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Positive solutions and eigenvalue intervals of nonlocal boundary value problems with delays ☆

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Abstract

The paper is concerned with the delay differential equation $u'' + \lambda b(t) f(u(t-\tau)) = 0$ satisfying $u(t) = 0$ for $-\tau \leqslant t \leqslant 0$ and $u(1) = g(\int_0^1 u(t)\,d\beta(t))$, where $\int_0^1 u(t)\,d\beta(t)$ denotes the Riemann–Stieltjes integral. By applying the fixed point theorem in cones, we show the relationship between the asymptotic behaviors of the quotient $\frac{f(u)}{u}$ (at zero and infinity) and the open intervals (eigenvalue intervals) of the parameter λ such that the problem has zero, one and two positive solution(s). If $g(t) = t$, by using a property of the Riemann–Stieltjes integral, the above nonlocal boundary value problem educes a three-point boundary value problem with delay, for which some similar results are established.

Keywords: Positive solution; Nonlocal boundary value problem; Delay differential equation; Fixed point; Cone

1. Introduction

In this paper we consider the following nonlocal boundary value problem of nonlinear delay differential equation

$$u'' + \lambda b(t) f\big(u(t-\tau)\big) = 0, \quad 0 < t < 1, \tag{1.1}$$

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$$\begin{cases} u(t) = 0, & -\tau \leqslant t \leqslant 0, \\ u(1) = g(\int_0^1 u(t)\, d\beta(t)), \end{cases} \tag{1.2}$$

where λ is a positive real parameter, and $\int_0^1 u(t)\, d\beta(t)$ denotes the Riemann–Stieltjes integral. We assume that

(A_1) β is an increasing nonconstant function defined on $[0, 1]$ with $\beta(0) = 0$;
(A_2) $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $g_M := \max\{g(t)\colon 0 \leqslant t \leqslant \beta(1)\} < 1$;
(A_3) $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $f(u) > 0$ for all $u > 0$;
(A_4) $0 < \tau < \frac{1}{2}$;
(A_5) $b \in C((0, 1), \mathbb{R}^+)$ and $\int_0^1 s(1 - s)b(s)\, ds < \infty$, and that there is $\theta \in (\tau, \frac{1}{2})$ such that $\int_\theta^{1-\theta} b(s)\, ds > 0$.

Note that (A_5) allows $b(t)$ to have a singularity at $t = 0$ and/or $t = 1$, and allows for $b(t) \equiv 0$ on some subinterval(s) of $[0, 1]$, such as

$$b(t) = t^{-1}(1 - t)^{-1}\left(\left|\ln\left(\frac{1}{2} + t\right)\right| \pm \ln\left(\frac{1}{2} + t\right)\right).$$

For the case that $\tau = 0$, the problem (1.1)–(1.2) is related to the nonlocal boundary value problem of ordinary differential equation. Nonlocal BVPs of ordinary differential equations arise in a variety of areas of applied mathematics and physics (see [1,2]). In recent years, more and more papers were devoted to deal with the existence of positive solutions of nonlocal BVPs since the existence problem of solutions for a linear nonlocal BVP had been studied for the first time by Il'in and Moiseev [3] in 1987. We refer the reader to [4–8] and the references therein.

However, there are relatively rare existence results of positive solutions for nonlocal BVPs of second-order differential equations with delays. The BVPs for second-order delay equations arise in many areas of applied mathematics, physics and variational problems of control theory (see [9]). Recently, local BVPs of second-order delay differential equations have received a lot of attention accompanied by the development of the theory of functional differential equations, see, for example [10–14], and the references therein. Therefore, in Section 2 of this paper we consider the positive solutions of the singular nonlocal boundary value problem (1.1)–(1.2). Our interest is the relationship between the asymptotic behaviors of the quotient of $\frac{f(u)}{u}$ (at zero and infinity) and the open intervals (eigenvalue intervals), which are correlative with delay τ, such that (1.1)–(1.2) has zero, one and two positive solution(s).

If $g(t) = t$, $t \in \mathbb{R}^+$, the nonlocal boundary value problem (1.1)–(1.2) educes a three-point boundary value problem with delay by applying the following well-known property of the Riemann–Stieltjes integral.

Lemma 1.1. *Assume that*

(1) *$u(t)$ is a bounded function valued on $[a, b]$, i.e., there exist $c, C \in \mathbb{R}$ such that $c \leqslant u(t) \leqslant C$, $\forall t \in [a, b]$;*
(2) *$\beta(t)$ is increasing on $[a, b]$;*
(3) *the Riemann–Stieltjes integral $\int_a^b u(t)\, d\beta(t)$ exists.*

Then there is a number $\nu \in \mathbb{R}$ with $c \leqslant \nu \leqslant C$ such that $\int_a^b u(t)\, d\beta(t) = \nu(\beta(b) - \beta(a))$.

For any continuous solution $u(t)$ of (1.1)–(1.2), by Lemma 1.1, there exists $\eta \in (0,1)$ such that

$$\int_0^1 u(t)\,d\beta(t) = u(\eta)\big(\beta(1) - \beta(0)\big) = u(\eta)\beta(1).$$

Let $\sigma = \beta(1)$ and $g(t) = t$, $\forall t \in \mathbb{R}^+$. Then the problem (1.1)–(1.2) can be rewritten as the following three-point boundary value problem of delay differential equation

$$u'' + \lambda b(t) f\big(u(t-\tau)\big) = 0, \quad 0 < t < 1, \tag{1.3}$$

$$\begin{cases} u(t) = 0, & -\tau \leqslant t \leqslant 0, \\ \sigma u(\eta) = u(1), & \eta \in (0,1). \end{cases} \tag{1.4}$$

In 1999, by using a fixed point theorem in cones, R. Ma [15] initiated the study of positive solutions for the problem (1.3)–(1.4) with $\lambda = 1$ and $\tau = 0$, in which f is superlinear or sublinear at zero and infinity and b is not singular. A key condition of discussing the existence of positive solutions for the three-point BVP (1.3)–(1.4) is put forward in [15], which is stated as follows:

(A_6) $0 < \sigma\eta < 1$.

Similar to the method of dealing with (1.1)–(1.2), in Section 3 of this paper, we establish the existence, no-existence and multiplicity of positive solutions for the problem (1.3)–(1.4). Our results extend and improve the results in [10,15].

