction.) So we may assume that P is pure, and hence $\dim A/\mathfrak{p} + \operatorname{ht}\mathfrak{p} = d$ for all $\mathfrak{p}$. Now, the assertion follows from (4.1). □

The proof of Theorem 4.4. We can prove (a)–(c) in the same way as [8, Theorems 6.4 and 6.7]. For (d), note that $d_j = \dim H^{-j-1}(I_A^\bullet) - 1$. So the assertion follows from Lemma 4.5. □

5. Further discussion

This section is a collection of miscellaneous results related to the arguments in the previous sections.

For an integer $i \leq d - 1$, the poset

$$P^{(i)} := \{\, x \in P \mid \rho(x) \leq i + 1\,\}$$

is called the *i-skeleton* of P. Clearly, $P^{(i)}$ is simplicial again, and set $A^{(i)} := A_{P^{(i)}}$. Then it is easy to see that the (Theorem 1.1 type) dualizing complex $I_{A^{(i)}}^\bullet$ of $A^{(i)}$ coincides with the brutal truncation

$$0 \to I_A^{-i-1} \to I_A^{-i} \to \cdots \to I_A^0 \to 0$$

of $I_A^\bullet$. Since $\operatorname{depth} A = \min\{\, i \mid H^{-i}(I_A^\bullet) \neq 0\,\}$ and $\dim A = \max\{\, i \mid H^{-i}(I_A^\bullet) \neq 0\,\}$, we have the equation

$$\operatorname{depth} A_P = 1 + \max\{\, i \mid A^{(i)} \text{ is Cohen–Macaulay}\,\}, \tag{5.1}$$

which is [1, Corollary 6.5].

Contrary to the Gorenstein* property, the Gorenstein property of A_P is *not* topological. This phenomenon occurs even for Stanley–Reisner rings. But there is a characterization of P such that A_P is Gorenstein. For posets P_1, P_2, we regard $P_1 \times P_2 = \{\, (x_1, x_2) \mid x_1 \in P_1,\ x_2 \in P_2\,\}$ as a poset by $(x_1, x_2) \geq (y_1, y_2) \overset{\text{def}}{\Longleftrightarrow} x_i \geq y_i$ in P_i for each $i = 1, 2$.

Proposition 5.1. *A_P is Gorenstein if and only if $P \cong 2^V \times Q$ as posets for a Boolean algebra 2^V and a simplicial poset Q such that A_Q is Gorenstein*.*

Proof. The sufficiency is clear. In fact, if $P \cong 2^V \times Q$, then $A := A_P$ is a polynomial ring over A_Q. So it remains to prove the necessity.

Recall that A is a squarefree module over the polynomial ring $T := \operatorname{Sym} A_1$ (Lemma 2.5). We say that $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ is *squarefree* if $a_i = 0, 1$ for all i. If this is the case, we identify $\mathbf{a}$ with the subset $\{\, i \mid a_i = 1\,\} \subset [n]$. Hence, a subset $F \subset [n]$ sometimes means the corresponding squarefree vector in $\mathbb{N}^n$.

Since A is Gorenstein (in particular, Cohen–Macaulay) now, a minimal $\mathbb{Z}^n$-graded T-free resolution of A is of the form

$$L_\bullet : 0 \to L_{n-d} \to \cdots \to L_1 \to L_0 \to 0 \quad \text{with } L_i = \bigoplus_{F \subset [n]} T(-F)^{\beta_{i,F}}$$

by [11, Corollary 2.4].

Let $\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{N}^n$. Note that $\underline{\operatorname{Hom}}_T^\bullet(L_\bullet, T(-\mathbf{1}))$ is a minimal $\mathbb{Z}^n$-graded T-free resolution of the canonical module $\omega_A = \underline{\operatorname{Ext}}_T^{n-d}(A, T(-\mathbf{1}))$ of A up to translation, and $\omega_A \cong A(-V)$ for some $V \subset [n]$. Set $W := [n] \setminus V$. Since $\underline{\operatorname{Hom}}_T(T(-F), T(-\mathbf{1})) \cong T(-([n] \setminus F))$, we have the following.

$(*)$ If $\beta_{i,F} \neq 0$ for some i, then $F \subset W$.

If $[V] = [\bigvee_{i \in V} y_i] = \emptyset$, then, by the construction of A, there is some $F \subset V$ with $\beta_{1,F} \neq 0$, and this contradicts the statement $(*)$. If $\#[V] \geq 2$, then $\beta_{0,V} \neq 0$, and this is a contradiction again. Hence, $[V] = \{x\}$ for some $x \in P$. We denote the closed interval $[\hat{0}, x]$ by 2^V.

