Computers and Mathematics with Applications 62 (2011) 1181–1199

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Existence of solutions for a class of fractional boundary value problems via critical point theory[☆]

Feng Jiao[a], Yong Zhou[b,*]

[a] School of Mathematics and Information Sciences, Guangzhou University, Guangdong 510006, PR China
[b] School of Mathematics and Computational Science, Xiangtan University, Hunan 411105, PR China

ARTICLE INFO

Keywords:
Fractional differential equations
Boundary value problem
Fractional advection–dispersion equation
Critical point theory
Existence

ABSTRACT

In this paper, by the critical point theory, a new approach is provided to study the existence of solutions to the following fractional boundary value problem:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \dfrac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\right) + \nabla F(t, u(t)) = 0, & \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0, \end{cases}$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann–Liouville fractional integrals of order $0 \leq \beta < 1$ respectively, $F : [0, T] \times \mathbf{R}^N \to \mathbf{R}$ is a given function and $\nabla F(t, x)$ is the gradient of F at x. Our interest in this problem arises from the fractional advection–dispersion equation (see Section 2). The variational structure is established and various criteria on the existence of solutions are obtained.

1. Introduction

Fractional differential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, neurons, electrochemistry, control, porous media, electromagnetism, etc., (see [1–6]). There has been significant development in fractional differential equations in recent years; see the monographs of Kilbas et al. [7], Miller and Ross [8], Podlubny [9], Samko et al. [10] and the papers [11–30] and the references therein.

In this paper, we consider the fractional boundary value problem (BVP) of the following form

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \dfrac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\right) + \nabla F(t, u(t)) = 0, & \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \tag{1}$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann–Liouville fractional integrals of order $0 \leq \beta < 1$ respectively, $F : [0, T] \times \mathbf{R}^N \to \mathbf{R}$ is a given function satisfying some assumptions and $\nabla F(t, x)$ is the gradient of F at x. In particular, if $\beta = 1$, BVP (1) reduces to the standard second-order boundary value problem.

Physical models containing fractional differential operators have recently renewed attention from scientists which is mainly due to applications as models for physical phenomena exhibiting anomalous diffusion. A strong motivation for

☆ This project is supported by the National Science Foundation of China (No. 10971173).
* Corresponding author.
E-mail addresses: jfmath@163.com (F. Jiao), yzhou@xtu.edu.cn (Y. Zhou).

0898-1221/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2011.03.086

investigating the fractional BVP (1) comes from the fractional advection–dispersion equation (ADE). A fractional ADE is a generalization of the classical ADE in which the second-order derivative is replaced with a fractional-order derivative. In contrast to the classical ADE, the fractional ADE has solutions that resemble the highly skewed and heavy-tailed breakthrough curves observed in field and laboratory studies [11,12], in particular in contaminant transport of ground-water flow [13]. In [13], Benson et al. state that solutes moving through a highly heterogeneous aquifer violations violate the basic assumptions of local second-order theories because of large deviations from the stochastic process of Brownian motion.

Let $\phi(t, x)$ represent the concentration of a solute at a point x at time t in an arbitrary bounded connected set $\Omega \subset \mathbf{R}^N$. According to [12,15], the N-dimensional form of the fractional ADE can be written as

$$\frac{\partial \phi}{\partial t} = -\nabla(\upsilon\phi) - \nabla(\nabla^{-\beta}(-k\nabla\phi)) + f, \quad \text{in } \Omega, \tag{2}$$

where υ is a constant mean velocity, k is a constant dispersion coefficient, $\upsilon\phi$ and $-k\nabla\phi$ denote the mass flux from advection and dispersion respectively. The components of $\nabla^{-\beta}$ in (2) are a linear combination of the left and right Riemann–Liouville fractional integral operators

$$(\nabla^{-\beta}(-k\nabla\phi))_i = (q_{-\infty}D_{x_i}^{-\beta} + (1-q)_{x_i}D_{\infty}^{-\beta})\left(-k\frac{\partial\phi}{\partial x_i}\right), \quad i = 1, \ldots, N, \tag{3}$$

where $q \in [0, 1]$ describes the skewness of the transport process, and $\beta \in [0, 1)$ is the order of the Riemann–Liouville left and right fractional integral operators on the real line (see Section 2, (12) and (13)). This equation may be interpreted as stating that the mass flux of a particle is related to the negative gradient via a combination of the left and right fractional integrals. Eq. (3) is physically interpreted as a Fick's law for concentrations of particles with a strong nonlocal interaction.

For discussions of Eq. (2), see [13,15]. When $\beta = 0$, the dispersion operators in (2) are identical and the classical ADE is recovered. In a more general version of (2), k is replaced by a symmetric positive definite matrix.

A special case of the fractional ADE (Eq. (2)) describes symmetric transitions, where $q = 1/2$. In this case, $\nabla^{-\beta}$ is equivalent to the symmetric operator

$$(\nabla^{-\beta})_i = \frac{1}{2}{}_{-\infty}D_{x_i}^{-\beta} + \frac{1}{2}{}_{x_i}D_{\infty}^{-\beta}, \quad i = 1, \ldots, N. \tag{4}$$

Combining (2) and (4) gives the mass balance equation for advection and symmetric fractional dispersion.

The fractional ADE has been studied in one dimension [13], and in three dimension [14], over infinite domains by using the Fourier transform of fractional differential operators to determine a classical solution. Variational methods, especially the Galerkin approximation has been investigated to find the solutions of fractional BVP [15] and fractional ADE [16] on a finite domain by establishing some suitable fractional derivative spaces. A Lagrangian structure for some partial differential equations is obtained by using the fractional embedding theory of continuous Lagrangian systems [17].

We note that for nonlinear fractional BVP, some fixed point theorems were already applied successfully to investigate the existence of solutions (e.g. [27–30]). However, it seems that fixed point theorem is not appropriate for discussing BVP (1) since the equivalent integral equation is not easy to be obtained. On the other hand, there is another effective approach, calculus of variation, which proved to be very useful in determining the existence of solutions for integer order differential equation provided that the equation with certain boundary value conditions possesses a variational structure on some suitable Sobolev spaces, for example, one can refer to [31–35] and the references therein for detailed discussions.

However, to the best of the author's knowledge, there are few results on the solutions to fractional BVP which were established by the critical point theory, since it is often very difficult to establish a suitable space and variational functional for fractional differential equations with some boundary conditions. These difficulties are mainly caused by the following properties of fractional integral and fractional derivative operators. These are

(i) the composition rule in general fails to be satisfied by fractional integral and fractional derivative operators (e.g. [7, Lemma 2.21]),
(ii) the fractional integral is a singular integral operator and fractional derivative operator is non-local (Section 2, Definitions 2.1–2.3), and
(iii) the adjoint of a fractional differential operator is not the negative of itself (e.g. [7, Lemma 2.7]).

It should be mentioned here that the fractional variational principles were started to be investigated deeply. The fractional calculus of variations was introduced by Riewe in [36] where he presented a new approach to mechanics that allows one to obtain the equations for a nonconservative system using certain functionals. Klimek [37] gave another approach by considering fractional derivatives, and corresponding Euler–Lagrange equations were obtained, using both Lagrangian and Hamiltonian formalisms. Agrawal [38] presented Euler–Lagrange equations for unconstrained and constrained fractional variational problems, and as a continuation of Agrawal's work, the generalized mechanics are considered to obtain the Hamiltonian formulation for the Lagrangian depending on fractional derivative of coordinates [39]. It is worth mentioning that the fractional Hamiltonian is not uniquely defined and many researchers have explored this area giving new insight into this problem (e.g., [40]).

In this paper, our main purpose is to investigate the existence of solutions for BVP (1). The technical tool is the critical point theory. The rest of this paper is organized as follows: in Section 2, we describe the fractional operators and some of their properties which will be used in this paper. In Section 3, we develop a fractional derivative space and some propositions are proven which will aid in our analysis, and in Section 4, we shall exhibit a variational structure for BVP (1). The results presented in Sections 3 and 4 are basic, but crucial to limpidly reveal that under some suitable assumptions, the critical points of the variational functional defined on a suitable Hilbert space are the solutions of BVP (1). In Section 5, we will introduce some critical point theorems. Also, various criteria on the existence of solutions for BVP (1) will be established.

We conclude this Introduction with some comments on BVP (1) when $\beta = 0$. As already mentioned, if $\beta = 0$, then BVP (1) reduces to the standard second-order boundary value problem of the following form

$$\begin{cases} u''(t) + \nabla F(t, u(t)) = 0, \quad \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0, \end{cases}$$

where $F: [0, T] \times \mathbf{R}^N \to \mathbf{R}$ is a given function and $\nabla F(t, x)$ is the gradient of F at x. Although many excellent results have been worked out on the existence of solutions for second-order BVP (e.g. [32,33]), it seems that no similar results were obtained in the literature for fractional BVP. The present paper is to show that the critical point theory is an effective approach to tackle the existence of solutions for fractional BVP.

2. Reminder about fractional calculus

A number of definitions for the fractional derivative have emerged over the years [9], and in this paper, we restrict our attention to the use of the Riemann–Liouville and Caputo fractional derivatives. In this section, we introduce some basic definitions and properties of the fractional calculus which are used further in this paper. For the proofs, which are omitted, we refer the reader to [7,9] or other texts on basic fractional calculus.

Definition 2.1 (*Left and Right Riemann–Liouville Fractional Integrals [7,9]*)**.** Let f be a function defined on $[a, b]$. The left and right Riemann–Liouville fractional integrals of order γ for function f denoted by ${}_aD_t^{-\gamma} f(t)$ and ${}_tD_b^{-\gamma} f(t)$, respectively, are defined by

$${}_aD_t^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s)\mathrm{d}s, \quad t \in [a, b],\ \gamma > 0,$$

and

$${}_tD_b^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} f(s)\mathrm{d}s, \quad t \in [a, b],\ \gamma > 0,$$

provided the right-hand sides are pointwise defined on $[a, b]$, where $\Gamma > 0$ is the gamma function.

Remark 2.1. For $n \in \mathbf{N}$, if $\gamma = -n$, Definition 2.1 coincides with nth integrals of the form [7,9]

$${}_aD_t^{-n} f(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s)\mathrm{d}s, \quad t \in [a, b],\ n \in \mathbf{N}$$

and

$${}_tD_b^{-n} f(t) = \frac{1}{(n-1)!} \int_t^b (s-t)^{n-1} f(s)\mathrm{d}s, \quad t \in [a, b],\ n \in \mathbf{N}.$$

Definition 2.2 (*Left and Right Riemann–Liouville Fractional Derivatives [7,9]*)**.** Let f be a function defined on $[a, b]$. The left and right Riemann–Liouville fractional derivatives of order γ for function f denoted by ${}_aD_t^{\gamma} f(t)$ and ${}_tD_b^{\gamma} f(t)$, respectively, are defined by

$${}_aD_t^{\gamma} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} {}_aD_t^{\gamma-n} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left(\int_a^t (t-s)^{n-\gamma-1} f(s)\mathrm{d}s \right)$$

and

$${}_tD_b^{\gamma} f(t) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} {}_tD_b^{\gamma-n} f(t) = \frac{1}{\Gamma(n-\gamma)} (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left(\int_t^b (s-t)^{n-\gamma-1} f(s)\mathrm{d}s \right),$$

where $t \in [a, b]$, $n - 1 \le \gamma < n$ and $n \in \mathbf{N}$. In particular, if $0 \le \gamma < 1$, then

$${}_aD_t^{\gamma} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} {}_aD_t^{\gamma-1} f(t) = \frac{1}{\Gamma(1-\gamma)} \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_a^t (t-s)^{-\gamma} f(s)\mathrm{d}s \right), \quad t \in [a, b] \tag{5}$$

and

$$ {}_tD_b^{\gamma}f(t) = -\frac{\mathrm{d}}{\mathrm{d}t}{}_tD_b^{\gamma-1}f(t) = -\frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_t^b (s-t)^{-\gamma}f(s)\mathrm{d}s\right), \quad t \in [a,b]. \tag{6} $$

Remark 2.2. For $n \in \mathbf{N}$, if γ becomes an integer $n-1$, according to Definition 2.2, we recover the usual definitions, namely

$$ {}_aD_t^{n-1}f(t) = f^{(n-1)}(t) \quad \text{and} \quad {}_tD_b^{n-1}f(t) = (-1)^{n-1}f^{(n-1)}(t), \quad t \in [a,b], $$

where $f^{(n-1)}(t)$ is the usual derivative of order $n-1$.