The main tool of this paper is the following fixed point index theorem [16–18].

Lemma 1.2. *Let $X = (X, \|\cdot\|)$ be a Banach space and $K \subset X$ a cone. For $r > 0$, define $K_r = \{u \in K\colon \|u\| < r\}$. Assume that $T : \overline{K}_r \to K$ is a completely continuous operator such that $Tu \neq u$ for $u \in \partial K_r = \{u \in K\colon \|u\| = r\}$.*

(1) *If $\|Tu\| \geqslant \|u\|$ for $u \in \partial K_r$, then $i(T, K_r, K) = 0$.*
(2) *If $\|Tu\| \leqslant \|u\|$ for $u \in \partial K_r$, then $i(T, K_r, K) = 1$.*

Also, the concavity of solution of (1.1)–(1.2) (and (1.3)–(1.4)) is sufficiently used in the proofs of our main results. The following lemma can be easily proved by the concavity of $u(t)$ on $[a,b]$ (see [12]).

Lemma 1.3. *Assume that $u \in C[a,b]$ $(a < b)$ is a nonnegative and concave function with $u(a) = 0$, $u(b) \geqslant 0$. Then for any fixed number δ: $a < \delta < \frac{a+b}{2}$,*

$$u(t) \geqslant \frac{\delta - a}{b - a}\|u\|_{[a,b]}, \quad t \in [\delta, b + a - \delta].$$

In particular, if $a = 0$, $b = 1$ and $0 < \tau < \delta < \frac{1}{2}$, then

$$u(t-\tau) \geqslant (\delta - \tau)\|u\|_{[0,1]}, \quad t \in [\delta, 1-\delta].$$

Here $\|\cdot\|_{[a,b]}$ stands for the sup-norm of $C[a,b]$.

Remark 1.1. The ideas of this paper could be extended so that some similar results may be established for the following more general functional differential equation

$$u'' + \lambda b(t) f\big(u\big(h(t)\big)\big) = 0, \quad 0 < t < 1, \tag{1.5}$$

$$\begin{cases} u(t) = \mu(t), & -\tau_0 \leqslant t \leqslant 0, \\ u(1) = g(\int_0^1 u(t)\, d\beta(t)). \end{cases} \tag{1.6}$$

Here $\mu \in C[-\tau_0, 0]$ with $\mu(0) = 0$ and $\mu > 0$ on $[-\tau_0, 0)$, $h(t)$ is a real-valued continuous function, $h(t) \leqslant t$ with h having a unique zero τ on $[0, 1)$ such that $h < 0$ on $[0, \tau)$, $h > 0$ strictly increasing on $(\tau, 1]$, $\tau_0 = -\min_{t \in [0,1]} h(t)$. If $0 < \tau < \frac{1}{2}$ and $u \in C[0, 1]$ is a nonnegative concave function with $u(0) = 0$, $u(1) \geqslant 0$, then for $\forall \delta$: $0 < \tau < \delta < \frac{1}{2}$, one can prove that

$$u\big(h(t)\big) \geqslant h(\delta)\|u\|, \quad t \in [\delta, 1 - \delta].$$

$u(t)$ is called a positive solution of (1.1)–(1.2) or (1.3)–(1.4) if it satisfies that

(1) $u \in C[-\tau, 1] \cap C^2(0, 1)$;
(2) $u(t) > 0$ for all $t \in (0, 1)$ and satisfies (1.2) or (1.4), respectively;
(3) $u'' = -\lambda b(t) f(u(t - \tau))$ for $t \in (0, 1)$.

2. Positive solutions of (1.1)–(1.2)

Let

$$X = \big\{u \in C[-\tau, 1]\colon u(t) = 0, \ \forall t \in [-\tau, 0]\big\}$$

with norm $\|\cdot\|$ given by $\|u\| = \sup\{|u(t)|\colon -\tau \leqslant t \leqslant 1\}$. Then $(X, \|\cdot\|)$ is a Banach space. It is obvious that $\|u\|_{[0,1]} = \|u\|$ for $u \in X\colon u \geqslant 0$. Here $\|\cdot\|_{[0,1]}$ stands for the sup-norm of $C[0, 1]$. Define K to be a cone in X by

$$K = \big\{u \in X\colon u(t) \text{ is concave and nonnegative on } [0, 1]\big\}.$$

Let $T_\lambda : K \to X$ be a map defined by

$$T_\lambda u(t) = \begin{cases} 0, & -\tau \leqslant t \leqslant 0, \\ \lambda \int_0^1 G(t, s) b(s) f(u(s - \tau))\, ds + g(\int_0^1 u(s)\, d\beta(s)) t, & 0 \leqslant t \leqslant 1. \end{cases}$$

It is easy to see that $T_\lambda(K) \subset K$. So the problem (1.1) and (1.2) is equivalent to the fixed point equation $T_\lambda u = u$, $u \in K$. Also, one can verify that T_λ is completely continuous by the Arzela–Ascoli theorem.

For convenience, denote that for $h \in C(\mathbb{R}^+, \mathbb{R}^+)$,

$$h^\alpha = \overline{\lim_{u \to \alpha}} \frac{h(u)}{u}, \quad h_\alpha = \underline{\lim_{u \to \alpha}} \frac{h(u)}{u}, \quad \alpha = 0^+, \infty,$$

$$\max\big\{f^\alpha\big\} = \max_{\alpha \in \{0^+, \infty\}} \big\{f^\alpha\big\}, \qquad \min\{f_\alpha\} = \min_{\alpha \in \{0^+, \infty\}} \{f_\alpha\},$$

and

$$M(r) = \max\big\{f(u) \bigm| 0 \leqslant u \leqslant r\big\}, \quad r > 0,$$

$$m(r) = \min\big\{f(u) \bigm| (\theta - \tau) r \leqslant u \leqslant r\big\}, \quad r > 0,$$

$$B_1 = \sup_{t \in [0,1]} \int_\theta^{1-\theta} G(t, s) b(s)\, ds, \qquad B_2 = \int_0^1 G(s, s) b(s)\, ds,$$

where $G(t,s)$ is given by

$$G(t,s)=\begin{cases}(1-t)s, & 0\leqslant s\leqslant t\leqslant 1,\\ t(1-s), & 0\leqslant t\leqslant s\leqslant 1.\end{cases}$$

Lemma 2.1. *Assume that* (A_1)–(A_5) *hold. Take* $r_0=1$. *Then*

(i) $i(T_\lambda, K_{r_0}, K)=0$ *for* $\lambda>\frac{1}{m(1)B_1}>0$,
(ii) $i(T_\lambda, K_{r_0}, K)=1$ *for* $0<\lambda<\frac{1-g_M}{M(1)B_2}$.