Set

$$Q := \{\, z \in P \mid z \not\geq y_i \text{ for all } i \in V\,\} = \coprod_{U \subset W} [U].$$

If $\#[x' \vee z] \neq 1$ for some $x' \in 2^V$ and $z \in Q$, then $\beta_{i,F} \neq 0$ for $i = 1$ or 0 and for some F with $F \cap V \neq \emptyset$, and it contradicts $(*)$. Hence, for all $x' \in 2^V$ and $z \in Q$, we have $\#[x' \vee z] = 1$. Denoting the element of $[x' \vee z]$ by $x' \vee z$, we have an order-preserving map

$$\psi : 2^V \times Q \ni (x', z) \longmapsto x' \vee z \in P.$$

Moreover, since $P = \coprod_{U \subset [n]} [U]$, ψ is an isomorphism of posets, and we have

$$P \cong 2^V \times Q.$$

Clearly, Q is a simplicial poset. Set $B := A_Q$. Since $A \cong B[\, t_i \mid i \in V\,]$ and

$$A(-V) \cong \omega_A \cong (\omega_B[\, t_i \mid i \in V\,])(-V),$$

B is Gorenstein*. □

Let Σ be a finite regular cell complex with the underlying topological space $X(\Sigma)$, and $Y, Z \subset X(\Sigma)$ closed subsets with $Y \supset Z \neq \emptyset$. Set $U := Y \setminus Z$, and let $h : U \hookrightarrow Y$ be the embedding map. We can define the Cohen–Macaulay property of the pair (Y, Z), which generalizes the Cohen–Macaulay property of a relative simplicial complex introduced in [10, III. Section 7]. See Lemma 5.3 below.

Definition 5.2. We say that the pair (Y, Z) is *Cohen–Macaulay* (over $\Bbbk$) if $H^i_c(U; \Bbbk) = 0$ for all $i \neq \dim U$ and $\mathrm{R}^i h_* \mathcal{D}^\bullet_U = 0$ for all $i \neq -\dim U$. Here, $\mathcal{D}^\bullet_U$ is the Verdier dualizing complex of U with coefficients in $\Bbbk$.

We say that an ideal $I \subset A$ is *squarefree* if it is squarefree as an A-module. For a squarefree ideal I, $\sigma(I) := \{\, x \in P \mid t_x \in I \,\}$ is an order filter (i.e., $x \in \sigma(I)$ and $y \geq x$ imply that $y \in \sigma(I)$), and $U_I := \bigcup_{x \in \sigma(I)} c(x)$ is an open set of X. The sheaf I^+ is (the extension to X of) the $\Bbbk$-constant sheaf on U_I.

Proposition 5.3. (1) *A squarefree ideal I with $I \subsetneq \mathfrak{m}$ is a Cohen–Macaulay module if and only if $(\overline{U_I}, \overline{U_I} \setminus U_I)$ is Cohen–Macaulay in the sense of Definition 5.2.*
(2) *The sequentially Cohen–Macaulay (see [10, III. Definition 2.9]) property of A depends only on X (and* $\mathrm{char}(\Bbbk)$*).*

Proof. (1) Set $U := U_I$, and let $h : U \to \overline{U}$ be the embedding map. The assertion follows from the fact that $\mathbf{T}^{-1}(\mathbb{D}(I)^+|_{\overline{U}}) \cong \mathrm{R}h_* \mathcal{D}^\bullet_U$ and $[H^{-i}(\mathbb{D}(I))]_{\mathbf{0}} \cong H^{i-1}_c(U; \Bbbk)$ by [13] (see also [12, Proposition 4.10] and its proof).
(2) Follows from (1) by the same argument as [14, Theorem 4.7]. □

Remark 5.4. While it is not stated in [8], the statements corresponding to Lemma 4.4, Eq. (5.1), and Proposition 5.3 hold for a cone-wise normal toric face ring.

As in [13], we regard the finite regular cell complex Σ as a poset by $\sigma \geq \tau \overset{\text{def}}{\Longleftrightarrow} \overline{\sigma} \supset \tau$. Here, we use the convention that $\emptyset \in \Sigma$. We say that Σ is a *meet-semilattice* (or, satisfies the *intersection property*) if the largest common lower bound $\sigma \wedge \tau \in \Sigma$ exists for all $\sigma, \tau \in \Sigma$. It is easy to see that Σ is a meet-semilattice if and only if $\#[\sigma \vee \tau] \leq 1$ for all $\sigma, \tau \in \Sigma$. The underlying cell complex of a toric face ring (especially, a simplicial complex) is a meet-semilattice.