Remark 2.3. If $f \in C([a,b],\mathbf{R}^N)$, it is obvious that Riemann–Liouville fractional integral of order $\gamma > 0$ exists on $[a,b]$. On the other hand, following [7, Lemma 2.2, p. 73], we know that the Riemann–Liouville fractional derivative of order $\gamma \in [n-1,n)$ exists a.e. on $[a,b]$ if $f \in AC^n([a,b],\mathbf{R}^N)$, where $C^k([a,b],\mathbf{R}^N)(k=0,1,\ldots)$ denotes the set of mappings having k times continuously differentiable on $[a,b]$, $AC([a,b],\mathbf{R}^N)$ is the space of functions which are absolutely continuous on $[a,b]$ and $AC^{(k)}([a,b],\mathbf{R}^N)$ $(k=1,\ldots)$ is the space of functions f such that $f \in C^{k-1}([a,b],\mathbf{R}^N)$ and $f^{(k-1)} \in AC([a,b],\mathbf{R}^N)$. In particular, $AC([a,b],\mathbf{R}^N) = AC^1([a,b],\mathbf{R}^N)$.

The left and right Caputo fractional derivatives are defined via the above Riemann–Liouville fractional derivatives (see [7, p. 91]). In particular, they are defined for the function belonging to the space of absolutely continuous functions.

Definition 2.3 (*Left and Right Caputo Fractional Derivatives [7]*)**.** Let $\gamma \geq 0$ and $n \in \mathbf{N}$.

(i) If $\gamma \in (n-1,n)$ and $f \in AC^n([a,b],\mathbf{R}^N)$, then the left and right Caputo fractional derivatives of order γ for function f denoted by ${}_a^cD_t^{\gamma}f(t)$ and ${}_t^cD_b^{\gamma}f(t)$, respectively, exist almost everywhere on $[a,b]$. ${}_a^cD_t^{\gamma}f(t)$ and ${}_t^cD_b^{\gamma}f(t)$ are represented by

$$ {}_a^cD_t^{\gamma}f(t) = {}_aD_t^{\gamma-n}f^{(n)}(t) = \frac{1}{\Gamma(n-\gamma)}\left(\int_a^t (t-s)^{n-\gamma-1}f^{(n)}(s)\mathrm{d}s\right) $$

and

$$ {}_t^cD_b^{\gamma}f(t) = (-1)^n{}_tD_b^{\gamma-n}f^{(n)}(t) = \frac{(-1)^n}{\Gamma(n-\gamma)}\left(\int_t^b (s-t)^{n-\gamma-1}f^{(n)}(s)\mathrm{d}s\right), $$

respectively, where $t \in [a,b]$. In particular, if $0 < \gamma < 1$, then

$$ {}_a^cD_t^{\gamma}f(t) = {}_aD_t^{\gamma-1}f'(t) = \frac{1}{\Gamma(1-\gamma)}\left(\int_a^t (t-s)^{-\gamma}f'(s)\mathrm{d}s\right), \quad t \in [a,b] \tag{7} $$

and

$$ {}_t^cD_b^{\gamma}f(t) = -{}_tD_b^{\gamma-1}f'(t) = -\frac{1}{\Gamma(1-\gamma)}\left(\int_t^b (s-t)^{-\gamma}f'(s)\mathrm{d}s\right), \quad t \in [a,b]. \tag{8} $$

(ii) If $\gamma = n-1$ and $f \in AC^{n-1}([a,b],\mathbf{R}^N)$, then ${}_a^cD_t^{n-1}f(t)$ and ${}_t^cD_b^{n-1}f(t)$ are represented by

$$ {}_a^cD_t^{n-1}f(t) = f^{(n-1)}(t), \quad \text{and} \quad {}_t^cD_b^{n-1}f(t) = (-1)^{(n-1)}f^{(n-1)}(t),\ t \in [a,b]. $$

In particular, ${}_a^cD_t^0f(t) = {}_t^cD_b^0f(t) = f(t),\ t \in [a,b]$.

With these definitions, we note some of the properties of the Riemann–Liouville fractional integral and derivative operators, as outlined in [7,9]. The first result yields the semigroup property of the Riemann–Liouville fractional integral operators.

Property 2.1 (*[7]*)**.** *The left and right Riemann–Liouville fractional integral operators have the property of a semigroup, i.e.*

$$ {}_aD_t^{-\gamma_1}({}_aD_t^{-\gamma_2}f(t)) = {}_aD_t^{-\gamma_1-\gamma_2}f(t) \quad \textit{and} \quad {}_tD_b^{-\gamma_1}({}_tD_b^{-\gamma_2}f(t)) = {}_tD_b^{-\gamma_1-\gamma_2}f(t), \quad \forall \gamma_1,\gamma_2 > 0 $$

in any point $t \in [a,b]$ for continuous function f and for almost every point in $[a,b]$ if the function $f \in L^1([a,b],\mathbf{R}^N)$.

Now we present the rule for fractional integration by parts, which were proved in [10].

Property 2.2 (*[7,10]*)**.** *We have the following property of fractional integration*

$$ \int_a^b \left[{}_aD_t^{-\gamma}f(t)\right]g(t)\mathrm{d}t = \int_a^b \left[{}_tD_b^{-\gamma}g(t)\right]f(t)\mathrm{d}t, \quad \gamma > 0, $$

provided that $f \in L^p([a,b],\mathbf{R}^N)$, $g \in L^q([a,b],\mathbf{R}^N)$ and $p \geq 1$, $q \geq 1$, $1/p+1/q \leq 1+\gamma$ or $p \neq 1$, $q \neq 1$, $1/p+1/q = 1+\gamma$.

The composition of the Riemann–Liouville fractional integration operator with the Caputo fractional differentiation operator is given by the following result.

Property 2.3 ([7]). *Let $n \in \mathbf{N}$ and $n-1 < \gamma \le n$. If $f \in AC^n([a,b], \mathbf{R}^N)$ or $f \in C^n([a,b], \mathbf{R}^N)$, then*

$$ {}_aD_t^{-\gamma}({}_a^cD_t^{\gamma} f(t)) = f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!}(t-a)^j $$

and

$$ {}_tD_b^{-\gamma}({}_t^cD_b^{\gamma} f(t)) = f(t) - \sum_{j=0}^{n-1} \frac{(-1)^j f^{(j)}(b)}{j!}(b-t)^j, $$

for $t \in [a,b]$. In particular, if $0 < \gamma \le 1$ and $f \in AC([a,b], \mathbf{R}^N)$ or $f \in C^1([a,b], \mathbf{R}^N)$, then

$$ {}_aD_t^{-\gamma}({}_a^cD_t^{\gamma} f(t)) = f(t) - f(a), \quad \text{and} \quad {}_tD_b^{-\gamma}({}_t^cD_b^{\gamma} f(t)) = f(t) - f(b). \tag{9} $$

The Riemann–Liouville fractional derivative and the Caputo fractional derivative are connected with each other by the following relations.

Property 2.4 ([7,9]). *Let $n \in \mathbf{N}$ and $n-1 < \gamma < n$. If f is a function defined on $[a,b]$ for which the Caputo fractional derivatives ${}_a^cD_t^{\gamma} f(t)$ and ${}_t^cD_b^{\gamma} f(t)$ of order γ exist together with the Riemann–Liouville fractional derivatives ${}_aD_t^{\gamma} f(t)$ and ${}_tD_b^{\gamma} f(t)$, then*

$$ {}_a^cD_t^{\gamma} f(t) = {}_aD_t^{\gamma} f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{\Gamma(j-\gamma+1)}(t-a)^{j-\gamma}, \quad t \in [a,b] $$

and

$$ {}_t^cD_b^{\gamma} f(t) = {}_tD_b^{\gamma} f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(b)}{\Gamma(j-\gamma+1)}(b-t)^{j-\gamma}, \quad t \in [a,b]. $$

In particular, when $0 < \gamma < 1$, we have

$$ {}_a^cD_t^{\gamma} f(t) = {}_aD_t^{\gamma} f(t) - \frac{f(a)}{\Gamma(1-\gamma)}(t-a)^{-\gamma}, \quad t \in [a,b] \tag{10} $$

and

$$ {}_t^cD_b^{\gamma} f(t) = {}_tD_b^{\gamma} f(t) - \frac{f(b)}{\Gamma(1-\gamma)}(b-t)^{-\gamma}, \quad t \in [a,b]. \tag{11} $$

The fractional integrals and derivatives (see Definitions 2.1–2.3), defined on a finite interval $[a,b]$ of the real line $\mathbf{R}$, are naturally extended to the real line $\mathbf{R}$.

Definition 2.4 (*Left and Right Riemann–Liouville Fractional Integrals and Fractional Derivatives on the Real Line [7,9]*). Let f be a function defined on $\mathbf{R}$.

The left and right Riemann–Liouville fractional integrals of order $\gamma > 0$ on the real line for function f denoted by ${}_{-\infty}D_t^{-\gamma} f(t)$ and ${}_tD_{\infty}^{-\gamma} f(t)$, respectively, are defined by

$$ {}_{-\infty}D_t^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_{-\infty}^{t} (t-s)^{\gamma-1} f(s)\mathrm{d}s \tag{12} $$

and

$$ {}_tD_{\infty}^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_{t}^{\infty} (s-t)^{\gamma-1} f(s)\mathrm{d}s, \tag{13} $$

where $t \in \mathbf{R}$ and $\gamma > 0$.

The left and right Riemann–Liouville fractional derivatives of order $\gamma > 0$ on the real line for function f denoted by ${}_{-\infty}D_t^{\gamma} f(t)$ and ${}_tD_{\infty}^{\gamma} f(t)$, respectively, are defined by

$$ {}_{-\infty}D_t^{\gamma} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} {}_{-\infty}D_t^{\gamma-n} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left(\int_{-\infty}^{t} (t-s)^{n-\gamma-1} f(s)\mathrm{d}s \right) $$

and

$$ {}_tD_\infty^\gamma f(t)=(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}t^n}{}_tD_\infty^{\gamma-n}f(t)=\frac{1}{\Gamma(n-\gamma)}(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\int_t^\infty (s-t)^{n-\gamma-1}f(s)\mathrm{d}s\right), $$

where $t\in\mathbf{R}$, $n-1\le\gamma<n$ and $n\in\mathbf{N}$. In particular, if γ becomes an integer $n-1$, then

$$ {}_{-\infty}D_t^{n-1}f(t)=f^{(n-1)}(t)\quad\text{and}\quad {}_tD_\infty^{n-1}f(t)=(-1)^{n-1}f^{(n-1)}(t),\quad t\in\mathbf{R},\ n\in\mathbf{N}, $$

where $f^{(n-1)}(t)$ is the usual derivative of order $n-1$.