Proof. (i) For $u\in\partial K_{r_0}$, we have from Lemma 1.3 that

$$u(t-\tau)\geqslant(\theta-\tau)\|u\|,\quad t\in[\theta,1-\theta],$$

which implies that

$$(\theta-\tau)r_0\leqslant u(t-\tau)\leqslant r_0,\quad t\in[\theta,1-\theta],$$

and consequently that

$$f\big(u(t-\tau)\big)\geqslant m(r_0)=m(1),\quad t\in[\theta,1-\theta].$$

Thus we have that for $u\in\partial K_{r_0}$,

$$\begin{aligned}\|T_\lambda u\|&=\sup_{t\in[0,1]}\left\{\lambda\int_0^1 G(t,s)b(s)f\big(u(s-\tau)\big)\,ds+g\left(\int_0^1 u(s)\,d\beta(s)\right)t\right\}\\&\geqslant\lambda\sup_{t\in[0,1]}\int_0^1 G(t,s)b(s)f\big(u(s-\tau)\big)\,ds\\&\geqslant\lambda\sup_{t\in[0,1]}\int_\theta^{1-\theta} G(t,s)b(s)f\big(u(s-\tau)\big)\,ds\\&\geqslant\lambda\sup_{t\in[0,1]}\int_\theta^{1-\theta} G(t,s)b(s)m(1)\,ds\\&>1=\|u\|.\end{aligned}$$

It follows from Lemma 1.2 that $i(T_\lambda, K_{r_0}, K)=0$ for $\lambda>\frac{1}{m(1)B_1}>0$.

(ii) For $u\in\partial K_{r_0}$, we have

$$f\big(u(t-\tau)\big)\leqslant M(r_0)=M(1),\quad t\in[0,1],$$

and

$$0\leqslant\int_0^1 u(t)\,d\beta(t)\leqslant\beta(1).$$

Thus we have that for $u\in\partial K_{r_0}$,

$$\|T_\lambda u\| \leqslant \lambda \int_0^1 G(s,s)b(s)f\big(u(s-\tau)\big)\,ds + g\left(\int_0^1 u(s)\,d\beta(s)\right)$$
$$\leqslant \lambda M(1)\int_0^1 G(s,s)b(s)\,ds + g_M$$
$$< 1 = \|u\|.$$

It follows from Lemma 1.2 that $i(T_\lambda, K_{r_0}, K) = 1$ for $0 < \lambda < \frac{1-g_M}{M(1)B_2}$. □

For $h \in C(\mathbb{R}^+, \mathbb{R}^+)$, define $\tilde{h}(t): \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\tilde{h}(t) = \max\big\{h(u):\ 0 \leqslant u \leqslant t\big\}.$$

Lemma 2.2. *For $h \in C(\mathbb{R}^+, \mathbb{R}^+)$, if $h^\alpha < \infty$, $h_\alpha > 0$, then*

$$\tilde{h}^\alpha = h^\alpha, \quad \tilde{h}_\alpha = h_\alpha, \quad \alpha = 0^+, \infty.$$

Proof. One can find some similar results in [12,19]. This lemma can be similarly proved as the methods in [19]. □

Throughout this section, we assume that p_1, p_2 are two positive numbers satisfying $\frac{1}{p_1} + \frac{1}{p_2} \leqslant 1$.

Theorem 2.1. *Assume* (A$_1$)–(A$_5$) *hold.*

(1) *If $f^\alpha < \infty$ and $g^\alpha < \frac{1}{p_2\beta(1)}$ for $\alpha = 0^+$ or ∞, then* (1.1)–(1.2) *has at least one positive solution for $\lambda \in (\frac{1}{m(1)B_1}, \frac{1}{f^\alpha B_2 p_1})$.*
(2) *If $f^\alpha < \infty$ and $g^\alpha < \frac{1}{p_2\beta(1)}$ for $\alpha = 0^+$ and ∞, then* (1.1)–(1.2) *has at least two positive solutions for $\lambda \in (\frac{1}{m(1)B_1}, \frac{1}{\max\{f^\alpha\}B_2 p_1})$.*
(3) *If $f_\alpha > 0$ for $\alpha = 0^+$ or ∞, then* (1.1)–(1.2) *has at least one positive solution for $\lambda \in (\frac{1}{(\theta-\tau)f_\alpha B_1}, \frac{1-g_M}{M(1)B_2})$.*
(4) *If $f_\alpha > 0$ for $\alpha = 0^+$ and ∞, then* (1.1)–(1.2) *has at least two positive solutions for $\lambda \in (\frac{1}{(\theta-\tau)\min\{f_\alpha\}B_1}, \frac{1-g_M}{M(1)B_2})$.*
(5) *If $f^0 < \infty$, $g^0 < \frac{1}{p_2\beta(1)}$, and $f_\infty > 0$, then* (1.1)–(1.2) *has at least one positive solution for $\lambda \in (\frac{1}{(\theta-\tau)f_\infty B_1}, \frac{1}{f^0 B_2 p_1})$.*
(6) *If $f^\infty < \infty$, $g^\infty < \frac{1}{p_2\beta(1)}$, and $f_0 > 0$, then* (1.1)–(1.2) *has at least one positive solution for $\lambda \in (\frac{1}{(\theta-\tau)f_0 B_1}, \frac{1}{f^\infty B_2 p_1})$.*
(7) *If $f_\alpha > 0$ for $\alpha = 0^+$ and ∞, then* (1.1)–(1.2) *has no positive solution for sufficiently large $\lambda > 0$.*
(8) *If $f^\alpha < \infty$ for $\alpha = 0^+$ and ∞, and $g(t) \leqslant t$ for $t > 0$, then* (1.3)–(1.4) *has no positive solution for sufficiently small $\lambda > 0$.*