For $\sigma \in \Sigma$, let U_σ be the open subset $\bigcup_{\tau \geq \sigma} \tau$ of $X(\Sigma)$. As shown in [13], if $X(\Sigma)$ is Cohen–Macaulay and Σ is a meet-semilattice, then $(\overline{U_\sigma}, \overline{U_\sigma} \setminus U_\sigma)$ is Cohen–Macaulay for all σ. (If Σ is not a meet-semilattice, we have an easy counterexample.) While a simplicial poset P is *not* a meet-semilattice in general, the above fact remains true. We also remark that an indecomposable projective in $\mathrm{Sq}\, A$ is isomorphic to the ideal $J_x := (t_y \mid y \geq x) \subset A$ for some $x \in P$.

Proposition 5.5. *If A is Cohen–Macaulay (resp. Buchsbaum), then the ideal $J_x := (t_y \mid y \geq x)$ is a Cohen–Macaulay module for all $x \in P$ (resp. for all $\hat{0} \neq x \in P$).*

Proof. Let $\underline{\mathbf{a}} \in \mathbb{M}$ with $\sigma(\underline{\mathbf{a}}) = y$. With the notation of Proposition 4.3, recall that

$$\mathrm{R}\Gamma_{\mathfrak{m}} A \cong (\mathbb{D}(A)^\vee)_{-\underline{\mathbf{a}}} \cong K^\bullet_y.$$

Similarly, we have

$$\mathrm{R}\Gamma_{\mathfrak{m}} J_x \cong (\mathbb{D}(J_x)^\vee)_{-\underline{\mathbf{a}}} \cong \bigoplus_{z \in [x \vee y]} K^\bullet_z.$$

To see the second isomorphism, note that, if $w \geq x, y$, then there exists a *unique* $z \in [x \vee y]$ such that $w \geq z$.

If A is Cohen–Macaulay (resp. Buchsbaum), then $H^i_{\mathfrak{m}}(A)_{-\underline{\mathbf{a}}} \cong H^i(\mathrm{R}\Gamma_{\mathfrak{m}} A)_{-\underline{\mathbf{a}}} = 0$ for all $i < d$ and all $\underline{\mathbf{a}} \in \mathbb{M}$ (resp. $\mathbf{0} \neq \underline{\mathbf{a}} \in \mathbb{M}$). Hence, we are done. □

Regarding $A = A_P$ as a $\mathbb{Z}$-graded ring, we have

$$\sum_{i \geq 0} (\dim_\Bbbk A_i) \cdot \lambda^i = \frac{h_0 + h_1 \lambda + \cdots + h_d \lambda^d}{(1-\lambda)^d},$$

for some integers $h_0, h_1, \ldots, h_d$ by [9, Proposition 3.8]. We call $(h_0, \ldots, h_d)$ the *h-vector* of P. The behavior of the h-vectors of simplicial complexes is an important subject of combinatorial commutative algebra. The h-vector of a simplicial poset has also been studied, and striking results are given in [10,4].

Recently, Murai and Terai gave nice results on the h-vector of a simplicial complex Δ such that the Stanley–Reisner ring $\Bbbk[\Delta]$ satisfies Serre's condition (S_r). We show that one of them also holds for simplicial posets.

Theorem 5.6 (*see Murai–Terai [7, Theorem 1.1]*). *Let P be a simplicial poset with the h-vector $(h_0, h_1, \ldots, h_d)$. If A satisfies Serre's condition (S_r), then $h_i \geq 0$ for all $i \leq r$.*

Proof. By virtue of Lemma 2.5, we can imitate the proof of [7, Theorem 1.1].

Let Δ be a simplicial complex with the vertex set $[n]$, $S = \Bbbk[x_1, \ldots, x_n]$ the polynomial ring, and $\Bbbk[\Delta] = S/I_\Delta$ the Stanley–Reisner ring of Δ. To prove our theorem, we replace $\Bbbk[\Delta]$ and S in their argument by $A = A_P$ and $T = \operatorname{Sym} A_1$, respectively. In the former half of the proof, they treat $\Bbbk[\Delta]$ as just a finitely generated graded S-module, and the argument is clearly applicable to our A and T. The latter half of their proof is based on the fact that $\underline{\operatorname{Ext}}_S^i(\Bbbk[\Delta], S(-\mathbf{1}))$ is a squarefree S-module of dimension at most $n - i - r$. Hence, if the following holds, the argument in [7] works in our case.

Claim. *$\underline{\operatorname{Ext}}_T^i(A, T(-\mathbf{1}))$ is a squarefree T-module of dimension at most $n - i - r$.*

The proof is easy. In fact, the squarefreeness follows from Lemma 2.5 and [11, Theorem 2.6]. Moreover, since $\underline{\operatorname{Ext}}_T^i(A, T(-\mathbf{1})) \cong \underline{\operatorname{Ext}}_A^{-n+i}(A, {}^*\mathcal{D}_A^\bullet) \cong H^{-n+i}(I_A^\bullet)$ by the local duality, we have $\underline{\operatorname{Ext}}_T^i(A, T(-\mathbf{1})) \leq n - i - r$ by Lemma 4.5. □

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