If $0\le\gamma<1$, then

$$ {}_{-\infty}D_t^\gamma f(t)=\frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^t (t-s)^{-\gamma}f(s)\mathrm{d}s\right),\quad t\in\mathbf{R} \tag{14} $$

and

$$ {}_tD_\infty^\gamma f(t)=-\frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_t^\infty (s-t)^{-\gamma}f(s)\mathrm{d}s\right),\quad t\in\mathbf{R}. \tag{15} $$

Remark 2.4. The main purpose of considering fractional derivatives on the real line in this paper is to exploit the properties of the Fourier transform of fractional differential operators to obtain some useful estimates for our investigation. We refer to [9] for detailed discussions about the Fourier transform of the fractional derivatives on the real line.

3. Fractional derivative space

Let us recall that for any fixed $t\in[0,T]$ and $1\le p<\infty$,

$$ \|u\|_{L^p([0,t])}=\left(\int_0^t |u(\xi)|^p\mathrm{d}\xi\right)^{1/p},\qquad \|u\|_{L^p}=\left(\int_0^T |u(t)|^p\mathrm{d}t\right)^{1/p}\quad\text{and}\quad \|u\|_\infty=\max_{t\in[0,T]}|u(t)|. $$

The following result yields the boundedness of the Riemann–Liouville fractional integral operators from the space $L^p([0,T],\mathbf{R}^N)$ to the space $L^p([0,T],\mathbf{R}^N)$, where $1\le p<\infty$. It should be mentioned here that the similar results have been presented in [7,10,15].

Lemma 3.1. *Let $0<\alpha\le 1$ and $1\le p<\infty$. For any $f\in L^p([0,T],\mathbf{R}^N)$, we have*

$$ \|{}_0D_\xi^{-\alpha}f\|_{L^p([0,t])}\le\frac{t^\alpha}{\Gamma(\alpha+1)}\|f\|_{L^p([0,t])},\quad \text{for } \xi\in[0,t],\ t\in[0,T]. \tag{16} $$

Proof. Inspired by the proof of Young's theorem [41], we can prove (16).

In fact, if $p=1$, we have

$$ \begin{aligned} \|{}_0D_\xi^{-\alpha}f\|_{L^1([0,t])} &= \frac{1}{\Gamma(\alpha)}\left|\int_0^t\int_0^\xi (\xi-\tau)^{\alpha-1}f(\tau)\mathrm{d}\tau\mathrm{d}\xi\right| \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^\xi (\xi-\tau)^{\alpha-1}|f(\tau)|\mathrm{d}\tau\mathrm{d}\xi \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t |f(\tau)|\mathrm{d}\tau\int_\tau^t (\xi-\tau)^{\alpha-1}\mathrm{d}\xi \\ &= \frac{1}{\Gamma(\alpha+1)}\int_0^t |f(\tau)|(t-\tau)^\alpha\mathrm{d}\tau \\ &\le \frac{t^\alpha}{\Gamma(\alpha+1)}\|f\|_{L^1([0,t])},\quad \text{for } t\in[0,T]. \end{aligned} \tag{17} $$

Now, suppose that $1<p<\infty$ and $g\in L^q([0,T],\mathbf{R}^N)$, where $1/p+1/q=1$. We have

$$ \begin{aligned} \left|\int_0^t g(\xi)\int_0^\xi (\xi-\tau)^{\alpha-1}f(\tau)\mathrm{d}\tau\mathrm{d}\xi\right| &= \left|\int_0^t g(\xi)\int_0^\xi \tau^{\alpha-1}f(\xi-\tau)\mathrm{d}\tau\mathrm{d}\xi\right| \\ &\le \int_0^t |g(\xi)|\int_0^\xi \tau^{\alpha-1}|f(\xi-\tau)|\mathrm{d}\tau\mathrm{d}\xi \end{aligned} $$

$$\begin{aligned} &= \int_0^t \tau^{\alpha-1}\mathrm{d}\tau \int_\tau^t |g(\xi)||f(\xi-\tau)|\mathrm{d}\xi \\ &\le \int_0^t \tau^{\alpha-1}\mathrm{d}\tau \left(\int_\tau^t |g(\xi)|^q\mathrm{d}\xi\right)^{1/q} \left(\int_\tau^t |f(\xi-\tau)|^p\mathrm{d}\xi\right)^{1/p} \\ &\le \frac{t^\alpha}{\alpha}\|f\|_{L^p([0,t])}\|g\|_{L^q([0,t])}, \quad \text{for } t\in[0,T]. \end{aligned} \tag{18}$$

For any fixed $t \in [0, T]$, consider the functional $H_{\xi*f} : L^q([0, T], \mathbf{R}^N) \to \mathbf{R}$

$$H_{\xi*f}(g) = \int_0^t \left[\int_0^\xi (\xi-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau\right] g(\xi)\mathrm{d}\xi. \tag{19}$$

According to (18), it is obvious that $H_{\xi*f} \in (L^q([0, T], \mathbf{R}^N))^*$, where $(L^q([0, T], \mathbf{R}^N))^*$ denotes the dual space of $L^q([0, T], \mathbf{R}^N)$. Therefore, by (18) and (19) and the Riesz representation theorem, there exists $h \in L^p([0, T], \mathbf{R}^N)$ such that

$$\int_0^t h(\xi)g(\xi)\mathrm{d}\xi = \int_0^t \left[\int_0^\xi (\xi-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau\right] g(\xi)\mathrm{d}\xi \tag{20}$$

and

$$\|h\|_{L^p([0,t])} \le \frac{t^\alpha}{\alpha}\|f\|_{L^p([0,t])} \tag{21}$$

for all $g \in L^q([0, T], \mathbf{R}^N)$. Hence, we have by (20)

$$\frac{1}{\Gamma(\alpha)}h(\xi) = \frac{1}{\Gamma(\alpha)}\int_0^\xi (\xi-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau = {}_0D_\xi^{-\alpha} f(\xi), \quad \text{for } \xi \in [0,t],$$

which means that

$$\|{}_0D_\xi^{-\alpha} f\|_{L^p([0,t])} = \frac{1}{\Gamma(\alpha)}\|h\|_{L^p([0,t])} \le \frac{t^\alpha}{\Gamma(\alpha+1)}\|f\|_{L^p([0,t])} \tag{22}$$

according to (21). Combining (17) and (22), we obtain inequality (16). The proof is complete. □

In order to establish a variational structure for BVP (1), it is necessary to construct appropriate function spaces. Denote by $C_0^\infty([0, T], \mathbf{R}^N)$ the set of all functions $h \in C^\infty([0, T], \mathbf{R}^N)$ with $h(0) = h(T) = 0$. According to Lemma 3.1, for any $h \in C_0^\infty([0, T], \mathbf{R}^N)$ and $1 < p < \infty$, we have $h \in L^p([0, T], \mathbf{R}^N)$ and ${}_0^cD_t^\alpha h \in L^p([0, T], \mathbf{R}^N)$. Therefore, one can construct a set of space $E_0^{\alpha,p}$, which depend on L^p-integrability of the Caputo fractional derivative of a function.

Definition 3.1. Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty([0, T], \mathbf{R}^N)$ with respect to the norm

$$\|u\|_{\alpha,p} = \left(\int_0^T |u(t)|^p\mathrm{d}t + \int_0^T |{}_0^cD_t^\alpha u(t)|^p\mathrm{d}t\right)^{1/p}, \quad \forall u \in E_0^{\alpha,p}. \tag{23}$$

Remark 3.1. (i) It is obvious that the fractional derivative space $E_0^{\alpha,p}$ is the space of functions $u \in L^p([0, T], \mathbf{R}^N)$ having an α-order Caputo fractional derivative ${}_0^cD_t^\alpha u \in L^p([0, T], \mathbf{R}^N)$ and $u(0) = u(T) = 0$.
(ii) For any $u \in E_0^{\alpha,p}$, noting the fact that $u(0) = 0$, we have ${}_0^cD_t^\alpha u(t) = {}_0D_t^\alpha u(t)$, $t \in [0, T]$ according to (10).
(iii) It is easy to verify that $E_0^{\alpha,p}$ is a reflexive and separable Banach space.

Proposition 3.1. *Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.*

Proof. In fact, owing to $L^p([0, T], \mathbf{R}^N)$ be reflexive and separable, the Cartesian product space

$$L_2^p([0,T],\mathbf{R}^N) = L^p([0,T],\mathbf{R}^N) \times L^p([0,T],\mathbf{R}^N)$$

is also a reflexive and separable Banach space with respect to the norm

$$\|v\|_{L_2^p} = \left(\sum_{i=1}^2 \|v_i\|_{L^p}^p\right)^{1/p}, \tag{24}$$

where $v = (v_1, v_2) \in L_2^p([0, T], \mathbf{R}^N)$.

Consider the space $\Omega = \{(u, {}_0^cD_t^\alpha u) : \forall u \in E_0^{\alpha,p}\}$, which is a closed subset of $L_2^p([0,T], \mathbf{R}^N)$ as $E_0^{\alpha,p}$ is closed. Therefore, Ω is also a reflexive and separable Banach space with respect to the norm (24) for $v = (v_1, v_2) \in \Omega$.

We form the operator $A : E_0^{\alpha,p} \to \Omega$ as follows

$$A : u \to (u, {}_0^cD_t^\alpha u), \quad \forall u \in E_0^{\alpha,p}.$$

It is obvious that

$$\|u\|_{\alpha,p} = \|Au\|_{L_2^p},$$

which means that the operator $A : u \to (u, {}_0^cD_t^\alpha u)$ is an isometric isomorphic mapping and the space $E_0^{\alpha,p}$ is isometric isomorphic to the space Ω. Thus $E_0^{\alpha,p}$ is a reflexive and separable Banach space, and this completes the proof. □

Applying Property 2.3 and Lemma 3.1, we can now give the following useful estimates.

Proposition 3.2. *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, we have*

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0^c D_t^\alpha u\|_{L^p}. \tag{25}$$

Moreover, if $\alpha > 1/p$ and $1/p + 1/q = 1$, then

$$\|u\|_\infty \leq \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \|{}_0^c D_t^\alpha u\|_{L^p}. \tag{26}$$

Proof. For any $u \in E_0^{\alpha,p}$, according to (9) and noting the fact that $u(0) = 0$, we have that

$${}_0D_t^{-\alpha}({}_0^cD_t^\alpha u(t)) = u(t), \quad t \in [0,T].$$

Therefore, in order to prove inequalities (25) and (26), we only need to prove that

$$\|{}_0 D^{-\alpha}{}_t({}_0^cD^\alpha{}_t u)\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0^c D_t^\alpha u\|_{L^p}, \tag{27}$$

where $0 < \alpha \leq 1$ and $1 < p < \infty$, and

$$\|{}_0D_t^{-\alpha}({}_0^cD_t^\alpha u)\|_\infty \leq \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \|{}_0^c D_t^\alpha u\|_{L^p}, \tag{28}$$

where $\alpha > 1/p$ and $1/p + 1/q = 1$.