Proof. (1) First we consider the case: $\alpha = 0^+$. Take a sufficiently small positive number ϵ: $0 < \epsilon < \frac{1}{p_2\beta(1)}$ such that $g^0 \leqslant \frac{1}{p_2\beta(1)} - \epsilon$ and $0 < \lambda < \frac{1}{(f^0+\epsilon)B_2 p_1}$. Then there exists η_1: $0 < \eta_1 < 1$, such that

$$g(t) \leqslant \left(\frac{1}{p_2\beta(1)} - \epsilon\right)t \quad \text{for } 0 < t \leqslant \eta_1.$$

By $f^0 < \infty$, we have from Lemma 2.2 that there exists η_2: $0 < \eta_2 < 1$, such that

$$\tilde f(t) < \big(\tilde f^0 + \epsilon\big)t = \big(f^0 + \epsilon\big)t \quad \text{for } 0 < t \leqslant \eta_2.$$

Take $r_1 = \min\{\eta_2, \frac{\eta_1}{\beta(1)}\}$. Then for $u \in \partial K_{r_1}$, we have $\int_0^1 u(t)\,d\beta(t) \leqslant r_1\beta(1) \leqslant \eta_1$, and

$$g\left(\int_0^1 u(t)\,d\beta(t)\right) \leqslant \left(\frac{1}{p_2\beta(1)} - \epsilon\right)\int_0^1 u(t)\,d\beta(t) \leqslant \frac{1}{p_2\beta(1)}\beta(1)r_1 = \frac{r_1}{p_2}.$$

Thus we have that for $u \in \partial K_{r_1}$,

$$\begin{aligned}
\|T_\lambda u\| &\leqslant \lambda\int_0^1 G(s,s)b(s)f\big(u(s-\tau)\big)\,ds + g\left(\int_0^1 u(t)\,d\beta(t)\right)\\
&\leqslant \lambda\int_0^1 G(s,s)b(s)\tilde f(r_1)\,ds + \frac{r_1}{p_2}\\
&\leqslant \lambda\big(f^0+\epsilon\big)r_1\int_0^1 G(s,s)b(s)\,ds + \frac{r_1}{p_2}\\
&< \left(\frac{1}{p_1} + \frac{1}{p_2}\right)r_1 \leqslant \|u\|
\end{aligned}$$

for $0 < \lambda < \frac{1}{f^0 B_2 p_1}$. It follows from Lemma 1.2 that

$$i(T_\lambda, K_{r_1}, K) = 1 \quad (0 < r_1 < 1) \text{ for } 0 < \lambda < \frac{1}{f^0 B_2 p_1}. \tag{2.1}$$

For the case $\alpha = \infty$, we can take a sufficiently small positive ϵ: $0 < \epsilon < \frac{1}{p_2\beta(1)}$ such that $g^\infty \leqslant \frac{1}{p_2\beta(1)} - \epsilon$ and $0 < \lambda < \frac{1}{(f^\infty+\epsilon)B_2 p_1}$. Then there is $\hat R_1 > 0$ such that $g(t) \leqslant (\frac{1}{p_2\beta(1)} - \epsilon)t$ for $t \geqslant \hat R_1$. By $f^\infty < \infty$, we have from Lemma 2.2 that there exists $\hat R_2 > 0$ such that $\tilde f(t) \leqslant (f^\infty + \epsilon)t$ for $t \geqslant \hat R_2$. Take $r_2 > \max\{\hat R_2, \frac{\hat R_1}{\theta(\beta(1-\theta)-\beta(\theta))}\} + 1$. Then for $u \in \partial K_{r_2}$, we have from Lemma 1.3 that

$$\int_0^1 u(t)\,d\beta(t) \geqslant \int_\theta^{1-\theta} u(t)\,d\beta(t) \geqslant \theta\|u\|\big(\beta(1-\theta) - \beta(\theta)\big) > \hat R_1,$$

which implies that

$$g\left(\int_0^1 u(t)\,d\beta(t)\right) \leqslant \left(\frac{1}{p_2\beta(1)} - \epsilon\right)\int_0^1 u(t)\,d\beta(t) \leqslant \left(\frac{1}{p_2\beta(1)} - \epsilon\right)r_2\beta(1) \leqslant \frac{r_2}{p_2}.$$

Thus we have that for $u \in \partial K_{r_2}$,

$$\begin{aligned}\|T_\lambda u\| &\leqslant \lambda \int_0^1 G(s,s)b(s) f\big(u(s-\tau)\big)\,ds + \frac{r_2}{p_2}\\ &\leqslant \lambda \int_0^1 G(s,s)b(s)\tilde{f}(r_2)\,ds + \frac{r_2}{p_2}\\ &\leqslant \lambda\big(f^\infty + \epsilon\big)r_2 \int_0^1 G(s,s)b(s)\,ds + \frac{r_2}{p_2}\\ &< \left(\frac{1}{p_1} + \frac{1}{p_2}\right) r_2 \leqslant \|u\|\end{aligned}$$

for $0 < \lambda < \frac{1}{f^\infty B_2 p_1}$. It follows from Lemma 1.2 that

$$i(T_\lambda, K_{r_2}, K) = 1 \quad (r_2 > 1) \text{ for } 0 < \lambda < \frac{1}{f^\infty B_2 p_1}. \tag{2.2}$$

On the other hand, Lemma 2.1(i) shows that

$$i(T_\lambda, K_{r_0}, K) = 0 \quad (r_0 = 1) \text{ for } \lambda > \frac{1}{m(1)B_1}. \tag{2.3}$$

Therefore we have from the additivity of fixed point index that for the case $\alpha = 0^+$,

$$i(T_\lambda, K_{r_0} \backslash \overline{K}_{r_1}, K) = i(T_\lambda, K_{r_0}, K) - i(T_\lambda, K_{r_1}, K) = -1 \tag{2.4}$$

for $\frac{1}{m(1)B_1} < \lambda < \frac{1}{f^0 B_2 p_1}$, and that for the case $\alpha = \infty$,

$$i(T_\lambda, K_{r_2} \backslash \overline{K}_{r_0}, K) = i(T_\lambda, K_{r_2}, K) - i(T_\lambda, K_{r_0}, K) = 1 \tag{2.5}$$

for $\frac{1}{m(1)B_1} < \lambda < \frac{1}{f^\infty B_2 p_1}$. Thus we can conclude that for the case $\alpha = 0^+$, T_λ has a fixed point u in $K_{r_0} \backslash \overline{K}_{r_1}$ with $r_1 < \|u\| < r_0$, which implies that (1.1) and (1.2) has a positive solution for $\frac{1}{m(1)B_1} < \lambda < \frac{1}{f^0 B_2 p_1}$, and that for the case $\alpha = \infty$, T_λ has a fixed point u in $K_{r_2} \backslash \overline{K}_{r_0}$ with $r_0 < \|u\| < r_2$, which implies that (1.1)–(1.2) has a positive solution for $\frac{1}{m(1)B_1} < \lambda < \frac{1}{f^\infty B_2 p_1}$.

(2) From (2.1)–(2.3) and the additivity of fixed point index, we can make (2.4) and (2.5) hold simultaneously for $\frac{1}{m(1)B_1} < \lambda < \frac{1}{\max\{f^\alpha\} B_2 p_1}$. It follows that T_λ has a fixed point u_1 in $K_{r_0} \backslash \overline{K}_{r_1}$ and a fixed point u_2 in $K_{r_2} \backslash \overline{K}_{r_0}$, satisfying $r_1 < \|u_1\| < r_0 < \|u_2\| < r_2$. Consequently, (1.1)–(1.2) has two positive solutions for $\frac{1}{m(1)B_1} < \lambda < \frac{1}{\max\{f^\alpha\} B_2 p_1}$.

(3) First we consider the case: $\alpha = 0^+$. Take a sufficiently small positive number ϵ: $0 < \epsilon < f_0$, such that

$$\frac{1}{(\theta - \tau)(f_0 - \epsilon)B_1} < \lambda < \frac{1 - g_M}{M(1)B_2}.$$

By $f_0 > 0$, there is r_3: $0 < r_3 < 1$ such that $f(u) \geqslant (f_0 - \epsilon)u$ for $0 < u \leqslant r_3$. Then for $u \in \partial K_{r_3}$, we have

$$f\big(u(t-\tau)\big) \geqslant (f_0 - \epsilon)u(t-\tau), \quad t \in [\theta, 1-\theta].$$

Thus we get from Lemma 1.3 that for $u \in \partial K_{r_3}$,

$$\begin{aligned}
\|T_\lambda u\| &\geqslant \lambda \sup_{t\in[0,1]} \int_0^1 G(t,s)b(s)f\big(u(s-\tau)\big)\,ds \\
&\geqslant \lambda \sup_{t\in[0,1]} \int_\theta^{1-\theta} G(t,s)b(s)(f_0-\epsilon)u(s-\tau)\,ds \\
&\geqslant \lambda(\theta-\tau)(f_0-\epsilon) \sup_{t\in[0,1]} \int_\theta^{1-\theta} G(t,s)b(s)\,ds\|u\| \\
&> \|u\|
\end{aligned}$$

for $\lambda > \frac{1}{(\theta-\tau)f_0 B_1} > 0$. It follows from Lemma 1.2 that

$$i(T_\lambda, K_{r_3}, K) = 0 \quad (r_3 < 1) \text{ for } \lambda > \frac{1}{(\theta-\tau)f_0 B_1} > 0. \tag{2.6}$$

For the case: $\alpha = \infty$, one can take $0 < \epsilon < f_\infty$ such that

$$\frac{1}{(\theta-\tau)(f_\infty-\epsilon)B_1} < \lambda < \frac{1-g_M}{M(1)B_2}.$$

By $f_\infty > 0$, there is $\hat{R} > 1$ such that $f(u) \geqslant (f_\infty - \epsilon)u$ for $u \geqslant \hat{R}$. Take $r_4 > \frac{\hat{R}}{\theta-\tau}$. Then for $u \in \partial K_{r_4}$, we have from Lemma 1.3 that

$$u(t-\tau) \geqslant (\theta-\tau)\|u\| \geqslant \hat{R}, \quad t \in [\theta, 1-\theta],$$

which implies that for $u \in \partial K_{r_4}$,

$$f\big(u(t-\tau)\big) \geqslant (f_\infty-\epsilon)u(t-\tau), \quad t \in [\theta, 1-\theta].$$

Thus we have that for $u \in \partial K_{r_4}$,

$$\begin{aligned}
\|T_\lambda u\| &\geqslant \lambda \sup_{t\in[0,1]} \int_0^1 G(t,s)b(s)f\big(u(s-\tau)\big)\,ds \\
&\geqslant \lambda \sup_{t\in[0,1]} \int_\theta^{1-\theta} G(t,s)b(s)(f_\infty-\epsilon)u(s-\tau)\,ds \\
&\geqslant \lambda(\theta-\tau)(f_\infty-\epsilon) \sup_{t\in[0,1]} \int_\theta^{1-\theta} G(t,s)b(s)\,ds\|u\| \\
&> \|u\|
\end{aligned}$$

for $\lambda > \frac{1}{(\theta-\tau)f_\infty B_1} > 0$. It follows again from Lemma 1.2 that

$$i(T_\lambda, K_{r_4}, K) = 0 \quad (r_4 > 1) \text{ for } \lambda > \frac{1}{(\theta-\tau)f_\infty B_1} > 0. \tag{2.7}$$

On the other hand, Lemma 2.1(ii) shows that

$$i(T_\lambda, K_{r_0}, K) = 1 \quad (r_0 = 1) \text{ for } 0 < \lambda < \frac{1 - g_M}{M(1)B_2}. \tag{2.8}$$