First, we note that ${}_0^cD_t^\alpha u \in L^p([0,T], \mathbf{R}^N)$, inequality (27) follows from (16) directly.

We are now in a position to prove (28). For $\alpha > 1/p$, choose q such that $1/p + 1/q = 1$. $\forall u \in E_0^{\alpha,p}$, we have

$$\begin{aligned}
|{}_0D_t^{-\alpha}({}_0^cD_t^\alpha u(t))| &= \frac{1}{\Gamma(\alpha)} \left| \int_0^t (t-s)^{\alpha-1} {}_0^cD_s^\alpha u(s) \mathrm{d}s \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{(\alpha-1)q} \mathrm{d}s \right)^{1/q} \|{}_0^cD_t^\alpha u\|_{L^p} \\
&\leq \frac{T^{1/q+\alpha-1}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \|{}_0^c D_t^\alpha u\|_{L^p} \\
&= \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \|{}_0^c D_t^\alpha u\|_{L^p},
\end{aligned}$$

and this completes the proof. □

According to (25), we can consider $E_0^{\alpha,p}$ with respect to the norm

$$\|u\|_{\alpha,p} = \|{}_0^c D_t^\alpha u\|_{L^p} = \left(\int_0^T |{}_0^c D_t^\alpha u(t)|^p \mathrm{d}t \right)^{1/p} \tag{29}$$

in the following analysis.

Proposition 3.3. *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. Assume that $\alpha > 1/p$ and the sequence $\{u_k\}$ converges weakly to u in $E_0^{\alpha,p}$, i.e. $u_k \rightharpoonup u$. Then $u_k \to u$ in $C([0,T], \mathrm{R}^N)$, i.e. $\|u - u_k\|_\infty = 0$, as $k \to \infty$.*

Proof. If $\alpha > 1/p$, then by (26) and (29), the injection of $E_0^{\alpha,p}$ into $C([0,T],\mathrm{R}^N)$, with its natural norm $\|\cdot\|_\infty$, is continuous, i.e. if $u_k \to u$ in $E_0^{\alpha,p}$, then $u_k \to u$ in $C([0,T],\mathrm{R}^N)$.

Since $u_k \rightharpoonup u$ in $E_0^{\alpha,p}$, it follows that $u_k \rightharpoonup u$ in $C([0,T],\mathrm{R}^N)$. In fact, for any $h \in (C([0,T],\mathrm{R}^N))^*$, if $u_k \to u$ in $E_0^{\alpha,p}$, then $u_k \to u$ in $C([0,T],\mathrm{R}^N)$, and thus $h(u_k) \to h(u)$. Therefore, $h \in (E_0^{\alpha,p})^*$, which means that $(C([0,T],\mathrm{R}^N))^* \subseteq (E_0^{\alpha,p})^*$.

Hence, if $u_k \rightharpoonup u$ in $E_0^{\alpha,p}$, then for any $h \in (C([0,T],\mathrm{R}^N))^*$, we have $h \in (E_0^{\alpha,p})^*$, and thus $h(u_k) \to h(u)$, i.e. $u_k \rightharpoonup u$ in $C([0,T],\mathrm{R}^N)$.

By the Banach–Steinhaus theorem, $\{u_k\}$ is bounded in $E_0^{\alpha,p}$ and, hence, in $C([0,T],\mathrm{R}^N)$. We are now in a position to prove that the sequence $\{u_k\}$ is equi-uniformly continuous.

Let $1/p+1/q=1$ and $0 \le t_1 < t_2 \le T$. $\forall f \in L^p([0,T],\mathbf{R}^N)$, by using the Hölder inequality and noting that $\alpha > 1/p$, we have

$$\begin{aligned}
|{}_0D_{t_1}^{-\alpha} f(t_1) - {}_0D_{t_2}^{-\alpha} f(t_2)| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}(t_1-s)^{\alpha-1}f(s)\mathrm{d}s - \int_0^{t_2}(t_2-s)^{\alpha-1}f(s)\mathrm{d}s\right| \\
&= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}(t_1-s)^{\alpha-1}f(s)\mathrm{d}s - \int_0^{t_1}(t_2-s)^{\alpha-1}f(s)\mathrm{d}s\right| + \frac{1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s)\mathrm{d}s\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^{t_1}((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1})|f(s)|\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f(s)|\mathrm{d}s \\
&\le \frac{1}{\Gamma(\alpha)}\left(\int_0^{t_1}((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1})^q\mathrm{d}s\right)^{1/q}\|f\|_{L^p} \\
&\quad + \frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(t_2-s)^{(\alpha-1)q}\mathrm{d}s\right)^{1/q}\|f\|_{L^p} \\
&\le \frac{1}{\Gamma(\alpha)}\left(\int_0^{t_1}((t_1-s)^{(\alpha-1)q}-(t_2-s)^{(\alpha-1)q})\mathrm{d}s\right)^{1/q}\|f\|_{L^p} \\
&\quad + \frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(t_2-s)^{(\alpha-1)q}\mathrm{d}s\right)^{1/q}\|f\|_{L^p} \\
&= \frac{\|f\|_{L^p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{1/q}}(t_1^{(\alpha-1)q+1} - t_2^{(\alpha-1)q+1} + (t_2-t_1)^{(\alpha-1)q+1})^{1/q} \\
&\quad + \frac{\|f\|_{L^p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{1/q}}((t_2-t_1)^{(\alpha-1)q+1})^{1/q} \\
&\le \frac{2\|f\|_{L^p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{1/q}}(t_2-t_1)^{\alpha-1+1/q} \\
&= \frac{2\|f\|_{L^p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{1/q}}(t_2-t_1)^{\alpha-1/p}. \qquad (30)
\end{aligned}$$

Therefore, the sequence $\{u_k\}$ is equi-uniformly continuous since, for $0 \le t_1 < t_2 \le T$, by applying (30) and in view of (29), we have

$$\begin{aligned}
|u_k(t_1) - u_k(t_2)| &= |{}_0D_{t_1}^{-\alpha}({}_0^cD_{t_1}^{\alpha}u_k(t_1)) - {}_0D_{t_2}^{-\alpha}({}_0^cD_{t_2}^{\alpha}u_k(t_2))| \\
&\le \frac{2(t_2-t_1)^{\alpha-1/p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{1/q}}\|{}_0^cD_t^{\alpha}u_k\|_{L^p} \\
&= \frac{2(t_2-t_1)^{\alpha-1/p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{1/q}}\|u_k\|_{\alpha,p} \\
&\le c(t_2-t_1)^{\alpha-1/p},
\end{aligned}$$

where $1/p+1/q=1$ and $c \in \mathbf{R}^+$ is a constant. By the Ascoli–Arzela theorem, $\{u_k\}$ is relatively compact in $C([0,T],\mathrm{R}^N)$. By the uniqueness of the weak limit in $C([0,T],\mathrm{R}^N)$, every uniformly convergent subsequence of $\{u_k\}$ converges uniformly on $[0,T]$ to u. The proof is complete. □

4. Variational structure

In this section, we will establish a variational structure which enables us to reduce the existence of solutions of BVP (1) to the one of finding critical points of corresponding functional defined on the space $E_0^{\alpha,p}$ with $p=2$ and $1/2 < \alpha \le 1$.

First of all, making use of Property 2.1, for any $u \in AC([0,T], \mathbf{R}^N)$, BVP (1) transforms to

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}{}_0D_t^{-\frac{\beta}{2}}\left({}_0D_t^{-\frac{\beta}{2}}u'(t)\right) + \frac{1}{2}{}_tD_T^{-\frac{\beta}{2}}\left({}_tD_T^{-\frac{\beta}{2}}u'(t)\right)\right) + \nabla F(t, u(t)) = 0, \\ u(0) = u(T) = 0, \end{cases} \tag{31}$$

for almost every $t \in [0,T]$, where $\beta \in [0,1)$.

Furthermore, in view of Definition 2.3, it is obvious that $u \in AC([0,T], \mathbf{R}^N)$ is a solution of BVP (31) if and only if u is a solution of the following problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}{}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - \frac{1}{2}{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\right) + \nabla F(t, u(t)) = 0, \\ u(0) = u(T) = 0, \end{cases} \tag{32}$$

for almost every $t \in [0,T]$, where $\alpha = 1 - \beta/2 \in (1/2, 1]$. Therefore, we seek a solution u of BVP (32) which, of course, corresponds to the solutions u of BVP (1) provided that $u \in AC([0,T], \mathbf{R}^N)$.

Let us denote by

$$D^{\alpha}(u(t)) = \frac{1}{2}{}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - \frac{1}{2}{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t)). \tag{33}$$

We are now in a position to give a definition of the solution of BVP (32).

Definition 4.1. A function $u \in AC([0,T], \mathbf{R}^N)$ is called a solution of BVP (32) if

(i) $D^{\alpha}(u(t))$ is derivative for almost every $t \in [0,T]$, and
(ii) u satisfies (32).

In what follows, we will treat BVP (32) in the Hilbert space $E^{\alpha} = E_0^{\alpha,2}$ with the corresponding norm $\|u\|_{\alpha} = \|u\|_{\alpha,2}$ which we defined in (29).

Consider the functional $u \to -\int_0^T ({}_0^cD_t^{\alpha}u(t), {}_t^cD_T^{\alpha}u(t))\mathrm{d}t$ on E^{α}. The following estimate is useful for our further discussion.

Proposition 4.1. *If $1/2 < \alpha \le 1$, then for any $u \in E^{\alpha}$, we have*

$$|\cos(\pi\alpha)|\|u\|_{\alpha}^2 \le -\int_0^T \left({}_0^cD_t^{\alpha}u(t), {}_t^cD_T^{\alpha}u(t)\right)\mathrm{d}t \le \frac{1}{|\cos(\pi\alpha)|}\|u\|_{\alpha}^2. \tag{34}$$

Proof. Let $u \in E^{\alpha}$ and $\tilde{u}$ be the extension of u by zero on $\mathbf{R}/[0,T]$. Then $\mathrm{supp}(\tilde{u}) \subseteq [0,T]$. However, as the left and right fractional derivatives are nonlocal,

$$\mathrm{supp}({}_{-\infty}D_t^{\alpha}\tilde{u}) \subseteq [0,\infty) \quad \text{and} \quad \mathrm{supp}({}_tD_{\infty}^{\alpha}\tilde{u}) \subseteq (-\infty, T].$$

Nonetheless, the product $({}_{-\infty}D_t^{\alpha}\tilde{u}, {}_tD_{\infty}^{\alpha}\tilde{u})$ has support in $[0,T]$.