Therefore we have from the additivity of fixed point index that for the case $\alpha = 0^+$,

$$i(T_\lambda, K_{r_0} \backslash \overline{K}_{r_3}, K) = i(T_\lambda, K_{r_0}, K) - i(T_\lambda, K_{r_3}, K) = 1 \tag{2.9}$$

for $\frac{1}{(\theta-\tau) f_0 B_1} < \lambda < \frac{1-g_M}{M(1)B_2}$, and that for the case $\alpha = \infty$,

$$i(T_\lambda, K_{r_4} \backslash \overline{K}_{r_0}, K) = i(T_\lambda, K_{r_4}, K) - i(T_\lambda, K_{r_0}, K) = -1 \tag{2.10}$$

for $\frac{1}{(\theta-\tau) f_\infty B_1} < \lambda < \frac{1-g_M}{M(1)B_2}$. Thus we can conclude that for the case $\alpha = 0^+$, T_λ has a fixed point u in $K_{r_0} \backslash \overline{K}_{r_3}$ with $r_3 < \|u\| < r_0$, which implies that (1.1)–(1.2) has a positive solution for $\frac{1}{(\theta-\tau) f_0 B_1} < \lambda < \frac{1-g_M}{M(1)B_2}$, and that for the case $\alpha = \infty$, T_λ has a fixed point u in $K_{r_4} \backslash \overline{K}_{r_0}$ with $r_0 < \|u\| < r_4$, which implies that (1.1)–(1.2) has a positive solution for $\frac{1}{(\theta-\tau) f_\infty B_1} < \lambda < \frac{1-g_M}{M(1)B_2}$.

(4) From (2.6)–(2.8) and the additivity of fixed point index, we can make (2.9) and (2.10) hold simultaneously for $\frac{1}{(\theta-\tau)\min\{f_\alpha\} B_1} < \lambda < \frac{1-g_M}{M(1)B_2}$. It follows that T_λ has a fixed point u_1 in $K_{r_0} \backslash \overline{K}_{r_3}$ and a fixed point u_2 in $K_{r_4} \backslash \overline{K}_{r_0}$, satisfying $r_3 < \|u_1\| < r_0 < \|u_2\| < r_4$. Consequently, (1.1)–(1.2) has two positive solutions for $\frac{1}{(\theta-\tau)\min\{f_\alpha\} B_1} < \lambda < \frac{1-g_M}{M(1)B_2}$.

(5) It can be proved by (2.1) and (2.7).

(6) It can be proved by (2.2) and (2.6).

(7) Let $0 < \epsilon < \min\{f_\alpha\}$. Choose $0 < r_5 < r_6$ such that

$$f(u) > (f_0 - \epsilon)u \quad \text{for } 0 \leqslant u \leqslant r_5,$$

and

$$f(u) > (f_\infty - \epsilon)u \quad \text{for } u \geqslant r_6.$$

Let

$$0 < \varrho < \min\left\{f_0 - \epsilon, f_\infty - \epsilon, \min\left\{\frac{f(u)}{u}\colon u \in \mathbb{R}^+,\ (\theta - \tau) r_5 \leqslant u \leqslant r_6\right\}\right\}.$$

Then we have that $f(u) > \varrho u$ for all $u \in \mathbb{R}^+$. Take

$$\lambda^* = \frac{1}{\varrho(\theta - \tau) B_1}.$$

Assume that v is a positive solution of (1.1)–(1.2) with $\lambda > \lambda^*$, then $v(t) = T_\lambda v(t)$ for $t \in [0, 1]$. If $\|v\| \leqslant r_5$, we have that

$$f\big(v(t-\tau)\big) \geqslant \varrho v(t-\tau), \quad t \in [\theta, 1-\theta].$$

If $\|v\| > r_5$, we have from Lemma 2.2 that

$$v(t-\tau) \geqslant (\theta - \tau)\|v\| > (\theta - \tau) r_5, \quad t \in [\theta, 1-\theta],$$

which implies that

$$f\big(v(t-\tau)\big) \geqslant \varrho v(t-\tau), \quad t \in [\theta, 1-\theta].$$

Thus we have that

$$\begin{aligned}\|v\| = \|T_\lambda v\| &\geqslant \lambda \sup_{t\in[0,1]} \int_0^1 G(t,s)b(s)f\bigl(v(s-\tau)\bigr)\,ds\\ &\geqslant \lambda \sup_{t\in[0,1]} \int_\theta^{1-\theta} G(t,s)b(s)f\bigl(v(s-\tau)\bigr)\,ds\\ &\geqslant \lambda \sup_{t\in[0,1]} \int_\theta^{1-\theta} G(t,s)b(s)\varrho v(s-\tau)\,ds\\ &\geqslant \lambda\varrho(\theta-\tau) \sup_{t\in[0,1]} \int_\theta^{1-\theta} G(t,s)b(s)\,ds\|v\|\\ &> \|v\|,\end{aligned}$$

which is a contradiction.

(8) Take $\epsilon > 0$. By Lemma 2.2 we have that $\tilde{f}^0 < \infty$, $\tilde{f}^\infty < \infty$. Choose $0 < r_7 < r_8$ such that $\tilde{f}(t) \leqslant (\tilde{f}^0 + \epsilon)t$ for $0 < t \leqslant r_7$, and $\tilde{f}(t) \leqslant (\tilde{f}^\infty + \epsilon)t$ for $t \geqslant r_8$. Let

$$\rho > \max\left\{\tilde{f}^0 + \epsilon,\ \tilde{f}^\infty + \epsilon,\ \max_{r_7\leqslant t\leqslant r_8} \frac{f(t)}{t}\right\} > 0.$$

Then $\tilde{f}(t) < \rho t$ for all $t > 0$. Take

$$\lambda^* = \frac{1-\beta(1)}{B_2\rho}.$$

Assume that the problem (1.1)–(1.2) has a positive solution v for $0 < \lambda < \lambda^*$, then $v(t) = T_\lambda v(t)$ for $t \in [0,1]$. Thus we have that for each $t \in [0,1]$,

$$\begin{aligned}\|v\| &\leqslant \lambda \int_0^1 s(1-s)b(s)f\bigl(v(s-\tau)\bigr)\,ds + g\left(\int_0^1 v(s)\,d\beta(s)\right)\\ &\leqslant \lambda \int_0^1 s(1-s)b(s)\tilde{f}\bigl(\|v\|\bigr)\,ds + \int_0^1 v(s)\,d\beta(s)\\ &\leqslant \lambda \int_0^1 s(1-s)b(s)\,ds\,\rho\|v\| + \beta(1)\|v\|\\ &< \|v\|,\end{aligned}$$

which is a contradiction.

The proof of Theorem 2.1 is completed. □

Remark 2.1. If we assume that

(A_2^*) $g \in C(\mathbb{R}^+, \mathbb{R}^+)$, $g(t) \leqslant t$ for $t > 0$, and $\beta(1) < 1$,

then by similar arguments we have the following result.

Theorem 2.2. *Assume that* (A_1), (A_2^*) *and* (A_3)–(A_5) *hold.*

(1) *If* $f^\alpha < \infty$ *for* $\alpha = 0^+$ *or* ∞, *then* (1.1)–(1.2) *has at least one positive solution for* $\lambda \in (\frac{1}{m(1)B_1}, \frac{1-\beta(1)}{f^\alpha B_2})$.
(2) *If* $f^\alpha < \infty$ *for* $\alpha = 0^+$ *and* ∞, *then* (1.1)–(1.2) *has at least two positive solutions for* $\lambda \in (\frac{1}{m(1)B_1}, \frac{1-\beta(1)}{\max\{f^\alpha\}B_2})$.
(3) *If* $f_\alpha > 0$ *for* $\alpha = 0^+$ *or* ∞, *then* (1.1)–(1.2) *has at least one positive solution for* $\lambda \in (\frac{1}{(\theta-\tau)f_\alpha B_1}, \frac{1-\beta(1)}{M(1)B_2})$.
(4) *If* $f_\alpha > 0$ *for* $\alpha = 0^+$ *and* ∞, *then* (1.1)–(1.2) *has at least two positive solutions for* $\lambda \in (\frac{1}{(\theta-\tau)\min\{f_\alpha\}B_1}, \frac{1-\beta(1)}{M(1)B_2})$.
(5) *If* $f^0 < \infty$ *and* $f_\infty > 0$, *then* (1.1)–(1.2) *has at least one positive solution for* $\lambda \in (\frac{1}{(\theta-\tau)f_\infty B_1}, \frac{1-\beta(1)}{f^0 B_2})$.
(6) *If* $f^\infty < \infty$ *and* $f_0 > 0$, *then* (1.1)–(1.2) *has at least one positive solution for* $\lambda \in (\frac{1}{(\theta-\tau)f_0 B_1}, \frac{1-\beta(1)}{f^\infty B_2})$.
(7) *If* $f_\alpha > 0$ *for* $\alpha = 0^+$ *and* ∞, *then* (1.1)–(1.2) *has no positive solution for sufficiently large* $\lambda > 0$.
(8) *If* $f^\alpha < \infty$ *for* $\alpha = 0^+$ *and* ∞, *then* (1.1)–(1.2) *has no positive solution for sufficiently small* $\lambda > 0$.

3. Positive solutions of (1.3)–(1.4)

In this section we consider the positive solutions of (1.3)–(1.4). The Banach space X and the cone $K \subset X$ are given by the forms in Section 2. The problem (1.3)–(1.4) is equivalent to the fixed point equation $T_\lambda u = u$, $u \in K$, where $T_\lambda : K \to X$ is defined by

$$T_\lambda u(t) = \begin{cases} 0, & -\tau \leqslant t \leqslant 0, \\ \lambda \int_0^1 G(t,s)b(s)f(u(s-\tau))\,ds + \sigma u(\eta)t, & 0 \leqslant t \leqslant 1. \end{cases} \tag{3.1}$$

The operator T_λ can be rewritten equivalently as the following form

$$T_\lambda u(t) = \begin{cases} 0, & -\tau \leqslant t \leqslant 0, \\ -\lambda \int_0^t (t-s)b(s)f(u(s-\tau))\,ds & \\ \quad - \frac{\sigma t}{1-\sigma\eta}\lambda \int_0^\eta (\eta-s)b(s)f(u(s-\tau))\,ds & \\ \quad + \frac{t}{1-\sigma\eta}\lambda \int_0^1 (1-s)b(s)f(u(s-\tau))\,ds, & 0 \leqslant t \leqslant 1. \end{cases} \tag{3.2}$$

Assume that

$$B_2^* = \int_0^1 (1-s)b(s)\,ds < \infty.$$

Lemma 3.1. *Assume that* (A_3)–(A_6) *hold. Take* $r_0 = 1$. *Then*

(i) $i(T_\lambda, K_{r_0}, K) = 0$ *for* $\lambda > \frac{1}{m(1)B_1} > 0$;
(ii) $i(T_\lambda, K_{r_0}, K) = 1$ *for* $0 < \lambda < \frac{1-\sigma\eta}{M(1)B_2^*}$.

Proof. We only show the proof of (ii). For $u \in \partial K_{r_0}$, since $f(u(t-\tau)) \leqslant M(r_0) = M(1)$, $t \in [0,1]$, we have by using the definition (3.2) of the operator T_λ that

$$\begin{aligned} T_\lambda u(t) &\leqslant \frac{1}{1-\sigma\eta}\lambda \int_0^1 (1-s)b(s)f\big(u(s-\tau)\big)\,ds \\ &\leqslant \frac{1}{1-\sigma\eta}\lambda \int_0^1 (1-s)b(s)\,ds M(1) \\ &< 1 = \|u\|, \quad t \in [0,1]. \end{aligned}$$

This implies that $\|T_\lambda u\| \leqslant \|u\|$, $u \in \partial K_{r_0}$. In view of Lemma 1.2, we have that $i(T_\lambda, K_{r_0}, K) = 1$ for $0 < \lambda < \frac{1-\sigma\eta}{M(1)B_2^*}$. □