On the other hand, according to [16, Theorem 2.3 and Lemma 2.4], we have

$$\begin{aligned} \int_{-\infty}^{\infty}({}_{-\infty}D_t^{\alpha}\tilde{u}(t), {}_tD_{\infty}^{\alpha}\tilde{u}(t))\mathrm{d}t &= \cos(\pi\alpha)\int_{-\infty}^{\infty}|{}_{-\infty}D_t^{\alpha}\tilde{u}(t)|^2\mathrm{d}t \\ &= \cos(\pi\alpha)\int_{-\infty}^{\infty}|{}_tD_{\infty}^{\alpha}\tilde{u}(t)|^2\mathrm{d}t, \end{aligned} \tag{35}$$

where ${}_{-\infty}D_t^{\alpha}$ and ${}_tD_{\infty}^{\alpha}$ are the Riemann–Liouville fractional derivatives on the real line (see Definition 2.4). Helpful in establishing (35) is the Fourier transform of the Riemann–Liouville fractional derivative on the real line [9]. Hence, according to Remark 3.1, (35) and noting that $\cos(\pi\alpha) \in [-1, 0)$ as $\alpha \in (1/2, 1]$, we have

$$\begin{aligned} -\int_0^T ({}_0^cD_t^{\alpha}u(t), {}_t^cD_T^{\alpha}u(t))\mathrm{d}t &= -\int_0^T ({}_0D_t^{\alpha}u(t), {}_tD_T^{\alpha}u(t))\mathrm{d}t \\ &= -\int_0^T ({}_{-\infty}D_t^{\alpha}\tilde{u}(t), {}_tD_{\infty}^{\alpha}\tilde{u}(t))\mathrm{d}t \\ &= -\int_{-\infty}^{\infty} ({}_{-\infty}D_t^{\alpha}\tilde{u}(t), {}_tD_{\infty}^{\alpha}\tilde{u}(t))\mathrm{d}t \\ &= -\cos(\pi\alpha)\int_{-\infty}^{\infty}|{}_{-\infty}D_t^{\alpha}\tilde{u}(t)|^2\mathrm{d}t \end{aligned}$$

$$\begin{aligned}
&= -\cos(\pi\alpha)\int_0^{\infty} |{}_0D_t^{\alpha}\tilde{u}(t)|^2 \mathrm{d}t \\
&\geq -\cos(\pi\alpha)\int_0^{T} |{}_0D_t^{\alpha}u(t)|^2 \mathrm{d}t \\
&= |\cos(\pi\alpha)|\int_0^{T} |{}_0^cD_t^{\alpha}u(t)|^2 \mathrm{d}t \\
&= |\cos(\pi\alpha)|\|u\|_{\alpha}^2.
\end{aligned} \tag{36}$$

On the other hand, by using Young's inequality, we obtain

$$\begin{aligned}
\left|\int_0^T ({}_0^cD_t^{\alpha}u(t), {}_t^cD_T^{\alpha}u(t))\mathrm{d}t\right| &= \left|\int_0^T ({}_0D_t^{\alpha}u(t), {}_tD_T^{\alpha}u(t))\mathrm{d}t\right| \\
&\leq \int_0^T \frac{1}{\sqrt{2}}|{}_0D_t^{\alpha}u(t)|\sqrt{2}|{}_tD_T^{\alpha}u(t)|\mathrm{d}t \\
&\leq \frac{1}{4\varepsilon}\int_0^T |{}_0D_t^{\alpha}u(t)|^2\mathrm{d}t + \varepsilon\int_0^T |{}_tD_T^{\alpha}u(t)|^2\mathrm{d}t \\
&= \frac{1}{4\varepsilon}\int_0^T |{}_0^cD_t^{\alpha}u(t)|^2\mathrm{d}t + \varepsilon\int_0^{\infty} |{}_tD_{\infty}^{\alpha}\tilde{u}(t)|^2\mathrm{d}t \\
&\leq \frac{1}{4\varepsilon}\|u\|_{\alpha}^2 + \varepsilon\int_{-\infty}^{\infty} |{}_tD_{\infty}^{\alpha}\tilde{u}(t)|^2\mathrm{d}t \\
&= \frac{1}{4\varepsilon}\|u\|_{\alpha}^2 + \frac{\varepsilon}{|\cos(\pi\alpha)|}\left|\int_{-\infty}^{\infty} ({}_{-\infty}D_t^{\alpha}\tilde{u}(t), {}_tD_{\infty}^{\alpha}\tilde{u}(t))\mathrm{d}t\right| \\
&= \frac{1}{4\varepsilon}\|u\|_{\alpha}^2 + \frac{\varepsilon}{|\cos(\pi\alpha)|}\left|\int_0^T ({}_0D_t^{\alpha}u(t), {}_tD_T^{\alpha}u(t))\mathrm{d}t\right| \\
&= \frac{1}{4\varepsilon}\|u\|_{\alpha}^2 + \frac{\varepsilon}{|\cos(\pi\alpha)|}\left|\int_0^T ({}_0^cD_t^{\alpha}u(t), {}_t^cD_T^{\alpha}u(t))\mathrm{d}t\right|.
\end{aligned}$$

Therefore, by taking $\varepsilon = |\cos(\pi\alpha)|/2$, we have

$$\left|\int_0^T ({}_0^cD_t^{\alpha}u(t), {}_t^cD_T^{\alpha}u(t))\mathrm{d}t\right| \leq \frac{1}{|\cos(\pi\alpha)|}\|u\|_{\alpha}^2. \tag{37}$$

Inequality (34) follows then from (36) and (37), and the proof is complete. □

Remark 4.1. According to (34) and (35), for any $u \in E^{\alpha}$, it is obvious that

$$\int_0^T |{}_t^cD_T^{\alpha}u(t)|^2\mathrm{d}t \leq \int_{-\infty}^{\infty} |{}_tD_{\infty}^{\alpha}\tilde{u}(t)|^2\mathrm{d}t = -\int_0^T \frac{({}_0^cD_t^{\alpha}u(t), {}_t^cD_T^{\alpha}u(t))}{|\cos(\pi\alpha)|}\mathrm{d}t \leq \frac{1}{|\cos(\pi\alpha)|^2}\|u\|_{\alpha}^2,$$

which means that ${}_t^cD_T^{\alpha}u \in L^2([0, T], \mathbf{R}^N)$.

Our task is now to establish a variational structure on E^{α} with $\alpha \in (1/2, 1]$. Also, we will show that the critical points of that functional are indeed solutions of BVP (32), and therefore, are solutions of BVP (1).

Theorem 4.1. *Let* $L : [0, T] \times \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ *be defined by*

$$L(t, x, y, z) = -\frac{1}{2}(y, z) - F(t, x)$$

where $F : [0, T] \times \mathbf{R}^N \to \mathbf{R}$ *satisfies the following assumption*

(H_1) $F(t, x)$ *is measurable in* t *for each* $x \in \mathbf{R}^N$, *continuously differentiable in* x *for almost every* $t \in [0, T]$ *and there exist* $m_1 \in C(\mathbf{R}^+, \mathbf{R}^+)$ *and* $m_2 \in L^1([0, T], \mathbf{R}^+)$ *such that*

$$|F(t, x)| \leq m_1(|x|)m_2(t), \qquad |\nabla F(t, x)| \leq m_1(|x|)m_2(t)$$

for all $x \in \mathbf{R}^N$ *and a.e.* $t \in [0, T]$.

If $1/2 < \alpha \leq 1$, then the functional defined by

$$\varphi(u) = \int_0^T L(t, u(t), {}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))\mathrm{d}t$$
$$= \int_0^T \left[-\frac{1}{2}({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)) - F(t, u(t))\right]\mathrm{d}t, \tag{38}$$

is continuously differentiable on E^α, and $\forall u, v \in E^\alpha$, we have

$$\langle\varphi'(u), v\rangle = \int_0^T (D_xL(t, u(t), {}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)), v(t))\mathrm{d}t + \int_0^T (D_yL(t, u(t), {}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)), {}_0^cD_t^\alpha v(t))\mathrm{d}t$$
$$+ \int_0^T (D_zL(t, u(t), {}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)), {}_t^cD_T^\alpha v(t))\mathrm{d}t$$
$$= -\int_0^T \frac{1}{2}[({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha v(t)) + ({}_t^cD_T^\alpha u(t), {}_0^cD_t^\alpha v(t))]\mathrm{d}t - \int_0^T (\nabla F(t, u(t)), v(t))\mathrm{d}t. \tag{39}$$

Proof. First, we note that for a.e. $t \in [0, T]$ and every $[x, y, z] \in \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N$, one has

$$|L(t, x, y, z)| \leq m_1(|x|)m_2(t) + \frac{1}{4}(|y|^2 + |z|^2), \tag{40}$$
$$|D_xL(t, x, y, z)| \leq m_1(|x|)m_2(t), \tag{41}$$
$$|D_yL(t, x, y, z)| \leq \frac{1}{2}|z| \quad \text{and} \quad |D_zL(t, x, y, z)| \leq \frac{1}{2}|y|. \tag{42}$$

Then, inspired by the proof of [31, Theorem 1.4], it suffices to prove that φ has at every point u a directional derivative $\varphi'(u) \in (E^\alpha)^*$ given by (39) and that the mapping

$$\varphi' : E^\alpha \to (E^\alpha)^*, \quad u \to \varphi'(u)$$

is continuous.

(1) It follows easily from Remark 4.1 and (40) that φ is everywhere finite on E^α. Let us define, for u and v fixed in E^α, $t \in [0, T]$, $\lambda \in [-1, 1]$,

$$G(\lambda, t) = L(t, u(t) + \lambda v(t), {}_0^cD_t^\alpha u(t) + \lambda {}_0^cD_t^\alpha v(t), {}_t^cD_T^\alpha u(t) + \lambda {}_t^cD_T^\alpha v(t))$$

and

$$\psi(\lambda) = \int_0^T G(\lambda, t)\mathrm{d}t = \varphi(u + \lambda v).$$

We shall apply the Leibniz formula of differentiation under integral sign to ψ. By (41) and (42), we have

$$|D_\lambda G(\lambda, t)| = |(D_xL(t, u(t) + \lambda v(t), {}_0^cD_t^\alpha u(t) + \lambda {}_0^cD_t^\alpha v(t), {}_t^cD_T^\alpha u(t) + \lambda {}_t^cD_T^\alpha v(t)), v(t))|$$
$$+ |(D_yL(t, u(t) + \lambda v(t), {}_0^cD_t^\alpha u(t) + \lambda {}_0^cD_t^\alpha v(t), {}_t^cD_T^\alpha u(t) + \lambda {}_t^cD_T^\alpha v(t)), {}_0^cD_t^\alpha v(t))|$$
$$+ |(D_zL(t, u(t) + \lambda v(t), {}_0^cD_t^\alpha u(t) + \lambda {}_0^cD_t^\alpha v(t), {}_t^cD_T^\alpha u(t) + \lambda {}_t^cD_T^\alpha v(t)), {}_t^cD_T^\alpha v(t))|$$
$$\leq m_1(|u(t) + \lambda v(t)|)m_2(t)|v(t)| + \frac{1}{2}|{}_t^cD_T^\alpha u(t) + \lambda {}_t^cD_T^\alpha v(t)||{}_0^cD_t^\alpha v(t)|$$
$$+ \frac{1}{2}|{}_0^cD_t^\alpha u(t) + \lambda {}_0^cD_t^\alpha v(t)||{}_t^cD_T^\alpha v(t)|$$
$$\leq m_0m_2(t)|v(t)| + \frac{1}{2}|{}_t^cD_T^\alpha u(t)||{}_0^cD_t^\alpha v(t)| + \frac{1}{2}|{}_0^cD_t^\alpha u(t)||{}_t^cD_T^\alpha v(t)| + |{}_0^cD_t^\alpha v(t)||{}_t^cD_T^\alpha v(t)|,$$

where

$$m_0 = \max_{(\lambda,t)\in[-1,1]\times[0,T]} m_1(|u(t) + \lambda v(t)|).$$

Since $m_2 \in L^1([0, T], \mathbf{R}^+)$, v is continuous on $[0, T]$, and in view of Remark 4.1, we have

$$|D_\lambda G(\lambda, t)| \leq d(t),$$

where $d \in L^1([0,T],\mathbf{R}^+)$. Thus the Leibniz formula is applicable and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\psi(0) &= \int_0^T D_\lambda G(0,t)\mathrm{d}t\\ &= \int_0^T (D_xL(t,u(t),{}_0^cD_t^\alpha u(t),{}_t^cD_T^\alpha u(t)),v(t))\mathrm{d}t + \int_0^T (D_yL(t,u(t),{}_0^cD_t^\alpha u(t),{}_t^cD_T^\alpha u(t)),{}_0^cD_t^\alpha v(t))\mathrm{d}t\\ &\quad + \int_0^T (D_zL(t,u(t),{}_0^cD_t^\alpha u(t),{}_t^cD_T^\alpha u(t)),{}_t^cD_T^\alpha v(t))\mathrm{d}t.\end{aligned}$$