Theorem 3.1. *Assume that* (A$_3$)–(A$_6$) *hold.*

(1) *If $f^\alpha < \infty$ for $\alpha = 0^+$ or ∞, then* (1.3)–(1.4) *has at least one positive solution for* $\lambda \in (\frac{1}{m(1)B_1}, \frac{1-\sigma\eta}{f^\alpha B_2^*})$.
(2) *If $f^\alpha < \infty$ for $\alpha = 0^+$ and ∞, then* (1.3)–(1.4) *has at least two positive solutions for* $\lambda \in (\frac{1}{m(1)B_1}, \frac{1-\sigma\eta}{\max\{f^\alpha\}B_2^*})$.
(3) *If $f_\alpha > 0$ for $\alpha = 0^+$ or ∞, then* (1.3)–(1.4) *has at least one positive solution for* $\lambda \in (\frac{1}{(\theta-\tau)f_\alpha B_1}, \frac{1-\sigma\eta}{M(1)B_2^*})$.
(4) *If $f_\alpha > 0$ for $\alpha = 0^+$ and ∞, then* (1.3)–(1.4) *has at least two positive solutions for* $\lambda \in (\frac{1}{(\theta-\tau)\min\{f_\alpha\}B_1}, \frac{1-\sigma\eta}{M(1)B_2^*})$.
(5) *If $f^0 < \infty$ and $f_\infty > 0$, then* (1.3)–(1.4) *has at least one positive solution for* $\lambda \in (\frac{1}{(\theta-\tau)f_\infty B_1}, \frac{1-\sigma\eta}{f^0 B_2^*})$.
(6) *If $f^\infty < \infty$ and $f_0 > 0$, then* (1.3)–(1.4) *has at least one positive solution for* $\lambda \in (\frac{1}{(\theta-\tau)f_0 B_1}, \frac{1-\sigma\eta}{f^\infty B_2^*})$.
(7) *If $f_\alpha > 0$ for $\alpha = 0^+$ and ∞, then* (1.3)–(1.4) *has no positive solution for sufficiently large* $\lambda > 0$.
(8) *If $f^\alpha < \infty$ for $\alpha = 0^+$ and ∞, then* (1.3)–(1.4) *has no positive solution for sufficiently small* $\lambda > 0$.

Proof. We only show the proofs of (1). (2)–(8) can be proved by the similar arguments of dealing with Theorem 2.1(2)–(8), respectively.

(1) First we consider the case: $\alpha = 0^+$. Take a sufficiently small positive number $\epsilon > 0$, such that $0 < \lambda < \frac{1-\sigma\eta}{(f^0+\epsilon)B_2^*}$. By $f^0 < \infty$, we have from Lemma 2.2 that there exists r_1: $0 < r_1 < 1$ such that

$$\tilde{f}(t) < \big(\tilde{f}^0 + \epsilon\big)t = \big(f^0 + \epsilon\big)t, \quad 0 < t \leqslant r_1.$$

Thus we have by using the definition (3.2) of the operator T_λ that for $u \in \partial K_{r_1}$,

$$\begin{aligned}
T_\lambda u(t) &\leqslant \frac{1}{1-\sigma\eta}\lambda\int_0^1 (1-s)b(s)f\big(u(s-\tau)\big)\,ds\\
&\leqslant \frac{1}{1-\sigma\eta}\lambda\int_0^1 (1-s)b(s)\tilde{f}(r_1)\,ds\\
&\leqslant \frac{1}{1-\sigma\eta}\lambda\big(f^0+\epsilon\big)r_1\int_0^1 (1-s)b(s)\,ds\\
&< r_1 = \|u\|.
\end{aligned}$$

It follows that $\|T_\lambda u\| \leqslant \|u\|$, $u \in \partial K_{r_1}$. In view of Lemma 1.2, we get that $i(T_\lambda, K_{r_1}, K) = 1$ $(0 < r_1 < 1)$ for $0 < \lambda < \frac{1-\sigma\eta}{f^0 B_2^*}$.

Now we consider the case: $\alpha = \infty$. Take a sufficiently small positive number $\epsilon > 0$ such that $0 < \lambda < \frac{1-\sigma\eta}{(f^\infty+\epsilon)B_2^*}$. By $f^\infty < \infty$, we have from Lemma 2.2 that there exists $r_2 > 1$ such that $\tilde{f}(t) \leqslant (f^\infty + \epsilon)t$, $t \geqslant r_2$. Then we have by using the definition (3.2) of the operator T_λ that for $u \in \partial K_{r_2}$,

$$\begin{aligned}
T_\lambda u(t) &\leqslant \frac{1}{1-\sigma\eta}\lambda\int_0^1 (1-s)b(s)f\big(u(s-\tau)\big)\,ds\\
&\leqslant \frac{1}{1-\sigma\eta}\lambda\int_0^1 (1-s)b(s)\tilde{f}(r_2)\,ds\\
&\leqslant \frac{1}{1-\sigma\eta}\lambda\big(f^\infty+\epsilon\big)r_2\int_0^1 (1-s)b(s)\,ds\\
&< r_2 = \|u\|,
\end{aligned}$$

which implies that $\|T_\lambda u\| \leqslant \|u\|$, $u \in \partial K_{r_2}$. Again by Lemma 1.2, we have $i(T_\lambda, K_{r_2}, K) = 1$ $(r_2 > 1)$ for $0 < \lambda < \frac{1-\sigma\eta}{f^\infty B_2^*}$.

On the other hand, Lemma 3.1(i) shows that $i(T_\lambda, K_{r_0}, K) = 0$ $(r_0 = 1)$ for $\lambda > \frac{1}{m(1)B_1}$. Thus we have from the additivity of fixed point index that for the case $\alpha = 0^+$,

$$i(T_\lambda, K_{r_0}\backslash\overline{K}_{r_1}, K) = i(T_\lambda, K_{r_0}, K) - i(T_\lambda, K_{r_1}, K) = -1,$$

for $\frac{1}{m(1)B_1} < \lambda < \frac{1-\sigma\eta}{f^0 B_2^*}$, and that for the case $\alpha = \infty$,

$$i(T_\lambda, K_{r_2}\backslash\overline{K}_{r_0}, K) = i(T_\lambda, K_{r_2}, K) - i(T_\lambda, K_{r_0}, K) = 1,$$

for $\frac{1}{m(1)B_1} < \lambda < \frac{1-\sigma\eta}{f^\infty B_2^*}$. Therefor for each case that $\alpha = 0^+$ and $\alpha = \infty$, the problem (1.3)–(1.4) has at least one positive solution for $\frac{1}{m(1)B_1} < \lambda < \frac{1-\sigma\eta}{f^0 B_2^*}$ and $\frac{1}{m(1)B_1} < \lambda < \frac{1-\sigma\eta}{f^\infty B_2^*}$, respectively. □

Remark 3.1. Clearly, Theorem 2.2 applies to the boundary value problem (1.3)–(1.4). Since the condition (A_6) allows of $\sigma \geqslant 1$, Theorem 3.1 is better than Theorem 2.2. However, as viewed from the length of interval of λ, neither is better than another.

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