Moreover,

$$\begin{aligned}&|D_xL(t,u(t),{}_0^cD_t^\alpha u(t),{}_t^cD_T^\alpha u(t))| \le m_1(|u(t)|)m_2(t),\\ &|D_yL(t,u(t),{}_0^cD_t^\alpha u(t),{}_t^cD_T^\alpha u(t))| \le \frac{1}{2}|{}_t^cD_T^\alpha u(t)|\end{aligned}$$

and

$$|D_zL(t,u(t),{}_0^cD_t^\alpha u(t),{}_t^cD_T^\alpha u(t))| \le \frac{1}{2}|{}_0^cD_t^\alpha u(t)|.$$

Thus, by Remark 4.1 and (26),

$$\begin{aligned}\langle\varphi'(u),v\rangle &= \int_0^T (D_xL(t,u(t),{}_0^cD_t^\alpha u(t),{}_t^cD_T^\alpha u(t)),v(t))\mathrm{d}t + \int_0^T (D_yL(t,u(t),{}_0^cD_t^\alpha u(t),{}_t^cD_T^\alpha u(t)),{}_0^cD_t^\alpha v(t))\mathrm{d}t\\ &\quad + \int_0^T (D_zL(t,u(t),{}_0^cD_t^\alpha u(t),{}_t^cD_T^\alpha u(t)),{}_t^cD_T^\alpha v(t))\mathrm{d}t\\ &\le c_1\|v\|_\infty + c_2\|{}_0^cD_t^\alpha v(t)\|_{L^2} + c_3\|{}_t^cD_T^\alpha v(t)\|_{L^2}\\ &\le c_1\|v\|_\infty + c_2\|v\|_\alpha + \frac{c_3}{|\cos(\pi\alpha)|}\|v\|_\alpha\\ &\le c_4\|v\|_\alpha,\end{aligned}$$

where c_1, c_2, c_3 and c_4 are some positive constants. Therefore, φ has, at u, a directional derivative $\varphi'(u) \in (E^\alpha)^*$ given by (39).

(2) By a theorem of Krasnosel'skii, (41) and (42) imply that the mapping form E^α into $L^1([0,T],\mathbf{R}^N) \times L^2([0,T],\mathbf{R}^N) \times L^2([0,T],\mathbf{R}^N)$ defined by

$$u \to (D_xL(\cdot,u,{}_0^cD_t^\alpha u,{}_t^cD_T^\alpha u), D_yL(\cdot,u,{}_0^cD_t^\alpha u,{}_t^cD_T^\alpha u), D_zL(\cdot,u,{}_0^cD_t^\alpha u,{}_t^cD_T^\alpha u))$$

is continuous, so that φ' is continuous from E^α into $(E^\alpha)^*$, and the proof is complete. □

Theorem 4.2. *Let $1/2 < \alpha \le 1$ and φ be defined by (38). If assumption (H_1) is satisfied and $u \in E^\alpha$ is a solution of the corresponding Euler equation $\varphi'(u) = 0$, then u is a solution of BVP (32) which, of course, corresponding to the solution of BVP (1).*

Proof. By Theorem 4.1, Property 2.2, (7) and (8), we have

$$\begin{aligned}0 = \langle\varphi'(u),v\rangle &= -\int_0^T \frac{1}{2}[({}_0^cD_t^\alpha u(t),{}_t^cD_T^\alpha v(t)) + ({}_t^cD_T^\alpha u(t),{}_0^cD_t^\alpha v(t))]\mathrm{d}t - \int_0^T (\nabla F(t,u(t)),v(t))\mathrm{d}t\\ &= \int_0^T \frac{1}{2}({}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)),v'(t)) - \frac{1}{2}({}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t)),v'(t))\mathrm{d}t - \int_0^T (\nabla F(t,u(t)),v(t))\mathrm{d}t \qquad (43)\end{aligned}$$

for all $v \in E^\alpha$.

Let us define $w \in C([0,T],\mathbf{R}^N)$ by

$$w(t) = \int_0^t \nabla F(s,u(s))\mathrm{d}s, \quad t \in [0,T],$$

so that

$$\int_0^T (w(t),v'(t))\mathrm{d}t = \int_0^T \left[\int_0^t (\nabla F(s,u(s)),v'(t))\mathrm{d}s\right]\mathrm{d}t.$$

By the Fubini theorem and noting that $v(T) = 0$, we obtain

$$\begin{aligned}\int_0^T (w(t), v'(t))\mathrm{d}t &= \int_0^T \left[\int_s^T (\nabla F(s, u(s)), v'(t))\mathrm{d}t\right]\mathrm{d}s \\ &= \int_0^T (\nabla F(s, u(s)), v(T) - v(s))\mathrm{d}s \\ &= -\int_0^T (\nabla F(s, u(s)), v(s))\mathrm{d}s.\end{aligned}$$

Hence, by (43) we have, for every $v \in E^\alpha$,

$$\int_0^T \left(\frac{1}{2}{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \frac{1}{2}{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t)) + w(t), v'(t)\right)\mathrm{d}t = 0 \tag{44}$$

If (e_j) denotes the Canonical basis of $\mathbf{R}^N$, we can choose $v \in E^\alpha$ such that

$$v(t) = \sin\frac{2k\pi t}{T}e_j \quad \text{or} \quad v(t) = e_j - \cos\frac{2k\pi t}{T}e_j, \quad k = 1, \ldots \text{ and } j = 1, \ldots, N.$$

The theory of Fourier series and (44) imply that

$$\frac{1}{2}{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \frac{1}{2}{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t)) + w(t) = C$$

a.e. on $[0, T]$ for some $C \in \mathbf{R}^N$. According to the definition of $w \in C([0, T], \mathbf{R}^N)$, we have

$$\frac{1}{2}{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \frac{1}{2}{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t)) = -\int_0^t \nabla F(s, u(s))\mathrm{d}s + C \tag{45}$$

a.e. on $[0, T]$ for some $C \in \mathbf{R}^N$.

In view of $\nabla F(\cdot, u(\cdot)) \in L^1([0, T], \mathbf{R}^N)$, we shall identify the equivalence class $D^\alpha(u(t))$ given by (33) and its continuous representant

$$D^\alpha(u(t)) = \frac{1}{2}{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \frac{1}{2}{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t)) = -\int_0^t \nabla F(s, u(s))\mathrm{d}s + C \tag{46}$$

for $t \in [0, T]$.

Therefore, it follows from (46) and a classical result of the Lebesgue theory that $-\nabla F(\cdot, u(\cdot))$ is the classical derivative of $D^\alpha(u(t))$ a.e. on $[0, T]$ which means that (i) in Definition 4.1 is verified.

Since $u \in E^\alpha$ implies that $u \in AC([0, T], \mathbf{R}^N)$, it remains to show that u satisfies (32). In fact, according to (46), we can get that

$$\frac{\mathrm{d}}{\mathrm{d}t}D^\alpha(u(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \frac{1}{2}{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\right) = -\nabla F(t, u(t)).$$

Moreover, $u \in E^\alpha$ implies that $u(0) = u(T) = 0$, and therefore (1) is verified. The proof is complete.

From now on, φ given by (38) will be considered as a functional on E^α with $1/2 < \alpha \leq 1$. □

5. The existence of solutions for BVP (1)

According to Theorem 4.2, we know that in order to find solutions of BVP (1), it suffices to obtain the critical points of functional φ given by (38). We need to use some critical point theorems. For instance, see [31–33]. However, for the reader's convenience, we still state some necessary definitions and theorems and skip the proofs.

Let H be a real Banach space and $C^1(H, \mathbf{R}^N)$ denote the set of functionals that are Fréchet differentiable and their Fréchet derivatives are continuous on H.

Definition A. Let $\psi \in C^1(H, \mathbf{R}^N)$. If any sequence $\{u_k\} \subset H$ for which $\{\psi(u_k)\}$ is bounded and $\psi'(u_k) \to 0$ as $k \to 0$ possesses a convergent subsequence, then we say that ψ satisfies the Palais–Smale condition (denoted by P.S. condition).

Theorem A ([31,33]). *Let H be a real reflexive Banach space. If the functional $\psi : H \to \mathbf{R}^N$ is weakly lower semi-continuous and coercive, i.e. $\lim_{\|z\|\to\infty} \psi(z) = +\infty$, then there exists $z_0 \in H$ such that $\psi(z_0) = \inf_{z\in H} \psi(z)$. Moreover, if ψ is also Fréchet differentiable on H, then $\psi'(z_0) = 0$.*

Theorem B (*Mountain Pass Theorem [32,33]*). *Let H be a real Banach space and $\psi \in C^1(H, \mathbf{R}^N)$ satisfying the P.S. condition. Suppose that*

(i) $\psi(0) = 0$,
(ii) *there exist $\rho > 0$ and $\sigma > 0$ such that $\psi(z) \geq \sigma$ for all $z \in H$ with $\|z\| = \rho$,*
(iii) *there exists z_1 in H with $\|z_1\| \geq \rho$ such that $\psi(z_1) < \sigma$.*

Then ψ possesses a critical value $c \geq \sigma$. Moreover, c can be characterized as

$$c = \inf_{g \in \bar{\Omega}} \max_{z \in g([0,1])} \psi(z),$$

where $\bar{\Omega} = \{g \in C([0,1], H) : g(0) = 0, g(1) = z_1\}$.

First, we use Theorem A to consider the existence of solutions for BVP (1). Assume that condition (H_1) is satisfied. Recall that, in our setting in (38), the corresponding functional φ on E^α given by

$$\varphi(u) = \int_0^T \left[-\frac{1}{2} ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)) - F(t, u(t)) \right] \mathrm{d}t$$

is continuously differentiable according to Theorem 4.1 and is also weakly lower semi-continuous functional on E^α as the sum of a convex continuous function [31, Theorem 1.2] and of a weakly continuous one [31, Proposition 1.2].

In fact, according to Proposition 3.3, if $u_k \rightharpoonup u$ in E^α, then $u_k \to u$ in $C([0, T], \mathbf{R}^N)$. Therefore, $F(t, u_k(t)) \to F(t, u(t))$ a.e. $t \in [0, T]$. By the Lebesgue dominated convergence theorem, we have $\int_0^T F(t, u_k(t))\mathrm{d}t \to \int_0^T F(t, u(t))\mathrm{d}t$, which means that the functional $u \to \int_0^T F(t, u(t))\mathrm{d}t$ is weakly continuous on E^α. Moreover, the following lemma implies that the functional $u \to -\int_0^T [({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))/2]\mathrm{d}t$ is convex and continuous on E^α.

Lemma 5.1. *Let $1/2 < \alpha \leq 1$ and assumption (H_1) be satisfied. If $u \in E^\alpha$, then the functional $H : E^\alpha \to \mathbf{R}^N$ denoted by*

$$H(u) = -\frac{1}{2} \int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))\mathrm{d}t$$

is convex and continuous on E^α.

Proof. The continuity follows from (34) and (29) directly. We are now in a position to prove the convexity of H.

Let $\lambda \in (0, 1)$, $u, v \in E^\alpha$ and $\tilde{u}, \tilde{v}$ be the extension of u and v by zero on $\mathbf{R}/[0, T]$ respectively. Since the Caputo fractional derivative operator is linear operator, we have by Remark 4.1 and (35) that

$$\begin{aligned}
H((1-\lambda)u + \lambda v) &= -\frac{1}{2} \int_0^T ({}_0^cD_t^\alpha((1-\lambda)u(t) + \lambda v(t)), {}_t^cD_T^\alpha((1-\lambda)u(t) + \lambda v(t)))\mathrm{d}t \\
&= -\frac{1}{2} \int_0^T ({}_0D_t^\alpha((1-\lambda)u(t) + \lambda v(t)), {}_tD_T^\alpha((1-\lambda)u(t) + \lambda v(t)))\mathrm{d}t \\
&= -\frac{1}{2} \int_{-\infty}^{\infty} ({}_{-\infty}D_t^\alpha((1-\lambda)\tilde{u}(t) + \lambda \tilde{v}(t)), {}_tD_\infty^\alpha((1-\lambda)\tilde{u}(t) + \lambda \tilde{v}(t)))\mathrm{d}t \\
&= \frac{|\cos(\pi\alpha)|}{2} \int_{-\infty}^{\infty} |{}_{-\infty}D_t^\alpha((1-\lambda)\tilde{u}(t) + \lambda \tilde{v}(t))|^2\mathrm{d}t \\
&\leq \frac{|\cos(\pi\alpha)|}{2} \int_{-\infty}^{\infty} [(1-\lambda)|{}_{-\infty}D_t^\alpha \tilde{u}(t)|^2 + \lambda |{}_{-\infty}D_t^\alpha \tilde{v}(t)|^2]\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \left[-\frac{1-\lambda}{2} ({}_{-\infty}D_t^\alpha \tilde{u}(t), {}_tD_\infty^\alpha \tilde{u}(t)) - \frac{\lambda}{2} ({}_{-\infty}D_t^\alpha \tilde{v}(t), {}_tD_\infty^\alpha \tilde{v}(t)) \right] \mathrm{d}t \\
&= \int_0^T \left[-\frac{1-\lambda}{2} ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)) - \frac{\lambda}{2} ({}_0^cD_t^\alpha v(t), {}_t^cD_T^\alpha v(t)) \right] \mathrm{d}t \\
&= (1-\lambda)H(u) + \lambda H(v),
\end{aligned}$$

which implies that H is a convex functional defined on E^α. The proof is complete. □

According to the arguments above, if φ is coercive, by Theorem A, φ has a minimum so that BVP (1) is solvable. It remains to find conditions under which φ is coercive on E^α, i.e. $\lim_{\|u\|_\alpha \to \infty} \varphi(u) = +\infty$, for $u \in E^\alpha$. We shall see that it suffices to require that $F(t, x)$ is bounded by a function for a.e. $t \in [0, T]$ and all $x \in \mathbf{R}^N$.

Theorem 5.1. *Let $\alpha \in (1/2, 1]$ and assume that F satisfies condition (H_1). If*

$$|F(t,x)| \le \bar{a}|x|^2 + \bar{b}(t)|x|^{2-\gamma} + \bar{c}(t), \quad t \in [0,T],\ x \in \mathbf{R}^N, \tag{47}$$

where $\bar{a} \in [0, |\cos(\pi\alpha)|\Gamma^2(\alpha+1)/2T^{2\alpha})$, $\gamma \in (0,2)$, $\bar{b} \in L^{2/\gamma}([0,T],\mathbf{R})$, and $\bar{c} \in L^1([0,T],\mathbf{R})$, then BVP (1) has at least one solution which minimizes φ on E^α.

Proof. According to the arguments above, Our problem reduces to prove that φ is coercive on E^α. For $u \in E^\alpha$, it follows from (34), (47) and (25) that

$$\begin{aligned}
\varphi(u) &= -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))\mathrm{d}t - \int_0^T F(t,u(t))\mathrm{d}t \\
&\ge \frac{|\cos(\pi\alpha)|}{2}\int_0^T |{}_0^cD_t^\alpha u(t)|^2\mathrm{d}t - \bar{a}\int_0^T |u(t)|^2\mathrm{d}t - \int_0^T \bar{b}(t)|u(t)|^{2-\gamma}\mathrm{d}t - \int_0^T \bar{c}(t)\mathrm{d}t \\
&= \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \bar{a}\|u\|_{L^2}^2 - \int_0^T \bar{b}(t)|u(t)|^{2-\gamma}\mathrm{d}t - \bar{c}_1 \\
&\ge \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \bar{a}\|u\|_{L^2}^2 - \left(\int_0^T |\bar{b}(t)|^{2/\gamma}\mathrm{d}t\right)^{\gamma/2}\left(\int_0^T |u(t)|^2\mathrm{d}t\right)^{1-\gamma/2} - \bar{c}_1 \\
&= \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \bar{a}\|u\|_{L^2}^2 - \bar{b}_1\|u\|_{L^2}^{2-\gamma} - \bar{c}_1 \\
&\ge \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \frac{\bar{a}T^{2\alpha}}{\Gamma^2(\alpha+1)}\|u\|_\alpha^2 - \bar{b}_1\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^{2-\gamma}\|u\|_\alpha^{2-\gamma} - \bar{c}_1 \\
&= \left(\frac{|\cos(\pi\alpha)|}{2} - \frac{\bar{a}T^{2\alpha}}{\Gamma^2(\alpha+1)}\right)\|u\|_\alpha^2 - \bar{b}_1\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^{2-\gamma}\|u\|_\alpha^{2-\gamma} - \bar{c}_1,
\end{aligned}$$

where $\bar{b}_1 = (\int_0^T |\bar{b}(t)|^{2/\gamma}\mathrm{d}t)^{\gamma/2}$ and $\bar{c}_1 = \int_0^T \bar{c}(t)\mathrm{d}t$.

Noting that $\bar{a} \in [0, |\cos(\pi\alpha)|\Gamma^2(\alpha+1)/2T^{2\alpha})$ and $\gamma \in (0,2)$, we have

$$\varphi(u) = +\infty \quad \text{as } \|u\|_\alpha \to \infty,$$

and hence φ is coercive, which completes the proof. □

Our task is now to use Theorem B (Mountain pass theorem) to find a nonzero critical point of functional φ on E^α.

Theorem 5.2. *Let $\alpha \in (1/2, 1]$ and suppose that F satisfies condition (H_1). If*

(H_2) *$F \in C([0,T]\times\mathbf{R}^N,\mathbf{R})$ and there exists $\mu \in [0,1/2)$ and $M > 0$ such that $0 < F(t,x) \le \mu(\nabla F(t,x),x)$ for all $x \in \mathbf{R}^N$ with $|x| \ge M$ and $t \in [0,T]$,*

(H_3) *$\limsup_{|x|\to 0} F(t,x)/|x|^2 < \Gamma^2(\alpha+1)/2T^{2\alpha}$ uniformly for $t \in [0,T]$ and $x \in \mathbf{R}^N$*

are satisfied, then BVP (1) has at least one nonzero solution on E^α.

Proof. We will verify that φ satisfies all conditions of Theorem B.

First, we will prove that φ satisfies the P.S. condition. Since $F(t,x) - \mu(\nabla F(t,x),x)$ is continuous for $t \in [0,T]$ and $|x| \le M$, there exists $c \in \mathbf{R}^+$, such that

$$F(t,x) \le \mu(\nabla F(t,x),x) + c, \quad t \in [0,T],\ |x| \le M.$$

By assumption (H_2), we obtain

$$F(t,x) \le \mu(\nabla F(t,x),x) + c, \quad t \in [0,T],\ x \in \mathbf{R}^N. \tag{48}$$

Let $\{u_k\} \subset E^\alpha$, $|\varphi(u_k)| \le K$, $k = 1,2,\ldots$, $\varphi'(u_k) \to 0$. Notice that

$$\langle \varphi'(u_k), u_k\rangle = -\int_0^T [({}_0^cD_t^\alpha u_k(t), {}_t^cD_T^\alpha u_k(t)) + (\nabla F(t,u_k(t)), u_k(t))]\mathrm{d}t \tag{49}$$

It follows from (48), (49) and (34) that

$$\begin{aligned}
K \ge \varphi(u_k) &= -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u_k(t), {}_t^cD_T^\alpha u_k(t))\mathrm{d}t - \int_0^T F(t,u_k(t))\mathrm{d}t \\
&\ge -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u_k(t), {}_t^cD_T^\alpha u_k(t))\mathrm{d}t - \mu\int_0^T (\nabla F(t,u_k(t)), u_k(t))\mathrm{d}t - cT
\end{aligned}$$

$$= \left(\mu - \frac{1}{2}\right) \int_0^T ({}_0^cD_t^\alpha u_k(t), {}_t^cD_T^\alpha u_k(t))dt + \mu\langle\varphi'(u_k), u_k\rangle - cT$$
$$\geq \left(\frac{1}{2} - \mu\right) |\cos(\pi\alpha)|\|u_k\|_\alpha^2 - \mu\|\varphi'(u_k)\|_\alpha\|u_k\|_\alpha - cT, \quad k = 1, 2, \ldots.$$

Since $\varphi'(u_k) \to 0$, there exists $N_0 \in \mathbf{N}$ such that

$$K \geq \left(\frac{1}{2} - \mu\right) |\cos(\pi\alpha)|\|u_k\|_\alpha^2 - \|u_k\|_\alpha - cT, \quad k > N_0,$$

and this implies that $\{u_k\} \subset E^\alpha$ is bounded. Since E^α is a reflexive space, going to a subsequence if necessary, we may assume that $u_k \rightharpoonup u$ weakly in E^α, thus we have

$$\begin{aligned}\langle\varphi'(u_k) - \varphi'(u), u_k - u\rangle &= \langle\varphi'(u_k), u_k - u\rangle - \langle\varphi'(u), u_k - u\rangle \\ &\leq \|\varphi'(u_k)\|_\alpha\|u_k - u\|_\alpha - \langle\varphi'(u), u_k - u\rangle \to 0,\end{aligned} \tag{50}$$

as $k \to \infty$. Moreover, according to (26) and Proposition 3.3, we have u_k is bounded in $C([0, T], \mathbf{R}^N)$ and $\|u_k - u\|_\infty = 0$ as $k \to \infty$. Hence, we have

$$\int_0^T \nabla F(t, u_k(t))dt \to \int_0^T \nabla F(t, u(t))dt \quad \text{as } k \to \infty. \tag{51}$$

Noting that

$$\begin{aligned}\langle\varphi'(u_k) - \varphi'(u), u_k - u\rangle &= -\int_0^T ({}_0^cD_t^\alpha(u_k(t) - u(t)), {}_t^cD_T^\alpha(u_k(t) - u(t)))dt \\ &\quad - \int_0^T (\nabla F(t, u_k(t)) - \nabla F(t, u(t)))(u_k(t) - u(t))dt \\ &\geq |\cos(\pi\alpha)|\|u_k - u\|_\alpha^2 - \left|\int_0^T (\nabla F(t, u_k(t)) - \nabla F(t, u(t)))dt\right| \|u_k - u\|_\infty.\end{aligned}$$

Combining (50) and (51), it is easy to verify that $\|u_k - u\|_\alpha^2 \to 0$ as $k \to \infty$, and hence that $u_k \to u$ in E_α. Thus, we obtain the desired convergence property.

From $\limsup_{|x|\to 0} F(t, x)/|x|^2 < \Gamma^2(\alpha + 1)/2T^{2\alpha}$ uniformly for $t \in [0, T]$, there exist $\epsilon \in (0, |\cos(\pi\alpha)|)$ and $\delta > 0$ such that $F(t, x) \leq (|\cos(\pi\alpha)| - \epsilon)(\Gamma^2(\alpha + 1)/2T^{2\alpha})|x|^2$ for all $t \in [0, T]$ and $x \in \mathbf{R}^N$ with $|x| \leq \delta$.

Let $\rho = \frac{\Gamma(\alpha)((\alpha-1)/2+1)^{1/2}}{T^{\alpha-1/2}}\delta$ and $\sigma = \epsilon\rho^2/2 > 0$. Then it follows from (26) that

$$\|u\|_\infty \leq \frac{T^{\alpha-1/2}}{\Gamma(\alpha)((\alpha - 1)/2 + 1)^{1/2}}\|u\|_\alpha = \delta$$

for all $u \in E^\alpha$ with $\|u\|_\alpha = \rho$. Therefore, we have

$$\begin{aligned}\varphi(u) &= -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))dt - \int_0^T F(t, u(t))dt \\ &\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - (|\cos(\pi\alpha)| - \epsilon)\frac{\Gamma^2(\alpha + 1)}{2T^{2\alpha}}\int_0^T |u(t)|^2dt \\ &\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \frac{1}{2}(|\cos(\pi\alpha)| - \epsilon)\|u\|_\alpha^2 \\ &= \frac{1}{2}\epsilon\|u\|_\alpha^2 \\ &= \sigma\end{aligned}$$

for all $u \in E^\alpha$ with $\|u\|_\alpha = \rho$. This implies that (ii) in Theorem B is satisfied.

It is obvious from the definition of φ and (H_3) that $\varphi(0) = 0$, and therefore, it suffices to show that φ satisfies (iii) in Theorem B.

Since $0 < F(t, x) \leq \mu(\nabla F(t, x), x)$ for all $x \in \mathbf{R}^N$ and $|x| \geq M$, a simple regularity argument then shows that there exists $r_1, r_2 > 0$ such that

$$F(t, x) \geq r_1|x|^{1/\mu} - r_2, \quad x \in \mathbf{R}^N,\ t \in [0, T].$$

For any $u \in E^\alpha$ with $u \neq 0$, $\kappa > 0$ and noting that $\mu \in [0, 1/2)$ and (34), we have

$$\begin{aligned}\varphi(\kappa u) &= -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha \kappa u(t), {}_t^cD_T^\alpha \kappa u(t))\mathrm{d}t - \int_0^T F(t, \kappa u(t))\mathrm{d}t \\ &\le \frac{\kappa^2}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - r_1\int_0^T |\kappa u(t)|^{1/\mu}\mathrm{d}t + r_2 T \\ &= \frac{\kappa^2}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - r_1\kappa^{1/\mu}\|u\|_{L^{1/\mu}}^{1/\mu} + r_2 T \to -\infty\end{aligned}$$

as $\kappa \to \infty$. Then there exists a sufficiently large κ_0 such that $\varphi(\kappa_0 u) \le 0$. Hence (iii) holds.

Finally, noting that $\varphi(0) = 0$ while for our critical point u, $\varphi(u) \ge \sigma > 0$. Hence u is a nontrivial weak solution of BVP (1), and this completes the proof. □

Corollary 5.2. $\forall \alpha \in (1/2, 1]$, *suppose that* F *satisfies conditions* (H_1) *and* (H_2). *If*

$(H_3)'$ $F(t, x) = o(|x|^2)$, *as* $|x| \to 0$ *uniformly for* $t \in [0, T]$ *and* $x \in \mathbf{R}^N$

is satisfied, then BVP (1) *has at least one nonzero solution on* E^α.

References

[1] K. Diethelm, A.D. Freed, On the solution of nonlinear fractional order differential equations used in the modeling of viscoelasticity, in: F. Keil, W. Mackens, H. Voss, J. Werther (Eds.), Scientific Computing in Chemical Engineering II—Computational Fluid Dynamics, Reaction Engineering and Molecular Properties, Springer-Verlag, Heidelberg, 1999, pp. 217–224.
[2] B.N. Lundstrom, M.H. Higgs, W.J. Spain, A.L. Fairhall, Fractional differentiation by neocortical pyramidal neurons, Nat. Neurosci. 11 (2008) 1335–1342.
[3] W.G. Glockle, T.F. Nonnenmacher, A fractional calculus approach of self-similar protein dynamics, Biophys. J. 68 (1995) 46–53.
[4] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
[5] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer-Verlag, Wien, 1997, pp. 291–348.
[6] J.W. Kirchner, X. Feng, C. Neal, Fractal streamchemistry and its implications for contaminant transport in catchments, Nature 403 (2000) 524–526.
[7] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.
[8] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, 1993.
[9] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[10] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integral and Derivatives: Theory and Applications, Gordon and Breach, Longhorne, PA, 1993.
[11] D.A. Benson, R. Schumer, M.M. Meerschaert, S.W. Wheatcraft, Fractional dispersion, Lévy motion, and the MADE tracer test, Transp. Porous Media 42 (2001) 211–240.
[12] D.A. Benson, S.W. Wheatcraft, M.M. Meerschaert, Application of a fractional advection–dispersion equation, Water Resour. Res. 36 (2000) 1403–1412.
[13] D.A. Benson, S.W. Wheatcraft, M.M. Meerschaert, The fractional-order governing equation of Lévy motion, Water Resour. Res. 36 (2000) 1413–1423.
[14] S. Lu, F.J. Molz, G.J. Fix, Possible problems of scale dependency in applications of the three-dimensional fractional advection–dispersion equation to natural porous media, Water Resour. Res. 38 (2002) 1165–1171.
[15] G.J. Fix, J.P. Roop, Least squares finite-element solution of a fractional order two-point boundary value problem, Comput. Math. Appl. 48 (2004) 1017–1033.
[16] V.J. Erwin, J.P. Roop, Variational formulation for the stationary fractional advection dispersion equation, Numer. Methods Partial Differential Equations 22 (2006) 58–76.
[17] J. Cresson, Inverse problem of fractional calculus of variations for partial differential equations, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 987–996.
[18] M. Belmekki, J.J. Nieto, R. Rodriguez-Lopez, Existence of periodic solution for a nonlinear fractional differential equation, Bound. Value Probl. (2009) Art. ID 324561, 18 pages.
[19] M.S. Tavazoei, M. Haeri, A proof for non existence of periodic solutions in time invariant fractional order systems, Automatica 45 (2009) 1886–1890.
[20] V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, Nonlinear Anal. TMA 69 (2008) 2677–2682.
[21] J. Vasundhara Devi, V. Lakshmikantham, Nonsmooth analysis and fractional differential equations, Nonlinear Anal. TMA 70 (2009) 4151–4157.
[22] G. Zaslavsky, Fractional kinetic equation for Hamiltonian chaos, chaotic advection, tracer dynamics and turbulent dispersion, Physica D 76 (1994) 110–122.
[23] Yong Zhou, Feng Jiao, Jing Li, Existence and uniqueness for p-type fractional neutral differential equations, Nonlinear Anal. TMA 71 (2009) 2724–2733.
[24] Yong Zhou, Feng Jiao, Jing Li, Existence and uniqueness for fractional neutral differential equations with infinite delay, Nonlinear Anal. TMA 71 (2009) 3249–3256.
[25] Yong Zhou, Feng Jiao, Nonlocal Cauchy problem for fractional evolution equations, Nonlinear Anal. RWA 11 (2010) 4465–4475.
[26] JinRong Wang, Yong Zhou, A class of fractional evolution equations and optimal controls, Nonlinear Anal. RWA 12 (2011) 262–272.
[27] R.P. Agarwal, M. Benchohra, S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, Acta Appl. Math. 109 (2010) 973–1033.
[28] M. Benchohra, S. Hamani, S.K. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, Nonlinear Anal. TMA 71 (2009) 2391–2396.
[29] Shuqin Zhang, Positive solutions to singular boundary value problem for nonlinear fractional differential equation, Comput. Math. Appl. 59 (2010) 1300–1309.
[30] B. Ahmad, J.J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, Comput. Math. Appl. 58 (2009) 1838–1843.
[31] J. Mawhin, M. Willem, Critical Point Theorey and Hamiltonian Systems, Springer, New York, 1989.
[32] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, in: CBMS, vol. 65, American Mathematical Society, 1986.
[33] Fuyi Li, Zhangping Liang, Qi Zhang, Existence of solutions to a class of nonlinear second order two-point boundary value problems, J. Math. Anal. Appl. 312 (2005) 357–373.
[34] J.-N. Corvellec, V.V. Motreanu, C. Saccon, Doubly resonant semilinear elliptic problems via nonsmooth critical point theory, J. Differential Equations 248 (2010) 2064–2091.

[35] Chun-Lei Tang, Xing-Ping Wu, Some critical point theorems and their applications to periodic solution for second order Hamiltonian systems, J. Differential Equations 248 (2010) 660–692.
[36] F. Riewe, Nonconservative Lagrangian and Hamiltonian mechanics, Phys. Rev. E 53 (1996) 1890–1899.
[37] M. Klimek, Lagrangian and Hamiltonian fractional sequential mechanics, Czech. J. Phys. 52 (2002) 1247–1253.
[38] O.P. Agrawal, Formulation of Euler–Lagrange equations for fractional variational problems, J. Math. Anal. Appl. 272 (2002) 368–379.
[39] E.M. Rabei, K.I. Nawafleh, R.S. Hijjawi, S.I. Muslih, D. Baleanu, The Hamilton formalism with fractional derivatives, J. Math. Anal. Appl. 327 (2007) 891–897.
[40] D. Baleanu, Ali Khalili Golmankaneh, Alireza Khalili Golmankaneh, The dual action of fractional multi time Hamilton equations, Internat. J. Theoret. Phys. 48 (2009) 2558–2569.
[41] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.